LOOP CORRECTIONS TO THE
UNIVERSAL HYPERMULTIPLET

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Abstract

The universal hypermultiplet arises as a subsector of every Calabi-Yau compactification of $M$-theory or Type II string theory. Classically its moduli space is the quaternionic space $SU(2,1)/U(2)$. We show that this moduli space receives a one-loop correction proportional to the Euler character of the Calabi-Yau. The correction vanishes in the limit that the Planck mass is taken to infinity, and hence is essentially gravitational in nature. An exactly quaternionic metric which reproduces the classical and one-loop results is exhibited.
1. Introduction

In the last several years there has been a great deal of progress in understanding quantum corrections to moduli spaces of string vacua. Most of the results can be understood in a field theory limit in which gravity is decoupled by taking the string or Planck mass to infinity. However, it is possible that there are qualitatively new quantum phenomena which occur only when gravity is included.

The universal hypermultiplet of $N = 2$ string compactifications [1] is an interesting place to search for such phenomena. This arises as a subsector in every $N = 2$ Calabi-Yau string compactification and classically parameterizes the quaternionic space $SU(2, 1)/U(2)$. Under type II mirror symmetry it transforms into the gravity multiplet rather than a vector multiplet. When the Planck mass is taken to infinity, the curvature of $SU(2, 1)/U(2)$ goes to zero, and it reduces to a free supermultiplet. There are then no quantum corrections. Hence any corrections to the classical moduli space must be essentially gravitational in nature.

In this paper we shall show, in the context of $M$-theory or IIA Calabi-Yau compactification, that the moduli space of the universal hypermultiplet indeed gets a one-loop correction proportional to the Euler character $\chi$ of the Calabi-Yau space. We shall further find an exactly quaternionic metric which reproduces the tree-level and one loop results and has non-trivial corrections at every order in perturbation theory. Quantum corrections to hypermultiplet moduli spaces have been analyzed in several essentially field-theoretic contexts [2-4], but it is not obvious how the present result could be interpreted in the low-energy field theory framework.

Looking beyond the scope of this paper, in [5] it was argued that the hypermultiplet metrics in general are non-perturbatively corrected by both membrane and fivebrane instantons\(^1\). Taken together with the perturbative corrections discussed herein, these could correct $SU(2, 1)/U(2)$ to a quaternionic analog of the Atiyah-Hitchin space.

In section 2 we describe how the universal hypermultiplet arises for $M$-theory or IIA compactification on a rigid Calabi-Yau. In section 3 one-loop (Riemann)\(^4\) corrections in $M$-theory are discussed. In section 4 it is shown that upon compactification this one-loop term corrects one component of the metric on the hypermultiplet moduli space. In section 5 the quaternionic geometry of the universal hypermultiplet is reviewed. In section

\(^1\) In interesting recent work [6], [7], nonperturbative D-instanton corrections in ten-dimensional IIB theory which should be relevant to this issue were found.
the results of section 4, together with the constraints of quaternionic geometry and some assumptions about the symmetries of string perturbation theory, are used to deduce the complete form of the one-loop correction (with some details relegated to an appendix). The quaternionic constraints imply that the corrections can not terminate at one loop. In section 7 an exactly quaternionic metric which reproduces the tree-level and one loop results, but has nontrivial corrections at every order, is presented.

2. The Universal Hypermultiplet in IIA String Theory and M-Theory

Compactification of M-Theory or IIA string theory on a rigid Calabi-Yau \((h_{21} = 0)\) leads to an \(N = 2\) theory with a single universal hypermultiplet [1], [8-12] parameterized by the complex fields \(S\) and \(C\). The leading low-energy action for M-theory contains the terms

\[
S^0_M = \frac{1}{2} \int d^{11}x \sqrt{-g} \ R - \frac{1}{4} \int [F \wedge \ast F + \frac{1}{3} A_3 \wedge F \wedge F],
\]

where \(F = dA_3\). In the M-theory case, the compactified theory is five-dimensional. The real part of \(S\) is related to the volume \(V_M\) of the Calabi-Yau by

\[
ReS = V_M = e^{2D}.
\]

We will be particularly interested in the kinetic term for \(D\). After rescaling the five-dimensional metric by a factor of \(e^{4D/3}\) to the Einstein frame, one finds a term

\[
- \int d^5x \sqrt{-g} (\nabla D)^2.
\]

The imaginary part of \(S\) derives from the dual of the four-form \(F = dA_3\), and \(C\) corresponds to expectation values for \(A_3\) proportional to the holomorphic three-form of the Calabi-Yau. The full \(S, C\) metric will be given in section 5 below.

For IIA compactification on a rigid Calabi-Yau, one classically obtains the same universal hypermultiplet in four dimensions. Now, however, one finds

\[
ReS = e^{-2\phi_4},
\]

where \(\phi_4\) is the four-dimensional string dilaton. It is related to the ten-dimensional string dilaton \(\phi_{10}\) and the string frame volume \(V_{str}\) by

\[
e^{-2\phi_4} = e^{-2\phi_{10}} V_{str}.
\]
A second scalar field, the volume $V_{str}$, is part of a vector multiplet. The imaginary part of $S$ is related to the dual of the three-form $H$, and $C$ is again proportional to $A_3$ expectation values. $S$ is a $NS-NS$ field while $C$ is a $R-R$ field.

The $M$-theory metric $ds_M^2$ is related to the string metric $ds_{str}^2$ by

$$ds_M^2 = e^{4\phi_{10}/3}(dx^{11})^2 + e^{-2\phi_{10}/3}ds_{str}^2,$$

(2.6)

where $x^{11} \sim x^{11} + 1$ and the IIA gauge field is suppressed. This implies that the $S$ defined in $M$-theory (2.2) and the $S$ defined in the IIA theory (2.4) are the same or, equivalently,

$$\phi_4 = -D.$$

(2.7)

The five-dimensional $M$-theory vacuum can be reached from the four-dimensional IIA vacuum by taking the radius of the eleventh dimension, or $\phi_{10}$, to infinity while keeping the Calabi-Yau volume in the $M$-theory frame, or $\phi_4$, fixed. This implies that $V_{str}$ must be taken to infinity. Since $V_{str}$ is part of a vector multiplet, the $M$-theory limit of IIA Calabi-Yau compactification is a boundary in the vector multiplet moduli space. Since neutral hyper and vector multiplets decouple, this implies that the same hypermultiplet moduli space is obtained for either IIA or $M$-theory on a given Calabi-Yau. Furthermore, the four-dimensional IIA dilaton $\phi_4$ becomes the $M$-theory six-volume $D$. Hence the IIA loop expansion will correspond to an expansion in higher dimension operators in the eleven-dimensional $M$-theory action.

The $M$-theory approach is in some ways simpler because the problem of untangling the radial and string dilaton is avoided. On the other hand, corrections are more readily calculated in the IIA picture. We shall find it useful to use both pictures in the following.

3. $R^4$ Corrections in Ten and Eleven Dimensions

The leading correction to the purely gravitational part of the IIA action has been inferred from four graviton scattering in [13], [14], [15]. The corrected action is given by, in the string frame ($\alpha' = 1$)

$$\frac{1}{2} \int d^{10}x \sqrt{-g} [e^{-2\phi_{10}}R - \frac{c_0}{3} \cdot \frac{2}{96} (e^{-2\phi_{10}} \zeta(3) + c_1)Y],$$

(3.1)
where we have included the tree-level \( \zeta(3) \) term [13] for comparison. \( Y \) here is the quartic curvature invariant defined by

\[
Y \equiv \hat{t}_8 \hat{t}_8 R^4 - \frac{1}{4} \varepsilon_{10 \varepsilon_{10}} R^4,
\]

\[
\hat{t}_8 \hat{t}_8 R^4 \equiv \hat{t}^{\mu_1 \nu_1 \cdots \mu_4 \nu_4} \hat{t}_{\alpha_1 \beta_1 \cdots \alpha_4 \beta_4} R_{\mu_1 \nu_1}^{\alpha_1 \beta_1} \cdots R_{\mu_4 \nu_4}^{\alpha_4 \beta_4},
\]

The tensor \( \hat{t} \) can be found in Appendix 9A of [16]. According to [7] the constants \( c_0 \) and \( c_1 \) are given by

\[
c_0 = 1,
\]

\[
c_1 = \frac{1}{3 \cdot 2^6 \pi^3}.
\]

There appears to be unresolved discrepancies in the values of these constants in the literature (see [15], [17], [18], [6], [7] for discussion). For now we will simply quote our results in terms of \( c_0 \) and \( c_1 \).

\( Y \) also arises at one loop for the heterotic string [15]. In that context it was shown [19] to be part of an \( N = 1 \) supermultiplet of terms containing the anomaly-canceling term \( \int B \wedge trR \wedge R \wedge R \wedge R \). It has been plausibly argued [18] that in the type II context it is also part of an \( N = 2 \) supermultiplet of terms containing \( \int B \wedge trR \wedge R \wedge R \wedge R \).

In the \( M \)-theory limit, \( \int B \wedge trR \wedge R \wedge R \wedge R \) goes over to \( \int A_3 \wedge trR \wedge R \wedge R \wedge R \). Because this term is connected to an inflow anomaly, the coefficient does not change in the transition from IIA to \( M \)-theory [20], [21], [22], [7]. In terms of the \( M \)-theory metric defined in (2.6), the corrected action becomes

\[
\frac{1}{2} \int d^{11}x \sqrt{-g} \left[ e^{2\phi_{10}/3} R - \frac{c_0}{3 \cdot 2^6} \left( e^{-4\phi_{10}/3} \zeta(3) + e^{2\phi_{10}/3} c_1 \right) Y \right],
\]

where we have suppressed terms involving derivatives of \( \phi_{10} \). The prefactor \( e^{2\phi_{10}/3} \) is just the radius of the eleventh dimension, so the \( M \)-theory action contains the terms (as similarly observed in [7])

\[
\frac{1}{2} \int d^{11}x \sqrt{-g} \left[ R - \frac{c_0 c_1}{3 \cdot 2^6} Y \right].
\]

In general one cannot naively extrapolate coefficients from IIA to \( M \)-theory in this fashion, but in this case the coefficient is protected by the anomaly. Note that the \( \zeta(3) \) term vanishes in the \( M \)-theory limit.

\[\text{In [7] } \alpha' = 1 \text{ and the dilaton is shifted so that } e^{-2\phi_{GV}} = e^{-2\phi} 2^6 \pi^7.\]
4. Compactification

In this section we consider the compactification of the $M$-theory terms (3.6) on a Calabi-Yau space. In the context of tree-level string compactification it has been shown [23] that the quartic invariant $Y$ leads to a correction to the kinetic term for the scalar $D$ governing the size of the Calabi-Yau. To see how this arises consider the ansatz

$$ds^2 = e^{2D(x^a)/3} \bar{g}_{MN} dx^M dx^N + \eta_{ab} dx^a dx^b,$$

where $a, b = 0, \cdots, 4$; $M, N = 5, \cdots, 10$, $\bar{g}_{MN}$ is the Ricci flat metric on the unit-volume Calabi-Yau and $D$ is allowed to depend on the Minkowski space coordinate $x^a$. The Riemann tensor then becomes

$$R_{\mu\nu\rho} = \bar{R}_{\mu\nu\rho} - 2\nabla_{[\mu} C^\rho_{\nu]} + 2C^\rho_{\lambda[\mu} C^\lambda_{\nu]\sigma},$$

where $\mu, \nu = 0, \cdots, 10$ and $\bar{R}$ is the Rienman tensor for $D = 0$. We are interested in terms descending from $Y$ proportional to $(\nabla D)^2$. These will involve the last term in the expansion (4.2) of the Riemann tensor and three powers of $\bar{R}_{MNP}$. Hence $Y$ will yield terms of the form

$$\nabla^a D \nabla_a D \bar{R}_{MN} P Q \bar{R}_{P Q} R S \bar{R}_{RS} M N.$$

Integrating over the Calabi-Yau and rescaling to the Einstein frame yields the loop-corrected action

$$- \int d^5 x \sqrt{-g} (\nabla D)^2 (1 - \frac{c_1 \chi}{40 \pi^3} e^{-2D}),$$

where $\chi$ is the Euler character of the Calabi-Yau.

In order to determine the correction in (4.4) we have employed a shortcut rather than the direct procedure outlined above. In [23] it was shown in the context of string theory compactification that the tree-level term corrects the metric for $D$ by a factor of

$$1 - \frac{\zeta(3) \chi}{40 \pi^3} e^{-2D}.$$

Comparing the coefficients of the one-loop and tree-level terms in (3.1) then yields (4.4). (4.4) is of course not the only one-loop correction to the metric. Corrections to other components could arise for example from terms of the form $F^2 R^3$ in eleven dimensions. It appears difficult to determine their coefficients directly. However, in section 6 we shall fix the full one-loop result from symmetries and constraints of quaternionic geometry. A direct IIA calculation, although possible in principle, is tedious because one must compute terms like $(\nabla \phi_10)^2 R^3$. These are determined from one-loop five point functions and are required in order to untangle the radial dilaton from the universal hypermultiplet.

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3 In the conventions of [23] $6e^{2D} = t^3$. 

6
5. Review of $SU(2,1)/U(2)$ and Quaternionic Geometry

An $n$-dimensional quaternionic space has an $Sp(1)$ triplet of almost complex structures obeying

$$J^i_a J^j_b J^k_c = -\delta^{ij} \delta_a^c + \varepsilon^{ijk} J^k_a J^c_j,$$  \hspace{1cm} (5.1)

where $i,j = 1,2,3$ and $a,b = 1, \cdots, 4n$. $J^i$ together with the metric define a triplet of two forms

$$\Omega^i = \frac{1}{2} J^i_a J^b_c dx^a \wedge dx^c.$$  \hspace{1cm} (5.2)

The holonomy group arising from $g$ is $Sp(1) \otimes Sp(n)$. $\Omega^i$ is the curvature of the $Sp(1)$ connection $p$

$$dp^i + \frac{1}{2} \varepsilon^{ijk} p^j \wedge p^k = \Lambda \Omega^i.$$  \hspace{1cm} (5.3)

In our conventions $\Lambda = 1$. In contrast, for a hyperkahler geometry the $Sp(1)$ curvature vanishes which corresponds to $\Lambda = 0$. In a more general set of conventions $\Lambda$ is Newton’s constant, and one sees directly that the quaternionic structure becomes hyperkahler when gravity is turned off.

For the universal hypermultiplet $n = 1$ and the classical quaternionic space is (locally) $SU(2,1)/U(2)$. The metric is explicitly

$$ds^2 = \bar{u}u + \bar{v}v,$$  \hspace{1cm} (5.4)

where

$$u \equiv e^\phi dC,$$

$$v \equiv e^{2\phi} \left( \frac{dS}{2} - \bar{C} dC \right),$$

$$\phi \equiv -\frac{1}{2} \ell \ln \left[ (S + \bar{S} - 2CC)/2 \right].$$ \hspace{1cm} (5.5)

$\phi$ here is $\phi_4$ or $-D$ with $R - R$ corrections. It is convenient to introduce the vierbein

$$V = \begin{pmatrix} u \\ v \\ \bar{v} \\ -\bar{u} \end{pmatrix}$$ \hspace{1cm} (5.6)

with metric

$$\frac{1}{2} \sigma_2 \otimes \sigma_2 = \begin{pmatrix} 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 \end{pmatrix}.$$ \hspace{1cm} (5.7)
The holonomy group $O(4)$ decomposes as $Sp(1) \otimes Sp(1)'$ with connections $p$ and $q$. $p$ acts on $v$ as

$$p = -\frac{i}{2} p^i \left( \begin{array}{cc} \sigma^i & 0 \\ 0 & \sigma^j \end{array} \right) = -\frac{i}{2} p^i \sigma^i \otimes 1_2,$$

so that $u$ and $v$ are an $Sp(1)$ doublet. The second $Sp(1)'$ connection $q$ acts as $1_2 \otimes q^k \sigma^k$, and commutes with $p$. We are primarily interested in $p$, rather than $q$, because it is subject to the constant curvature constraint (5.3). $p$ and $q$ are determined from $V$ by

$$dV + p \wedge V + q \wedge V = 0 .$$

One finds

$$p + q = \left( \begin{array}{cccc} \frac{1}{2}(\bar{v} - v) & -u & 0 & 0 \\ -\bar{u} & \bar{v} - v & 0 & 0 \\ 0 & 0 & v - \bar{v} & -u \\ 0 & 0 & \bar{u} & \frac{1}{2}(v - \bar{v}) \end{array} \right).$$

The two-form triplet $\Omega^i$ is given by

$$\Omega^i = \frac{i}{2} \bar{V} \wedge \Sigma^i V,$$

where $\Sigma^i \equiv \sigma^i \otimes 1_2$ and $\bar{V}$ is constructed with (5.7). Explicitly

$$\begin{align*}
\Omega^1 &= i(\bar{u} \wedge v + \bar{v} \wedge u), \\
\Omega^2 &= (\bar{u} \wedge v - \bar{v} \wedge u), \\
\Omega^3 &= i(\bar{u} \wedge u - \bar{v} \wedge v).
\end{align*}$$

One may verify that

$$dp^i + \frac{1}{2} \epsilon^{ijk} p^j \wedge p^k = \Omega^i,$$

and the geometry is therefore quaternionic.

Some useful relations are

$$\begin{align*}
dv &= v \wedge \bar{v} + u \wedge \bar{u}, \\
du &= \frac{1}{2} u \wedge (v + \bar{v}), \\
d\phi &= -\frac{1}{2} (v + \bar{v}).
\end{align*}$$
6. Quaternionic Perturbations

The $SU(2,1)/U(2)$ vierbein in (5.6) cannot be the exact answer because at one loop the correction (4.4) is encountered. In addition to (4.4) there may be further one-loop corrections to other metric components. These should combine into a linearized quaternionic perturbation. A perturbation $\delta V$ is quaternionic to first order if and only if the variation $\delta \Omega^i$ of $\Omega^i$ induced from (5.11) and the variation $\delta p^i$ of $p^i$ induced from (5.9) are related by the linearization of (5.13):

$$d\delta p^i + \frac{1}{2} \varepsilon^{ijk} \delta p^j \wedge p^k + \frac{1}{2} \varepsilon^{ijk} p^j \wedge \delta p^k = \delta \Omega^i .$$  (6.1)

$\delta V$ is further constrained by the following observations:

1. At string tree level there are three relevant Peccei-Quinn symmetries corresponding to constant shifts of the $NS-NS$ axion and the two $R-R$ three-form potentials. These act as

$$S \rightarrow S + i\theta + 2\varepsilon C,$$

$$C \rightarrow C + \varepsilon ,$$

where $\theta$ is real and $\varepsilon$ is complex, and generate a subgroup of $SU(2,1)$. Note that $u, v$ and $\phi$ are all invariant under (6.2). We have not proven but will assume that these symmetries are unmodified in string perturbation theory and hence $\delta V$ must be invariant as well.

2. Perturbative string amplitudes with an odd number of $R-R$ fields vanish. Hence all terms in the perturbed metric must have an even number of $R-R$ fields. This rules out for example $\bar{u}v$ corrections.

3. The perturbative theory is parity invariant. Parity changes the sign of the $NS-NS$ axion, and hence rules out $vv$ corrections to the metric.

4. As we are interested in one-loop corrections, $\delta V$ should scale like $\lambda^{-2}$ under $S \rightarrow \lambda^2 S, C \rightarrow \lambda C$.

5. The coefficient of $(\nabla D)^2$ must agree with (4.4).

One may verify the following perturbations obey the linearized quaternionic relations and are consistent with (1)-(4)

$$\delta V = e^{2\phi} \begin{pmatrix} u \\ 2v \\ 2\bar{v} \\ -\bar{u} \end{pmatrix}$$  (6.3)

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Charge quantization implies discrete identifications along these shifts [12].
\[ \delta p = e^{2\phi} \begin{pmatrix} \frac{1}{2}(v - \bar{v}) & -u & 0 & 0 \\ \bar{u} & \frac{1}{2}(\bar{v} - v) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(v - \bar{v}) & -u \\ 0 & 0 & \bar{u} & \frac{1}{2}(v - \bar{v}) \end{pmatrix}, \quad (6.4) \]

\[ \delta q = e^{2\phi} \begin{pmatrix} \frac{3}{2}(\bar{v} - v) & 0 & 0 & 0 \\ 0 & \frac{3}{2}(\bar{v} - v) & 0 & 0 \\ 0 & 0 & \frac{3}{2}(v - \bar{v}) & 0 \\ 0 & 0 & 0 & \frac{3}{2}(v - \bar{v}) \end{pmatrix}, \quad (6.5) \]

\[ \delta \Omega^1 = 3ie^{2\phi}(\bar{u} \wedge v + \bar{v} \wedge u), \]

\[ \delta \Omega^2 = 3e^{2\phi}(\bar{u} \wedge v - \bar{v} \wedge u), \quad (6.6) \]

\[ \delta \Omega^3 = 2i(\bar{u} \wedge u - 2\bar{v} \wedge v). \]

In the appendix we show that this is the unique perturbation consistent with (1)-(4). Finally matching to the computed coefficient of \((\nabla D)^2\) one finds

\[ \delta V = -\frac{c_1 \chi e^{2\phi}}{160\pi^3} \begin{pmatrix} u \\ 2v \\ 2\bar{v} \\ -\bar{u} \end{pmatrix}. \quad (6.7) \]

The quaternionic metric is then given, through one loop order, by

\[ ds^2 = \bar{u}u + \bar{v}v - \frac{c_1 \chi e^{2\phi}}{80\pi^3} (\bar{u}u + 2\bar{v}v) + O(e^{4\phi}) \]. \quad (6.8) \]

### 7. All Orders in Perturbation Theory

In this section we present an exactly quaternionic (but singular) metric which agrees perturbatively with (6.8). Let

\[ ds'^2 = \bar{u}'u' + \bar{v}'v', \quad (7.1) \]

where

\[ u' \equiv \frac{1}{\sqrt{e^{-2\phi} + \tilde{A}}} dC, \]

\[ v' \equiv \frac{1}{e^{-2\phi} + \tilde{A}} \left( \frac{dS}{2} - \tilde{C} dC \right) . \quad (7.2) \]
This agrees with (5.5) when $A = 0$, and amounts to a shift in the inverse coupling $e^{-2\phi} \rightarrow e^{-2\phi} + A$. For any value of the constant $A$, $(u', v')$ obey

$$
\begin{align*}
    dv' &= v' \land \bar{v'} + u' \land \bar{u'}, \\
    du' &= \frac{1}{2} u' \land (v' + \bar{v'}),
\end{align*}
$$

(7.3)
as in (5.14). This turns out to be sufficient to guarantee that the primed versions of all the equations in section 5 are valid, and that (7.1) is a quaternionic metric for any constant value of $A$. Comparison with (6.8) yields

$$
A = \frac{c_1 \chi}{80\pi^3}.
$$

(7.4)

By analogy with the hyperkahler case - in which the one loop result uniquely determines all orders in perturbation theory - one expects that this is the unique perturbative metric. On the other hand for large $\phi$ there may well be nonperturbative corrections to (7.1) which modify the geometry, much as in the hyperkahler case instanton corrections can modify the singular negative-mass Taub-Nut to the smooth Atiyah-Hitchin metric [3]. We leave this issue for future exploration.

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Appendix A. Linearized Quaternionic Perturbations

In this appendix we show that

\[ \delta V = e^{2\phi} \begin{pmatrix} u \\ 2v \\ 2\bar{v} \\ -\bar{u} \end{pmatrix} \]  

(A.1)

is the unique one-loop quaternionic perturbation, up to local \( Sp(1) \otimes Sp(1)' \) rotations (which give phase transformations of \( u \) and \( v \)), consistent with the perturbative symmetries discussed in Section 5.

Considerations (1), (2), (3) and (4) limit \( \delta V \) to the general form, up to \( Sp(1) \otimes Sp(1)' \) transformations,

\[ \delta V = e^{2\phi} \begin{pmatrix} au + \beta\bar{u} \\ av \\ a\bar{v} \\ -a\bar{u} - \bar{\beta}u \end{pmatrix}, \]  

(A.2)

where \( a \) is real and \( \beta \) is complex. One then finds

\[ e^{-2\phi} \delta \Omega^3 = 2ia(\bar{u}u - \bar{v}v), \]
\[ e^{-2\phi} \delta \Omega^+ = 2ia\bar{u}v + i\bar{\beta}uv, \]  

(A.3)

where \( \Omega^\pm = \frac{1}{2}(\Omega^1 \pm i\Omega^2) \). The deformation of the \( Sp(1) \) connection \( \delta p \) is constrained by the linearized equations

\[ d\delta p^+ + i\delta p^3 p^+ + ip^3 \delta p^+ = \delta \Omega^+, \]
\[ d\delta p^3 + 2i\delta p^+ p^- + 2ip^+ \delta p^- = \delta \Omega^3. \]  

(A.4)

Substituting the known quantities yields

\[ d\delta p^+ - \delta p^3 \bar{u} - \frac{1}{2}(v - \bar{v})\delta p^+ = e^{2\phi}(2ia\bar{u}v + i\bar{\beta}uv), \]  

(A.5)

and

\[ d\delta p^3 + 2\delta p^+ u - 2\bar{u}\delta p^- = 2iae^{2\phi}(\bar{u}u - \bar{v}v). \]  

(A.6)

First we show \( \beta = 0 \). The only way to obtain a \( \bar{\beta}uv \) term on the RHS of (A.5) is if \( \delta p^+ \) has a term proportional to \( u e^{2\phi} \). But then an unwanted \( u\bar{v} \) term will appear on the LHS, which is not on the RHS. Hence we conclude

\[ \beta = 0. \]  

(A.7)

\footnotesize
\textsuperscript{5} The \( \wedge \) symbol is suppressed in the following.
Once $\beta = 0$ (A.2) reduces to a scale transformation which is clearly not quaternionic (for fixed $\Lambda$). To see this explicitly consider (A.6). In order to reproduce the $\bar{v}v$ term on the LHS, $\delta p_3$ must be of the form

$$e^{-2\phi} \delta p_3 = \frac{ia}{2} (v - \bar{v}) + c_0 (v + \bar{v}) + c_1 u + \bar{c}_1 \bar{u}, \quad (A.8)$$

where $c_0$ is real but $c_1$ is complex. Cancellation of $\bar{u}u$ terms in (A.6) then requires

$$e^{-2\phi} \delta p^+ = \frac{3ia}{4} \bar{u} + d_1 v + d_2 \bar{v}, \quad (A.9)$$

for $d_1, d_2$ complex. Now let us consider $\bar{u}v$ and $\bar{u}\bar{v}$ terms in (A.5). They must obey

$$- \frac{9ia}{8} (v + \bar{v})\bar{u} - \frac{ia}{2} (v - \bar{v})\bar{u} - c_0 (v + \bar{v})\bar{u} - \frac{3ia}{8} (v - \bar{v})\bar{u} = 2ia\bar{u}v. \quad (A.10)$$

Cancellation of $\bar{v}u$ terms implies $a = c_0 = 0$. Hence no perturbation of the form (A.2) can be quaternionic, and we conclude that the unique quaternionic perturbation is given by (6.7) up to local $Sp(1) \otimes Sp(1)'$ transformations.

References