Sigma Model of Near-Extreme Rotating Black Holes and Their Microstates

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Abstract

Five-dimensional non-extreme rotating black holes with large NS-NS five-brane and fundamental string charge are shown to be described by a conformal sigma model, which is a marginal integrable deformation of six-dimensional $SL(2,R) \times SU(2)$ WZW model. The two WZW levels are equal to the five-brane charge, while the parameters of the two marginal deformations generated by the left and right chiral $SU(2)$ currents are proportional to the two angular momentum components of the black hole. The near-horizon description is effectively in terms of a free fundamental string whose tension is rescaled by the five-brane charge. The microstates are identified with those of left and right moving superconformal string oscillations in the four directions transverse to the five-brane. Their statistical entropy reproduces precisely the Bekenstein-Hawking entropy of the rotating black hole.
I. INTRODUCTION

String theory, as a quantum theory of gravity, should provide a framework within which one should be able to address quantum aspects of strong gravitational fields, in particular the issue of black hole information loss and related issues of microscopic structure of black holes. While the information loss is still an open unresolved problem, an important progress has been made in clarifying the black hole microscopics. In the days of perturbative string physics Sen made a pioneering proposal [1] to identify the microstates of extreme (BPS-saturated) electrically charged black holes with perturbative excitations of string theory. This idea was clarified and put on a firmer ground in [2,3] by interpreting the extreme electric black hole states as oscillating modes of underlying macroscopic string.

However, only after the discovery of BPS-saturated multi-charge dyonic black holes with regular horizons [4,5] and thus finite Bekenstein-Hawking (BH) entropy, that attempts to establish a quantitative agreement between the microscopic and macroscopic entropy in string theory became feasible. Such black holes are necessarily non-perturbative objects, and were originally specified as solutions of effective four-dimensional toroidally compactified string theory with charges from the Neveu-Schwarz–Neveu-Schwarz (NS-NS) sector [5,6]. Their microscopic features were captured by string theory in curved space-time geometry of the near-horizon region which is that of an $SL(2, R) \times SU(2)$ Wess-Zumino-Witten (WZW) model [7,8]. In particular, the small-scale string oscillations [9,8] were shown to reproduce [8,10,11] the extreme black hole entropy directly from the near-horizon geometry. The key point was that the BH entropy should represent the leading term in the statistical entropy in the limit of large charges (in which the characteristic scale of black hole is large and thus string corrections can be ignored). For large charges it should be sufficient to consider only the near-horizon region where the corresponding string sigma-model takes the WZW form.

It was suggested in [9] that the BH entropy of these BPS-saturated black holes can be interpreted as the entropy of left-moving supersymmetric oscillating states of a free string with tension renormalised by the (product of) “magnetic” charges, i.e. non-perturbative charges of the NS-NS sector (solitonic five-brane and Kaluza-Klein monopole charges). The origin of this renormalisation was explained in [6,10] starting with the conformal sigma-model which describes the embedding [6,8,10] of the dyonic black holes into string theory. The marginal supersymmetric deformations in the left-moving sector of the conformal sigma-model were interpreted, in the spirit of [2,3], as microstates describing degenerate black holes with the same asymptotic charges but different short distance structure. Since these perturbations are important only at small scales they can be effectively counted near the horizon ($r \sim r_+$). The crucial observation [8] is that for $r \to r_+$ the model reduces to a six-dimensional WZW theory with level proportional to (product of) magnetic charges, so that for large charges its spectrum is thus effectively that of the free fundamental string with tension rescaled by the magnetic charges. While the subleading terms in the statistical entropy may depend on embedding of the black hole into a particular string theory, the leading term should be universal.

As was explained in [11], the leading term receives contributions only from certain universal types of perturbations which are common to heterotic and type II embeddings. Other perturbations, in particular, string oscillations in internal toroidal directions, which do not carry charges, contribute only to subleading terms in the entropy. Indeed, the relevant
perturbations turn out to correspond to string oscillations in the four spatial directions transverse to the NS-NS five-brane. These dimensions are “intrinsic” to the black hole and do not depend on a choice of a superstring theory. It is only these four directions that get multiplied by the (product of) magnetic charges in the near-horizon region. In terms of the free-oscillator description of perturbations (which indeed applies in the near-horizon region for large charges, i.e. large level $\kappa \sim P$ of the underlying WZW model) this corresponds to the effective number of degrees of freedom $c_{eff} = 4(1 + \frac{1}{2}) = 6$, where we have included the contributions of superpartners of the four bosonic string coordinates. The statistical entropy obtained by counting the supersymmetric string oscillations in the four transverse directions with the tension or oscillator level rescaled by $\kappa$ exactly matches the BH entropy [11].

These developments were overshadowed by the advent of D-branes – nonperturbative objects in string theory with Ramond-Ramond (R-R) charges [12]. Black holes with NS-NS charges can be mapped, by duality symmetry, onto black holes with R-R charges which have the same space-time metric and thus BH entropy. The higher-dimensional interpretation of these black holes in terms of intersecting D-branes lead to a counting of black hole quantum states that agrees precisely with the BH entropy for both the extreme [13] and near-extreme [14,15] static, as well as rotating [16,17] black holes.

Recently, Strominger [18] (see also [19,20]) has given an alternative derivation of the BH entropy, by again employing the near-horizon geometry. The central observation is that, when embedded in a higher dimensional space, the near-horizon geometry contains locally the three-dimensional anti de Sitter space-time ($AdS_3$), whose quantum states are described by a two-dimensional conformal field theory (CFT) at the asymptotic boundary [21]. The counting of states in this CFT is then used to reproduce the BH entropy. While apparently elegant and compelling, this approach does not provide a detailed understanding of microstates involved and recently came under some criticism [22]. The method nevertheless reproduces the BH entropy of the static near-extreme black holes in four [18] and five [23] dimensions as well as that of the corresponding rotating black holes [24,25].

The robustness of these results for both extreme and near-extreme black holes renewed the interest in trying to relate the details of black hole microscopics to features of the black hole near-horizon geometry [26–28]. One of the aims of the present paper is to generalize the earlier approach [8,10,11] to counting of microstates for BPS-saturated black-holes with NS-NS charges as perturbations of the WZW models, describing the curved space-time geometry of the near-horizon region, to the case of non-extreme black holes. (An early attempt in that direction was made in [29].)

Our approach is different from that of [18,23–25] as it does not use the local equivalence of the Banados-Teitelboim-Zanelli (BTZ) black hole [30–32] and $AdS_3$, and the fact that the asymptotic boundary of the latter corresponds to $SL(2, R)_L \times SL(2, R)_R$ CFT [21]. In our case the microstates are counted directly as string states at the horizon and not at the asymptotic boundary. The identification of the underlying CFT is more explicit and relies on the correspondence (matching) between the solitonic and fundamental string states, extending the discussion in [8,10,11] to non-BPS states.

The crucial observation is that in the large charge limit the string sigma-model representing a non-extremal black hole with NS-NS charges is still equivalent to a (version of) $SL(2, R) \times SU(2)$ WZW model. Thus the near-horizon region remains to be effectively
described by the free fundamental string whose tension is again rescaled by the magnetic charges. However, now the black-hole microstates are identified not only with the left-moving, but also right-moving superconformal string oscillations, which can be of the same order of magnitude.

As a prototype example we consider the five-dimensional non-extreme rotating black holes and demonstrate that for a large NS-NS five-brane \((P)\) and fundamental string \((Q)\) charge this black hole is described by a marginal integrable perturbation of the \(SL(2,R) \times SU(2)\) WZW model generated by the left- and right-moving Cartan chiral \(SU(2)\) currents with the perturbation parameters proportional to the two angular momentum components. We shall demonstrate that the non-trivial expression for the BH entropy of these rotating black holes \([33,17]\) can be reproduced precisely by identifying the microstates as the left- and right-moving oscillations of the effective fundamental string.

The rest of the paper is organized as follows. In Section II some essential features of the \(SL(2,R) \times SU(2)\) WZW model and its deformations are discussed. In Section III we consider the statistical entropy of microstates represented by left-moving and right-moving marginal perturbations in the four transverse directions of the effective string described by a large-level WZW model. Section IV is devoted to the sigma-model corresponding to the non-extreme five-dimensional black holes. In Subsection IV A the sigma-model for the non-extreme static five-dimensional black hole is derived, and it is shown that in the case of the large \(P\) and \(Q\) charges it reduces to the \(SL(2,R) \times SU(2)\) WZW model with equal levels \(\kappa = P/\alpha'\). In Subsection IV B the sigma-model in the case of the rotating black hole with large \(P\) and \(Q\) is written down and is shown to be equivalent to the deformed WZW model with the coefficients of the left- and right-moving deformations proportional to the two angular momentum components. We establish the precise relation between the standard light-cone coordinates of the effective fundamental string and the canonical coordinates of the \(SL(2,R) \times SU(2)\) WZW model. In Subsection IV C it is demonstrated that the counting of the relevant states of the obtained fundamental string reproduces the BH entropy of the black hole.

II. \(SL(2,R) \times SU(2)\) WZW THEORY AND ITS MARGINAL DEFORMATIONS

The central role in our discussion will be played by the \(SL(2,R) \times SU(2)\) WZW model with equal levels \(\kappa\) of the two factors. This unique model has several remarkable features. Since the curvatures of the \(AdS_3\) and \(S^3\) factors are equal and opposite in sign, the leading correction to its central charge vanishes. Moreover, in the supersymmetric case the total central charge has the free-theory value (see also \([34,6]\)):

\[
c = \frac{3(\kappa - 2)}{(\kappa - 2) + 2} + \frac{3(\kappa + 2)}{(\kappa + 2) - 2} = 6.
\]  

Thus multiplied, e.g., by a four-torus or \(K3\), this CFT represents an exact solution of the \(D = 10\) (type II or heterotic) superstring theory describing \(AdS_3 \times S^3\) space supported by NS-NS two-form-tensor and constant dilaton.

This WZW theory, its orbifolds and marginal deformations turn out to describe the near-horizon regions or large charge limits of the basic examples of the BPS-saturated \(D = 4\) static \([7,6,8]\) and \(D = 5\) static and rotating \([10,11]\) black holes with regular horizons.
As we shall demonstrate in Section IV, this remarkable model describes also the near-horizon limit of the non-extreme (both static and rotating) $D = 5$ black holes with three NS-NS charges (and two rotation parameters). (This approach can be generalized to the case of the four-dimensional black holes specified by four charges and one angular momentum parameter as will be discussed elsewhere.)

The Lagrangian of the $SL(2,R) \times SU(2)$ model can be written as

\[ L = \kappa \left[ L_{SL(2)} + L_{SU(2)} \right], \quad (2) \]

\[ L_{SL(2)} = \frac{1}{4}(2 \cosh z \partial u \bar{\partial} v + \partial u \bar{\partial} u + \partial v \bar{\partial} v + \partial z \partial \bar{z}), \quad (3) \]

\[ L_{SU(2)} = \frac{1}{4}(\partial \theta \partial \bar{\theta} + \partial \varphi \partial \bar{\varphi} + \partial \psi \partial \bar{\psi} + 2 \cos \theta \partial \psi \partial \bar{\varphi}), \quad (4) \]

where $x, u, v$ are the $AdS_3$ coordinates and $\theta, \psi, \varphi$ are the $S^3$ coordinates (Euler angles). Note that in the Gauss parametrisation of $SL(2)$

\[ L'_{SL(2)} = e^{-z} \partial u' \bar{\partial} v' + \frac{1}{4} \partial z' \partial \bar{z}' . \quad (5) \]

The near-horizon limit of the extreme $D = 5$ three-charge static black hole turns out to be represented by the following Lagrangian [10]:

\[ L = L''_{SL(2)} + \kappa L_{SU(2)}, \quad L''_{SL(2)} = e^{-z} \partial U \bar{\partial} V + \tilde{Q} Q^{-1} \partial U \bar{\partial} U + \frac{1}{4} P \partial z \partial \bar{z} . \quad (6) \]

Here $P = \kappa$ is the NS-NS five-brane charge, $Q$ is the fundamental string charge and $\tilde{Q}$ is momentum along the string. The radial coordinate $r$ of the black hole is expressed in terms of $z$ by $\frac{Q}{2} = e^z$. The coordinates $(U, V) = \pm t + y$ are interpreted as the light-cone coordinates of the fundamental string.

The Lagrangians (6), (5) and (3) are related by coordinate transformations. These transformations may be well-defined only locally if some of the coordinates are compact. This is indeed the case for the solitonic string case where the coordinate $y = \frac{1}{2}(U + V)$ along the string is compact.

Making a formal coordinate transformation that mixes the parameters of $SL(2,R)$ and $SU(2)$ (but may not respect periodicities of the coordinates and thus, in particular, break supersymmetry [36,35]) one obtains an exact conformal model which is a marginal deformation of the original WZW model. This deformation itself represents a near-horizon region of a different string solution. Previous examples were considered in [8,10].

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1The action is given by $I = \frac{1}{\alpha'} \int d^2 \sigma \ L$. We shall often set $\alpha' = 1$ in what follows.

2The group element of $SL(2,R)$ in the Gauss decomposition parametrisation is $g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{z} & 0 \\ 0 & e^{-z} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ v & 1 \end{pmatrix}$, while in the ‘Euler angle’ parametrisation is $g = e^{\frac{i}{2} \sigma_{2} e^{\frac{1}{2} x \sigma_{1} e^{\frac{1}{2} v}}.}$ The transformation between $e^{-2x} \partial u \bar{\partial} v + n \partial u \bar{\partial} u + \partial x \partial x$ and $e^{-2x} \partial u' \bar{\partial} v' + \partial x' \partial x'$ is $u' = \frac{1}{2} \sqrt{n} e^{2x} v', \quad v' = v - \sqrt{n} e^{2x}, \quad x' = x + \sqrt{n} u.$
A particularly simple deformation which will be relevant in the present paper is generated by the Cartan $SU(2)$ chiral currents and is obtained by the following redefinition\(^3\)

$$\varphi \rightarrow \varphi + q_2 v, \quad \psi \rightarrow \psi + q_1 u,$$

(7)

where $q_1, q_2$ are two arbitrary constants. This leads (after integrating by parts) to the following model

$$L = \frac{1}{4}\kappa \left[ 2(\cosh z + q_1 q_2 \cos \theta)\partial u \bar{\partial} v + (1 + q_1^2)\partial u \bar{\partial} u + (1 + q_2^2)\partial v \bar{\partial} v + \partial z \bar{\partial} z \right]$$

$$+ \frac{1}{2}\kappa (q_1 \partial u \bar{J}_3 + q_2 J_3 \bar{\partial} v) + L_{SU(2)},$$

(8)

where

$$J_3 = \partial \varphi + \cos \theta \partial \psi, \quad \bar{J}_3 = \bar{\partial} \psi + \cos \theta \bar{\partial} \varphi.$$  

If a combination of $u$ and $v$ is periodic, the transformation (7) does not preserve periodicities but the model (8) is still well-defined. While locally it is a direct product, globally it is not for generic values of $q_1, q_2$.

In Section IV we shall show that essentially the same WZW theory (8) describes the large $P$ and $Q$ charge limit of the non-extreme rotating $D = 5$ black hole, with $q_1, q_2$ being proportional to the two angular momentum parameters (see Subsection IV B). (Although this model has two “rotational” parameters, in contrast to the models considered in [37], it has a natural extension to finite radial distance which is asymptotically flat.) The original undeformed $SL(2, R) \times SU(2)$ model in the form written in (2),(3),(4) describes the large $P, Q$ charge limit of the static non-extreme black hole (see Subsection IV A). We shall then demonstrate (see Subsection IV C) that the non-trivial expression for the non-extreme rotating black hole entropy [33,17] can be reproduced by applying the counting of microstates discussed in Section III. This generalizes the previous analysis in the case of extreme black holes [8,10,11] to the non-extremal case.

### III. NEAR-HORIZON STRING PERTURBATIONS AND STATISTICAL ENTROPY

In this section we shall generalize the counting of microstates of the effective fundamental string, as initiated in [8,10] and further clarified in [11], to the case that along with the left-moving marginal perturbations includes also the right-moving marginal perturbations. The main idea is that in the large charge limit the $SL(2, R) \times SU(2)$ WZW model describes a fundamental string with a tension rescaled by the NS-NS five-brane charge $P$, and whose relevant marginal deformations are only those of the four transverse directions to the NS-NS five-brane. (Oscillations within 5-brane are suppressed by a large charge factor and

\(^3\)The corresponding marginal deformation can be interpreted also as a certain gauging of WZW model.
contribute only to subleading terms in the entropy \([11]\)). Since the charges are large, the conformal sigma-model is weakly coupled, so one is to count string states in nearly-flat space. To relate the oscillation numbers of the fundamental string and thus its entropy to the global parameters of the black hole one is to use the “matching conditions” analogous to the ones in \([2,3,11]\).

Let us combine the \(z\)-direction of \(SL(2)\) in \((3)\), related to the radial coordinate \(r\) of the black hole and three angular coordinates of \((\theta, \phi, \psi)\) of \(SU(2)\) in \((4)\) into four transverse coordinates \(x^i\). The key point is that since we are interested in the large charge or large WZW level limit, interactions in WZW theory are suppressed by \(1/\kappa\), and thus the count of perturbations should be essentially the same as in the theory of free fields \(x^i\). The only difference compared to the flat space case is the presence of the factor \(\kappa\) in front of the kinetic term of \(x^i\). To make this more explicit, let us consider the following sigma model which represents the perturbed version of the model \((2)\) (cf. \([38,8]\))

\[
L = F(x)\partial u \bar{\partial} v + K_0 \partial u \bar{\partial} u + M_0 \partial v \bar{\partial} v \\
+ A_i(x, u) \partial u \bar{\partial} x^i + \bar{A}_i(x, v) \partial x^i \bar{\partial} v + \frac{1}{4} \kappa h_{ij}(x) \partial x^i \bar{\partial} x^j .
\] (10)

Here \(u, v\) correspond to the light-cone coordinates \((u, v) = \mp t + y\) of the effective fundamental string. We have included along with the “left-moving” perturbations, \(\sim \partial u\), also the “right-moving” ones, \(\sim \bar{\partial} v\).

Since the level \(\kappa\) is large, we may assume that \(h_{ij} = G_{ij} + B_{ij}\) is approximately flat and also set \(F(x) \approx 1\) (the large charge limit corresponds to the small curvature of \(AdS_3\) space). As a result, the perturbations \(A_i(x, u)\) and \(\bar{A}_i(x, v)\) are marginal to the leading order in their strength. Indeed, integrating out \(u\) “freezes” \(v\) and thus makes \(\bar{A}_i\) marginal and vice versa for \(A_i\). In general, there is also a constraint on their \(x\)-dependence, and the solution which is relevant in the large charge limit is

\[
A_i(x, u) \sim F(x) a_i(u) \rightarrow a_i(u) , \quad \bar{A}_i(x, v) \sim F(x) b_i(v) \rightarrow b_i(v) .
\] (11)

Integrating out the four transverse fields \(x^i\) in the large \(\kappa\) limit we find that \(K_0\) and \(M_0\) in \((10)\) are replaced by

\[
K(u) = K_0 - \frac{1}{4} \kappa^{-1} a_i^2(u) + O(k^{-2}) , \quad M(v) = M_0 - \frac{1}{4} \kappa^{-1} b_i^2(v) + O(k^{-2}) .
\] (12)

In general, there is also a \(\mathcal{A}\)\(\mathcal{A}\)-correction to the \(\partial u \bar{\partial} v\) term.

Generalising the discussion in \([11]\) to include also the right-moving oscillations, we assume that the analogue of the level matching condition for the free fundamental string \([2,3]\) which relates the oscillation level numbers \(N_L\) and \(N_R\) to the background charges is as follows: the coefficients of both \(\partial u \bar{\partial} u\) and \(\partial v \bar{\partial} v\) terms should vanish on average to allow matching onto a fundamental string source, i.e.

\[
\overline{K} = 0 , \quad \overline{M} = 0 , \quad \overline{f} \equiv \frac{1}{2\pi R} \int_0^{2\pi R} dy \, f(y) ,
\] (13)

where \(R\) is the radius of a compact direction along which the string is wound.

Then, to the leading order in \(1/\kappa\),
\[ a_i^2 = 4\kappa K_0 = qN_L , \quad b_i^2 = 4\kappa M_0 = qN_R , \quad (14) \]

where \(N_L\) and \(N_R\) are interpreted as the left- and the right-moving string oscillation numbers and \(q\) is a proportionality constant, depending on the tension of the fundamental string \((q = \frac{\pi}{4G_5} = 1\) in the units used below).

The statistical entropy should, in general, be obtained by counting all near-horizon perturbations in all possible directions, but the oscillations in the four “external” spatial dimensions \(x^i\) have dominant statistical weight for large charges \((\kappa \gg 1)\) \([11]\). Taking into account the superpartners of the four bosonic oscillation directions\(^4\) so that the effective central charge is \(c = 4(1 + \frac{3}{2}) = 6\), the leading term in the statistical entropy is then given by

\[ S_{stat} = 2\pi \left( \sqrt{N_L} + \sqrt{N_R} \right) . \quad (15) \]

**IV. SIGMA-MODEL AND MICROSTATES OF FIVE-DIMENSIONAL NON-EXTREME ROTATING BLACK HOLE**

In this section we shall derive the large-charge sigma-model for five-dimensional rotating non-extreme black holes and count the corresponding microstates.

**A. Sigma-model for non-extreme static black hole**

The static non-extreme black hole is specified by three NS-NS charges: the NS-NS five-brane charge \(P\), the fundamental string charge \(Q\) and the string momentum \(\tilde{Q}\). The five-dimensional solution (as found in [33]) can be written as a six-dimensional black string with the following string-frame space-time metric:

\[ ds_6^2 = H_{1}^{-1} (-f dt^2 + dy^2) + H_{5}(f^{-1}dr^2 + r^2 d\Omega_3) , \quad (16) \]

where

\[ d\Omega_3 = \frac{1}{4}(d\theta^2 + d\varphi^2 + d\psi^2 + 2 \cos \theta \ d\varphi d\psi) \quad (17) \]

is the line element of a three-sphere, expressed in terms of the Euler angles. The two-form field \(B_{mn}\) has the form:

\[ B_{yt} = m \sinh 2\delta_Q r^{-2} H_{1}^{-1} - 1 , \quad B_{\psi\varphi} = \frac{1}{4} m \sinh 2\delta_P \cos \theta , \quad (18) \]

and the dilaton field \(\Phi\) is given by:

\[ \Phi = \ln \left( \frac{\sqrt{N_L} + \sqrt{N_R}}{2\pi} \right) . \]

\(^4\)We are assuming that the world-sheet theory is \((1, 1)\) supersymmetric as in type II theory, but the same conclusions are true in the heterotic case [11] (provided the black hole solution is embedded into the heterotic theory in the manifestly conformally-invariant “symmetric” way).
\[ e^{2\Phi} = H_5 H_1^{-1}. \] (19)

Here
\[ f = 1 - \frac{2m}{r^2}, \quad H_1 = 1 + \frac{2m \sinh^2 \delta_Q}{r^2}, \quad H_5 = 1 + \frac{2m \sinh^2 \delta_P}{r^2}. \] (20)

The momentum along the string can be introduced by the boost transformation along the \( y \) direction, i.e.
\[ U \equiv t + y \rightarrow e^{\delta \tilde{Q}} U, \quad V \equiv -t + y \rightarrow e^{-\delta \tilde{Q}} V. \] (21)

Here the parameterization is in terms of the three “boost” parameters \( \delta_{P,Q,\tilde{Q}} \) and the non-extremality parameter \( m \), which are related to the charges and the mass of the black hole in the following way:
\[ P = m \sinh 2\delta_P, \quad Q = m \sinh 2\delta_Q, \quad \tilde{Q} = m \sinh 2\delta_{\tilde{Q}}, \]
\[ M = m (\cosh 2\delta_P + \cosh 2\delta_Q + \cosh 2\delta_{\tilde{Q}}). \] (22)

We work in the Planck units where the gravitational coupling constant in five dimensions is \( G_5 = \frac{\pi}{4} \) (for relation to conventional units see [24]).

The string sigma-model corresponding to this background is of the following complicated form:
\[ L = H_1^{-1} \left[ \left( 1 - \frac{m}{r^2}(1 - \frac{1}{2}e^{-2\delta_{\tilde{Q}}}) \right) \partial U \partial \bar{V} + \frac{m}{2r^2} \left( e^{-2\delta_{\tilde{Q}}} \partial \bar{V} \partial U + e^{2\delta_{\tilde{Q}}} \partial U \partial \bar{V} + e^{-2\delta_{\tilde{Q}}} \partial V \partial \bar{V} \right) \right] 
+ H_5 r^2 \left[ \frac{\partial r \partial \bar{r}}{r^2 - 2m} + \frac{1}{4} \left( \partial \theta \partial \theta + \partial \varphi \partial \varphi + \partial \psi \partial \psi + \cos \theta \left( \partial \psi \partial \varphi + \partial \varphi \partial \psi \right) \right) \right] 
+ \frac{1}{4} P \cos \theta \left( \partial \psi \partial \varphi - \partial \varphi \partial \psi \right), \] (23)

where one should also add the dilaton term \( \sim \Phi R^{(2)} = \frac{1}{2} \log(H_1^{-1}H_5)R^{(2)}. \)

In the limit of large \( P \) and \( Q \) charges, i.e. \( \delta_{P,Q} \gg 1 \), the (static) non-extreme \( D = 6 \) conformal model (23) takes the following simpler form
\[ L_{\text{static}} \equiv (L)_{\delta_{P,Q} \gg 1} = Q^{-1} \left[ \left( r^2 - \frac{1}{2}(r_+^2 + r_-^2) \right) \partial U \partial \bar{V} \right] 
+ \frac{1}{2}(r_+ + r_-)^2 e^{2\delta_{\tilde{Q}}} \partial U \partial \bar{U} + \frac{1}{2}(r_+ - r_-)^2 e^{-2\delta_{\tilde{Q}}} \partial V \partial \bar{V} \right] 
+ P \left[ \frac{r^2 \partial r \partial \bar{r}}{(r^2 - r_+^2)(r^2 - r_-^2)} + \frac{1}{4} \left( \partial \theta \partial \theta + \partial \varphi \partial \varphi + \partial \psi \partial \psi + 2 \cos \theta \partial \psi \partial \varphi \right) \right] \] (24)

with the constant dilaton \( e^{2\Phi} = \frac{P}{\tilde{Q}} \). Here
\[ r_+ = \sqrt{2m}, \quad r_- = 0, \] (25)

which correspond to the location of the inner and outer horizons of the static black hole.

The Lagrangian (24) is indeed the \( SL(2,R) \times SU(2) \) WZW model (2), discussed in Section II, with the level \( \kappa = P \) ! The coordinate \( z \) in (2) and the radial coordinate \( r \) in (24) are related by
The coordinates $u, v$ of the WZW model (2) are proportional to the light-cone coordinates $U, V$ in (24)

$$u = \frac{1}{\sqrt{PQ}} (r_+ + r_-) e^{\delta \tilde{Q}} U, \quad v = \frac{1}{\sqrt{PQ}} (r_+ - r_-) e^{-\delta \tilde{Q}} V. \quad (27)$$

It is instructive to recall the BPS-saturated limit of (24) [10]. This limit is obtained by taking $m \to 0$ and $\delta_{P,Q,\tilde{Q}} \to \infty$ while keeping the charges $P, Q, \tilde{Q}$ fixed. The Lagrangian (24) then becomes:

$$L_{\text{BPS static}} = Q - 1 \left( r^2 \partial U \bar{\partial} V + \tilde{Q} \partial U \bar{\partial} U \right) + P \left( r^{-2} \partial r \bar{\partial} r + L_{SU(2)} \right). \quad (28)$$

After the transformation of the radial coordinate $r \to z = -\log \frac{r^2}{Q}$ becomes that of (6) [10]. Note that the terms $\sim \partial V \bar{\partial} V$ are absent in the BPS limit.

**B. The large charge limit of the sigma model**

**for non-extreme rotating black hole**

The $D = 5$ rotating black hole is specified, in addition to the three charges and the mass (22), by the two angular momentum parameters

$$J_{L,R} = m(l_1 \pm l_2) \left( \cosh \delta_P \cosh \delta_Q \cosh \delta_{\tilde{Q}} \pm \sinh \delta_P \sinh \delta_Q \sinh \delta_{\tilde{Q}} \right), \quad (29)$$

where $l_{1,2}$ are the angular momenta of the corresponding neutral black hole.

Its Einstein-frame $D = 5$ metric, dilaton and the antisymmetric tensor field $B_{mn}$ found in [33] are of very complicated form. The metric of the associated six-dimensional rotating black string solution given in [24] is more transparent. (By duality, this Einstein-frame metric is the same as that of the six-dimensional rotating string with R-R charges.) The string sigma-model which still looks rather involved, simplifies dramatically in the large $P,Q$ charge limit, i.e. $\delta_{P,Q} \gg 1$ (see eqs. (12)–(15) in [24] for the Einstein-frame metric). In this limit it takes the following form (cf. (24))

$$L_{rot} = (L)_{\delta_{P,Q} \gg 1} = Q - 1 \left[ \left( r^2 - \frac{1}{2}(r_+^2 + r_-^2) + \frac{1}{2}(r_+^2 - r_-^2) q_1 q_2 \cos \theta \right) \partial U \bar{\partial} V 
+ \frac{1}{4} (1 + q_2^2) (r_+ - r_-) e^{2\delta \tilde{Q}} \partial U \bar{\partial} U + \frac{1}{4} (1 + q_2^2) (r_+ - r_-) e^{-2\delta \tilde{Q}} \partial V \bar{\partial} V 
- \frac{1}{2} (l_1 + l_2) \frac{P}{Q} e^{\delta \tilde{Q}} \partial U \bar{J}_3 - \frac{1}{2} (l_1 - l_2) \frac{P}{Q} e^{-\delta \tilde{Q}} \partial V \bar{J}_3 
+ P \left[ \frac{r^2 \partial r \bar{\partial} r}{(r^2 - r_+^2)(r^2 - r_-^2)} + L_{SU(2)} \right] \right], \quad (30)$$

with the constant dilaton $e^{2\Phi} = \frac{P}{Q}$. Here
\[ r_{\pm} = \frac{1}{2} \left[ \sqrt{2m - (l_1 + l_2)^2} \pm \sqrt{2m - (l_1 - l_2)^2} \right] , \tag{31} \]

and
\[ q_1 = -\frac{l_1 + l_2}{r_+ + r_-} , \quad q_2 = -\frac{l_1 - l_2}{r_+ - r_-} . \tag{32} \]

The \( SU(2) \) chiral currents \( J_3, \bar{J}_3 \) are defined in (9).

Using the transformation (26) between \( r \) and \( z \) and the transformation (27) between \( (U, V) \) and \( (u, v) \) (now with \( r_{\pm} \) defined by (31)) it is easy to see that (30) is precisely the integrable marginal deformation (8) of the \( SL(2, R) \times SU(2) \) WZW model discussed in Section II!

This can be seen also by noting that after the coordinate transformation \([24]\)
\[ \varphi \to \varphi - (l_1 - l_2)(PQ)^{-\frac{1}{2}} e^{-\delta \tilde{Q}} V , \quad \psi \to \psi - (l_1 + l_2)(PQ)^{-\frac{1}{2}} e^{\delta \tilde{Q}} U , \tag{33} \]
i.e. the shift (7), the Lagrangian (30) becomes that of (24), but now with the locations \( r_{\pm} \) of the inner and outer horizons related to the non-extremality parameter \( m \) and the angular momentum parameters \( l_1, l_2 \) by (31), i.e. as for rotating black hole. Thus after the formal field redefinition (33) the sigma-model Lagrangian (30) can be locally written as (24), i.e. as a direct-product \( SL(2, R) \times SU(2) \) WZW model (2) with level \( \kappa = P \). Further redefinition of \( r \) and \( U, V \) which leaves the form of (24) invariant but effectively replaces \( r_{\pm} \) in (31) by the \( l_1, l_2 \)-independent ones in (25) explains why (30) may be interpreted also as a deformation of the static \((l_1, l_2 = 0)\) WZW model (24).

The BPS-saturated limit of the black hole solution is found by taking \( m \sim l_1^2 \sim l_2^2 \to 0, \delta_{P,Q,\tilde{Q}} \to \infty \), with the charges \( P, Q, \tilde{Q} \) and \( J_L \) fixed \((J_R \to 0)\). Then the Lagrangian (30) becomes \([10]\)
\[ L_{BPS}^{\text{rot}} = Q^{-1} \left( r^2 \partial U \tilde{Q} V + \tilde{Q} \partial U \partial V - J_L \partial U \bar{J}_3 \right) + P \left( r^{-2} \partial r \tilde{Q} r + L_{SU(2)} \right) , \tag{34} \]
which can be interpreted as a marginal deformation of the sigma-model (28) representing the static black hole \([10]\). Note that in this case the perturbation involves only “right” \((\bar{J}_3)\) chiral \( SU(2) \) Cartan current.

C. Statistical entropy

The general formula for the BH entropy of rotating black holes in five dimensions is \([33]\):
\[ S_{BH} = 2\pi m \left[ \left( \prod_{i=1}^{3} \cosh \delta_i + \prod_{i=1}^{3} \sinh \delta_i \right) \sqrt{2m - (l_1 + l_2)^2} \right. \]
\[ + \left( \prod_{i=1}^{3} \cosh \delta_i - \prod_{i=1}^{3} \sinh \delta_i \right) \sqrt{2m - (l_1 - l_2)^2} \right] . \tag{35} \]
where \( \delta_{1,2,3} \) is a short notation for \( \delta_{P,Q,\tilde{Q}} \).
In the limit \( \delta_{P,Q} \gg 1 \) this becomes [33,17]:

\[
S_{BH} = \pi \sqrt{PQ} \left[ \sqrt{2m - (l_1 + l_2)^2} e^{\delta Q} + \sqrt{2m - (l_1 - l_2)^2} e^{-\delta Q} \right]
= \pi \sqrt{PQ} \left[ (r_+ + r_-)e^{\delta \tilde{Q}} + (r_+ - r_-)e^{-\delta \tilde{Q}} \right].
\] (36)

In the extreme case the entropy takes the form [16,10]

\[
S_{BH}^{BPS} = 2\pi \sqrt{PQ\tilde{Q} - J_L^2}.
\] (37)

Our aim will be to employ the method of Section III to give a microscopic interpretation of the BH entropy (36). For that we will need to identify properly the oscillation numbers \( N_L \) and \( N_R \) which determine the statistic entropy (15).

It is instructive to recall first the explicit expressions one finds in the extreme case [10,11], where there is just one rotational parameter \( J_L \) (see (34). In this case \( M \partial v \bar{\partial} v \) term in (10) is absent, and so one is to ignore the right-moving (\( \sim \bar{\partial} v \)) perturbations: their contribution will vanish according to the matching condition in (12), i.e. one counts essentially only the BPS-saturated states. For the \( D = 6 \) string \( \tilde{Q} = \frac{n}{R}, Q = wR, \) where \( w \) and \( m \) are integer winding and momentum numbers, \( R \) is the radius of the compact direction \( y \) and as above we assume \( G_5 = \frac{\pi}{4}, \alpha' = 1 \). Special rotational perturbation of the \( A_i \)-type in (10) which is proportional to the chiral \( SU(2) \) current \( \tilde{J}_3, A_i \sim a(u) \), contains the constant part, \( \tilde{\alpha} = J_L \) (see [11] for details). One finds that the corresponding rotational parameter should be quantised [10]: \( J_L R \tilde{P} - Q^{-1} = l = \text{integer} \). The quantisation of \( Q = wR \) and \( P = \kappa \) implies that the angular momentum takes integer values \( J_L = \kappa \tilde{w} l = j \). Expressing the “left” matching condition in (14) (relating the throat region perturbations to the charges) in terms of the integer quantum numbers one finds [10,11]

\[
N_L = \tilde{\alpha}_i^2 = PQQ - J_L^2 = \kappa \tilde{w} \tilde{n} - j^2,
\] (38)

so that the statistical entropy (15) reproduces indeed the Bekenstein-Hawking entropy (37).

(The same result was reached using D-brane counting in [16].)

Let us now turn to the discussion of the entropy in the non-extreme rotating case (with large \( P \) and \( Q \) charges). Here there are two angular momenta, and the Lagrangian (30) contains deformation terms of both chiralities. Similar left-moving (\( \sim \partial u \)) and right-moving (\( \sim \bar{\partial} v \)) terms will appear in the perturbed version (10) of (30). The conditions of matching onto the string source (13),(14) will then determine \( N_L \) and \( N_R \).

As in the extreme case, it is useful to interpret (30) as a special case of the general deformation (10) of the static model (24) by assuming that the special rotational perturbations among \( A_i \) and \( \tilde{A}_i \) terms in (10), which are proportional to the chiral \( SU(2) \) currents \( \tilde{J}_3 \) and \( J_3 \), respectively, \( A_i \sim a(u), \tilde{A}_i \sim b(v) \), contain the constant parts \( \tilde{\alpha} = J_L \) and \( \tilde{\beta} = J_R \).

From the “left” and “right” matching conditions in (14) one finds

\[
^5\text{Note that after integrating out the deformations in (30) as discussed in Section III, the Lagrangian becomes that of (24) with coefficients in front of } \partial \bar{U} \bar{\partial} U \text{ and } \partial \bar{V} \bar{\partial} V \text{ being proportional to the coefficients in the l.h.s. of (14), i.e. to } N_L \text{ and } N_R, \text{ respectively.}
\]
\[ N_L = \frac{1}{4} PQ (r_+ + r_-)^2 e^{2\delta Q} , \quad N_R = \frac{1}{4} PQ (r_+ - r_-)^2 e^{-2\delta Q} . \] (39)

Inserting these expressions into the statistical entropy (15), we reproduce precisely the (large charge limit of) Bekenstein-Hawking entropy (36) of the non-extreme rotating black hole.

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