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CLASSIFYING ALGEBRAS FOR BOUNDARY CONDITIONS AND TRACES ON SPACES OF CONFORMAL BLOCKS

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Abstract
The boundary conditions of a non-trivial string background are classified. To this end we need traces on various spaces of conformal blocks, for which generalizations of the Verlinde formula are presented.

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Introduction.

It has been known for a long time that the low energy effective action of superstring theories has solitonic solutions. Recently, it has become apparent that string perturbation theory in such a background can be formulated in terms of world sheets with boundaries, where one imposes certain non-trivial boundary conditions. Therefore, theories of open strings and conformal field theories on two-dimensional surfaces with boundaries have received renewed interest. A central problem in these theories is the classification of all consistent boundary conditions. So far, however, most investigations have been limited either to models based on free world sheet theories or on orbifolds of such theories, or to BPS-sectors of models with extended supersymmetry. In this contribution we discuss the structure of boundary conditions in an arbitrary rational conformal field theory with a specific type of non-diagonal modular invariant. For a general discussion of boundary conditions in two-dimensional conformal field theory we refer to [1].

Modular invariants.

A chiral conformal field theory typically admits several consistent torus partition functions. Any non-trivial modular invariant of a rational conformal field theory can be obtained [2] by first extending the chiral algebra and then superposing an automorphism of the fusion rules. The extension of chiral algebras is by now fairly well understood, at least in the case of extensions by so-called simple currents [3, 4, 5], and can be described entirely in terms of a chiral half of the theory. As a consequence, such extensions do not raise any problem in the construction of open string theories that was not already encountered for closed strings so that we can assume that the modular invariant in question is of pure automorphism type. Hence the modular invariant describes the pairing between left-moving and right-moving fields, or, more precisely [1], the pairing between the two chiral conformal field theories on the oriented cover of the world sheet.

For simplicity, we also make a few more assumptions on the boundary conditions we consider: first, we assume that the pairing for bulk fields is the same as in the case of closed orientable surfaces. In the terminology of Ref. [1], this means that we choose a trivial automorphism type, i.e. generalized Neumann boundary conditions. Next, we assume that the boundary preserves all symmetries of the bulk. Finally, we assume that we are dealing with one and the same chiral conformal field theory on every type of two-dimensional surface. (This implies for instance [1] that D-brane configurations with multiple wrapping are excluded.)

Under these assumptions, the possible boundary conditions have been classified for a theory with the charge conjugation modular invariant [6]. A first investigation in the case of non-trivial modular invariants has been undertaken in [7] for WZW–models based on SU(2).

The type of modular invariant we focus on generalizes the modular invariants that in the A-D-E classification of SU(2) modular invariants are of $D_{\rm odd}$-type. This modular invariant exists for level $k = 4l + 2$ with l integer:

$$Z(\tau) = \sum_{l=0}^{k/2} |\chi_{2l}|^2 + \sum_{l=0}^{k/2} \chi_{2l+1}\chi^*_{k-2l-1} \,. \tag{1}$$

The full conformal field theory described by this modular invariant can be regarded as a $\mathbb{Z}_2$-orbifold of the WZW–theory on SU(2) with the diagonal modular invariant. There are three types of primary fields:

- Primary fields with integral isospin form the untwisted sector of the orbifold. In the full theory, they are paired with themselves.
- The twisted sector consists of the primary fields with half-integral isospin. These fields Φ_l are paired with some other primary field Φ_{k-l} which is obtained by taking the fusion product with the primary field Φ_k of highest possible isospin, $\Phi_{k-l} = \Phi_k \star \Phi_l$. The primary field Φ_k is a *simple current*: its fusion product with any other primary field contains just one primary field with multiplicity one.
- The twisted sector contains in particular the *fixed point* $\Phi_{k/2}$ which is mapped by the simple current to itself, $\Phi_k \star \Phi_{k/2} = \Phi_{k/2}$. In the twisted sector, we distinguish between fixed points and non-fixed points.

In this note we consider conformal field theories with a similar $\mathbb{Z}_2$ symmetry: we assume that the theory contains a simple current J, i.e. a primary field such that its fusion product has the form $J \star \Phi_\Lambda = \Phi_{J\Lambda}$, for any primary field Φ_Λ. Moreover, we assume that J squares to the vacuum primary field, $J^2 = \Phi_0$, and that it has conformal weight $\Delta_J \in \mathbb{Z} + 1/2$. Given such a simple current J, we associate to every primary field Φ_Λ its *monodromy charge*

$$Q_J(\Lambda) := \Delta_\Lambda + \Delta_J - \Delta_{J\Lambda} \mod \mathbb{Z} \tag{2}$$

which generalizes the conjugacy class and is conserved in operator products. One can show that in this situation the following expression gives a modular invariant partition function.

$$Z = \sum_{Q(\Lambda)=0} \chi_\Lambda \chi^*_{\Lambda^+} + \sum_{Q(\Lambda)=1/2} \chi_\Lambda \chi^*_{J\Lambda^+} \tag{3}$$

(This partition function looks like a $\mathbb{Z}_2$ orbifold of the original theory. It has a similar structure as the one of the type IIA superstring in light cone gauge.) Again we have three types of primary fields: N_0 primary fields in the untwisted sector with monodromy charge $Q(\Lambda) = 0$. They all form orbits of length 2 under the action of J. In the twisted sector, all fields have $Q(\Lambda) = 1/2$. Among these we have N_1 fields on full orbits and N_f fixed points.

Construction of the classifying algebra.

It has been shown in [1] that a consistent boundary condition in a conformal field theory can be described by an automorphism of the fusion rules which preserves conformal weights, the *automorphism type* of the boundary condition, and a further degeneracy label, the *Chan-Paton type.* Moreover, it has been shown [1] that each Chan-Paton type corresponds to an irreducible representation of a so-called *classifying algebra.* This classifying algebra $\tilde{\mathcal{A}}$ has been computed in [7] in the special case of WZW–theories based on SU(2), using the explicit form of fusing matrices and operator product coefficients. It was observed that the possible Chan-Paton types are in one-to-one correspondence to the orbits of the simple current J, where the fixed points of J are counted with multiplicity two.

Traces on the space of conformal blocks.

The construction of the correct classifying algebra in the general case (3) requires the knowledge of some traces on the spaces of conformal blocks. In general, we consider the finite-dimensional vector space $B_{\vec{\Lambda}}$ of conformal blocks, where $\vec{\Lambda} = (\Lambda_1, \ldots, \Lambda_n)$ stands for a finite sequence of primary fields. In the case of three-point blocks $B_{\lambda\mu\nu}$ its dimension is given by the Verlinde formula

$$\mathcal{N}_{\lambda\mu\nu} = \dim B_{\lambda\mu\nu} = \sum_\rho S_{\lambda\rho} S_{\mu\rho} S_{\nu\rho} \,/\, S_{0\rho} \,. \tag{4}$$

We now consider a collection $\vec{J} = (J_1, \ldots, J_n)$ of simple currents which fulfils $J_1 \star J_2 \star \cdots \star J_n = 1$. In this situation one can define a natural isomorphism between the spaces of conformal blocks

$$\Theta_{\vec{J}} : \quad B_{\vec{\Lambda}} \to B_{\vec{J}\vec{\Lambda}} \,, \tag{5}$$

where we introduced the short hand $\vec{J}\vec{\Lambda} = (J_1\Lambda_1, \ldots, J_n\Lambda_n)$. In the case of a simple current of order two, there are in particular isomorphisms:

$$\Theta_J : \quad B_{\lambda\mu\nu} \to B_{J\lambda\, J\mu\, \nu} \,. \tag{6}$$

If $\lambda = f$ and $\mu = g$ are fixed points of J, then Θ_J is an *endomorphism* and we can consider the trace

$$\check{\mathcal{N}}_{fg\nu} := \mathrm{Tr}_{B_{fg\nu}} \Theta_J \,. \tag{7}$$

The trace $\check{\mathcal{N}}_{fg\nu}$ is an integer (surprisingly enough this is also true for simple currents of any arbitrary order), and can be used to compute the dimension of the eigenspaces of Θ_J to the eigenvalues ± 1:

$$\mathrm{Tr}_{B_{fg\mu}}(\tfrac{1}{2}(1 \pm \Theta_J)) = \tfrac{1}{2}(\mathcal{N}_{fg\mu} \pm \check{\mathcal{N}}_{fg\mu}) \,, \tag{8}$$

which are manifestly non-negative integers.

These traces have already played an important role in chiral conformal field theory, in the analysis of the twisted sector of extensions [3]. One finds that there is a *fixed point theory* with modular matrix $\check{S}$ whose primary fields are in one-to-one correspondence with the fixed points. The traces are then given by a generalization of the Verlinde formula:

$$\check{\mathcal{N}}_{fg\nu} = \sum_{h \text{ fix}} \check{S}_{fh} \check{S}_{gh} S_{\nu h} \,/\, S_{0h} \,. \tag{9}$$

For WZW–theories (and also for coset conformal field theories) the fixed point theories are obtained by folding Dynkin diagrams [8]. In general, it is conjectured that they describe the modular properties of the one-point blocks on the torus with insertion the simple current J. The structure of fixed point theories is found in many places: it occurs in the twisted sector of extension modular invariants [3], in the very definition of coset conformal field theories [9], in the Verlinde formula for WZW-theories based on non-simply connected Lie groups [3] and in the topologically non-trivial components of the moduli spaces of holomorphic principal bundles with non-simply connected structure groups over an elliptic curve [10].

As a side remark we mention some other traces on the space of conformal blocks which can be computed explicitly: on a four-point block on the sphere with two identical primary fields one can consider the trace $Y_{i,i,j,k}$ of the permutation acting on the two identical insertions. Such traces appear in the theory of permutation orbifolds [11] as well as in the description of amplitudes on the Möbius strip [12]. Again, one has a generalization of the Verlinde formula:

$$Y_{i,i,j,k} = \sum_n P_{jn} P_{kn} S_{in} \,/\, S_{0n} \,, \tag{10}$$

where we have introduced the matrix $P = T^{1/2}ST^2ST^{1/2}$.

The classifying algebra and its representation theory.

We are now in a position to display the classifying algebra $\tilde{\mathcal{A}}$ [13] for the conformal field theory with torus partition function (3). The dimension of $\tilde{\mathcal{A}}$ equals the number of bulk fields, $\dim\tilde{\mathcal{A}} = N_0 + N_f$. Moreover $\tilde{\mathcal{A}}$ is $\mathbb{Z}_2$-graded and contains the fusion algebra of fields in the untwisted sector as a subalgebra, since the operator products in the untwisted sector of an orbifold theory are the same as in the original theory. This leads to the following structure constants for $\tilde{\mathcal{A}}$:

$$\tilde{\mathcal{N}}_{\lambda\mu}^{\nu} = \begin{cases} \mathcal{N}_{\lambda\mu}^{\nu} & \text{if } Q(\lambda)=Q(\mu)=Q(\nu)=0 \\ \check{\mathcal{N}}_{\lambda\mu}^{\nu} & \text{if there is precisely one full orbit} \\ 0 & \text{else} \end{cases} \tag{11}$$

One easily checks that this classifying algebra $\tilde{\mathcal{A}}$ is commutative and associative, that Φ_0 is a unit element and that the evaluation on the identity gives a conjugation on $\tilde{\mathcal{A}}$: $\check{\mathcal{N}}_{fg}^{0} = \delta_{f^+,g}$. As a consequence, $\tilde{\mathcal{A}}$ is still semi-simple, but in contrast to fusion algebras, its structure constants can also be negative. Notice that $\tilde{\mathcal{A}}$ is *not* a subalgebra of the fusion algebra $\mathcal{A}$.

The representation theory of $\tilde{\mathcal{A}}$ has the following structure: there are $N_0+N_f = \frac{1}{2}(N_0+N_1)+2N_f$ irreducible representations, which are all one-dimensional. They are in correspondence to orbits of the simple current J: each of the orbits α of length two gives rise to a single irreducible representation $\mathcal{R}_{(\alpha)}$:

$$\mathcal{R}_{(\alpha)}(\Phi_\mu) = \begin{cases} S_{\alpha\mu}\,/\,S_{0\alpha} & \text{for} \quad Q(\mu)=0\,, \\ 0 & \text{for} \quad J\mu=\mu\,. \end{cases} \tag{12}$$

Each of the orbits f of length one, the fixed points, gives rise to *two* different irreducible representations $\mathcal{R}_{(f+)}$ and $\mathcal{R}_{(f-)}$:

$$\mathcal{R}_{(f\pm)}(\Phi_\mu) = \begin{cases} S_{f\mu}\,/\,S_{0f} & \text{for} \quad Q(\mu)=0\,, \\ \pm\check{S}_{f\mu}\,/\,S_{0f} & \text{for} \quad J\mu=\mu\,. \end{cases} \tag{13}$$

Notice that fixed points are fields in the twisted sector, and accordingly the modular matrix $\check{S}$ of the fixed point theory appears.

The annulus amplitude, consistency checks.

To be able to perform several consistency checks, we compute the amplitude $A_{ab}(t) = \sum_\mu A_{ab}^{\mu}\chi_\mu(\frac{it}{2})$ for an annulus, where we impose boundary condition a respectively b on the two boundaries. We obtain the following result for the tensor A_{ab}^{μ}:

$$\begin{aligned} A_{\alpha\beta}^{\mu} &= \mathcal{N}_{\beta\mu}^{\alpha} + \mathcal{N}_{\beta\mu}^{J\alpha}\,, & A_{\alpha(f\pm)}^{\mu} &= \mathcal{N}_{f\mu}^{\alpha}\,, \\ A_{(f\pm)(g\pm)}^{\mu} &= \tfrac{1}{2}(\mathcal{N}_{f^+g\mu} + \check{\mathcal{N}}_{f^+g\mu})\,, & A_{(f\pm)(g\mp)}^{\mu} &= \tfrac{1}{2}(\mathcal{N}_{f^+g\mu} - \check{\mathcal{N}}_{f^+g\mu})\,. \end{aligned} \tag{14}$$

The tensor A_{ab}^{μ} allows to perform the following checks. We first remark that all A_{ab}^{μ} are non-negative integers as it befits for an expansion of an open string partition function. This result is particularly non-trivial for fixed points, where it follows from (8). Next we observe that the fact that the conjugation on the classifying algebra that is the evaluation on the identity implies that the multiplicities of the vacuum in the open string partition functions are either 0 or 1. This was a consistency requirement in [6]. Finally we can check the consistency of several factorizations. In the case of a sphere with four boundary circles with boundary conditions a, b, c and d we have $\sum_\mu A^\mu_{ab} A^{\mu^+}_{cd} = \sum_\mu A^\mu_{ac^+} A^{\mu^+}_{b^+d}$; also, $A^\mu A^\nu = \sum_\lambda \mathcal{N}_{\mu\nu}^{\lambda} A^\lambda$, which gives the correct factorization of the annulus. We also remark that the heuristic argument used in [6] to derive the classifying algebra for the charge conjugation modular invariant can be generalized to the case of our interest; for details we refer to [13].

Conclusions.

The structure of the classifying algebra $\tilde{\mathcal{A}}$ is actually closely related to the fusion algebra of another type of modular invariants, namely those of '$D_{\rm even}$-type' (which are also known as integer spin simple current extensions). In particular, our results for the classifying algebra look as if the boundary theory were extended by the *half*-integer spin simple current J.

This is indeed most remarkable, because in the case of extensions, this structure is a consequence of the powerful consistency requirements of modular invariance. But for the crosscap as well as for the annulus and the Möbius strip, there is no analogue of a modular group. In string theory it is usually argued that tadpole cancellation provides a substitute for such consistency conditions. Note, however, that for our investigations we did not have to assume that the conformal field theory is part of a string compactification (e.g., the central charge is not restricted), so that the conditions of tadpole cancellation cannot even be formulated. Still, it seems that already on a pure conformal field theory level there are similar powerful constraints; to unravel the underlying structure seems to be a promising task.

Finally we mention that the construction of a classifying algebra for modular invariants of automorphism type that are not simple current automorphisms is still an open problem. A particularly interesting case is the one of generalized Neumann boundary conditions for the true diagonal modular invariant, where the relevant automorphism is just charge conjugation.

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