Turbulence and Particle Heating in Advection-Dominated Accretion Flows

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ABSTRACT

We extend and reconcile recent work on turbulence and particle heating in advection-dominated accretion flows. For approximately equipartition magnetic fields, the turbulence primarily heats the electrons. For weaker magnetic fields, the protons are primarily heated. The division between electron and proton heating occurs between $\beta \sim 5$ and $\beta \sim 100$ ($\beta \equiv$ ratio of gas to magnetic pressure), depending on unknown details of how Alfvén waves are converted into whistlers on scales of the proton Larmor radius. We also discuss the possibility that magnetic reconnection could be a significant source of electron heating.

Subject headings: accretion – hydromagnetics – plasmas – turbulence

1. Introduction

Phenomenological models of accretion flows should predict a radiation spectrum for a central object of a given mass using a few relevant dimensionless parameters (e.g., the mass accretion rate in Eddington units, the Shakura-Sunyaev viscosity parameter, $\alpha$, and the ratio of the gas to the magnetic pressure, $\beta$). In advection-dominated accretion flows (ADAFs), which are hot collisionless plasmas that have been argued to form around some accreting black holes, an additional dimensionless parameter, the fraction of the turbulent energy which heats the electrons ($\equiv \delta$), becomes important. In this paper, we discuss the electron heating mechanisms in ADAFs; in particular, we show that $\delta$ should grow as $\beta$ decreases.

In ADAFs (Ichimaru 1977; Rees et al. 1982; Narayan & Yi 1994, 1995; Abramowicz et al. 1995), the accreting gas is unable to cool efficiently and most of the energy generated by viscous stresses is stored as thermal energy of the gas and is advected onto the central object. As a result, the gas heats up to nearly virial temperatures and adopts a two-temperature configuration, with the protons significantly hotter than the radiating electrons (Rees et al. 1982; Narayan & Yi 1995).
In the context of ADAFs, the issue of which particles (electrons or protons) receive the viscous energy acquires particular importance. In “standard” ADAF models (see Narayan et al. 1998 for a review) inefficient cooling of the gas occurs because the viscously generated energy is assumed to primarily heat the protons. Since the accretion flow is effectively collisionless (e.g., Rees et al. 1982) only a small fraction of this energy is transferred to the electrons via Coulomb collisions; the total energy radiated by the gas (almost all by the electrons) is therefore much less than the total energy generated by viscosity. If viscosity were to predominantly heat the electrons, an accretion flow could be advection-dominated only at very low densities (and thus very low accretion rates) when the electrons themselves are unable to cool efficiently. As a result, the optically thin ADAF formalism would probably not be relevant for observed systems.

Recently there has been some theoretical progress in addressing the issue of particle heating in ADAFs. The basic physical picture is that ADAFs, like thin disks, are magnetized turbulent plasmas. On small scales (much less than the outer scale of the turbulence, which is ~the local radius in the accretion flow), the turbulence will be approximately incompressible. Since the Alfvén wave is the incompressible MHD mode, turbulence on relatively small scales can be described as a spectrum of nonlinearly interacting Alfvén waves. These nonlinear interactions transfer energy from large scale perturbations (waves) to smaller scales until it is dissipated. The relative heating of protons and electrons is determined by which particle species is responsible for dissipating the turbulence.

There are (at least) two physical processes which are neglected by simply equating particle heating in ADAFs with particle heating by Alfvénic turbulence. The first (which, in our view, is potentially important) is magnetic reconnection, which may occur at energetically important levels in the accretion flow. This is discussed in §5 and in Appendix B.

The second (which, in our view, is less important) is that, on large scales, the turbulence may have a compressible component. This possibility has been considered (implicitly) by Blackman (1998), who argues that Fermi acceleration by relatively large scale (much larger than \( \rho_p \equiv \text{the Larmor radius of thermal protons} \)) magnetic perturbations associated with MHD turbulence preferentially heats the protons to the degree required by ADAF models. This conclusion is, however, equivalent to considering the collisionless dissipation of the fast mode component of MHD turbulence (which is compressible), and thus does not apply to the entire turbulent cascade (Achterberg 1981). The incompressible component of MHD turbulence, i.e., the Alfvén waves, is undamped on spatial scales much larger than \( \rho_p \) (see §2).

Two qualitative considerations suggest to us that compressible turbulence may be energetically less important than Alfvénic turbulence: (1) the Balbus-Hawley instability, which is likely relevant for the generation of magnetic fields and turbulence in the accretion flow, is non-compressible to linear order (Balbus & Hawley 1991). (2) if the
turbulent velocity is subsonic, the generation of compressive turbulence may be severely suppressed. Throughout this paper, we therefore focus exclusively on Alfvénic turbulence. Our results can, of course, be trivially recast in terms of the fraction (likely $\sim 1$) of the viscously generated energy residing in Alfvénic turbulence.

Gruzinov (1998; hereafter G98) and Quataert (1998; hereafter Q98) analyzed particle heating by the Alfvénic component of MHD turbulence. In this paper we provide a synthesis and extension of their work. We first review Alfvénic turbulence and wave dissipation on length scales comparable to or larger than $\rho_p$ (§2). The primary difference between Gruzinov and Quataert’s analysis is that Quataert’s was restricted to these “large” length scales, while Gruzinov included a preliminary analysis of what happens on smaller scales. In §3 (and Appendix A) we provide a more detailed discussion of turbulence and wave dissipation on length scales smaller than $\rho_p$. We then present a model for how the turbulent energy cascades past $\rho_p$ (§4). The principle results of this paper are given in §5. This contains our best estimate, along with what we take to be plausible uncertainties, of the electron heating rate in ADAFs.

2. Kolmogorov-Goldreich-Sridhar Turbulence

To investigate particle heating by turbulence, we must understand both the dynamics of the turbulent cascade, as well as the dissipation mechanisms which operate on it. Recent work by Goldreich & Sridhar (1995; hereafter GS) provides the necessary characteristics of Alfvénic turbulence. In the MHD limit, linear Alfvén waves satisfy the dispersion relation $\omega = v_A |k_\parallel|$, where $\omega$ is the mode frequency, $v_A$ is the Alfvén speed and $k_\parallel$ is the component of the wavevector along the mean magnetic field.

GS argue (and recent numerical calculations confirm; Maron 1998) that Alfvénic turbulence naturally evolves into a critically balanced state in which the timescale for nonlinear effects to transfer energy from a wavevector $\sim k$ to a wavevector $\sim 2k$ (≡ the cascade time, $T_c$) is comparable to the linear wave period at that scale, $T \equiv 2\pi/\omega$; this determines how rapid the dissipation must be to halt the cascade. It also implies that the cascade is highly anisotropic, with the energy cascading primarily perpendicular to the local magnetic field; the parallel and perpendicular sizes of a wave at any scale are correlated, with $k_\parallel \sim k_\perp^{2/3} R^{-1/3}$, where $R$, the outer scale of the turbulence, is $\sim$ the local radius in the accretion flow. Since the dissipation of an Alfvén wave depends on its direction with respect to the background magnetic field (see below), the path of the cascade in wavevector space is crucial.

In collisionless plasmas, such as those in ADAFs, the wave dissipation mechanisms of principle importance are wave-particle resonances (molecular viscosity, thermal conductivity, electrical resistivity, etc. are entirely unimportant). Resonance occurs when the frequency of the wave, in the frame moving with the particle along the field line, is
an integer multiple of the particle’s cyclotron frequency (\(\Omega\)),

\[
\omega - k\parallel v\parallel = n\Omega,
\]

(1)

where \(v\parallel\) is the particle’s velocity along the magnetic field. For \(n = 0\), resonance occurs when the wave’s phase speed along the field line, \(v = \omega/k\parallel\), equals \(v\parallel\). A necessary (but not sufficient) condition for strong damping is that \(v\) be comparable to the thermal speed of the particles, so that there are a large number of resonant particles.

The \(n = 0\) resonance corresponds to two physically distinct wave-particle interactions. In Landau damping (LD), particle acceleration is due to the wave’s longitudinal electric field perturbation (i.e., the usual electrostatic force, \(E_z\)). The ratio of the electron to the proton heating rates (\(\equiv P\)) for a wave (with \(k_\perp \rho_p \lesssim 1\)) damped solely by LD is \(P_{LD} \approx (m_e T_e^3/m_p T_p^3)^{1/2}\), which is \(\gg 1\) for the \(T_p \gg T_e\) plasmas of interest to us. In transit-time damping (TTD), the magnetic analogue of LD, the interaction is between the particle’s effective magnetic moment (\(\mu = mv_\perp^2/2B\)) and the wave’s longitudinal magnetic field perturbation, \(B_z\) (Stix 1992). For a wave (with \(k_\perp \rho_p \lesssim 1\)) damped solely by TTD, the protons are preferentially heated: \(P_{TTD} \approx (m_e T_e/m_p T_p)^{1/2} \ll 1\). This is because in plasmas with \(T_p \gg T_e\), the protons have the larger magnetic moment and so couple better to a wave’s magnetic field perturbation.

The Alfvén wave has \(|v| = v_A\); in plasmas appropriate to ADAFs (\(T_p \gg T_e\) and \(\beta \gtrsim 1\), where \(\beta\) is the ratio of the gas pressure to the magnetic pressure), \(v_A\) is comparable to the electron and proton thermal speeds and so there are a large number of particles available to resonate with the wave. In the MHD limit, however, the Alfvén wave has both \(E_z = 0\) and \(B_z = 0\) and so is undamped by linear collisionless effects.\(^1\) For larger wavevectors, the MHD approximations are less applicable and kinetic theory corrections to \(E_z\) and \(B_z\) become important, leading to finite dissipation. For the perpendicular cascade of Alfvén waves due to GS, however, when \(k_\perp \rho_p \sim 1\), \(\omega \sim \Omega_p (\rho_p/R)^{1/3} \beta^{-1/2} \ll \Omega_p\) and so \(n \neq 0\) resonances can be satisfied only by particles with \(|v\parallel| \sim |n\Omega| v_A/\omega \gg v_A\), of which there are a negligible number. The cyclotron resonance is thus unimportant in dissipating the turbulent energy in the GS cascade.

In order to accurately assess the properties of Alfvén waves with \(k_\perp \sim \rho_p^{-1} \gg k\parallel\), we have solved the full kinetic theory dispersion relation for linear perturbations to a warm plasma. These calculations are utilized throughout this paper, but we refer the reader to G98 or Q98 for details. Here, and in Figure 1, we summarize the primary results. Alfvén waves with \(k_\perp \sim \rho_p^{-1} \gg k\parallel\) are damped primarily by the protons (via TTD). The dissipation rate, \(\gamma\), is essentially independent of \(T_p/T_e\), but is an increasing

\(^1\)In this and the previous expression we have taken \(v \lesssim \) the electron and proton thermal speeds; see Q98 for a more general expression.

\(^2\)By contrast, the fast mode has \(B_z \neq 0\) in the MHD limit and is thus strongly damped by TTD. This is why protons are preferentially heated by compressive MHD turbulence in ADAFs.
function of $\beta$ ($\gamma/\omega \approx 0.16\beta^{1/2}(k_{\perp}p_p)^2$ for $\beta \gg 1$ and $k_{\perp}p_p \lesssim 1$). For plasmas with $\beta \gg 1$, Alfvénic turbulence is therefore dissipated on length scales $\gtrsim \rho_p$, with most of the turbulent energy heating the protons (this is quantified in §5).

For $\beta = 1$, however, the maximal dissipation rate for an Alfvén wave in the GS cascade (obtained at $k_{\perp}p_p = 1$) is $\gamma T \approx 0.1$ (see Figure 1). Since the timescale for energy to cascade through the inertial range is $T_c \approx T$, this suggests that, for plasmas with equipartition magnetic fields ($\beta \approx 1$), very little of the turbulent energy is dissipated on scales comparable to or greater than $\rho_p$. Consequently, we must investigate the flux of turbulent energy past $k_{\perp}p_p \approx 1$, as well as turbulence and wave dissipation on scales smaller than $\rho_p$.

3. Whistler Turbulence

Alfvén waves only exist for $k_{\perp}p_p \lesssim 1$. For $k_{\perp}p_p \gtrsim 1$, the same mode is called the whistler. In Appendix A we show that all of the turbulent energy which is not damped on scales of $k_{\perp}p_p \lesssim 1$ is transformed into whistlers at $k_{\perp}p_p \gtrsim 1$ (in particular, we argue that there are no other channels through which the energy can travel).

The following argument regarding particle heating by whistler turbulence can be provided. For $k_{\perp}p_p \gtrsim 1$, but $k_{\perp} \gg k_{\parallel}$, whistlers have $\omega \ll \Omega_p$. Thus, in a mode period, a particle undergoes many Larmor orbits. Since $k_{\perp}p_p \gtrsim 1$, the protons (but not the electrons) sample a rapidly varying electro-magnetic field in the course of a Larmor orbit. As a result, the protons are “frozen out” and become dynamically unimportant (they provide a uniform background of positive charge). In particular, they cannot contribute to damping the whistler energy. Whistler energy therefore cascades to smaller length scales until it is damped by the electrons.

The more detailed analysis of whistler turbulence and particle heating given in Appendix A confirms this picture. The result hinges crucially on our estimate that whistler turbulence maintains $k_{\parallel} \ll k_{\perp}$ (recall that the GS cascade gives $k_{\parallel} \sim k_{\perp}(p_p/R)^{1/3} \sim 10^{-3}k_{\perp}$ at $k_{\perp}p_p \sim 1$). In this case Larmor circle averaging is efficient and the whistler energy is dissipated by the electrons (via Landau damping) on scales of $k_{\perp}p_p \sim 30$ (see Fig. 1). If whistler turbulence were to reach the proton cyclotron frequency before $k_{\perp}p_p \sim 10$ (which we argue is unlikely), the whistler energy would heat the protons via the cyclotron resonance (see Appendix A.3).

4. Energy Flux through the Damping Barrier

We therefore assume that most of the energy dissipated on scales $\gtrsim \rho_p$ heats the protons, while the energy that gets past $k_{\perp}p_p \sim 1$ heats the electrons. Here we give a
crude model for calculating the damping of the turbulent energy flux; this provides an estimate of the fraction of the energy flux that gets through the proton damping barrier at \( k_\perp \rho_p \sim 1 \), which is needed to calculate the electron heating rate (\( \delta \)).

We define the full energy spectrum of the turbulence (normalized to the plasma density) as \( E(k_\perp, k_\parallel) \) and the perpendicular energy spectrum as \( E(k_\perp) \):

\[
\langle v^2 \rangle = \int dk_\perp dk_\parallel E(k_\perp, k_\parallel) = \int dk_\perp E(k_\perp),
\]

where \( v \) is the plasma velocity. As we are interested in the flux of energy past \( k_\perp \rho_p = 1 \), we only need \( E(k_\perp) \) for \( k_\perp \rho_p \lesssim 1 \), which is given by the Kolmogorov-Goldreich-Sridhar spectrum

\[
E(k_\perp) = C_1 \epsilon^{2/3} k_\perp^{-5/3},
\]

where \( \epsilon = \frac{cm^2}{s^3} \) is the energy flux in wavenumber space and \( C_1 \) is a dimensionless constant of order unity (a Kolmogorov constant). There is essentially no dissipation at \( k_\perp \rho_p \ll 1 \), and so \( \epsilon \) is constant (\( \equiv \epsilon_0 \)) in this region. At larger \( k_\perp \), the energy flux \( \epsilon \) is a monotonically decreasing function of \( k_\perp \). The decrease of the energy flux is described by

\[
\frac{d\epsilon}{dk_\perp} = -2 \langle \gamma(k_\perp) \rangle E(k_\perp),
\]

where \( \langle \gamma(k_\perp) \rangle \) is the parallel wave number averaged damping rate,

\[
\langle \gamma(k_\perp) \rangle = \frac{\int dk_\parallel \gamma(k_\perp, k_\parallel) E(k_\perp, k_\parallel)}{E(k_\perp)}.
\]

For the GS cascade, the turbulent energy lies inside the cone \( k_\parallel \lesssim \omega_{nl}/v_A \), where \( \omega_{nl} = C_2 \epsilon^{1/3} k_\perp^{2/3} \) is the nonlinear frequency of the turbulence (at scale \( k_\perp \)) and \( C_2 \) is another dimensionless constant of order unity. We model this using a simple expression for \( E(k_\perp, k_\parallel) \), namely,

\[
E(k_\perp, k_\parallel) = E(k_\perp) \delta(k_\parallel - \omega_{nl}/v_A).
\]

This corresponds to all of the turbulent energy flowing along the line in \( k \) space for which the linear mode frequency (\( \omega = v_A|k_\parallel| \)) is equal to the nonlinear frequency of the turbulence (which captures the key physical result of the GS cascade). Using equations (3)-(6), it is straightforward to show that

\[
\epsilon = \epsilon_0 \exp \left[ -2C_1 \int d\log k_\perp D(k_\perp) \right],
\]

where \( D(k_\perp) = \gamma(k_\perp, k_\parallel)/\omega(k_\perp, k_\parallel) \) (with \( k_\parallel = \omega_{nl}/v_A \)) is the dimensionless damping rate of the mode (the inverse of the quality factor; see Fig. 1) and \( C = C_1 C_2 \).

Recent numerical simulations confirm the GS picture of Alfvénic turbulence (Maron 1998). They also give values for the dimensionless constants in the above expressions, namely \( C_1 = 2.5 \pm 0.6 \) and \( C_2 = 2.2 \pm 0.4 \), so that \( C \approx 6 \).
We are interested in the energy flux past \( k_\perp \rho_p \sim 1 \). Since \( D \propto k_\perp^2 \), the integral in equation (7) is dominated by the contribution from \( k_\perp \rho_p \sim 1 \), which lies outside the region where the MHD simulations are valid. Consequently, \( C \approx 6 \) must be taken as a crude approximation, subject to significant uncertainties. The basic uncertainty is that we do not understand the details of how Alfvénic turbulence is converted into whistler turbulence on scales of \( \rho_p \). For example, Alfvén waves with \( k_\perp \rho_p \lesssim 1 \) can excite whistlers with \( k_\perp \rho_p \gtrsim 1 \) by three wave interactions (see Appendix A). However, the effective \( k_\perp \rho_p \) at which this occurs is uncertain. Consequently, so is the upper limit for the integral in equation (7). This uncertainty can be absorbed into an uncertainty in \( C \). In addition, given that Alfvén waves excite whistlers, the precise timescale on which this occurs is uncertain (i.e., is the cascade time at \( k_\perp \rho_p \sim 1 \) the same as it is in the MHD regime?). Small uncertainties in \( C \) translate directly (and exponentially) into large uncertainties in the energy flux past \( k_\perp \rho_p \sim 1 \) and thus into large uncertainties in the predicted electron heating rate.

It should be noted that equation (6) is only a crude approximation for \( E(k_\perp, k_\parallel) \). Given the uncertainty in \( C \), however, alternative expressions do not give significantly different results for the energy flux past \( k_\perp \rho_p \sim 1 \).

5. Electron Heating Rate

As before, we define \( \delta \) to be the fraction of the turbulent energy which heats the electrons; for the model in this paper, it is given by \( \delta \approx P_{TTD} + \epsilon / \epsilon_0 \). \( P_{TTD} \) is the (generally small) contribution to the electron heating from Alfvén wave energy dissipated at \( k_\perp \rho_p \lesssim 1 \). \( \epsilon / \epsilon_0 \) is the fraction of the turbulent energy which cascades past the damping barrier at \( k_\perp \rho_p \sim 1 \) (essentially all of which heats the electrons). This is given by equation (7) of the previous section.

Figure 2 shows our best estimates of \( \delta \) as a function of \( \beta \) for \( C = 24, 6, \) and 1.5 (from left to right). For small \( \beta \), the electron heating is dominated by the energy that cascades past \( k_\perp \rho_p \sim 1 \). This is independent of the proton to electron temperature ratio (since the mode dissipation rate at \( k_\perp \rho_p \lesssim 1 \) is; see §2), but is a strong function of \( \beta \). For larger \( \beta \) the dominant contribution to the electron heating is from the dissipation of Alfvén waves at \( k_\perp \rho_p \lesssim 1 \) (since almost no energy cascades past \( k_\perp \rho_p \sim 1 \)). Since \( P_{TTD} \approx (m_e T_e / m_p T_p)^{1/2} \), this is a strong function of \( T_p / T_e \), but is independent of \( \beta \). In Figure 2, we have taken \( T_p / T_e = 100 \), appropriate to accretion near the Schwarzschild radius of a black hole (where most of the observed radiation originates). Variations in \( T_p / T_e \) vertically shift the value of \( \delta \) when it plateaus (i.e., the high \( \beta \) values), but do not significantly modify the \( \beta \) at which the plateau occurs.

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\(^3\)For example, as noted above, the GS cascade actually lies inside the cone \( k_\parallel \lesssim \omega_{nl} / v_A \), not on the line \( k_\parallel = \omega_{nl} / v_A \).
From Figure 2 we infer that turbulence in ADAFs predominantly heats the protons \((\delta \lesssim 10^{-2}, \text{say})\) only for \(\beta\) larger than some critical value, which lies between \(\sim 5\) and \(\sim 100\) for the values of \(C\) taken in Figure 2. These values encompass what we feel to be a reasonable estimate of the uncertainty in how the turbulent energy cascades past \(k_\perp \rho_p \sim 1\). The corresponding uncertainty in \(\delta\) is extremely large, but this is an accurate reflection of the (exponential) sensitivity of our results to unknown details of turbulence on scales of the proton Larmor radius.

The conclusions to be drawn from Figure 2 depend, of course, on the degree of one’s confidence that all possible sources of electron heating have been accounted for. The most accurate interpretation of Figure 2 is that it represents the fraction of the small scale (Alfvénic) turbulent energy which heats the electrons. In §1, we argued that this interpretation can be broadened to be (within a factor of few) the fraction of both compressible and incompressible turbulent energy which heats the electrons (since compressible turbulence is unlikely to energetically dominate Alfvénic turbulence and, in any case, does not significantly heat the electrons).

In Appendix B, however, we argue that, at large \(\beta\), the physical picture and calculations leading to Figure 2 neglect a source of electron heating (possibly the most important one), namely magnetic reconnection. The reasoning is as follows. Proton damping of the turbulent energy at large \(\beta\) is essentially a viscous dissipation mechanism (since the protons carry the momentum of the plasma). For a nontrivial topology, the magnetic free energy of the turbulence cannot be fully damped by viscosity. This is because changes in the magnetic topology, magnetic flux, magnetic helicity, etc., can only be due to resistive effects (regardless of how small the resistivity is). This suggests that coincident with the dissipation of turbulent energy by Alfvén wave damping is the formation of current sheets in which the topology of the magnetic field changes. Crude estimates suggest that the energy dissipation in current sheets may be \(\sim\) that of Alfvén wave damping. Unfortunately, we cannot assess whether \(\sim\) corresponds to \(1\) or \(10^{-2}\), etc., and thus we cannot assess whether reconnection is energetically important (that it is required for topological changes is clear). One way to address this question is through numerical simulations of MHD turbulence with varying magnetic Prandtl numbers (ratio of viscosity to resistivity). The relative contributions of Joule heating and viscous heating in such simulations could potentially provide important information on the energetic importance of magnetic reconnection in large magnetic Prandtl number turbulence.

6. Conclusions

Spectral models of ADAFs generally assume that (see, e.g., Narayan et al. 1998) (1) magnetic fields are amplified until they are in strict equipartition with gas pressure
(\(\beta = 1\))\(^4\) and that (2) the energy generated by viscous stresses predominantly heats the protons (\(\delta \approx 10^{-2}\) is a typical value).

We suggest that these assumptions may be incompatible. Based on an analysis of incompressible turbulence (Alfvénic and whistler) and collisionless wave dissipation, we find predominantly proton heating only for \(\beta\) greater than some critical value, which lies between \(\sim 5\) and \(\sim 100\) depending on unknown details of how Alfvén waves are converted into whistlers on scales of the proton Larmor radius. This does not, of course, imply that ADAF models are untenable. Rather, their sensitivity to changes in input microphysics (e.g., \(\beta\) and \(\delta\)) should be carefully assessed. This will be pursued in a future paper.

We cannot overemphasize the uncertainty in the numerical values given in this paper (which are, e.g., based on the assumption of a uniform thermal plasma). Nonetheless, we believe that the basic physical picture of particle heating (wave damping + reconnection), and the general conclusions drawn from it (\(\delta\) increases as \(\beta\) decreases), are essentially correct. One point which clearly requires additional investigation is the energetic importance of magnetic reconnection (§5 and Appendix B).

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\(^4\)Narayan et al. define the magnetic pressure as \(B^2/24\pi\) while the \(\beta\) used in this paper defines magnetic pressure as \(B^2/8\pi\). Thus equipartition for Narayan et al. actually corresponds to \(\beta = 1/3\).
A. Whistler Turbulence

The purpose of this appendix is three-fold. First, we argue that Alfvénic turbulence is converted into whistler turbulence on scales of the proton Larmor radius. While the details of this process are uncertain, all of the turbulent energy not damped from Alfvén waves on scales of \( \sim \rho_p \) is transformed into whistler energy on smaller scales (there are no other channels through which the energy can travel). Second, we discuss the turbulent cascade of whistlers. Finally, we discuss the dissipation of whistler turbulence. This, together with §2 on Alfvénic turbulence, provides a picture (if approximate and uncertain in detail) of turbulence and particle heating from the outer to the inner scale.

A.1. Alfvén → Whistler: \( k_\perp \rho_p \sim 1 \)

For \( \beta \sim 1 \), the full kinetic theory dispersion relation evolves continuously from Alfvén waves to whistlers as we pass \( k_\perp \rho_p \sim 1 \). This is because whistlers are the natural generalization of Alfvén waves once protons drop out of the small-scale dynamics due to Larmor circle averaging. Consequently, Alfvén waves at \( k_\perp \rho_p \lesssim 1 \) can excite whistlers at \( k_\perp \rho_p \gtrsim 1 \) by three-wave interactions (i.e., the resonance conditions can be satisfied).\(^5\) For \( \beta \gtrsim 100 \), the situation is more complicated because there is a region (the “damping barrier”) of wavevector space (0.85 \( \lesssim k_\perp \rho_p \lesssim 1.15 \) for \( \beta \sim 100 \) and larger for larger \( \beta \)) in which Alfvén waves do not propagate (see G98). Because this region is rather narrow for the \( \beta \) of interest to us, Alfvén waves can excite whistlers across the damping barrier (narrow meaning that the jump in \( k_\perp \) is less than a factor of \( \sim 2 \)). This establishes that the turbulent energy not damped on scales of \( k_\perp \rho_p \sim 1 \) can be converted to whistler energy on smaller scales (although, as discussed in §4, the precise details – e.g., the timescale – are uncertain).

To establish that no other channels of energy travel are possible is straightforward. Whistlers are the only modes with \( k_\perp \rho_p \gtrsim 1 \) and \( \omega \ll \Omega_p \) (“sound waves” are too strongly damped to be excited in collisionless plasmas; see Q98). They are therefore the only possible sink of the Alfvénic energy not damped on scales of \( k_\perp \rho_p \sim 1 \).

A.2. Whistler Cascade

Like collisionless Alfvén waves, collisionless whistlers can be described hydrodynamically. Hydrodynamic equations valid at \( k_\perp \rho_p \gtrsim 1 \) can be obtained as follows. The protons are dynamically frozen; their sole function is to create a positive charge back-

\(^5\) Note that because the GS cascade is strong, the frequency “resonance” condition is actually quite broad.
ground on which the electrons and magnetic fields evolve. Electrons move freely along magnetic field lines and so $E_\parallel = 0$. In the perpendicular direction, electrons move with the $\text{E} \times \text{B}$-drift velocity, $v_\perp = cE \times B/B^2$. These two equations give

$$E + \frac{1}{c}v \times B = 0.$$  \hspace{1cm} (A1)

Neglecting displacement currents,

$$\nabla \times B = -\frac{4\pi}{c} nev.$$  \hspace{1cm} (A2)

From equations (A1) and (A2), we obtain the Electron Magnetohydrodynamics (EMHD) equation (Kingsep, Chukbar, & Yan’kov 1990)

$$\partial_t B = \frac{c}{4\pi ne} \nabla \times (B \times \nabla \times B).$$  \hspace{1cm} (A3)

The linear waves in equation (A3) are whistlers; their dispersion relation can be presented as

$$\omega = \frac{v_p^2}{v_p^2} |(k_\perp \rho_p)k_\parallel|,$$

where $v_p$ is the proton thermal speed.

To see at what scales the protons freeze out, we compared the whistler dispersion relation obtained numerically from a full plasma permittivity tensor (including both electrons and protons) to the analytical dispersion relation (A4). At $\beta = 100$, the dispersion relations agree to 30% at $k_\perp \rho_p = 1.5$, to 10% at $k_\perp \rho_p = 2$, and to 3% at $k_\perp \rho_p = 3$.\textsuperscript{6}

The turbulent cascade in (A3) was described by Kingsep, Chukbar, & Yan’kov (1990). A very clear discussion is given by Goldreich & Reisenegger (1992). These authors assumed that whistler turbulence is roughly isotropic in wavenumber space, and that the turbulence is weak. Their argument for the turbulence being weak is analogous to the standard argument for the “weakness” of Alfvénic turbulence. Assuming strong isotropic turbulence, one calculates a nonlinear frequency at a given scale. This frequency turns out to be smaller than the linear wave frequency; the turbulence must therefore be weak.

As explained by GS for the MHD case, however, the above “explanation” does not necessarily work. Instead of being weak and isotropic, the turbulence is strong in a narrow cone in wavenumber space ($k_\parallel \ll k_\perp$), for which the linear frequency is smaller than the nonlinear frequency. Numerical simulations seem to confirm the GS picture in the Alfvén wave case (Maron 1998). But the same picture must hold true

\textsuperscript{6}Other relevant plasma parameters in this example are $v_e/c = 0.5$, $v_p/c = 0.33$, $k_\parallel \ll k_\perp$. For details see Q98 or G98.
(to some extent) in the whistler case. In a narrow cone in wavenumber space, the
 turbulent cascade can proceed at a (fast) nonlinear rate. Outside of the cone, the cascade
 slows down (the turbulence is weak). This is because nonlinear interactions outside the
 cone must satisfy the frequency resonance condition (interactions are restricted to the
 resonant manifold).

For our problem, energy is injected into whistler turbulence in the narrow cone for
 which the turbulence is strong (because the Alfvénic turbulence which excites whistler
 turbulence is itself strong). This suggests that a large part of the energy will continue
to cascade in the strong cone, maintaining $k_\parallel \ll k_\perp$. Numerical simulations in two di-
mensions (Biskamp, Schwarz & Drake 1996) confirm that strong Kolmogorov turbulence
results in both the Alfvén and the whistler cases.

We have argued that the energy in whistler turbulence injected at $k_\parallel \ll k_\perp$ preserves
this anisotropy because the turbulent energy cascade is faster in the strong cone where
the nonlinear frequency shifts exceed the linear frequency. We cannot, however, rule
out that the strong cone may be leaky. It is a peculiar feature of Alfvénic turbulence,
not shared by its whistler counterpart, that the 3-wave (and 4-wave) resonance condi-
tions drive the turbulent energy into the strong cone, regardless of their starting point
in wavevector space. Our analysis of whistler turbulence is therefore only suggestive.
Thankfully, the precise path of whistler turbulence in wavevector space is not needed to
assess its particle heating properties. All that is needed is that it maintains $k_\parallel \ll k_\perp$
(this is quantified below).

### A.3. Whistler Dissipation

In this subsection, we assess the dissipation of whistler turbulence. Assuming that
whistler turbulence maintains $k_\parallel \ll k_\perp$, we show that the protons cannot be heated by
whistlers.

For a (subthermal) wave with $k_\perp \rho_p \gtrsim 1$ damped solely by transit time damping,
the relative heating of electrons and protons is given by

$$P_{TTD} \sim \left( \frac{m_e T_e}{m_p T_p} \right)^{1/2} (k_\perp \rho_p)^3.$$ \hfill (A5)

The $(k_\perp \rho_p)^3$ factor describes quantitatively the effects of Larmor circle averaging (com-
pare with the $k_\perp \rho_p \lesssim 1$ expression in §2). For $T_p \sim 100 T_e$, electrons are heated more
than protons for $k_\perp \rho_p \gtrsim 7$. The decrease in the whistler dissipation rate for $k_\perp \rho_p \gtrsim 1$
in Figure 1 is due to the freezing out of the protons. Even though the protons are in
principle heated more than the electrons for $k_\perp \rho_p \lesssim 7$, the decrease in the dissipation

\footnote{Note that for all $k_\perp \rho_p$ Landau damping preferentially heats the electrons for $T_p \gg T_e$.}
rate ensures that very little energy is dissipated between $1 \lesssim k_{\perp} \rho_p \lesssim 7$. For length scales smaller than $k_{\perp} \rho_p \sim 7$, Larmor circle averaging entails that protons cannot be efficiently heated by the $n = 0$ Landau resonance (see eq. [1]). This conclusion is essentially independent of the details of the whistler cascade.

For $k_{\perp} \rho_p \gtrsim 1$, protons can also be heated by the cyclotron resonance (for wave frequencies $\sim$ the proton cyclotron frequency; the $|n| = 1$ resonance in eq. [1]). For large $k_{\perp} \rho_p$, the dominant electron heating mechanism is Landau damping. The relative importance of proton heating by the cyclotron resonance and electron heating by Landau damping is given roughly by $(m_p/m_e)(k_{\perp} \rho_p)^{-3}$, where we have taken $v_A \sim v_p \sim v_e \sim v \sim c$ and $\omega \sim \Omega_p$.\(^8\)

For $k_{\perp} \rho_p \gtrsim 10$, Larmor circle averaging is sufficiently strong that proton heating by the cyclotron resonance is less important than electron heating by Landau damping.\(^9\) Only if whistler turbulence reaches $\omega \sim \Omega_p$ before $k_{\perp} \rho_p \sim 10$ can the protons be heated by whistler turbulence. This would require a complete reversal in the direction of the turbulent cascade since at $k_{\perp} \rho_p \sim 1$ the GS cascade yields $\omega \sim 10^{-3}\Omega_p$. Our estimates of whistler turbulence in the previous subsection suggest that it reaches $\omega \sim \Omega_p$ only for $k_{\perp} \rho_p \gg 10$. Consequently, Larmor circle averaging of the protons is efficient and the whistler energy is damped by the electrons (by Landau damping) on scales of $k_{\perp} \rho_p \sim 30$ (see Figure 1).

**B. Reconnection**

In this appendix, we discuss why reconnection (Joule heating) may be the primary heating mechanism for the electrons, especially at low magnetic fields (when whistlers are not significantly excited). We argue that, to order of magnitude, comparable amounts of energy are damped by viscous heating of the protons and by Joule heating of the electrons. Bisnovatyi-Kogan & Lovelace (1997) first suggested that Joule heating of the electrons could be important in ADAFs (we do not, however, agree with the particular scenario they propose; see Q98 or Blackman (1998)). Blandford (1998) also suggested that reconnection could be important. The following analysis attempts to assess the potential significance of reconnection in a semi-rigorous manner.

For $\beta \gg 1$, Alfvén waves are damped at $k_{\perp} \rho_p < 1$ and the plasma dynamics can be described by the MHD equations. There are two heating mechanisms in the framework of MHD - viscous heating and Joule heating. Roughly speaking, viscosity ($\nu$) heats the particles that carry the momentum, i.e., the protons, and resistivity ($\eta$) heats the

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\(^8\)In deriving this expression we have used an analytical approximation for the relative strengths of the perpendicular and parallel electric field components of the whistler wave (from G98).

\(^9\)Our numerical calculations with the full plasma permittivity tensor confirm this result.
particles that carry the current, i.e., the electrons. We know that $\nu \gg \eta$ in a $\beta \gg 1$ plasma - protons are heated when Alfvén waves are damped by viscosity.

Is it true, however, that Joule (electron) heating dissipates much less energy than viscous (proton) heating in a $\nu \gg \eta$ plasma (i.e., for large magnetic Prandtl numbers)? In other words, can Alfvén wave damping actually dissipate all of the turbulent energy?

The magnetic field of the accreting plasma is ostensibly self-generated. The magnetic field energy is constantly being created, destroyed, and created again. To demonstrate that reconnection is unavoidable, consider the relaxation of a magnetic field perturbation created “by hand” at time zero. This initial magnetic field is assumed to be a generic solenoidal field with a characteristic strength $B_0$ and a characteristic scale $L$; we also assume that the normal component of the field is specified at the boundary of a box with side $\sim L$. Now let the field evolve according to MHD equations with $\nu \gg \eta$.

Magnetic pressure and tension cause plasma motions on scales $\sim L$. A Kolmogorov-GS cascade develops and magnetic energy is converted into kinetic energy; viscous heating by the protons dissipates the kinetic energy. In a time $\sim L/v_A$, the kinetic energy goes to zero and the magnetic field reaches an equilibrium state defined by

$$\nabla \times \mathbf{B} \times \mathbf{B} = 4\pi \nabla p,$$  \hspace{1cm} (B1)

where $p$ is the plasma pressure. Generically $B^2 \sim B_0^2 \sim (B_0 - B)^2$ so that the energy heating the protons is $\sim B_0^2 L^3$.

As explained by Arnold (1986), the equilibrium magnetic field lines described by equation (B1) have a very special topology. They lie on a set of nested tori with $p = \text{const}$. For $\eta = 0$, however, the magnetic field is frozen into the plasma (magnetic flux, helicity, and the topology of the field lines are all conserved). Since the initial magnetic field topology was a generic one (by assumption), it is impossible to reach the equilibrium configuration with $\eta = 0$ (since that requires changing the topology of the field). On the other hand, the equilibrium must be reached, because out of equilibrium the magnetic field causes plasma motions and therefore viscous heating.

The resolution of the paradox is the formation of current sheets. Topological barriers on the way to equilibrium are squeezed to zero volume. No matter how small, resistivity becomes important in the current sheets thus allowing reconnection ($\equiv$ changes in the magnetic field topology). Generically the magnetic free energy released at this stage is $\sim B_0^2 L^3$.

The only way to avoid reconnection is to assume that the magnetic field has a trivial topology. For example, one can assume that magnetic field lines constantly lie on surfaces that are topologically equivalent to a set of nested tori. The dynamo nature of the magnetic field in accretion flows, however, seems incompatible with a trivial topology (cf the complex structures observed in numerical simulations of the Balbus-Hawley instability).
While we do not know how to rigorously estimate the fraction of the turbulent energy damped in reconnection events, the potential importance of reconnection for electron heating, and its unavoidability even in large magnetic Prandtl number plasmas, seems clear.

References
Maron, J. 1998, private communication
Fig. 1.— Dimensionless dissipation rate of the Alfvén ($k_{\perp}\rho_p < 1$) and whistler ($k_{\perp}\rho_p > 1$) modes, taking $k_{||} \ll k_{\perp}$. The solid (dashed) curve is for $\beta = 1(10)$, while both curves take $T_p = 100T_e$. The peak in the dissipation at $k_{\perp}\rho_p \sim 1$ corresponds to proton heating while the strong damping at $k_{\perp}\rho_p > 30$ corresponds to electron heating.
Fig. 2.— Estimates of $\delta$, the fraction of the turbulent energy which heats the electrons, versus $\beta$ (for $T_p = 100 T_e$). The three curves correspond to (from left to right) $C = 24$, 6, and 1.5 (see eq. [7]).