Analytic Torsion on Hyperbolic Manifolds and the Semiclassical Approximation for Chern-Simons Theory

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Abstract

The invariant integration method for Chern-Simons theory for gauge group $SU(2)$ and manifold $\Gamma \backslash H^3$ is verified in the semiclassical approximation. The semiclassical limit for the partition function associated with a connected sum of hyperbolic 3-manifolds is presented. We discuss briefly $L^2$—analytical and topological torsions of a manifold with boundary.

1 Introduction

It is known that topological invariants associated with 3-manifolds can be constructed within the framework of Chern-Simons gauge theory [1]. These

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values were specified in terms of the axioms of topological quantum field theory [2], whereas equivalent derivation of invariants was also given combinatorially in [3, 4], where modular Hopf algebras related to quantum groups have been used. The Witten’s (topological) invariants have been explicitly calculated for a number of 3-manifolds and gauge groups [5, 6, 7, 8, 9, 10, 11, 12].

The semiclassical approximation for the Chern-Simons partition function $Z_W(k)$ can be given by the asymptotic $k \to \infty$ of Witten’s invariant of a 3-manifold $M$ and a gauge group $G$. Typically this expression is a partition function of quadratic functional. The asymptotic leads to a series of $C^\infty$– invariants associated to triplets $\{M; F; \xi\}$ with $M$ a smooth homology 3–sphere, $F$ a homology class of framings of $M$, and $\xi$ an acyclic conjugacy class of orthogonal representations of fundamental group $\pi_1(M)$ [13]. In addition the cohomology $H(M; Ad \xi)$ of $M$ with respect to the local system related to $Ad \xi$ vanishes. Description of an invariant of rational homology 3– spheres can be found in [14, 15, 16].

This note is an extension of the previous paper [17]. Our aim here will be to use the invariant integration method [18, 19] in its simplest form for the semiclassical approximation in Chern-Simons theory. We do this analysing the partition function as well as the partition function related to a connected sum of hyperbolic 3-manifolds. The resulting expressions are evaluated for gauge group $SU(2)$ and manifold $M = \Gamma \backslash H^3$, where $H^3$ is the Lobachevsky space and $\Gamma$ is a co-compact discrete group of isometries (see for detail [20]).

In Sect. 2 we review the semiclassical approximation, involving the partition function and the canonical elliptic resolvent of a quadratic functional. The explicit calculation of the Chern-Simons partition function with gauge group $SU(2)$ is presented in Sect. 3. In Sect. 4 we discuss the $L^2$– topological torsion of a compact manifold with boundary. The semiclassical approximation for the partition function relating to a connected sum of hyperbolic 3– manifold is calculated in Sect. 5. Sect. 6 contains some remarks in summary.

2 The Semiclassical Approximation for the Partition Function

The partition function associated to Chern-Simons gauge theory has the form

$$Z_W(k) = \int DA \exp[ikCS(A)], \quad k \in \mathbb{Z}, \quad (2.1)$$
\[ CS(A) = \frac{1}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \quad (2.2) \]

The quantity \( Z_W(k) \) is a (well-defined) topological invariant of \( M \). The formal integration in (2.1) is over the gauge fields \( A \) in a trivial bundle, i.e. 1-forms on a 3-dimensional manifold \( M \) with values in Lie algebra \( \mathfrak{g} \) of gauge group \( G = SU(N) \).

In the limit \( k \to \infty \) Eq. (2.1) is given by its semiclassical approximation, involving only partition functions of quadratic functionals [1]:

\[
\sum_{[A_\theta]} \exp[ikCS(A_\theta)] \int DB \exp[ikCS^{(2)}_{A_\theta}(B)],
\]

\[
CS^{(2)}_{A_\theta}(B) = \frac{1}{4\pi} \int_M \text{Tr}(B \wedge dA_\theta B).
\] (2.4)

In Eq. (2.3) the sum is taken over representatives \( A_\theta \) for each point \([A_\theta]\) in the moduli-space of flat gauge fields on \( M \). In addition, \( B \) is Lie-algebra-valued 1-form and \( dA_\theta \) is the covariant derivative determined by \( A_\theta \), namely \( dA_\theta = dB + [A_\theta, B] \). We shall use the invariant integration method [18, 19], which enables the partition functions in Eq. (2.3) to be evaluated in complete generality.

Let \( X \) be a compact oriented Riemannian manifold without boundary, and \( \dim(X) = (2m + 1) \) is the dimension of manifold. A quadratic functional (like functional (2.4)) can be defined on the space \( \mathcal{G} = \mathcal{G}(X, \xi) \) of smooth sections in a real Hermitian vector bundle \( \xi \) over \( X \). Let \( D_q \) denote the restriction of a flat connection map \( D \) on the space \( \Omega(X, \xi) \) of differential forms on \( X \) with values in \( \xi \) to the space \( \Omega^q(X, \xi) \) of \( q \)-forms and \( H^q(X; \xi) = H^q(D) = \ker(D_q) / \Im(D_q - 1)^{-1} \) are the cohomology spaces. One can construct the inner products \( \langle \cdot, \cdot \rangle_m \) in the space \( \Omega^m(X, \xi) \) and the quadratic functional on this space, \( S_D(B) = \langle B, DB \rangle_m \), (*) is the Hodge-star map. A canonical topological elliptic resolvent \( R(S_D) \) for the quadratic functional can be written as follows:

\[
0 \xrightarrow{D_0} \Omega^0(M, \xi) \xrightarrow{D_0} \cdots \xrightarrow{D_{m-2}} \Omega^{m-1}(M, \xi) \xrightarrow{D_{m-1}} \ker(S_D) \xrightarrow{0} 0. \quad (2.5)
\]

From Eq. (2.5) it follows that for the resolvent \( R(S_D) \) we have \( \mathcal{G}_q = \Omega^{m-q}(X, \xi) \) and \( H^q(R(S_D)) = H^{m-q}(D) \). Note that \( S \geq 0 \) and therefore \( \ker(S_D) \equiv \ker(D_m) \).
Let \( X = M \) be a closed hyperbolic 3-manifold, \( G \) a compact simple Lie group, \( g \) the Lie algebra of \( G \). The inner products in the space \( \Omega^q(M, g) \) of \( g \)-valued \( q \)-forms naturally can be chosen as \( \langle B, *d_1 A^B \rangle \), \( D_1 \equiv *d_1 A \). Thus the canonical elliptic resolvent for quadratic functional (2.4) takes the form:

\[
0 \longrightarrow \Omega^0(M, g) \overset{d_0}{\longrightarrow} \ker(d_1 A) \overset{\longrightarrow}{\longrightarrow} 0,
\]
and the resolvent has cohomology spaces \( H^0(R(S)) = H^1(d^A \theta) \), \( H^1(R(S)) = H^0(d^A \theta) \).

3 Gauge Group \( G = SU(2) \)

The invariant integration method [18] leads to the following expression for the Chern-Simons partition function [21, 19]:

\[
Z_{sc}(k) = \int_M D[A_0] V_G(H_{A_0})^{-1} e^{i(\frac{k}{4\pi^2} \eta(A_0) + kS(A_0))} \left( \frac{kl}{4\pi^2} \right)^{-\zeta(A_0)/2} |T_{an}(M)|^{1/2},
\]
where \( T_{an}(M) \) is the Ray-Singer analytic torsion of \( D_1 \) [22], \( \eta(A_0) \) and \( \zeta(A_0) \) are the analytic continuations to \( s = 0 \) of the eta and the zeta functions respectively [23, 20], \( \zeta(A_0) = \dim H^0(A_0) - \dim H^1(A_0) \) and \( H^q(A_0) \) the \( q \)-th cohomology space of \( d^A \).

Let \( G = SU(2) \) and \( M = \Gamma \setminus H^3 \). A convenient choice of orthogonal basis (determining a left invariant metric on \( SU(2) \)) for \( g = su(2) \) is

\[
a_1 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad a_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad a_3 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix},
\]
then it follows \( \lambda = 1/2 \). We define \( U(1) \in SU(2) \) by [19]

\[
U(1) \overset{def}{=} e^{a_3 \theta} = \begin{pmatrix} e^{i\frac{\theta}{2}} & 0 \\ 0 & e^{-i\frac{\theta}{2}} \end{pmatrix}, |\theta| \in [0, 4\pi[.
\]
For any \( U(1) \) flat connection on a manifold with Betti number \( b_1(M) = 0 \) we have

\[
H_{A_0} = U(1), \quad \dim H^0_{A_0} = \dim H_{A_0} = 1, \quad \dim H^1_{A_0} = 0,
\]
while for trivial $SU(2)$ connection,

$$H_{A_0} = SU(2), \quad \dim H^0_{A_0} = \dim H^1_{A_0} = 3, \quad \dim H^2_{A_0} = 0,$$

and $\zeta(0) = 3 - 0 = 3$. The volumes of $U(1)$ and $SU(2)$ are equal to $V(U(1)) = 4\pi$ and $V(SU(2)) = 16\pi^2$ respectively. Thus we get

$$Z_{sc}(k) = \int_M \langle \Phi V(SU(2))^{-1} e^{ikS(0)/4} \left( \frac{k\lambda}{4\pi^2} \right)^{-\zeta(0)/2} [T_{an}(M)]^{1/2}$$

$$= \sqrt{2}\pi k^{-3/2} [T_{an}(M)]^{1/2}. \quad (3.6)$$

Using the Hodge decomposition, the cohomology $H(X; \xi)$ can be embedded into $\Omega(X; \xi)$ as the space of harmonic forms. This embedding induces a norm $||\cdot||^{RS}$ on the determinant line $\det H(X; \xi)$. The Ray-Singer norm $||\cdot||^{RS}$ on $\det H(X; \xi)$ is defined by [22]

$$||\cdot||^{RS} \overset{def}{=} |\cdot| \prod_{q=0}^{\dim X} \exp \left( -\frac{d}{ds} \zeta_q(s)|_{s=0} \right) \right)^{(-1)^q/2}, \quad (3.7)$$

where the zeta function $\zeta_q(s)$ of the Laplacian acting on the space of $q$-forms orthogonal to the harmonic forms has been used. For a closed connected orientable smooth manifold of odd dimension and for Euler structure $\eta \in \text{Eul}(X)$ the Ray-Singer norm of its cohomological torsion $T_{an}(X; \eta) = T_{an}(X) \in \det H(X; \xi)$ is equal to the positive square root of the absolute value of the monodromy of $\xi$ along the characteristic class $c(\eta) \in H^1(X)$ [24]: $||T_{an}(X)||^{RS} = |\det \xi c(\eta)|^{1/2}$. In the special case where the flat bundle $\xi$ is acyclic ($H^q(X; \xi) = 0$) we have

$$[T_{an}(X)]^2 = |\det \xi c(\eta)| \prod_{q=0}^{\dim X} \exp \left( -\frac{d}{ds} \zeta_q(s)|_{s=0} \right) \right)^{(-1)^{q+1}/2}. \quad (3.8)$$

For odd-dimensional manifold the Ray-Singer norm is topological invariant: it does not depend on the choice of metric on $X$ and $\xi$, used in the construction. But for even-dimensional $X$ this is not the case [25].
4 Manifold with Boundary

The topological Chern-Simons action on a manifold with boundary is

\[
CS(A) = \frac{1}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right) + \frac{1}{4\pi} \int_{\partial M} \text{Tr} A_z A_{\bar{z}}. \tag{4.1}
\]

For closed manifold the boundary term in Eq. (4.1) disappears and the integrand in Eq. (2.1), \( \exp(ikCS(A)) \), is gauge invariant. This invariance is broken for manifold with boundary since under the decomposition \( A = g^{-1}dg + g^{-1}\tilde{A}g \) the action becomes \cite{26, 27}: \( CS(A) = CS(\tilde{A}) + I_{WZW}(g, \tilde{A}_z) \)

where the action \( I_{WZW}(g, \tilde{A}_z) \) of a chiral WZW model on the boundary is given by

\[
I_{WZW}(g, \tilde{A}_z) = \frac{1}{4\pi} \int_{\partial M} \text{Tr} \left( g^{-1}\partial_z gg^{-1}\partial_{\bar{z}}g - 2g^{-1}\partial_z g\tilde{A}_z \right) + \frac{1}{12\pi} \int_M \text{Tr} \left( g^{-1}dg \right)^3. \tag{4.2}
\]

The gravitational analog of the WZW action is especially difficult to construct. Gravity is not quite a pure Chern-Simons theory, however for Euclidean gravity in three spacetime dimensions with negative cosmological constant \( \Lambda = -1 \) one can define an \( SL(2, \mathbb{C}) \) gauge field \( A^a \overset{def}{=} \omega^a + ie^a \), \( \tilde{A}^a \overset{def}{=} \omega^a - ie^a \), where \( e^a = \epsilon^a_{\mu}dx^\mu \) is a triad and \( \omega^a = (1/2)e^{abc}\omega_{\mubc}dx^\mu \) is a spin connection. The standard Einstein action then is easily found to be

\[
I_{GR} = CS(A) - CS(\tilde{A}). \tag{4.3}
\]

It has been shown however \cite{28, 29} that the covariant Chern-Simons action supplemented with boundary conditions \( A_z = \tilde{A}_z = 0 \) does not require any boundary terms. Nevertheless, in this case the analytic torsion of a Riemannian compact manifold with boundary has to be computed. We recall here the definition of \( L^2 \)-analytic torsion (see also \cite{30, 31}). Let \( \mathcal{X} \mapsto X \) be the universal covering of a compact connected Riemannian manifold \( X \), which is of determinant-class. Define the \( L^2 \)-analytic torsion of \( X \) by

\[
T^{(2)}_{an}(X) \overset{def}{=} \sum_{q \geq 0} (-1)^q q \left[ \frac{d}{ds} \Gamma(s) \right]_0^1 t^{s-1} \left. \text{Tr} e^{-t\Delta_+^\frac{1}{2}|\mathcal{X}|} dt \right|_{s=0} + \int_1^\infty \left. \text{Tr} e^{-t\Delta_+^\frac{1}{2}|\mathcal{X}|} \frac{dt}{t} \right|.
\tag{4.4}
\]
Here $\triangle_q[X]$ is the Laplacian on $q-$ forms on the universal covering $X$ considered as an unbounded self-adjoint operator, $\triangle_q^+ [X]$ is the operator from the orthogonal complement of the kernel of $\triangle_q[X]$ to itself which is obtained from $\triangle_q[X]$ by restriction, and $\text{Tr}_\Gamma \exp(-t\triangle_q^+[X])$ is the normalized trace for the $\Gamma = \pi_1(X)$—action (see for detail [32]). The first integral in Eq. (4.4) is defined for $\Re s > (\dim X)/2$, but it has a meromorphic extension to the complex plane $\mathbb{C}$ with no pole in $s = 0$. The second integral converges since $X$ is of determinant-class by assumption. Let \( \{g_u\}_{0 \leq u \leq 1} \) be a smooth family of Riemannian metric on $X$. Let \( V_q = V_q(g_u) \) be defined as $d_{\nu} - \text{Tr}_\Gamma V_q|_{\ker \triangle}$. It can be shown [33, 32] that analytic torsion \( T_{\text{an}}(X) = T_{\text{an}}(X; g_u) \) and $L^2-$analytic torsion \( T_{\text{an}}^{(2)}(X) = T_{\text{an}}^{(2)}(X; g_u) \) are smooth function of $u$, whereas

$$
\frac{d}{du} \left[ T_{\text{an}}^{(2)}(X) - T_{\text{an}}(X) \right]_{u=0} = \sum_q (-1)^{q+1} \left( \text{Tr}_\Gamma V_q|_{\ker \triangle} - \text{Tr}_\Gamma V_q|_{\ker \triangle} \right),
$$

(4.5)

$$
\frac{d}{du} [T_{\text{an}}^{(2)}(X)]_{u=0} = \sum_q (-1)^q \left( d_q - \text{Tr} V_q|_{\ker \triangle} \right).
$$

(4.6)

Let $\bar{X}$ be a compact connected manifold with boundary whose interior $X$ comes with a complete hyperbolic metric of finite volume. It can be shown [32] that the $L^2$- topological torsion of $\bar{X}$ and the $L^2$-analytical torsion of Riemannian manifold $X$ are equal. The $L^2$- topological torsion $T_{\text{top}}^{(2)}(\bar{M})$ in dimension 3 is proportional to the hyperbolic volume of $M$, with a constant of proportionality which depends only on the dimension [32, 34]. This gives a complete calculation of the $L^2$- topological torsion of a compact $L^2$-acyclic 3-manifolds which admit a geometric JSJT- decomposition, i.e. the decomposition of Jaco-Shalen and Johannson by a minimal family of pairwise non-isotopic incompressible not boundary-parallel embedded 2-tori into Seifert pieces and atoroidal pieces. Moreover, there is a dimension constant $C_N$ [32] such that $T_{\text{an}}^{(2)}(X) = C_N \cdot \text{vol}(M)$, and $C_N = 0$ for $N$ even, $C_3 = -1/(3\pi), C_5 = -3/(\pi^2), C_7 = 11/(2\pi^3)$.

5 Connected Sum of 3-Manifolds

The analytic torsion $T_{\text{an}}(\Gamma \backslash H^3)$ can be expressed in terms of the Selberg zeta functions $Z_p(s)$. Let the flat bundle $\xi$ is acyclic. The Ruelle's zeta function
in three dimension associated with closed oriented hyperbolic manifold $M = \Gamma \backslash H^3$ has the form

$$\mathcal{R}_\chi(s) = \prod_{p=0}^{2} Z_p(p + s)^{(-1)^p} = \frac{Z_0(s)Z_2(2 + s)}{Z_1(1 + s)}. \quad (5.1)$$

The function $\mathcal{R}_\chi(s)$ extends meromorphically to the entire complex plane $\mathbb{C}$ [35]. For the Ray-Singer torsion one gets [17]

$$[T_{an}(\Gamma \backslash H^3)]^2 = \mathcal{R}_\chi(0) = \frac{|Z_0(2)|^2}{Z_1(1)} \exp \left( -\frac{V(\mathcal{F})}{3\pi} \right). \quad (5.2)$$

In the presence of non-vanishing Betti numbers $b_j = b_j(M)$ we have

$$[T_{an}(\Gamma \backslash H^3)]^2 = \frac{(b_1 - b_0)!|Z_0^{(b_0)}(2)|^2}{[b_0]!^2 Z_1^{(1)}(b_1 - b_0)(1)} \exp \left( -\frac{V(\mathcal{F})}{3\pi} \right), \quad (5.3)$$

and using Eq. (3.6) one obtains,

$$Z_{sc}(k) = \sqrt{2\pi} k^{-\frac{3}{2}} \left[ \frac{(b_1 - b_0)!|Z_0^{(b_0)}(2)|^2}{[b_0]!^2 Z_1^{(1)}(b_1 - b_0)(1)} \right]^\frac{1}{4} \exp \left( -\frac{V(\mathcal{F})}{12\pi} \right). \quad (5.4)$$

In Chern-Simons theory the partition function for a connected sum $\mathcal{M} = M_1 \# M_2 \# \ldots \# M_N$ can be written as follows [1]

$$Z(\mathcal{M}) = \bigotimes_{\ell=1}^{N} Z(M_{\ell}) \left[ Z(S^3) \right]^{N-1}. \quad (5.5)$$

The fundamental group for 3-sphere $\pi_1(S^3)$ is trivial and $M$ consists of a single point corresponding to $A_\theta$. Since the Ray-Singer torsion is to be equal one (cf. [19]), using Eq. (5.1) we get $Z_{sc}(k) = \sqrt{2\pi} k^{-3/2}$. The partition function associated with the semiclassical approximation then takes the form

$$Z_{sc}(\mathcal{M}) = \left( \frac{k^3}{2\pi^2} \right)^{\frac{N-1}{2}} \bigotimes_{\ell=1}^{N} Z_{sc}(M_{\ell}) = \sqrt{2\pi} k^{-\frac{3}{2}} \bigotimes_{\ell=1}^{N} |\mathcal{R}_{\chi(\ell)}(0)|^\frac{1}{4}, \quad (5.6)$$

while in the presence of non-vanishing Betti numbers $b_{j\ell} = b_j(M_{\ell})$ one gets

$$Z_{sc}(\mathcal{M}) = \sqrt{2\pi} k^{-\frac{3}{2}} \bigotimes_{\ell=1}^{N} \left[ \frac{(b_{1\ell} - b_{0\ell})!|Z_0^{(b_0)}(2)|^2}{(b_0!)^2 Z_1^{(1)}(b_{1\ell} - b_{0\ell})(1)} \right]^\frac{1}{4} \exp \left[ -\frac{1}{12\pi} \bigoplus_{\ell=1}^{N} V(\mathcal{F}_{\ell}) \right]. \quad (5.7)$$
In the case of non-trivial characters $b_0(M) = 0$. If $b_1 = 0$ then Eq. (5.2) holds.

For trivial character one has $b_0 = 1$ (for any closed manifold) and $b_1 = 0$ for an infinite number of $M = \Gamma \setminus H^3$. The function $\mathcal{R}(s)$ has a zero at $s = 0$ of order 4 [36]. However, there is a class of compact sufficiently large hyperbolic manifolds which admit arbitrarily large value of $b_1(M)$. Sufficiently large manifold $X$ contains a surface $S$ whereas $\pi_1(S)$ is infinite and $\pi_1(S) \subset \pi_1(X)$. In general, hyperbolic manifolds have not been completely classified and therefore a systematic computation is not yet possible. However it is not the case for sufficiently large manifolds, classification of which we mention here. These manifolds give an essential contribution to the partition functions (5.4) and (5.7).

It is known that any 3-manifold can be triangulated, and hence can be partitioned into handles. Since the existence and uniqueness of a decomposition of an orientable manifold as the sum of simple orientable parts have been established (see [37, 38]), the question of homeomorphy can be considered only for irreducible manifolds. The method proposed by Haken [39] permits to describe all normal surfaces of a 3-manifold $X$ which has been partitioned on handles previously. The Haken’s theory of normal surfaces was further verified for the procedure of geometric summation of surfaces. As a result the classification theorem [40] says: there exists an algorithm for enumerating of all the Haken manifolds and there exists an algorithm for recognizing homeomorphy of the Haken manifolds.

6 Conclusions

We have derived explicit formulae for the semiclassical approximation for the Chern-Simons partition functions, using the invariant integration method. The resulting expressions are explicitly evaluated for gauge group $SU(2)$ and manifold $\Gamma \setminus H^3$. Manifolds with boundary and $L^2$—topological torsion of such manifolds has been discussed briefly. It will be interesting to obtain the exact formulae for the partition function (and for the partition function of connected sum of manifolds) in the large $k$—limit for more complicate situations, e.g. when a discrete group $\Gamma$ contains elliptic or parabolic elements, and for gauge groups other than $SU(2)$. We hope that proposed discussion of the topological invariants will be interesting in view of future applications.
to concrete problems in quantum field theory.

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