Damping rate of plasmons and photons
in a degenerate nonrelativistic plasma

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(October 3, 1997)

Abstract

A calculation is presented of the plasmon and photon damping rates in a dense nonrelativistic plasma at zero temperature, following the resummation program of Braaten-Pisarski. At small soft momentum $k$, the damping is dominated by $3 \to 2$ scattering processes corresponding to double longitudinal Landau damping. The dampings are proportional to $(\alpha / v_F)^{3/2} k^2 / m$, where $v_F$ is the Fermi velocity.

11.10.Wx, 12.20.Ds, 12.38.Mh, 71.45.Gm

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Over the last few years, there has emerged a systematic perturbation scheme developed mainly by Braaten and Pisarski [1] and by Frenkel and Taylor [2], which has finally allowed the calculation of some thermodynamic and dissipative properties of quasiparticles in ultrarelativistic plasmas (for a recent overview, see [3]). It is based on a resummation method in which the distinction between soft scales of order $gT$ or smaller and hard scales of order $T$ is completely crucial. Basically, when the quantities to be computed depend on soft scales, the resummed perturbative expansion makes use of effective propagators and vertices accounting for long-distance medium effects as such Debye screening and Landau damping. The independence on the choice of gauge is guaranteed by the general properties of the hard thermal loops entering into the effective propagators and vertices [1,2].

One of the greatest success of these calculations has been the consistent calculation of the gluon damping rate at zero momentum [4], thus solving the plasmon puzzle in perturbative QCD at high temperature. Also, the fermion damping rate for an excitation at rest has been computed [5]. Other interesting quantities as the damping rate of hard photons in the quark-gluon plasma [6], and the lifetime of a moving fermion in QED plasmas also have been considered within this framework [7]. Only very recently, it has been shown by Blaizot and Iancu that the infrared divergences which plague this last quantity can be eliminated by a resummation based on the Bloch-Nordsieck mechanism at finite temperature [8]. All of these developments refer to a ultrarelativistic, high-temperature, zero-density regime. However, although degenerate plasmas (relativistic or not) are relevant in a wide area of applications ranging from metals to heavy ion collisions, relatively little work has been devoted to the computation of the damping of their excitations within the new developments [9,10]. In both of these references, the authors compute the damping of hard relativistic modes of momentum $k \sim k_F$, where $k_F$ is the Fermi momentum.

Also, it is interesting to consider the problem of the damping rate of soft modes of momentum and energy of order $gk_F$ and $gE_F$ respectively. They correspond to collective excitations in the plasma. Actually, the perturbative calculation of damping rates of these quasiparticles in the nonrelativistic regime is an old question. Over the sixties, there was
some controversy concerning the damping rate of plasmons. A preliminar calculation was carried out by DuBois [11] who neglected screening effects. Ninham, Powell and Swanson [12] repeated the calculation by including screening but missing out effective vertices which contribute to the same order in $e$ as the tree-level diagrams. Lately, DuBois and Kivelson [13] corrected that procedure by taking into account the screening and the effective vertices and this seemed to be the final perturbative answer. However, the consistency of their method is not guaranteed because the mixing of the soft scales $(gE_F, gk_F)$ and the hard scales $(E_F, k_F)$ which is present in their calculations. Moreover, these authors do not report the numerical values of dampings.

In this paper, I will largely follow the resummation program of Braaten and Pisarski to explicitly compute the plasmon and the photon damping rates at nearly zero momentum in a non relativistic degenerate electron gas to leading order in the electromagnetic fine structure constant $\alpha = g^2/4\pi \simeq 1/137$. In this context, it is usual to introduce another coupling constant $r_s = (\alpha/v_F)(9\pi/4)^{1/3}$. The degenerate limit is then $r_s \to 0$. Resummation is required in this limit because the plasma frequency $\Omega_p = g_e k_F \sqrt{v_F/3\pi}$ is soft versus the Fermi energy $E_F = k_F^2/2m$, their ratio, of order $r_s^{1/2}$, playing the role of the coupling constant $g$ in the Braaten-Pisarski terminology.

The main results of this paper are the formulae for dampings rates listed below in Eqs. (20)–(22). The calculation, which is exact at leading order in $r_s$, does not include the effects of current-density and current-current interactions which lie outside the non relativistic approximation and are suppressed by powers of the Fermi velocity over $c$. However, it is would be noted that the non-leading terms can be computed also within the framework described below.

Let us proceed to outline our calculations. The scattering processes in the plasma contributing to the plasmon and photon damping rate to lowest order in $r_s$ can be identified by cutting the one-loop self-energy graphs with effective propagators and vertices drawn in Fig. 1. There are three types of contributions depending on which piece of the imaginary part of effective propagators is picked. The pole-pole term corresponds to the decay of the
plasmon or photon in a pair of plasmons which is forbidden by kinematics. The pole-cut term describes $2 \rightarrow 2$ contributions which are zero since $k^0 = \Omega_p$ is at or below threshold for these processes. The only nonzero contribution is the cut-cut. It comes from $3 \rightarrow 2$ processes: $\gamma^*(k) e(p_1) e(p_2) \rightarrow e(p_3) e(p_4)$, two electrons from the Fermi sea end up above the Fermi surface through absorption of the plasmon. The damping arises entirely from the double Landau damping in the longitudinal exchange between the electrons outside and inside the Fermi sea as can be seen in Fig. 2. It should be noted that the first four scattering diagrams arise from cuts in both effective three-vertices and effective four-vertex. The remaining five diagrams obtained by exchange of particles $P_3 \leftrightarrow P_4$ do not enter to lowest order in $r_s$, since they come from cuts of higher-loop self-energy graphs.

The damping rate $\Gamma_{l,t}(k)$ corresponding to this process is

$$\Gamma_{l,t}(k) = \frac{1}{2\omega_{l,t}(k)} \left| \frac{1}{2\pi^2} \int \frac{d^3p_1}{(2\pi)^3} \frac{d^3p_2}{(2\pi)^3} \frac{d^3p_3}{(2\pi)^3} \frac{d^3p_4}{(2\pi)^3} n_F(p_1)n_F(p_2)[1 - n_F(p_3)][1 - n_F(p_4)]  
\times (2\pi)^4 \delta(\omega_{l,t}(k) + E_1 + E_2 - E_3 - E_4) \delta^{(3)}(k + p_1 + p_2 - p_3 - p_4) \right| \left| \mathcal{M}_{l,t} \right|^2, \quad (1)$$

where $|\mathcal{M}_{l,t}|^2$ denotes the scattering amplitude squared, averaged over the helicity states of the incoming photon and summed over the spin states of the initial and final electrons. A double factor $1/2!$ is included corresponding to the two pairs of identical particles present in the initial and final states. The electron energies are $E_i = p_i^2/2m$ and the Fermi distribution is $n_F(p_i) = \theta(k_F - p_i)$. Now it is easy to write the matrix element from the diagrams in Fig. 2,

$$\mathcal{M}_l = g^3_e g^2_e \Pi^{(3)}(P_3 - P_1, P_4 - P_2) D(\omega, p_3 - p_1) D(\omega', p_4 - p_2) + \mathcal{J}_0 \epsilon_0(k), \quad (2)$$

$$\mathcal{M}_t(\lambda) = g^3_e g^2_e \Pi^{(3)}(P_3 - P_1, P_4 - P_2) D(\omega, p_3 - p_1) D(\omega', p_4 - p_2) + \mathcal{J}_j \epsilon_j(k, \lambda). \quad (3)$$

The alert reader may wonder about the different orders of perturbation theory which seem to be mixed in the previous equations. In fact, we shall see below that the scattering amplitudes are of order $g^3_e$ due to cancellations between the effective vertices and propagators.

In the Coulomb gauge, the longitudinal effective propagator is

$$D(\omega, k) = \frac{v_F^2}{v_F^2 k^2 - 3\Omega_p^2 Q_1(\omega/v_F k)}, \quad (4)$$
where $Q_1$ is a Legendre function of the second kind,

$$Q_1(x) = -1 + \frac{x}{2} \ln \left| \frac{1 + x}{1 - x} \right| - \frac{|x|}{2} \theta(1 - |x|).$$  \hfill (5)

This is not the retarded propagator but coincides with it for $\omega > 0$. Since the Fermi numbers in Eq. (1) and the energy conservation condition $\omega_{l,t}(k) = \omega + \omega'$ ensure that energy transfer variables $\omega = E_3 - E_1$ and $\omega' = E_4 - E_2$ are both positive, the effective vertices appearing in the amplitudes can be computed from their euclidean counterparts depending on imaginary frequencies. The complete calculation of the $\Pi$’s is rather involved, but fortunately only the ‘hard dense loop’ part of them is required. In this approximation, one considers $\omega, \omega', p, k \ll E_F, k_F$. Besides, the leading contribution to the $\Pi$’s comes from a region in phase space where all internal fermionic momenta are in a narrow shell around the Fermi sphere, so that one can make an expansion of the integrand around the Fermi momentum after the imaginary frequency integration is performed. Then, the analytic continuation with the retarded prescription from the imaginary frequencies to $\omega + i\varepsilon$ and $\omega' + i\varepsilon$ gives

$$\Pi^{(3)}(\omega, \omega', p, k) = \frac{k_F k \cdot p}{\pi^2 \omega_l^2} \left[ -Q_1(\omega/v_F p) + Q_1(\omega'/v_F p) \right]$$

$$+ \frac{k_F k \cdot p^2}{2 \pi^2 p^2 \omega_l^2} \left[ \omega \frac{\partial}{\partial \omega} Q_1(\omega/v_F p) + \omega' \frac{\partial}{\partial \omega'} Q_1(\omega'/v_F p) \right]$$

$$- \frac{2 k_F k \cdot p^2}{\pi^2 p^2 \omega_l^2} \left[ \omega Q_1(\omega/v_F p) + \omega' Q_1(\omega'/v_F) \right]$$

$$+ \frac{k_F k^2 p^2}{2 \pi^2 p^2 \omega_l^2} \left[ Q_1(\omega/v_F p) + Q_1(\omega'/v_F p) \right] + O(k^3),$$  \hfill (6)

and

$$\Pi_j^{(3)}(\omega, \omega', p, k) = \frac{k_F p_j}{\pi^2 \omega_l} \left[ -Q_1(\omega/v_F p) + Q_1(\omega'/v_F p) \right]$$

$$+ \frac{k_F k \cdot p p_j}{2 \pi^2 p^2 \omega_l^2} \left[ \omega \frac{\partial}{\partial \omega} Q_1(\omega/v_F p) + \omega' \frac{\partial}{\partial \omega'} Q_1(\omega'/v_F p) \right]$$

$$- \frac{2 k_F k \cdot p p_j}{\pi^2 p^2 \omega_l^2} \left[ \omega Q_1(\omega/v_F p) + \omega' Q_1(\omega'/v_F) \right] + O(k^2),$$  \hfill (7)

where $p = 1/2(p_3 - p_1 + p_2 - p_4)$. Terms proportional to $k_j$ do not appear in Eq. (7)
since they do not contribute because the polarization vectors satisfy the Coulomb gauge constraint $k \cdot \epsilon(\lambda) = 0$.

The terms $J_0$ and $J_j$ in the amplitudes are easily written from the real part of the non-relativistic electron propagator because the virtual fermion is not on shell. They are of the form

$$J_0 = \frac{1}{E(p_3) + \omega - E(p_2 + k)} + \text{three similar terms}, \quad (8)$$

$$J_j = \frac{(2p_2^j + k^j)(2m)^{-1}}{E(p_3) + \omega - E(p_2 + k)} + \text{three similar terms}. \quad (9)$$

Putting this together, we find the amplitudes to lowest order in $k$

$$M_l = g_3^3 \left[ \frac{k^2 p^2 - 4 k \cdot p^2}{m \omega^2} D(\omega, p) D(\omega', p) \right. \\
+ \frac{k \cdot p (2 k \cdot p m \omega + 2 p_2 \cdot k p^2 - k \cdot p p^2)}{m^2 p^2 \omega^3} D(\omega, p) \\
+ \left. \frac{k \cdot p (2 k \cdot p m \omega - 2 p_1 \cdot k p^2 - k \cdot p p^2)}{m^2 p^2 \omega^3} D(\omega', p) \right] \epsilon_0(k), \quad (10)$$

$$M_t(\lambda) = g_3^3 \left[ -\frac{4 k \cdot p^2 p^j}{m \omega_t} D(\omega, p) D(\omega', p) \right. \\
+ \frac{2 k \cdot p \omega \hat{p}^j + k \cdot p p_2^j - k \cdot p p^j + k \cdot p_2 p^j}{m \omega_t^2} D(\omega, p) \\
+ \left. \frac{2 k \cdot p \omega \hat{p}^j - k \cdot p p_1^j - k \cdot p p^j - k \cdot p_1 p^j}{m \omega_t^2} D(\omega', p) \right] \epsilon_t(k, \lambda). \quad (11)$$

Because of simplifications between Legendre functions, these expressions are far simpler than one might have expected given the complicated form for the effective vertices. It should be noted the cancellation of terms of first and zeroth order in $k$ in $M_l$ and $M_t$, respectively. Also, they agree with those previously computed, long time ago, in Ref. [13]. The terms proportional to $k \cdot p p^2$ and $k \cdot p p^j$ can be dropped because they are subleading versus $p_1, p_2 \cdot k p^2$ and $k \cdot p_1, p_2 p^j$ respectively. The reason is that both $p_1, p_2$ and the final momenta, which for $k \to 0$, can be approximated for $p_1 + p$ and $p_2 - p$, are close to the Fermi surface, which means that $p \ll k_F$.

We still have to give the expressions for the polarization terms,
\[ \epsilon_0(\mathbf{k}) = \frac{\omega(k)}{k} \sqrt{Z_i(k)}, \]
\[ \epsilon_j(\mathbf{k}, \lambda) = \sqrt{Z_i(k)} \epsilon_j(\mathbf{k}, \lambda), \]

where \( Z_i(k) \) and \( Z_i(k) \) define the residue functions at the poles in the external effective propagators [14] and \( \epsilon_j(\mathbf{k}, \lambda) \) are orthogonal to \( \mathbf{k} \) and normalized so that \( \mathbf{e}(\mathbf{k}, \lambda) \cdot \mathbf{e}(\mathbf{k}, \lambda)^* = 1 \).

To lowest order needed, these are \( Z_i = Z_i + \mathcal{O}(k^2) \). The sum over transverse polatization vectors is

\[ \sum_{\lambda = \pm} \epsilon_i(\mathbf{k}, \lambda) \epsilon_j(\mathbf{k}, \lambda)^* = \delta_{ij} - \hat{k}_i \hat{k}_j. \]

Now, it is convenient to make use of the fact that \( \Gamma_{l,t} \) are independent of the direction of \( \mathbf{k} \). We can therefore average the squares of \( M_{l,t} \) over the directions of \( \mathbf{k} \). Then, the \( p_1 \) and \( p_2 \) integrals can be done using the 'hard dense loop' results

\[
\int \frac{d^3p_1}{(2\pi)^3} n_F(\mathbf{p}_1)(1 - n_F(\mathbf{p}_1 + \mathbf{p})) \delta \left( \omega - \frac{\mathbf{p}_1 \cdot \mathbf{P}}{m} - \frac{p^2}{2m} \right) = \frac{m^2 \omega}{4\pi^2 p} \theta(\omega) \theta \left( 1 - \frac{\omega^2}{v_F^2 p^2} \right),
\]
\[
\int \frac{d^3p_1}{(2\pi)^3} p_1 n_F(\mathbf{p}_1)(1 - n_F(\mathbf{p}_1 + \mathbf{p})) \delta \left( \omega - \frac{\mathbf{p}_1 \cdot \mathbf{P}}{m} - \frac{p^2}{2m} \right) = \frac{m^2 \omega^2 p}{4\pi^2 p^3} \theta(\omega) \theta \left( 1 - \frac{\omega^2}{v_F^2 p^2} \right),
\]
\[
\int \frac{d^3p_1}{(2\pi)^3} p_1^2 n_F(\mathbf{p}_1)(1 - n_F(\mathbf{p}_1 + \mathbf{p})) \delta \left( \omega - \frac{\mathbf{p}_1 \cdot \mathbf{P}}{m} - \frac{p^2}{2m} \right) = \frac{k_F^2 m^2 \omega}{4\pi^2 p} \theta(\omega) \theta \left( 1 - \frac{\omega^2}{v_F^2 p^2} \right),
\]
valid for \( p \ll k_F \). Lastly, the spin and polarization summations give

\[
\Gamma_l(k) = \frac{g_0^6 m^2 k^2}{480 \pi^5 \Omega_p^3} \int_0^\infty dp \int_0^{v_F p} d\omega \int_0^{v_F p} d\omega' \delta(\Omega_p - \omega - \omega') \times [23 \omega \omega' p^2 \Omega_p^2 |D(\omega, p)|^2 |D(\omega', p)|^2 + 4 \omega \omega' (p^2 v_F^2 - \omega^2) |D(\omega', p)|^2]
\]
\[
\Gamma_t(k) = \frac{g_0^6 m^2 k^2}{480 \pi^5 \Omega_p^3} \int_0^\infty dp \int_0^{v_F p} d\omega \int_0^{v_F p} d\omega' \delta(\Omega_p - \omega - \omega') \times [16 \omega \omega' p^4 \Omega_p^2 |D(\omega, p)|^2 |D(\omega', p)|^2 + 3 \omega \omega' (p^2 v_F^2 - \omega^2) |D(\omega', p)|^2 + 3 \omega \omega' (p^2 v_F^2 - \omega^2) |D(\omega, p)|^2].
\]

The integrals can be computed through the introduction of the dimensionless variables \( x = \omega / v_F p, x' = \omega' / v_F p \) and \( \overline{q} = v_F q / \Omega_p \). The final results are
\[ \Gamma_{i,t}(k) = a_{i,t} \left( \frac{\alpha}{v_F} \right)^{3/2} \frac{k^2}{200m} \]

(20)

where the constants are determined by numerical integration to be

\[ a_i \simeq 6.23387, \]

(21)

\[ a_t \simeq 4.63608. \]

(22)

In conclusion, I have computed the leading contribution in \( v_F/c \) to the damping rates of soft quasiparticles in a degenerate non relativistic plasma for small momentum \( k \). This contribution comes from density-density interactions in the plasma and vanishes as \( k \to 0 \). For \( k = 0 \), there are also contributions of higher orders in \( v_F/c \) coming from velocity dependent interactions. These could be computed within the same framework by including one transverse effective interaction which would give the subleading correction.

**Acknowledgements**

This work has been supported in part by funds provided by CICYT, AEN96-1668.
REFERENCES


FIGURES

FIG. 1. One-loop self-energy diagrams and effective vertices.

FIG. 2. Direct scattering graphs for $\gamma^* (k) e(p_1) e(p_2) \rightarrow e(p_3) e(p_4)$. 
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FIG. 2. Direct scattering graphs for $\gamma^*(k) e(p_1) e(p_2) \rightarrow e(p_3) e(p_4)$. 