

Symmetries, Lie
Algebras and
Representations
A graduate course for physicists

JÜRGEN FUCHS
DESY

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