QCD SUM RULES AND THE LOWEST-LYING I = 0 SCALAR RESONANCE

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A QCD Laplace sum rule framework is employed to investigate the relationship between the mass and the decay width of the sigma, the lowest-lying isoscalar resonance in the spin-zero meson channel. Our analysis differs from prior analyses, not only by incorporating finite-width departures from the narrow resonance approximation, but also through the inclusion of direct-instanton contributions and three-loop-order perturbative effects. Assuming that the sigma is the dominant subcontinuum resonance in this channel, the least upper bound we obtain for the sigma-mass is shown to increase with the sigma-width, and suggests that a sigma lighter than 800 MeV should have a total width substantially smaller than its mass.

1 Introduction: The $\sigma$-Meson and Chiral Symmetry Breaking

A light $I = 0$ scalar meson [denoted henceforth as the $\sigma$] with a mass approximately equal to twice the constituent mass of u and d quarks is common to a number of models for the dynamical breakdown of SU(2) × SU(2) chiral symmetry. Such a particle is seen to occur within quark-model and QCD adaptations of the Nambu Jona-Lasinio mechanism for chiral symmetry breaking, as well as in models for the quark-antiquark scattering amplitude in an instanton background. A light scalar particle with a mass around 500-600 MeV is also a familiar feature of one- boson exchange models of the nuclear potential. More recent work has shown that the instanton vacuum structure predicted for QCD necessarily contains such a meson in the 500-600 MeV range.

Despite its long history in theoretical models for chiral symmetry breakdown, experimental evidence for the $\sigma$ has been both equivocal and controversial. Only in the last year have several independent analyses of $\pi - \pi$ scattering data been shown to be consistent with a low-mass isoscalar scalar...
resonance, providing sufficient experimental evidence for the $\sigma$ to be listed as the $f_0(400-1200)$ lowest-lying scalar resonance in the present edition of the Particle Data Guide. Although recent studies clearly indicate a mass for this particle in the 500-600 MeV range, controversy remains as to whether this resonance exhibits...

1) ... a very broad decay width comparable to its mass,$^{6,9}$ as anticipated ($\Gamma_\sigma \approx 9\Gamma_\rho / 2$) from underlying chiral symmetry,$^{10}$

2) ... a somewhat narrower ($\approx 300$ MeV) width,$^{7,8}$ as anticipated $^{11}$ from identifying the $\sigma$ with the near-Goldstone particle associated with the (partially-conserved) dilatation current when the QCD coupling approaches criticality, or

3) ... the very narrow width anticipated from the medium-range nucleon-nucleon interaction that has been recently extracted via inversion potentials of phase shift data.$^{12}$

In view of the importance of the $\sigma$ to our present understanding of chiral symmetry breaking, as well as the controversy and uncertainty concerning its mass, width, or even its existence,$^{13}$ there is more than ample motivation to seek a QCD-based understanding of this lowest-lying scalar resonance.

2 Field Theoretical Contribution to $I = 0$ Scalar Channel Sum Rules

QCD Laplace sum rules are well suited for determining properties of the lowest-lying resonance in a given channel (e.g., the textbook$^{14}$ QCD sum-rule extraction of the $\rho$-meson mass and decay constant in the vector channel), because they lead to an exponential suppression of higher-mass resonances not already absorbed into the QCD-continuum.$^{15}$ QCD sum-rule methods relate phenomenological hadronic physics to perturbative QCD and order-parameters of the QCD vacuum through comparison of hadronic and field-theoretical contributions to appropriate two-current correlation functions.$^{15}$

The field theoretical contributions to the Laplace sum rules $R_{0,1}$,

$$
\left[ \begin{array}{c}
R_0(\tau) \\
R_1(\tau)
\end{array} \right] \equiv \frac{1}{\pi} \int_0^\infty ds \left[ \begin{array}{c}
1 \\
s
\end{array} \right] Im [\Pi(s)] e^{-s\tau} \\
- \int_{s_0}^\infty ds \left[ \begin{array}{c}
1 \\
s
\end{array} \right] Im [\Pi^{pert}(s)] e^{-s\tau},
$$

are defined to be integrals over the scalar-current correlation function

$$
\Pi(p^2) = i \int d^4x \ e^{ip\cdot x} < 0|Tj(x)j(0)|0 >, 
$$
\[ j(x) \equiv \left[ m_u \bar{u}(x) u(x) + m_d \bar{d}(x) d(x) \right] / 2. \]  

(3)

The definition (1) substracts off the hadronic continuum of states when \( s > s_0 \), which is assumed to be equivalent (modulo duality) to the purely-perturbative contribution \( \Pi^{\text{pert}} \) of QCD. Thus \( R_0 \) and \( R_1 \) are to be identified with corresponding expressions

\[
\begin{bmatrix} R_0(\tau) \\ R_1(\tau) \end{bmatrix} = \frac{1}{\pi} \int_0^{s_0} ds \left[ \frac{1}{s} \right] \text{Im} \left[ \Pi^{\text{res}}(s) \right] e^{-s\tau}
\]

(4)

obtained from phenomenological subcontinuum-resonance contributions (\( \Pi^{\text{res}} \)) to the correlation function in the \( I = 0 \) scalar channel, contributions which are discussed in the section that follows.

Noting that \( R_1(\tau) = -\frac{d}{d\tau} R_0(\tau) \) (before introducing renormalization-group improvement), we obtain respectively the following field-theoretical QCD-vacuum-condensate, direct single-instanton, and purely-perturbative QCD contributions to the \( R_0 \) sum rule [we work in the global-SU(2)\(_f\) limit to set \( m_u = m_d \equiv m \)]:

\[
R_0(\tau) = R_0^{\text{cond}}(\tau) + R_0^{\text{inst}}(\tau) + R_0^{\text{pert}}(\tau),
\]

(5)

\[
R_0^{\text{cond}}(\tau) = m^2 \left[ \frac{3}{2} < m\bar{q}q > + \frac{1}{16\pi} < \alpha_s G^2 > - \frac{88\pi\tau}{27} < \alpha_s(\bar{q}q)^2 > \right].
\]

(6)

\[
R_0^{\text{inst}}(\tau) = \frac{3\rho^2 m^2}{16\pi^2 \tau^3} e^{-\rho^2 \tau} \left[ K_0 \left( \frac{\rho^2}{2\tau} \right) + K_1 \left( \frac{\rho^2}{2\tau} \right) \right],
\]

(7)

\[
R_0^{\text{pert}}(\tau) = \frac{3m^2}{16\pi^2 \tau^2} \left[ \left( 1 - \left( 1 + s_0 \tau \right) e^{-s_0 \tau} \right) \left( 1 + \left( \frac{\alpha_s}{\pi} \right) \frac{17}{3} + \left( \frac{\alpha_s}{\pi} \right)^2 \right) - \left( \frac{\alpha_s}{\pi} \right) \left( 2 + \left( \frac{\alpha_s}{\pi} \right) \frac{95}{3} \right) \right] \int_0^{s_0 \tau} w \ln(w) e^{-w} dw
\]

\[ + \left( \frac{\alpha_s}{\pi} \right)^2 (4.25) \int_0^{s_0 \tau} w \left[ \ln(w) \right]^2 e^{-w} dw \].

(8)

In the QCD sum-rule approach, long-distance effects are characterized by QCD-vacuum condensates as in (6), local matrix elements of quark and gluon operators averaged over the physical vacuum. The QCD vacuum condensate contributions (6), which have been known for some time, \(^{15,16}\) do not distinguish between \( I = 0 \) and \( I = 1 \) in either the scalar or pseudoscalar channels.
However, such local condensates, corresponding to vacuum fluctuations with infinite correlation length, are insufficient in the scalar and pseudoscalar channels to account for the full nonperturbative content of the QCD vacuum—they do not take into account the nonlocal contributions to current correlation functions arising from instanton-induced vacuum fluctuations. The failure to find a light scalar resonance in previous QCD sum-rule treatments, can be attributed in part to the failure to incorporate an explicit (or sufficiently well-understood) direct instanton contribution to the two-scalar-current correlation function. The instanton contribution (7) to \( R_0 \) is the same as the known instanton contribution to \( R_0 \) in the \( I = 1 \) pseudoscalar channels. The equivalence to the \( I = 0 \) scalar channel follows from cancelling sign changes between the isostructure and the \( \gamma \)-matrix traces in going from \( I = 1 \) pseudoscalar to \( I = 0 \) scalar channels. The parameter \( \rho \) represents the single-instanton size. Finally, the purely perturbative contribution (8) is obtained by substituting the explicit three-loop perturbative QCD contribution to the scalar-current correlation function, which can be extracted from an expression calculated by Gorishny et al, directly into eq. (1).

3 Finite-Width Corrections to Lowest-Lying Resonance Masses

In the usual narrow resonance approximation, resonance contributions to \( Im\Pi(s) \) appearing in (1) are proportional to \( \delta \)-functions at the resonance mass:

\[
Im\Pi^{res}(s) = \sum_r \pi g_r \delta(s - m^2_r). \tag{9}
\]

It is evident from substitution of (9) into (1) that higher-mass resonance contributions to \( R_k \) are exponentially suppressed:

\[
R_k^{res}(\tau) = \sum_r g_r m^2_r e^{-m^2_r \tau}. \tag{10}
\]

Moreover, if \( m_\ell \) denotes the lightest subcontinuum resonance mass, then

\[
R_1(\tau)/R_0(\tau) \geq m^2_\ell. \tag{11}
\]

In standard sum rule methodology (such as in using the \( I = 1 \) vector channel sum rules to extract the \( \rho \) mass), one can minimize field-theoretical expressions for \( R_1(\tau)/R_0(\tau) \) over an appropriate range of \( \tau \) to obtain via (11) a least upper-bound on the lowest-lying resonance mass \( m_\ell \).

Relations (9-11) require modification to take into account nonzero resonance widths. The \( \delta \)-functions in (9) should be understood to be the narrow
width limit of Breit-Wigner resonances:
\[
\pi g_r \delta(s - m_r^2) = \lim_{\Gamma_r \to 0} Im[-g_r/(s - m_r^2 + im_r \Gamma_r)]. \tag{12}
\]
The Breit-Wigner shape on the right hand side of (12) may be expressed as a Riemann sum of unit-area pulses \(P_m\), centred at \(s = m_r^2\):
\[
P_M(s, \Gamma) \equiv \frac{1}{2M \Gamma} \left[ \Theta(s - M^2 + iM \Gamma) - \Theta(s - M^2 - iM \Gamma) \right]. \tag{13}
\]
In an \(n = 4\) (four-pulse) approximation with \(f\) chosen to be 0.70 to ensure that the area under the four pulses is equal to \(\pi\) (the area under the Breit-Wigner curve), we find for nonzero widths that the lowest-lying resonance contributions to \(R_{0,1}\) are given by
\[
[R_0(\tau)]_\ell = g_\ell m_\ell^2 e^{-m_\ell^2 \tau} W_0(m_\ell, \Gamma_\ell, \tau), \tag{15}
\]
\[
[R_1(\tau)]_\ell = g_\ell m_\ell^2 e^{-m_\ell^2 \tau} W_1(m_\ell, \Gamma_\ell, \tau_\ell), \tag{16}
\]
where
\[
W_k[M, \Gamma, \tau] = 0.5589 \Delta_k(M, 3.5119 \Gamma, \tau) + 0.2294 \Delta_k(M, 1.4412 \Gamma, \tau)
+ 0.1368 \Delta_k(M, 0.8597 \Gamma, \tau) + 0.0733 \Delta_k(M, 0.4606 \Gamma, \tau) \tag{17}
\]
\[
\Delta_k(M, \Gamma, \tau) = \int_{-\infty}^{\infty} P_M(s, \Gamma) s^k e^{-s \tau} ds. \tag{18}
\]
Width effects are then seen to alter the expression (11) for the lowest-lying resonance mass. If the lowest-lying resonance is the dominant subcontinuum resonance in a given channel, we find that
\[
m_\ell^2 \leq \left( \frac{R_1(\tau)}{R_0(\tau)} \right) \left( \frac{W_0(m_\ell, \Gamma_\ell, \tau)}{W_1(m_\ell, \Gamma_\ell, \tau)} \right). \tag{19}
\]
Thus, for a given choice of width \(\Gamma_\ell\), one can use the field-theoretical expressions for \(R_{0,1}(\tau)\) to obtain via (19) a self-consistent least upper bound for \(m_\ell^2\).
4 Application to the I = 0 Scalar Channel

For a given choice of $\Gamma_\sigma$, $s_0$, and the Borel-parameter mass-scale $M(\equiv \tau^{-1/2})$, one can use the I = 0 scalar channel expressions for $R_{0,1}$ in conjunction with (19) to obtain a sum-rule estimate of $m_\sigma$. In Figure 1, we have displayed such estimates as a function of $M$ for various choices of $s_0 \geq 1$ GeV$^2$, assuming values for $\Gamma_\sigma$ of zero, 300, 400, and 500 MeV. We have used standard values for the parameters appearing in (6-8): 

\begin{align*}
<\bar{m}qq> &= -f_\pi^2 \pi^2/4 (f_\pi = 131 \text{ MeV}), \\
<\alpha_s G^2> &= 0.045 \text{ GeV}^2, \\
<\alpha_s(\bar{q}q)^2> &= 0.00018 \text{ GeV}^6, \\
\rho &= (600 \text{ MeV})^{-1}.
\end{align*}

Factors of $\alpha_s$ that are not absorbed in (approximately-) RG-invariant condensates are replaced with 3-loop running ($\Lambda_{QCD} = 150$ MeV) coupling constants $\alpha_s(M)$, consistent with $R_k$ being solutions of RG equations in the Borel parameter $M$.\textsuperscript{23} The Borel scale $M$ is itself allowed to vary over the range $0.4 \text{ GeV}^2 \leq M^2 \leq s_0$, the upper bound being a necessary condition for the subcontinuum character of the lowest-lying resonance. A property common to all four graphs of Fig 1 is an increase in sum-rule-estimated values of $m_\sigma$ with increasing $s_0$. A comparison of the four graphs also shows, for a given value of $s_0$, that sum-rule estimates of $m_\sigma$ increase with width. We observe from the final graph, for example, that the global minimum value of $m_\sigma$ is larger than $800$ MeV if $\Gamma = 500$ MeV. If we assume, as is usual in sum-rule methodology, that a physical result should be locally insensitive to the Borel parameter, then we must consider only those curves for which a local minimum occurs at a value of $M$ less than $s_0^{1/2}$ [all curves displayed in Fig 1 are cut off at $M^2 = s_0$]. As is evident from the Fig 1 graphs, such a local minimum does not occur unless $s_0 > 1.6$ GeV$^2$. Moreover, the value of $m_\sigma$ at this local minimum increases with increasing $\Gamma$. In Table I are tabulated the lowest values of the continuum threshold $s_0$ for the onset of a local minimum, as well as the value of $m_\sigma$ associated with this local minimum (i.e., the minimum minimizing $m_\sigma$). The results are clearly indicative of a sigma mass that increases with width. We have also performed a weighted least squares fit of $R_0(\tau)$, as given in (5-8) to the $\tau$-dependence (15) anticipated for a single lowest-lying resonance. The fit is performed utilizing the Monte Carlo simulation of uncertainties described in ref. 22 over the Borel-parameter range $0.4 \text{ GeV}^{-2} \leq \tau \leq 2.2 \text{ GeV}^{-2}$. Results of the fit are as follows (uncertainties are 90% confidence levels): 

\begin{align*}
&\ m_\sigma = 0.93 \pm 0.11 \text{ GeV}, \\
&\ \Gamma \leq 260 \text{ MeV}, \\
&\ s_0 = 3.15 \pm 0.96 \text{ GeV}^2.
\end{align*}

As a final note, we stress that the above results are obtained by assuming that only the lowest-lying resonance $\sigma$ contributes to $R_{0,1}$. While I = 0 scalar resonances with masses much larger than 1 GeV may safely be regarded as exponentially suppressed (if not massive enough to be absorbed in the $s > s_0$ hadronic continuum), the $f_0(980)$ resonance must be regarded as subcontinuum
Table 1: Sigma mass lower-bounds associated with the onset of a local minimum

<table>
<thead>
<tr>
<th>$s_0 (GeV^2)$</th>
<th>0.3</th>
<th>1.3</th>
<th>2.3</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
<th>1.2</th>
</tr>
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<tbody>
<tr>
<td>$m_\sigma (MeV)$</td>
<td>680</td>
<td>687</td>
<td>716</td>
<td>778</td>
<td>884</td>
<td>1087</td>
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References