The Schrödinger functional running coupling with staggered fermions

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Abstract

We discuss the Schrödinger functional in lattice QCD with staggered fermions including its order $O(a)$ boundary counterterms. We relate it, in the classical continuum limit, to the Schrödinger functional as obtained in the same limit with Wilson fermions. We compute the strong coupling constant defined via the Schrödinger functional with staggered fermions at one loop and show that it agrees with the continuum running coupling constant in the Schrödinger functional formalism.

1 Introduction

From the pioneering work of the ALPHA collaboration [1–7] it has become clear that the Schrödinger functional in lattice QCD is a useful setup and tool for nonperturbative computations in QCD. Through the boundary conditions in the Schrödinger functional one can introduce, in a gauge invariant way, non-vanishing background fields and use those to probe the model. The response to a constant chromoelectric background field, for example, allows the nonperturbative computation of a well-defined renormalized running strong coupling constant with the scale given by the spatial size of the system [1–3,5]. The Schrödinger functional has also proved useful for the nonperturbative computation of the “clover” coefficient for $O(a)$ improved Wilson fermions [6] and the gauge independent computation of current renormalization constants [7].

In this paper we shall discuss the formulation of the Schrödinger functional with staggered fermions. The basic steps have already been done by Miyazaki and Kikukawa [8]. We shall extend their work in several directions: discuss the form of $O(a)$ boundary counterterms, compute the one-loop fermionic contribution to the pure gauge $O(a)$ boundary counterterm $\text{tr}(F_k F_k)$, and most importantly, discuss the relation between the lattice Schrödinger functional with staggered fermions and the one with Wilson fermions [9] in the continuum limit. In particular we will compute the contribution from the staggered
fermions to the running coupling constant defined through the Schrödinger functional at one loop and show that it agrees with the computations using other regularization schemes. As a by-product of this computation we obtain, we believe for the first time, an explicit calculation, at one loop, of lattice artefacts from staggered fermions in a physical observable.

In the next section we will briefly review the formulation of the Schrödinger functional with staggered fermions. We shall consider massless fermions, and discuss inclusion of a mass term later in section 5. In section 3 we will discuss the Schrödinger functional in the continuum limit in terms of four flavors of Dirac fermions. In section 4 we will compute the one-loop contribution from the staggered fermions to the Schrödinger functional running coupling for massless fermions. In section 5 we discuss inclusion of a mass term and compute the one-loop contribution to the coupling from massive fermions. Section 6 contains some concluding remarks. The symmetry properties of staggered fermions are briefly reviewed in appendix A, and a “two–time–slice” transfer matrix for staggered fermions is sketched in appendix B.

2 The Schrödinger functional with staggered fermions

The Schrödinger functional describes the evolution of a state at (Euclidean) time \( t = 0 \) to another state at time \( t = T \). Using the transfer matrix it can be written as a path integral with fixed boundary conditions at time \( t = 0 \) and \( T \). For fermions, described classically by a first order differential equation, actually only half the degrees of freedom can be specified at each boundary [9,10]. The transfer matrix for staggered fermions has been worked out in [11]. It is non positive, but its square is positive. This leads to a doubling of degrees of freedom and it is better to think of this square as a transition amplitude from two time-slices to two time-slices (see appendix B for a “two–time–slice” transfer matrix that corresponds explicitly to the square of the transfer matrix derived in [11]). As shown in [8] this leads to the fact that for staggered fermions all degrees of freedom can be fixed at both boundaries. The Schrödinger functional can thus be represented as the path integral

\[
Z[W, \zeta, \zeta'; W', \zeta', \zeta''] = \int [DU] \int \prod_{\vec{x}} \prod_{x_4=1}^{T-1} [d\bar{\chi}(\vec{x}, x_4) d\chi(\vec{x}, x_4)] \exp\{-S_G - S_{SF}\}.
\]  

(2.1)

Here \( W \) and \( W' \) represent the boundary values of the gauge fields, \( S_G \) is the pure gauge action and \([DU]\) the Haar measure over gauge fields, both appropriate for the Schrödinger functional formulation as in [1]. The fermionic part of the action is given by [8]

\[
S_{SF} = \sum_{\vec{x}} \sum_{x_4=1}^{T-1} \sum_{\mu} \frac{1}{2} \eta_{\mu}(x) \left[ U_{\mu}(x) \chi(x + \mu) - U_{\mu}(x - \mu) \chi(x - \mu) \right] + S_B^{(0)} + S_B^{(T)}
\]  

(2.2)

with \( \eta_{\mu}(x) = (-1)^{\sum_{\nu < \mu} x_\nu} \) the usual staggered phase factors. At the boundaries the fields take on their boundary values

\[
\chi(\vec{x}, 0) = \zeta(\vec{x}), \quad \bar{\chi}(\vec{x}, 0) = \bar{\zeta}(\vec{x}), \quad \chi(\vec{x}, T) = \zeta'(\vec{x}), \quad \bar{\chi}(\vec{x}, T) = \bar{\zeta}'(\vec{x}),
\]  

(2.3)
with $\zeta, \zeta', \zeta$ and $\zeta'$ independent complex Grassmann fields. The additional boundary terms of the action are

$$S_B^{(0)} = \sum_{\vec{x}} \sum_{k=1}^3 \frac{1}{2} \eta_k(\vec{x}, 0) \bar{\zeta}(\vec{x}) \left( W_k(\vec{x}) \zeta(\vec{x} + \hat{k}) - W_k^\dagger(\vec{x} - \hat{k}) \zeta(\vec{x} - \hat{k}) \right)$$

$$+ \sum_{\vec{x}} \frac{1}{2} \eta_4(\vec{x}, 0) \bar{\zeta}(\vec{x}) \chi(\vec{x}, 1).$$

(2.4)

and

$$S_B^{(T)} = \sum_{\vec{x}} \sum_{k=1}^3 \frac{1}{2} \eta_k(\vec{x}, 0) \bar{\zeta'}(\vec{x}) \left( W_k^\prime(\vec{x}) \zeta'(\vec{x} + \hat{k}) - W_k^\prime\dagger(\vec{x} - \hat{k}) \zeta'(\vec{x} - \hat{k}) \right)$$

$$- \sum_{\vec{x}} \frac{1}{2} \eta_4(\vec{x}, T) \bar{\zeta'}(\vec{x}) \chi(\vec{x}, T - 1).$$

(2.5)

where $W_k$ and $W_k'$ are the gauge fields at the boundaries.

Note that for staggered fermions the total number of time-slices has to be even. Therefore, in the labeling adopted in this paper for the lattice sites, which agrees with that of [1,5], the time extent $T$ has to be odd. In the spatial direction we take the lattice to be of size $L$ (even!) and impose the generalized periodic boundary conditions [5]

$$\chi(x + L\hat{k}) = e^{i\theta_k} \chi(x), \quad \bar{\chi}(x + L\hat{k}) = \bar{\chi}(x)e^{-i\theta_k}.$$  

(2.6)

These boundary conditions are easily implemented by transforming to periodic fermion fields in a constant abelian background field $u_4 = 1$, $u_k = e^{i\theta_k/L}$.

3 The Schrödinger functional in terms of four-component spinors

We construct four-component spinors from the one-component Grassmann fields $\chi$ and $\bar{\chi}$ in the standard way, following Kluberg-Stern et al. [12]. We discuss here only free staggered fermions. Including the coupling to gauge fields is quite straightforward, though notationally somewhat cumbersome [12,13]. Since we will be interested in the classical continuum limit, we shall display the lattice spacing explicitly (except in the labelling of the lattice sites). We divide the lattice into $2^4$ hypercubes in which the four-component spinors reside.\(^1\) Thus we set $x = 2y + \xi$ with $\xi_\mu = 0, 1$ and define

$$\chi(2y + \xi) = \chi_\xi(y), \quad \bar{\chi}(2y + \xi) = \bar{\chi}_\xi(y).$$

(3.1)

We also introduce

$$\Gamma_\xi = \gamma_1^{\xi_1} \gamma_2^{\xi_2} \gamma_3^{\xi_3} \gamma_4^{\xi_4}.$$  

(3.2)

\(^1\)This division of the lattice into $2^4$ hypercubes requires that in each direction we have an even number of sites. With our labeling convention in the time direction $T$ is odd and $0 \leq x_4 \leq T$.  

3
We use hermitian Euclidean Dirac matrices. The $\Gamma_\xi$ matrices satisfy the orthogonality and completeness relations
\[
\text{tr} \left( \Gamma_\xi^\dagger \Gamma_{\xi'} \right) = 4 \delta_{\xi\xi'}, \quad \sum_\xi \left( \Gamma_\xi^\dagger \right)^{a\alpha} \Gamma_\xi^{b\beta} = 4 \delta^{ab} \delta^{\alpha\beta}. \tag{3.3}
\]
The four flavors of four-component Dirac spinors are then constructed as
\[
\psi^{\alpha a}(y) = \frac{1}{8} \sum_\xi \Gamma_\xi^{\alpha a} \chi_\xi(y); \quad \bar{\psi}^{a\alpha}(y) = \frac{1}{8} \sum_\xi \bar{\chi}_\xi(y) \left( \Gamma_\xi^\dagger \right)^{a\alpha}. \tag{3.4}
\]
where greek superscripts denote spin indices and roman superscripts flavor indices. These relations can be inverted:
\[
\chi_\xi(y) = 2 \sum_{\alpha a} \left( \Gamma_\xi^\dagger \right)^{a\alpha} \psi^{\alpha a}(y); \quad \bar{\chi}_\xi(y) = 2 \sum_{a\alpha} \bar{\psi}^{a\alpha}(y) \Gamma_\xi^{a\alpha}. \tag{3.5}
\]
It is useful to introduce some more notation. Let $\Lambda = \Gamma_S \otimes \Gamma_F$ for some gamma matrix $\Gamma_S$ acting in Dirac space and some gamma matrix $\Gamma_F$ acting in flavor space. Then
\[
\Lambda_1 \cdot \Lambda_2 = (\Gamma_S_1 \Gamma_S_2) \otimes (\Gamma_F_1 \Gamma_F_2)^T \tag{3.6}
\]
In the second and third equations above, just as in the definition of the $\psi$-fields, (3.4), $\Gamma_\xi$ is a “mixed” matrix with the first index a spinor index, and the second one a flavor index. We also define
\[
\check{1} = 1 \otimes 1, \quad \check{\Gamma}_\mu = \gamma_\mu \otimes 1, \quad \check{\Gamma}_5 = \gamma_5 \otimes (\gamma_5 \gamma_\mu)^T \tag{3.7}
\]
and the projectors
\[
P^{(\mu)}_0 = \frac{1}{2} \left[ 1 \otimes 1 + (\gamma_\mu \gamma_5) \otimes (\gamma_5 \gamma_\mu)^T \right] = \frac{1}{2} \check{\Gamma}_\mu \left( \check{\Gamma}_\mu + \check{\Gamma}_5 \right), \tag{3.8}
\]
These projectors are useful since
\[
P^{(\mu)}_0 \cdot \Gamma_\xi = \delta_{\xi,0} \Gamma_\xi; \quad \Gamma_\xi^\dagger \cdot P^{(\mu)}_0 = \delta_{\xi,0} \Gamma_\xi; \quad P^{(\mu)}_1 \cdot \Gamma_\xi = \delta_{\xi,1} \Gamma_\xi; \quad \Gamma_\xi^\dagger \cdot P^{(\mu)}_1 = \delta_{\xi,1} \Gamma_\xi. \tag{3.9}
\]
Thus they project onto the one-component fields with $\xi_\mu = 0$ or 1.

Inserting the relations (3.5) into the free staggered action, it can be written as
\[
S_F = (2a)^4 \sum_{y,\mu} \bar{\psi}(y) \left[ \Gamma_\mu D_\mu \psi(y) + \check{\Gamma}_5 a \Delta_\mu \psi(y) \right] \tag{3.10}
\]
\[ D_\mu f(y) = \frac{1}{4a} [f(y + \mu) - f(y - \mu)] \]
\[ \Delta_\mu f(y) = \frac{1}{4a^2} [f(y + \mu) - 2f(y) + f(y - \mu)]. \]  

(3.11)

The expression for the action can be compactified further by defining \( D_\mu \) as
\[ D_\mu \psi(y) = (\mathbf{1} \otimes \mathbf{1})D_\mu \psi(y) + (\gamma_\mu \gamma_5) \otimes (\gamma_5 \gamma_\mu)^T a \Delta_\mu \psi(y). \]  

(3.12)

To deal with the boundaries in the Schrödinger functional we note that the projectors \( P_{0,1} \equiv P_{0,1}^{(4)} \) project onto the boundary fields
\[ P_0 \psi(\vec{y}, 0) = \rho(\vec{y}), \quad \tilde{\psi}(\vec{y}, 0) P_0 = \tilde{\rho}(\vec{y}), \quad P_1 \psi(\vec{y}, T') = \rho'(\vec{y}), \quad \tilde{\psi}(\vec{y}, T') P_1 = \tilde{\rho}'(\vec{y}), \]  

(3.13)

where we set \( T' = (T - 1)/2 \) for the upper boundary hypercubes (recall that \( T \) has to be odd). The boundary four-component spinors \( \rho \) and \( \rho' \) are related to the boundary one-component spinors \( \zeta \) and \( \zeta' \) as in eq. (3.4),
\[ \rho^{\alpha a}(\vec{y}) = \frac{1}{8} \sum_\xi \Gamma^{\alpha a}_{(\xi,0)} \zeta(\vec{y}), \quad \rho'^{\alpha a}(\vec{y}) = \frac{1}{8} \sum_\xi \Gamma^{\alpha a}_{(\xi,1)} \zeta'(\vec{y}) \]  

(3.14)

and analogously for \( \tilde{\rho} \) and \( \tilde{\rho}' \). The action appropriate for the Schrödinger functional can now be written as [8]
\[ S_{SF} = (2a)^4 \sum_\vec{y} \sum_{\gamma=1}^{T' - 1} \sum_\mu \tilde{\psi}(\vec{y}) \tilde{\Gamma}_\mu D_\mu \psi(y) + S_B^{(0)} + S_B^{(T)} \]  

(3.15)

with the boundary contributions
\[ S_B^{(0)} = (2a)^4 \sum_\vec{y} \sum_{k=1}^3 \tilde{\psi}(\vec{y}, 0) \tilde{\Gamma}_k D_k \psi(\vec{y}, 0) + (2a)^3 \sum_\vec{y} \tilde{\psi}(\vec{y}, 0) P_1 \tilde{\Gamma}_4 \psi(\vec{y}, 1) \]
\[ - (2a)^3 \sum_\vec{y} \tilde{\psi}(\vec{y}, 0) \tilde{\Gamma}_4^5 \psi(\vec{y}, 0) \]  

(3.16)

and
\[ S_B^{(T)} = (2a)^4 \sum_\vec{y} \sum_{k=1}^3 \tilde{\psi}(\vec{y}, T') \tilde{\Gamma}_k D_k \psi(\vec{y}, T') - (2a)^3 \sum_\vec{y} \tilde{\psi}(\vec{y}, T') P_0 \tilde{\Gamma}_4 \psi(\vec{y}, T' - 1) \]
\[ - (2a)^3 \sum_\vec{y} \tilde{\psi}(\vec{y}, T') \tilde{\Gamma}_4^5 \psi(\vec{y}, T'). \]  

(3.17)

Using \( \tilde{\Gamma}_4^5 = P_1 \tilde{\Gamma}_4 P_0 - P_0 \tilde{\Gamma}_4 P_1 \) we see that the last term in both \( S_B^{(0)} \) and in \( S_B^{(T)} \) involves a boundary field, and hence vanishes for homogeneous boundary conditions.
Miyazaki and Kikukawa already addressed the question as to whether there are additional boundary counterterms contributing in the continuum limit. They would have to be operators of dimension three, \(i.e.\) of the form
\[
\Delta S_{B} = (2a)^{3} \sum \sum \left\{ c_{i}^{(0)} \overline{\psi(y,0)} \Lambda_i \psi(y,0) + c_{i}^{(T)} \overline{\psi(y,T')} \Lambda_i \psi(y,T') \right\}.
\] (3.18)

Taking into account the discrete spatial rotational symmetry, parity — the projectors \(P_{0,1}\) appearing in the boundary conditions (3.13) are invariant under parity — and the chiral \(U(1)\) symmetry of massless staggered fermions, Miyazaki and Kikukawa concluded that [8]
\[
\Lambda_i = \tilde{\Gamma}_4, \quad \tilde{\Gamma}_4, \quad (\gamma_4 \otimes (\gamma_4 \tilde{\gamma}))^T \quad \text{or} \quad (\gamma_5 \otimes (\gamma_5 \tilde{\gamma}))^T,
\] (3.19)
where \(\tilde{\gamma} \equiv \sum_{j=1}^{3} \gamma_j\). Writing these possible boundary contributions in terms of the one-component fields \(\chi\) and \(\bar{\chi}\) they then argue that the last two terms contain derivatives and are of order \(O(a)\) and therefore do not contribute in the continuum limit. Staggered fermions, however, have additional symmetries, overlooked in [8], namely shift invariance by one (fine) lattice spacing and a charge conjugation symmetry (see the appendix A for a summary of the symmetries of staggered fermions). Shift invariance (in spatial directions) excludes the last two possibilities in (3.19), making the more indirect argument in [8] unnecessary. \(\tilde{\Gamma}_4\), on the other hand, is excluded by the charge conjugation symmetry. Thus the only possible dimension three boundary counterterm is already present in the action. As mentioned above it involves a boundary field and it can therefore be absorbed into a renormalization of the boundary field, just as in the continuum (or Wilson fermion) Schrödinger functional [9].

### 3.1 Boundary counterterms at order \(a\)

Though the bulk part of the action, eq. (3.15), appears to have order \(O(a)\) lattice effects — the \(a\Delta_\mu\) part of \(D_\mu\) — this is not so. The apparent \(O(a)\) effect just stems from a “bad” choice in the construction of the four-component spinors in eq. (3.4). It can be transformed away by using “improved” fields [14]
\[
\chi_\xi(y) = \chi_\xi(y) - a \sum \delta_{\xi,1} D_\nu \chi_\xi(y), \quad \bar{\chi}_\xi(y) = \bar{\chi}_\xi(y) - a \sum \delta_{\xi,1} \bar{\chi}_\xi \bar{D}_\nu(y).
\] (3.20)

For our Dirac spinors this transformation becomes
\[
\psi_I(y) = \psi(y) - a \sum \nu P^{(\nu)}_1 \cdot D_\nu \psi(y), \quad \bar{\psi}_I(y) = \bar{\psi}(y) - a \sum \bar{\psi} \bar{D}_\nu(y) \cdot P^{(\nu)}_1.
\] (3.21)

This choice of improved fields is not unique. We prefer a somewhat more symmetric form, which turns out to treat the fields near the boundaries in the Schrödinger functional more equally,
\[
\psi_I(y) = \psi(y) - a \sum \nu \frac{1}{2} \left( P^{(\nu)}_1 - P^{(\nu)}_0 \right) \cdot D_\nu \psi(y)
\]
\[
\bar{\psi}_I(y) = \bar{\psi}(y) - a \sum \nu \bar{\psi} \bar{D}_\nu(y) \cdot \frac{1}{2} \left( P^{(\nu)}_1 - P^{(\nu)}_0 \right).
\] (3.22)
To apply the transformation, eq. (3.22), near the boundaries it is convenient to extend the fields \( \psi \) and \( \bar{\psi} \) to the region \( y_4 < 0 \) and \( y_4 > T' \) by setting them to zero there. Inserting the improved fields into the action and doing a partial resummation one finds the additional term in the bulk \[14\]

\[-(2a)^4 a \sum_{y,\mu} \bar{\psi}^I(y) D_\mu^2 \bar{\Gamma}^5 \psi^I(y)\]

which cancels the \( \mathcal{O}(a) \) term in eq. (3.15), up to higher orders in \( a \). Luo \[14\] shows in addition that no dimension five operators (in the bulk) are allowed by the staggered symmetries and hence that no bulk \( \mathcal{O}(a) \) artefacts will appear at the quantum level.

At the boundaries, however, \( \mathcal{O}(a) \) effects may occur. These come from dimension four operators at the boundary. Operators allowed by the staggered symmetries in the massless case under consideration are (see appendix A for a review of those symmetries)

\[
\begin{align*}
\mathcal{O}_1 &= \sum_{k=1}^3 \bar{\psi} \tilde{\Gamma}_k D_k \psi \\
\mathcal{O}_2 &= \sum_{k=1}^3 \bar{\psi}(\gamma_k \gamma_4 \gamma_5) \otimes (\gamma_5 \gamma_4)^T \tilde{D}_k \psi \\
\mathcal{O}_3 &= \bar{\psi}(0) \tilde{\Gamma}_4 \frac{1}{a} [\bar{\psi}(1) - \bar{\psi}(0)] - \frac{1}{a} [\bar{\psi}(1) - \bar{\psi}(0)] \tilde{\Gamma}_4 \psi(0) \\
\mathcal{O}_4 &= \bar{\psi}(0) \tilde{\Gamma}_5 \frac{1}{a} [\bar{\psi}(1) - \bar{\psi}(0)] + \frac{1}{a} [\bar{\psi}(1) - \bar{\psi}(0)] \tilde{\Gamma}_5 \psi(0)
\end{align*}
\]

The term \( \mathcal{O}_1 \) and the combination \( \mathcal{O}_3 + \mathcal{O}_4 \) already occur in the action, eq. (3.15). \( \mathcal{O}_2 \) and the combination \( \mathcal{O}_3 - \mathcal{O}_4 \), both flavor symmetry breaking — they break the shift symmetry in the time direction which is broken already, of course, by the presence of the boundary in the Schrödinger functional — appear at the quantum level.

### 3.2 Relation to the usual Schrödinger functional

Starting from the transfer matrix for Wilson fermions, Sint arrived at fermionic boundary conditions that are different from eq. (3.13) \[9\].\(^2\) For the four-flavor Dirac spinors considered here they would read

\[
P_+ \psi(\vec{y},0) = \rho(\vec{y}), \quad \bar{\psi}(\vec{y},0) P_- = \bar{\rho}(\vec{y}), \quad P_- \psi(\vec{y},T') = \rho'(\vec{y}), \quad \bar{\psi}(\vec{y},T') P_+ = \bar{\rho}'(\vec{y}), \quad (3.25)
\]

with \( P_\pm = \frac{1}{2} \left[ \hat{1} \pm \tilde{\Gamma}_4 \right] \). Hence the Schrödinger functional with staggered fermions, defined thus far, does not seem to agree with the “usual” definition of the Schrödinger functional in the presence of fermions. However, Sint noted that the staggered action and boundary

\(^2\)Yet other boundary conditions were considered by Symanzik \[10\]. However, those boundary conditions explicitly break parity.
conditions in the classical continuum limit derived in [8] could be brought into the form he
derived for Wilson fermions by a “chiral rotation” [15]
\[ \psi' = R \cdot \psi \quad \text{and} \quad \bar{\psi}' = \bar{\psi} \cdot \bar{\bar{R}} \]  
(3.26)
where \( \bar{\bar{R}} = \tilde{\Gamma}_4 \cdot R^\dagger \cdot \tilde{\Gamma}_4 \) and
\[ R = R_4^5(\theta_5) = \exp\{i\theta_5(i\tilde{\Gamma}_4)\} = \cos \theta_5 \mathbb{1} - \sin \theta_5 \tilde{\Gamma}_4^5 = \bar{R}_4^5(\theta_5). \]  
(3.27)
Defining
\[ P_{0,1}(\theta_5) = R_4^5(\theta_5) \cdot P_{0,1} \cdot (R_4^5(\theta_5))^{-1}. \]  
(3.28)
one finds
\[ P_{0,1}(\theta_5) = \frac{1}{2} \left[ \mathbb{1} \pm \cos 2\theta_5 (\gamma_4 \gamma_5) \otimes (\gamma_5 \gamma_4)^T \mp \sin 2\theta_5 \tilde{\Gamma}_4 \right]. \]  
(3.29)
Hence one can smoothly go from the “natural” Schrödinger boundary conditions for staggered
fermions to the conventional ones for continuum (and Wilson) fermions, at least in the
massless case under discussion. Since the rotation needed, eq. (3.27), is chiral this will not
be true for massive fermions. We shall discuss massive fermions later on, in section 5.

The “rotation” (3.26) reflects an arbitrariness in the assignment of Dirac and flavor
indices in the construction of the four-component spinors in eq. (3.4). Such a flavor-spinor
rotation could be inserted directly into (3.4). An allowed rotation should leave the kinetic
term of the fermion action in the continuum limit unchanged. Thus we require that
\[ \bar{\bar{R}}^{-1} \cdot \tilde{\Gamma}_\mu \cdot R^{-1} = \tilde{\Gamma}_\mu \quad \forall \mu. \]  
(3.30)
Therefore the generators of \( R \) need to be
\[ \Gamma_S \otimes \Gamma_F^T = [\mathbb{1}, \gamma_5] \otimes [\mathbb{1}, \gamma_\mu, \gamma_5, i(\gamma_5 \gamma_\mu), i(\gamma_\mu \gamma_\nu)]^T. \]  
(3.31)
Note that all these generators are hermitian. In particular the generator \( i\tilde{\Gamma}_4 \), and hence the
rotation \( R_4^5 \), (3.27), is allowed.

4 The Schrödinger coupling in the massless case

The first real application of the Schrödinger functional formalism in lattice gauge theory
was the computation of a well–defined renormalized coupling [1–3]. The special boundary
conditions allow introduction of an external field to probe the system, and the finite size of
the system gives a definite scale. In this section we will compute the one-loop contribution
from the staggered fermions to this coupling and show that it agrees with the result using
(improved) Wilson fermions [5] or a continuum regularization. This serves as a test for the
correctness of the Schrödinger functional setup with staggered fermions. As a by-product
we will also obtain the contribution from the staggered fermions to the pure gauge \( \mathcal{O}(a) \)
boundary counterterm with coefficient \( c_t \) in the notation of [1,3,5].

8
The external field is introduced via the boundary gauge fields taken as abelian with
\[ W_k(\vec{x}) = \text{diag} \left( e^{i\phi_1/L}, e^{i\phi_2/L}, e^{i\phi_3/L} \right) \]
\[ W_k'(\vec{x}) = \text{diag} \left( e^{i\phi'_1/L}, e^{i\phi'_2/L}, e^{i\phi'_3/L} \right). \]  (4.1)

They lead to the classical gauge fields
\[ U_4^cl(x) = 0, \quad [U_k^cl(\vec{x}, x_4)]_{ij} = \delta_{ij} e^{i(x_4\phi'_j + (T-x_4)\phi_j)/(LT)}. \]  (4.2)

As [3,5] we choose the boundary fields to depend on a parameter, which we denote by \( \omega \), through
\[ \phi_1 = -\frac{\pi}{3} + \omega, \quad \phi_2 = -\frac{1}{2} \omega, \quad \phi_3 = \frac{\pi}{3} - \frac{1}{2} \omega, \]
\[ \phi'_1 = -\pi - \omega, \quad \phi'_2 = \frac{\pi}{3} + \frac{1}{2} \omega, \quad \phi'_3 = \frac{2\pi}{3} + \frac{1}{2} \omega. \]  (4.3)

The “Schrödinger functional coupling constant” is then defined as
\[ \frac{k}{g^2} = -\frac{\partial}{\partial \omega} \log Z \bigg|_{\omega=0}, \quad k = 12 \left( \frac{L}{a} \right)^2 \left[ \sin \left( \frac{2\pi a^2}{3LT} \right) + \sin \left( \frac{\pi a^2}{3LT} \right) \right], \]  (4.4)

with \( T = L \) such that \( g^2 \) depends only on one scale, \( g^2 = g^2(L) \). The normalization \( k \) has been chosen such that \( \bar{g} \) equals the bare coupling at tree–level without any cutoff effects.

The one–loop contribution from the staggered fermions to the coupling constant eq. (4.4) comes from the derivative of the fermion fluctuation determinant. The fermion boundary fields here are set to zero. Then the fermion action can be written schematically as
\[ S_{SF} = \sum_{\vec{x}} \sum_{x_4=1}^{T-1} \sum_z \bar{\chi}(x) M_{x,z} \chi(z). \]  (4.5)

As usual with staggered fermions, one easily sees that
\[ (-1)^{|x|} M_{x,z} (-1)^{|z|} = M^\dagger_{z,x}. \]  (4.6)

Thus \((-1)^{|x|} M_{x,z} \equiv M_{x,z} \) is hermitian, and has the same determinant as \( M_{x,z} \). The one-loop contribution to the running coupling is given by [5]
\[ g^2 = g_0^2 + p_1 g_0^4 + \mathcal{O}(g_0^6), \quad p_1 = p_{1,0} + n_f p_{1,1}, \]  (4.7)

with the fermionic contribution
\[ p_{1,1} = \frac{1}{kn_f} \frac{\partial}{\partial \omega} \log \det M \bigg|_{\omega=0}. \]  (4.8)

Here \( n_f = 4 \) for one flavor of staggered fermions, since they correspond to four flavors of continuum fermions.
The eigenfunctions of $\mathcal{M}$ are of the form

$$\chi(x) = e^{i(\vec{p}+\vec{\alpha}\pi)\vec{x}} f_{\vec{\alpha}}(x_4), \quad f_{\vec{\alpha}}(x_4 = 0) = f_{\vec{\alpha}}(x_4 = T) = 0,$$

(4.9)

where $\vec{\alpha}$ is a 3-dimensional vector with $\alpha_k = 0, 1$ and

$$p_k = \frac{2\pi n_k}{L} + \frac{\theta_k}{L}, \quad n_k = 0, \ldots, \frac{1}{2}L - 1.$$

(4.10)

As usual for staggered fermions the momentum components go only over half the Brillouin zone interval. The remainder, $\alpha_k\pi$, becomes an “internal” index, loosely corresponding to spin/flavor.

Introducing $\eta^{(\mu)}$ as

$$\eta^{(\mu)} = \begin{cases} 1 & \text{for } \nu < \mu \\ 0 & \text{for } \nu \geq \mu \end{cases}$$

(4.11)

such that the staggered phase factors become $\eta_\mu(x) = (-1)^{\eta_{(\mu)}x}$ and realizing that $(-1)^{|x|} = (-1)^{\eta(x)x + x_4}$ we find that $f_{\vec{\alpha}}(x_4)$ satisfies

$$(-1)^{x_4} M_{\vec{\alpha},\vec{\beta}} f_{\vec{\beta}}^{(i)}(x_4) = \frac{1}{2} f_{\vec{\alpha}}^{(i)}(x_4 + 1) - \sum_{k=1}^{3} i \sin(r_k^{(i)} + \varphi^{(i)}x_4)(-1)^{\alpha_k \delta_{\vec{\alpha} + \vec{\eta}^{(\mu)} + \vec{\eta}^{(\nu)}},\vec{\beta}} f_{\vec{\beta}}^{(i)}(x_4)$$

$$- \frac{1}{2} f_{\vec{\alpha}}^{(i)}(x_4 - 1).$$

(4.12)

Here $\delta$ means $\delta$ modulo 2, $r_k^{(i)} = p_k + \phi_k/L$ and $\varphi^{(i)} = (\phi'_i - \phi_i)/(LT)$.

Essentially, due to the species doubling of staggered fermions in the time direction, even though we are dealing with fermions, eq. (4.12) is a (hermitian) second difference equation. We can therefore directly apply the recursive method of appendix B of [1] to compute the fluctuation determinant for fixed $\vec{p}$.

To be precise, the coupling is defined in [3,5] as eq. (4.4) with $T = L$ so that it depends on a single scale, $L$. Unfortunately this is not possible for staggered fermions, since, as we have seen, $L/a$ must be even but $T/a$ must be odd (we have restored here the dimensions of $L$ and $T$). The closest we can get to $T = L$ is therefore $T = L \pm a$. They all coincide in the continuum limit, but at finite $a$ taking $T = L \pm a$ introduces additional $O(a)$ effects. Averaging the couplings obtained with $T = L + a$ and $T = L - a$ cancels this additional $O(a)$ effect.

We have evaluated $p^{(\pm)}_{1,1}$ of eq. (4.8) for $L/a$ ranging from 4 to 64, in steps of 2. Here the superscript $\pm$ stands for the choices $T = L \pm a$. One expects $p_{1,1}(L/a)$ (we omit the superscript for the generic case) to be given by an asymptotic series of the form [1,5]

$$p_{1,1}(L/a) = r_0 + s_0 \log(L/a) + (r_1 + s_1 \log(L/a))(a/L) + O(\log(L/a)(a/L)^2).$$

(4.13)

$^3$The fermion fluctuation determinant would be zero for $T/a$ even, yet another confirmation that $T/a$ must be odd.
The first few coefficients in eq. (4.13) can be extracted by first cancelling higher order terms in \( a/L \) through numerical differentiation and then checking for stability as \( L/a \) is increased [16]. \( s_0 \), the coefficient of the logarithmically divergent term in the continuum limit, should just be \( 2b_{0,1} = -1/(12\pi^2) \), the fermionic contribution, per flavor, to the \( \beta \)-function [5], and thus absorbed by renormalization. We indeed found this result to about 6 digits accuracy for all cases considered.

\( s_1 \) was found to be compatible with zero, albeit only with an accuracy of about 4 digits. To extract \( r_0 \) we assume the exact value for \( s_0 \); to obtain \( r_1 \) we assume in addition that \( s_1 = 0 \). Then we found the values listed in Table 1. \( r_0 \) obtained from the choices \( T = L \pm a \) always agreed to the accuracy given.

Table 1: The first two ‘non-log’ terms in the expansion eq. (4.13) of \( p_{1,1} \). \( r_1^{(\pm)} \) comes from the choices \( T = L \pm a \) and \( r_1^{(av)} \) is the average of the two.

<table>
<thead>
<tr>
<th>( \theta )</th>
<th>( r_0 )</th>
<th>( r_1^{(+)} )</th>
<th>( r_1^{(-)} )</th>
<th>( r_1^{(av)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-0.004416(1)</td>
<td>-0.02003(1)</td>
<td>0.03897(1)</td>
<td>0.00947(1)</td>
</tr>
<tr>
<td>( \pi/5 )</td>
<td>-0.00579695(2)</td>
<td>-0.023113(5)</td>
<td>0.042061(5)</td>
<td>0.009477(5)</td>
</tr>
<tr>
<td>1.0</td>
<td>-0.0068642(1)</td>
<td>-0.02552(1)</td>
<td>0.04447(1)</td>
<td>0.00947(1)</td>
</tr>
</tbody>
</table>

\( r_0 \) of eq. (4.13) is the finite part of the fermionic contribution to the one-loop relation between bare lattice and running Schrödinger functional coupling. Using the known one-loop relation between lattice and \( \overline{\text{MS}} \) coupling we can obtain a relation between the Schrödinger functional coupling \( \alpha \) and \( \alpha_{\overline{\text{MS}}} \)

\[
\alpha_{\overline{\text{MS}}}(q) = \alpha(q) + c_1 \alpha^2(q) + \mathcal{O}(\alpha^3), \quad \alpha(q = 1/L) = \bar{g}^2(L)/(4\pi). \tag{4.14}
\]

This relation between two renormalized continuum couplings has to be independent of the regularization used to obtain it. Thus, we should find the same result for \( c_{1,1} \) in \( c_1 = c_{1,0} + n_f c_{1,1} \) as [5]. \( c_{1,1} \) is given by

\[
c_{1,1} = -4\pi[P_4 + r_0] \tag{4.15}
\]

where \( P_4 = 0.0026247371 \) is the finite contribution from staggered fermions to the one-loop relation between lattice and \( \overline{\text{MS}} \) coupling, computed in [11].\(^4\) Using \( r_0 \) from Table 1 we obtain

\[
c_{1,1} = \begin{cases} 
0.02251(1) & \text{for } \theta = 0 \\
0.0398632(2) & \text{for } \theta = \pi/5
\end{cases} \tag{4.16}
\]

in good agreement with the results of Sint and Sommer for Wilson fermions [5]. This confirms that our Schrödinger functional for massless staggered fermions is a correct regularization of the continuum Schrödinger functional with massless fermions.

\(^4\)The better accuracy quoted here for \( P_4 \) is obtained from eq. (6.12) of [11] using the more accurate values for \( P_1 \) and \( P_2 \) from [17].
4.1 Lattice artefacts

The vanishing of \( s_1 \) indicates the absence of bulk \( \mathcal{O}(a) \) artefacts [5], as expected for staggered fermions [14]. The non-vanishing of \( r_1 \) reflects the presence of boundary \( \mathcal{O}(a) \) effects. As mentioned before, the fact that we are not allowed to take \( T = L \) for the computation of the running coupling introduces additional \( \mathcal{O}(a) \) effects that can not be cancelled by boundary counterterms. They can, however, be cancelled by averaging over the choices \( T = L \pm a \), as can be seen from Table 1.\(^5\) \( r_1^{(av)} \) is independent of the spatial boundary conditions parameterized by \( \theta \). This remaining \( \mathcal{O}(a) \) effect can be absorbed into the pure gauge boundary counterterm \( (c_t - 1) \sum_{k=1}^{3} \text{tr}(F_k F_k) [/5] \) by choosing \( c_t = 1 + (c_t^{(1,0)} + n_f c_t^{(1,1)})g_s^2 \) with the pure gauge part \( c_t^{(1,0)} = -0.08900(5) [3] \) and the fermionic contribution

\[
(c_t^{(1,1)} = \frac{1}{2} r_1^{(av)} = 0.00474(1).
\]

(4.17)

After cancelling the \( \mathcal{O}(a) \) boundary lattice artefact by the boundary counterterm, higher order lattice artefacts, both from the bulk and the boundary, remain. With our results we can study them for the fermionic contribution to the step scaling function [3,5], \( \Sigma(s, u, a/L) \) which is the coupling \( g^2(sL) \) at scale \( sL \) when keeping the coupling at scale \( L \) fixed at \( g^2(L) = u \). In the continuum, the one-loop fermionic contribution to \( \sigma(s, u) = \Sigma(s, u, 0) \) is \( 2b_{0,1} n_f \log(s) \) \( u^2 \), while on the lattice it is given by \( n_f [p_{1,1}(2L/a) - p_{1,1}(L/a)] u^2 \). We take \( s = 2 \) and compare the lattice result (per continuum flavor) — we consider here the average between the choices \( T = L \pm a \) — with its continuum limit:

\[
\tilde{\delta}_{1,1}(a/L) = \frac{p_{1,1}^{(av)}(2L/a) - p_{1,1}^{(av)}(L/a)}{2b_{0,1} \log 2}.
\]

(4.18)

Deviation of \( \tilde{\delta}_{1,1}(a/L) \) from 1 at finite \( a/L \) is a lattice artefact. Note that we define the lattice artefact from the fermions, \( \tilde{\delta}_{1,1}(a/L) \), with respect only to the fermionic contribution in the continuum limit, in contrast to [5]. \( \tilde{\delta}_{1,1}(a/L) \) is shown in Figs. 1 and 2 for \( \theta = 0 \) and \( \pi/5 \), respectively. Shown in both cases is the result with and without the cancellation of the \( \mathcal{O}(a) \) part of the lattice artefact by the pure gauge boundary counterterm. As can be seen from the two cases, the higher order lattice artefacts can depend very sensitively on the observable considered, here the step \( \beta \)-function for different values of the spatial boundary conditions. After cancellation of the \( \mathcal{O}(a) \) part by the boundary counterterm, one expect the remainder to go asymptotically (up to logs) like \( (a/L)^2 \). Rough estimates of the coefficient of the \( (a/L)^2 \) contribution from the figures are -7.5 and -0.5 respectively. For \( \theta = 1 \) the coefficient becomes about 4.5. These rather large variations should serve as a caution to drawing conclusions about the order of magnitude of cut-off effects from studying just one observable.

\(^5\) We expect this to be true also for the pure gauge part that, in a simulation with dynamical staggered fermions, would obviously have to come from an average of systems with \( T = L \pm a \). We have not checked this explicitly. But S. Sint has checked that the statement holds for the one-loop contribution from Wilson fermions. In addition we have performed a pure gauge MC simulation for \( \beta = 5.9044 \) with \( L/a = 4 \) and \( T = L \pm a \) and verified that the average of the inverse of the non-perturbatively computed coupling constants agrees, within errors, with the result for \( T/a = L/a = 4 \) in [3].
Figure 1: The ratio of the fermionic contribution to the step scaling function at one-loop from the lattice to its continuum limit, (4.18), for $\theta = 0$. Crosses show the result obtained from the fermion fluctuation determinant without cancellation of the $\mathcal{O}(a)$ part by the pure gauge boundary counterterm $c_t^{(1,1)}$, and octagons the result after the cancellation. The line shows the $\mathcal{O}(a)$ part of the lattice artefact that is cancelled by the pure gauge boundary counterterm.

5 The Schrödinger functional for massive staggered fermions

We have seen that with a chiral rotation, eq. (3.27) with $\theta_5 = \pi/4$, we can turn the “natural” staggered boundary conditions, in terms of four-component Dirac spinors, into the usual boundary conditions obtained when starting with Wilson fermions. Since this is a chiral rotation, the mass term, omitted so far, will not remain invariant. Thus, to obtain the usual Schrödinger functional boundary conditions with the usual mass term we have two choices. Either we change the boundary conditions for the staggered fermions, or we use an unconventional mass term.

Here we consider the second alternative. Thus, instead of the usual mass term $m\bar{\psi}\psi$ in terms of the four-component spinors before the chiral rotation we consider $m\bar{\psi}\Gamma_5^5\psi$, which becomes the usual mass term after the chiral rotation. Using eq. (3.4) this mass term can be written in terms of the one-component spinors as:

$$S_{m_5} = m \sum_x \eta_4(x) \bar{\chi}(x) \left\{ \frac{1}{2} [1 - (-1)^{x_4}] U_4^\dagger(x - \hat{4}) \chi(x - \hat{4}) - \frac{1}{2} [1 + (-1)^{x_4}] U_4(x) \chi(x + \hat{4}) \right\},$$

(5.1)
Figure 2: Same as Fig. 1, but for $\theta = \pi/5$.

where we included the gauge fields to make this non-local (on the fine lattice) mass term gauge invariant.

However, this new mass term, eq. (5.1), introduces an $\mathcal{O}(a)$ term in the bulk. This can be seen by writing it in terms of the improved four-component spinors, eq. (3.22),

$$
m(2a)^4 \sum_y \bar{\psi}(y) \tilde{\Gamma}_5 \psi(y) = m(2a)^4 \sum_y \left[ \bar{\psi}'(y) \tilde{\Gamma}_5 \psi'(y) + a \bar{\psi}'(y) \tilde{\Gamma}_4 D_4 \psi'(y) \right] + \mathcal{O}(a^2). \tag{5.2}
$$

To the same order, we can replace $D_4$ in the second term by $\hat{D}_4$. Subtracting this term, we arrive at the $\mathcal{O}(a)$ improved mass term

$$
S_{m_5} = -m \sum_x \frac{1}{2} (-1)^{x_4} \eta_4(x) \bar{\chi}(x) \left( U_4(x) \chi(x + \hat{4}) + U_4^t(x - \hat{4}) \chi(x - \hat{4}) \right). \tag{5.3}
$$

A few comments are in order here: the mass term (5.3) obviously singles out the time direction. But the time direction is singled out in the Schrödinger functional formalism anyway. This mass term is invariant under the “chiral” $U(1)_c$ symmetry, but it breaks the shift symmetry in the time direction. Indeed, a shift by one lattice unit in the time direction reverses the sign of the mass. Hence, for the usual setting with (anti-) periodic boundary conditions, the model is invariant under a change of sign of the mass. The free spectrum, for example, is a function of $m^2$. However, such a shift in the time direction is not possible at the boundary. The Schrödinger functional is thus sensitive to the sign of the mass, and we therefore expect the Schrödinger coupling to depend linearly on the mass for small masses. In the continuum limit, the shift symmetries become part of the spin and vector and axial flavor
symmetries. Thus breaking of the shift symmetry implies breaking of the chiral symmetry in the continuum, just as is expected of a mass term.

Since a mass term can break a symmetry, counterterms have to be re-examined in its presence. The mass term (5.3) only breaks the shift symmetry in the time direction, but leaves the “chiral” $U(1)_c$ symmetry unbroken. In the bulk, the only allowed counterterm multiplicatively renormalizes the mass. On the boundary, the only $O(a)$ term allowed in addition to those in eq. (3.24) is

$$O_5 = m\bar{\psi}\Gamma^5\psi,$$

\[ (5.4) \]

\[ i.e., \] the new term now already appearing in the action.

Having found a mass term that is designed to correspond to the usual mass term in the Schrödinger functional formalism, we can check this at one–loop by computing the contribution to the running coupling. The computation goes as in the previous section. The eigenfunctions now satisfy

\[ \begin{align*}
(-1)^{x_4}M_{\alpha,\beta}f^{(i)}_{\beta}(x_4) &= \frac{1}{2} \left[ 1 + (-1)^{x_4}m \right] f^{(i)}_{\alpha}(x_4 + 1) \\
&- \sum_{k=1}^{3} i \sin(r_k^{(i)} + \varphi^{(i)}x_4)(-1)^{\alpha
\bar{\delta}_{\alpha+i\bar{\eta}^{(4)}+\bar{\eta}^{(5)}}\beta}f^{(i)}_{\beta}(x_4)
\end{align*} \]

\[ (5.5) \]

Since the scale for the running coupling is given by the spatial system size $L$ we want to keep $z = mL$ fixed when varying $L/a$. Again, we compute the contribution $p_{1,1}(z, L/a)$ for $L/a = 4$ up to 64 and extract the leading terms in the asymptotic expansion eq. (4.13). A few examples are listed in Table 2. Again we find $s_0 = -1/(12\pi^2)$ to our accuracy, and $s_1$ compatible with zero.\(^6\)

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$z$</th>
<th>$r_0$</th>
<th>$r_1^{(+)i}$</th>
<th>$r_1^{(-)}$</th>
<th>$r_1^{(av)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>0.0024408(1)</td>
<td>-0.00426(1)</td>
<td>0.02319(1)</td>
<td>0.00947(1)</td>
</tr>
<tr>
<td>$\pi/5$</td>
<td>1.0</td>
<td>0.00179605(5)</td>
<td>-0.00617(1)</td>
<td>0.02511(1)</td>
<td>0.00947(1)</td>
</tr>
<tr>
<td>0</td>
<td>2.0</td>
<td>0.0064310(1)</td>
<td>0.00223(1)</td>
<td>0.01671(1)</td>
<td>0.00947(1)</td>
</tr>
<tr>
<td>$\pi/5$</td>
<td>2.0</td>
<td>0.00632882(5)</td>
<td>0.00203(1)</td>
<td>0.01691(1)</td>
<td>0.00947(1)</td>
</tr>
</tbody>
</table>

Table 2: $r_0$ and $r_1$’s as in Table 1 but in the massive case with $z = mL$ held fixed.

The difference $c_{1,1}(z) - c_{1,1}(0) = -4\pi[r_0(z) - r_0(0)]$ should be regularization independent. Comparing with the Wilson fermion results of [5] we indeed find agreement within the accuracy given in Tables 1 and 2.

\[ ^6 \text{We actually discovered the presence of a bulk } O(a) \text{ contribution for the unimproved mass term, eq. (5.1), by finding a non-vanishing } s_1. \]
The value of $r_{1}^{(av)}$ is independent of $z$ and hence cancelled by the pure gauge boundary counterterm with coefficient (4.17). No mass dependent $O(a)$ bulk term of the form $\text{tr}(F_{\mu\nu}F_{\mu\nu})$ is needed, in contrast to the case of Wilson fermions [5].

If we do not insist on reproducing the conventional (i.e. Wilson fermion inspired) massive Schrödinger functional in the continuum limit, we can use the “natural” staggered boundary conditions and the usual (degenerate) staggered mass term without chiral rotation

$$S_{m} = m \sum_{x} \bar{\chi}(x)\chi(x) = (2a)^{4} m \sum_{y} \bar{\psi}(y)\psi(y).$$

(5.6)

This leads to a well–defined Schrödinger functional in the continuum, albeit with boundary conditions that mix the four continuum flavors. In contrast to the “conventional” case now both action and boundary conditions are invariant under $m \rightarrow -m$ and thus the Schrödinger functional running coupling defined by these conventions is an even function of $z = mL$. Inclusion of the mass term (5.6) into eq. (4.12) is straightforward, giving

$$(-1)^{x_{4}} \mathcal{M}_{\bar{\alpha},\bar{\beta}} f_{\bar{\beta}}^{(i)}(x_{4}) = \frac{1}{2} f_{\alpha}^{(i)}(x_{4} + 1) - \sum_{k=1}^{3} i \sin(r_{k}^{(i)} + \varphi^{(i)} x_{4}) (-1)^{\alpha_{k}} \tilde{\delta}_{\bar{\alpha} + \bar{\eta}^{(i)} + \bar{\eta}^{(4)}, \bar{\beta}} f_{\bar{\beta}}^{(i)}(x_{4})$$

$$+ \ m \tilde{\delta}_{\bar{\alpha} + \bar{\eta}^{(i)} + \bar{\eta}^{(4)}, \bar{\beta}} f_{\bar{\beta}}^{(i)}(x_{4}) - \frac{1}{2} f_{\alpha}^{(i)}(x_{4} - 1).$$

(5.7)

Proceeding as before, we now obtain the sample results listed in Table 3. We note in particular that $r_{1}^{(av)}$ is unchanged from before and hence cancelled by the pure gauge boundary counterterm with coefficient (4.17).

<table>
<thead>
<tr>
<th>$\theta$</th>
<th>$z^{2}$</th>
<th>$r_{0}$</th>
<th>$r_{1}^{(+)}$</th>
<th>$r_{1}^{(-)}$</th>
<th>$r_{1}^{(av)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1.0</td>
<td>-0.0004189(1)</td>
<td>-0.00853(1)</td>
<td>0.02747(1)</td>
<td>0.00947(1)</td>
</tr>
<tr>
<td>$\pi/5$</td>
<td>1.0</td>
<td>-0.00195976(3)</td>
<td>-0.013405(3)</td>
<td>0.032350(3)</td>
<td>0.009473(3)</td>
</tr>
</tbody>
</table>

Table 3: $r_{0}$ and $r_{1}$’s as in Table 1 but in the massive case with the standard staggered mass term, (5.6).

The mass term (5.6) breaks the “chiral” $U(1)_{c}$ symmetry. As a consequence the $O(a)$ boundary counterterms

$$O_{6} = m \bar{\psi}\psi$$

$$O_{7} = m \bar{\psi}(\gamma_{4}\gamma_{5}) \otimes (\gamma_{5}\gamma_{4})^{T}\psi.$$

(5.8)

are allowed in addition to those in eq. (3.24) and the term in (5.4).

---

7We do not consider mass terms that break some of the staggered symmetries (other than the chiral symmetry) which might be used to lift the degeneracy among the four continuum flavors.
6 Conclusions

We have discussed the Schrödinger functional with staggered fermions, extending and completing previous work by Miyazaki and Kikukawa. In particular, we have shown that for massless fermions the Schrödinger functional constructed agrees, in the continuum limit, with the conventional Schrödinger functional for fermions as described by Sint, by examining the Schrödinger functional for staggered fermions in terms of four flavors of Dirac spinors. We found one boundary counterterm that contributes in the continuum limit, just as for Wilson fermions and for continuum fermions with dimensional regularization. It can be absorbed by a renormalization of the boundary fields.

We have computed the one-loop contribution from massless staggered fermions to the running Schrödinger functional coupling and found agreement, in the continuum limit, with the results of Sint and Sommer for Wilson fermions. We computed the one–loop contribution from staggered fermions to the pure gauge $\mathcal{O}(a)$ boundary counterterm, which cancels the $\mathcal{O}(a)$ lattice artefact in the running coupling. We discussed the higher order lattice artefacts in the step scaling function at one loop. They can be as large as 20% on an $L/a = 6$ lattice, and they can depend rather sensitively on the observable considered, in our case the coupling defined with different fermionic boundary conditions.

A chiral, flavor changing rotation was needed to bring the boundary condition of staggered fermions, expressed in terms of four flavors of Dirac fermions, to the conventional form. The mass term is not invariant under such a rotation. We therefore considered an unconventional mass term for the staggered fermions, constructed to give the conventional Schrödinger functional for massive fermions in the continuum limit. The construction was verified by computing the one-loop contribution to the Schrödinger coupling constant for fixed $z = mL$ and reproducing results by Sint and Sommer.

One unpleasantness in defining the Schrödinger functional running coupling constant with dynamical staggered fermions is the fact that we cannot take $T = L$, since $L/a$ has to be even while $T/a$ has to be odd. Therefore the coupling does not strictly depend on a single scale. Taking $T = L \pm a$ gives a single scale in the continuum limit ($a \to 0$), but introduces $\mathcal{O}(a)$ effects at finite lattice spacing. As discussed, these $\mathcal{O}(a)$ effects, which can not be cancelled by boundary counterterms, can be cancelled by averaging over the coupling obtained with $T = L \pm a$. However, since simulations with staggered fermions tend to be less costly than simulations with Wilson-type fermions — staggered fermions have four times fewer degrees of freedom, and no fine–tuning is required to make them massless — having to do two simulations for each $L/a$ might not be such a big price to pay.

Having established the Schrödinger functional with dynamical staggered quarks opens the possibility to use it for the non-perturbative computation of improvement coefficients and current renormalization constants for gauge field ensembles with dynamical staggered fermions, just as it is done for quenched simulations by the ALPHA collaboration [6,7]. We envision here the computation of the clover coefficient and current renormalization constants for (improved) Wilson valence quarks. These can then be used to compute phenomenologically interesting quantities, such as $\epsilon_B$ and $f_D$, with Wilson valence fermions on gauge
configurations generated with dynamical staggered fermions [18]. Comparison with quenched results then allows an estimation of quenching errors [19] in those quenched computations.

Acknowledgements

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Appendix A

In this appendix we briefly review the symmetries of staggered fermions [20,13]. Unless otherwise noted, the one-component spinors transform under a symmetry transformation $T$ as

\[
\chi(x) \rightarrow \chi'(x) = \eta_T(T^{-1}x)\chi(T^{-1}x)
\]

\[
\bar{\chi}(x) \rightarrow \bar{\chi}'(x) = \eta_T(T^{-1}x)\bar{\chi}(T^{-1}x).
\]

The gauge fields transform as

\[
U(x, z) \rightarrow U(T^{-1}x, T^{-1}z)
\]

with

\[
U(x, z) = \begin{cases} 
U_\mu(x) & \text{for } z = x + \mu \\
U_\mu^\dagger(z) & \text{for } x = z + \mu.
\end{cases}
\]

(i) Rotations by $\pi/2$:

Here we consider rotations around the center of a hypercube ($\rho < \sigma$):

\[
R_{H}^{\rho,\sigma} : 
\begin{align*}
    x_\rho &\rightarrow x_\sigma \\
x_\sigma &\rightarrow -x_\rho + 1 \\
x_\mu &\rightarrow x_\mu \text{ for } \mu \neq \rho, \sigma.
\end{align*}
\]

Then we have

\[
\eta_{R_{H}}(x)(-1)^{(x_\rho+x_\sigma)(x_{\rho+1}+\ldots+x_{\sigma-1})+x_\rho(x_\sigma+1)+x_{\sigma+1}+\ldots+x_4}.
\]

In terms of the hypercube block coordinates:

\[
y_\rho \rightarrow y_\sigma, \quad y_\sigma \rightarrow -y_\rho, \quad y_\mu \rightarrow y_\mu \text{ for } \mu \neq \rho, \sigma,
\]

18
and the fields $\psi$ and $\bar{\psi}$ transform as

\[
\begin{align*}
\psi(y) & \rightarrow \frac{1}{2}(1 + \gamma_\rho \gamma_\sigma) \otimes (\gamma_\sigma^T - \gamma_\rho^T) \cdot \psi(R_{H}^{-1}y) \\
\bar{\psi}(y) & \rightarrow \bar{\psi}(R_{H}^{-1}y) \cdot \frac{1}{2}(1 - \gamma_\rho \gamma_\sigma) \otimes (\gamma_\sigma^T - \gamma_\rho^T).
\end{align*}
\] (A.7)

(ii) Reflections on hyperplanes:

Here we consider reflections on hyperplanes through the center of a hypercube:

\[
I_{H}^\rho : \quad x_\rho \rightarrow -x_\rho + 1 \quad \text{for } \mu \neq \rho.
\] (A.8)

Then we have

\[
\eta_{I_{H}}(x)(-1)^{x_\rho+...+x_4}.
\] (A.9)

In terms of the hypercube block coordinates:

\[
y_\rho \rightarrow -y_\rho, \quad y_\mu \rightarrow y_\mu \quad \text{for } \mu \neq \rho,
\] (A.10)

and the fields $\psi$ and $\bar{\psi}$ transform as

\[
\psi(y) \rightarrow (\gamma_\rho \gamma_5) \otimes \gamma_5^T \cdot \psi(I_{H}^{-1}y), \quad \bar{\psi}(y) \rightarrow \bar{\psi}(I_{H}^{-1}y) \cdot (\gamma_5 \gamma_\rho) \otimes \gamma_5^T.
\] (A.11)

Combining 3 reflections, one orthogonal to each spatial direction gives the parity transformation

\[
P : \quad \vec{y} \rightarrow -\vec{y}, \quad y_4 \rightarrow y_4,
\] (A.12)

with

\[
\psi(y) \rightarrow \gamma_4 \otimes \gamma_5^T \cdot \psi(Py), \quad \bar{\psi}(y) \rightarrow \bar{\psi}(Py) \cdot \gamma_4 \otimes \gamma_5^T.
\] (A.13)

(iii) Shift invariance:

\[
T^\rho : \quad x_\rho \rightarrow x_\rho + 1 \quad \text{for } \mu \neq \rho.
\] (A.14)

Then we have

\[
\eta_{T}(x)(-1)^{x_\rho + 1 + ... + x_4},
\] (A.15)

and the fields $\psi$ and $\bar{\psi}$ transform as

\[
\psi(y) \rightarrow \frac{1}{2}\delta_{y,y'}(1 \otimes \gamma_\rho^T - \gamma_\rho \gamma_5 \otimes \gamma_5^T) \cdot \psi(y') + \frac{1}{2}\delta_{y+\rho,y'}(1 \otimes \gamma_\rho^T + \gamma_\rho \gamma_5 \otimes \gamma_5^T) \cdot \psi(y')
\]

\[
\bar{\psi}(y) \rightarrow \frac{1}{2}\delta_{y,y'}\bar{\psi}(y') \cdot (1 \otimes \gamma_\rho^T + \gamma_\rho \gamma_5 \otimes \gamma_5^T) + \frac{1}{2}\delta_{y,y'-\rho}\bar{\psi}(y') \cdot (1 \otimes \gamma_\rho^T - \gamma_\rho \gamma_5 \otimes \gamma_5^T).
\] (A.16)
Symmetries (i) through (iii) are the space-time symmetries of staggered fermions. In addition we have the invariances:

(iv) $U(1)$-invariance:
\[
\chi(x) \to e^{i\alpha} \chi(x), \quad \bar{\chi}(x) \to e^{-i\alpha} \bar{\chi}(x),
\] (A.17)
which just becomes
\[
\psi(y) \to e^{i\alpha} \psi(y), \quad \bar{\psi}(y) \to e^{-i\alpha} \bar{\psi}(y).
\] (A.18)

and

(v) $U(1)_\epsilon$-invariance:
\[
\chi(x) \to e^{i\beta \epsilon(x)} \chi(x), \quad \bar{\chi}(x) \to e^{i\beta \epsilon(x)} \bar{\chi}(x),
\] (A.19)
where $\epsilon(x) = (-1)^{|x|}$. This becomes the “chiral” transformation
\[
\psi(y) \to e^{i\beta \gamma_5 \otimes \gamma_5^T} \psi(y), \quad \bar{\psi}(y) \to \bar{\psi}(y) \cdot e^{i\beta \gamma_5 \otimes \gamma_5^T}.
\] (A.20)
This chiral symmetry protects the zero-mass limit for staggered fermions, since the usual mass term
\[
m \sum_x \bar{\chi}(x) \chi(x) = m 2^4 \sum_y \bar{\psi}(y) \psi(y)
\] is not invariant.

Finally, staggered fermions have the discrete

(vi) Interchange symmetry:
\[
\chi(x) \to \epsilon(x) \bar{\chi}^T(x), \quad \bar{\chi}(x) \to -\chi^T(x) \epsilon(x),
\] (A.21)
where $T$ stands for transpose (as a color vector). The gauge fields transform as
\[
U_\mu(x) \to U_\mu^*(x).
\] (A.22)
In terms of the four-component spinors this becomes charge conjugation symmetry
\[
\psi(y) \to C \bar{\psi}^T(y), \quad \bar{\psi}(y) \to -\psi^T(y) C.
\] (A.23)
Here $C = C \otimes (C^{-1})^T$ where $C$ is the usual Euclidean charge conjugation symmetry matrix satisfying
\[
C \gamma_\mu C^{-1} = -\gamma_\mu^T \quad C \gamma_5 C^{-1} = \gamma_5^T \quad C \gamma_5 \gamma_\mu C^{-1} = (\gamma_5 \gamma_\mu)^T \quad -C = C^T = C^{-1} = C^\dagger.
\] (A.24)
Appendix B

In this appendix we sketch the construction of a “two–time–slice” transfer matrix for staggered fermions, following quite closely the construction in [11] for the “reduced” staggered fermions. We consider staggered fermions with the standard (flavor degenerate) mass term. The partition function is

\[ Z = \int [DU][d\bar{\chi}d\chi] \exp\{-S_G - S_F\}. \]  

(B.1)

with \( S_G \) the usual Wilson gauge action and

\[ S_F = \sum_x \sum_{\mu} \frac{1}{2} \eta_\mu(x) \bar{\chi}(x) \left[ U_\mu(x) \chi(x + \mu) - U_\mu^\dagger(x - \mu) \chi(x - \mu) \right] + \sum_x m \bar{\chi}(x) \chi(x). \]  

(B.2)

We now change integration variables for each pair of time slices \( x_4 = 2\tau \) and \( 2\tau + 1 \)

\[
\begin{align*}
\alpha_\tau^\dagger(\bar{x}) &= P_\tau(\bar{x}) \frac{1}{2} \eta_4(\bar{x}) \bar{\chi}(\bar{x}, 2\tau) + P_\tau(\bar{x}) \chi(\bar{x}, 2\tau) \\
\alpha_\tau(\bar{x}) &= P_\tau(\bar{x}) \chi(\bar{x}, 2\tau + 1) + P_\tau(\bar{x}) \frac{1}{2} \eta_4(\bar{x}) \bar{\chi}(\bar{x}, 2\tau + 1) \\
\beta_\tau^\dagger(\bar{x}) &= P_\tau(\bar{x}) \chi(\bar{x}, 2\tau) + P_\tau(\bar{x}) \frac{1}{2} \eta_4(\bar{x}) \bar{\chi}(\bar{x}, 2\tau) \\
\beta_\tau(\bar{x}) &= P_\tau(\bar{x}) \frac{1}{2} \eta_4(\bar{x}) \bar{\chi}(\bar{x}, 2\tau + 1) + P_\tau(\bar{x}) \chi(\bar{x}, 2\tau + 1)
\end{align*}
\]

where \( P_\tau, o(\bar{x}) = \frac{1}{2} \left( 1 \pm (-1)^{|k|} \right) \) are the projectors onto even and odd spatial sites \( \bar{x} \). We have used here that staggered phase factors \( \eta_\mu \) are independent of the time coordinate; in particular, \( \eta_4(\bar{x}, x_4) = \eta_4(\bar{x}) = (-1)^{|k|} \).

Introducing

\[ \bar{\eta}_k(\bar{x}) = \eta_4(\bar{x}) \eta_k(\bar{x}), \quad \tilde{\eta}_k(\bar{x} + \hat{k}) = -\tilde{\eta}_k(\bar{x}), \]

we can write the staggered action in temporal gauge, \( U_4(\bar{x}) = 1 \), as

\[
\begin{align*}
S_F &= \sum_\tau \sum_{\bar{x}} \left\{ \left[ \alpha_\tau^\dagger(\bar{x}) \alpha_\tau(\bar{x}) - \alpha_{\tau+1}^\dagger(\bar{x}) \alpha_\tau(\bar{x}) + \beta_\tau^\dagger(\bar{x}) \beta_\tau(\bar{x}) - \beta_{\tau+1}^\dagger(\bar{x}) \beta_\tau(\bar{x}) \right] \\
&\quad + P_\tau(\bar{x}) \sum_k \left[ \alpha_\tau^\dagger(\bar{x}) \tilde{\eta}_k(\bar{x}) U_k(\bar{x}, 2\tau) \alpha_\tau^\dagger(\bar{x} + \hat{k}) + \beta_\tau^\dagger(\bar{x} + \hat{k}) \tilde{\eta}_k(\bar{x}) U_k(\bar{x}, 2\tau) \beta_\tau^\dagger(\bar{x}) \right] \\
&\quad + P_\tau(\bar{x}) \sum_k \left[ \beta_\tau^\dagger(\bar{x}) \tilde{\eta}_k(\bar{x}) U_k(\bar{x}, 2\tau + 1) \beta_\tau^\dagger(\bar{x} + \hat{k}) + \alpha_\tau^\dagger(\bar{x} + \hat{k}) \tilde{\eta}_k(\bar{x}) U_k(\bar{x}, 2\tau + 1) \alpha_\tau^\dagger(\bar{x}) \right] \\
&\quad + 2m \left[ \alpha_\tau^\dagger(\bar{x}) \beta_\tau(\bar{x}) + \beta_\tau^\dagger(\bar{x}) \alpha_\tau(\bar{x}) \right] \right\}.
\end{align*}
\]

(B.4)
that all daggered operators appear to the left of all non-daggered operators, 

\[ \{ \hat{\alpha}(\vec{x}), \hat{\alpha}^\dagger(\vec{x}) \} = \{ \hat{\beta}(\vec{x}), \hat{\beta}^\dagger(\vec{x}) \} = \delta_{\vec{x}\vec{x}} \]  

(B.5)

and all other anticommutators vanishing. On the Fock space, spanned by these operators, we consider the coherent states

\[ |\alpha_\tau, \beta_\tau\rangle \equiv \exp \left\{ \sum_{\vec{x}} \left[ \hat{\alpha}^\dagger(\vec{x}) \alpha_\tau(\vec{x}) + \hat{\beta}^\dagger(\vec{x}) \beta_\tau(\vec{x}) \right] \right\} |0\rangle \]  

(B.6)

and

\[ \langle \alpha^\dagger_\tau, \beta^\dagger_\tau | = \langle 0 | \exp \left\{ \sum_{\vec{x}} \left[ \alpha^\dagger_\tau(\vec{x}) \hat{\alpha}(\vec{x}) + \beta^\dagger_\tau(\vec{x}) \hat{\beta}(\vec{x}) \right] \right\} \]  

(B.7)

where \(|0\rangle\) is the Fock vacuum. It can be shown that these coherent states satisfy the completeness relation

\[ 1 = \int [d\alpha^\dagger_\tau] [d\beta^\dagger_\tau] \exp \left\{ - \sum_{\vec{x}} \left[ \alpha^\dagger_\tau(\vec{x}) \alpha_\tau(\vec{x}) + \beta^\dagger_\tau(\vec{x}) \beta_\tau(\vec{x}) \right] \right\} |\alpha_\tau, \beta_\tau\rangle \langle \alpha^\dagger_\tau, \beta^\dagger_\tau |, \]  

(B.8)

and that for a normal ordered operator \( \hat{A} = A(\hat{\alpha}^\dagger, \hat{\beta}^\dagger; \hat{\alpha}, \hat{\beta}) \), where normal ordering means that all daggered operators appear to the left of all non-daggered operators,

\[ \langle \alpha^\dagger_{\tau+1}, \beta^\dagger_{\tau+1} | \hat{A} \alpha_\tau, \beta_\tau \rangle = A(\alpha^\dagger_{\tau+1}, \beta^\dagger_{\tau+1}; \alpha_\tau, \beta_\tau) \exp \left\{ \sum_{\vec{x}} \left[ \alpha^\dagger_{\tau+1}(\vec{x}) \alpha_\tau(\vec{x}) + \beta^\dagger_{\tau+1}(\vec{x}) \beta_\tau(\vec{x}) \right] \right\}. \]  

(B.9)

We can see that the terms in the exponentials in (B.8) and (B.9) reproduce the terms from the first line of the action \( S_F \), eq. (B.4). The other terms can be reproduced from the transfer matrix (in temporal gauge)

\[ \hat{T} = \hat{T}_G^{1/2} \hat{T}_F \hat{T}_G \hat{T}_F^{1/2} \]  

(B.10)

with \( \hat{T}_G \) the pure gauge transfer matrix (in temporal gauge) [21] — we need a total of \( \hat{T}_G^2 \) since we move two time slices forward. For \( \hat{T}_F \) we find

\[ \hat{T}_F = \exp \left\{ - \sum_{\vec{x}} \left( P_e(\vec{x}) \left[ \hat{\alpha}(\vec{x} + \hat{k}) \tilde{\eta}_k(\vec{x}) \hat{U}_k(\vec{x}) \hat{\alpha}(\vec{x}) + \hat{\beta}(\vec{x}) \tilde{\eta}_k(\vec{x}) \hat{U}_k(\vec{x}) \hat{\beta}(\vec{x} + \hat{k}) \right] \right. \right. \]

\[ + P_o(\vec{x}) \left[ \hat{\beta}(\vec{x} + \hat{k}) \tilde{\eta}_k(\vec{x}) \hat{U}_k(\vec{x}) \hat{\beta}(\vec{x}) + \hat{\alpha}(\vec{x}) \tilde{\eta}_k(\vec{x}) \hat{U}_k(\vec{x}) \hat{\alpha}(\vec{x} + \hat{k}) \right] \]

\[ + \left. \left. m \hat{\beta}(\vec{x}) \hat{\alpha}(\vec{x}) \right] \right\} . \]  

(B.11)

With some algebra one can show that \( \hat{T}_F^\dagger \hat{T}_F \) is equal to the square of the “one–time–slice” transfer matrix of [11] when gauge fields and the mass term are included in the latter.
The “two–time–slice” transfer matrix $T$, (B.10), is self–adjoint and positive. Therefore, restoring the lattice spacing in the time direction, $a_t$, we can define a Hamiltonian by

$$T = e^{-2a_t\mathcal{H}}. \tag{B.12}$$

$\mathcal{H}$ is a complicated function of the fermion and gauge field operators. It simplifies in the naive time continuum limit, $a_t \to 0$, where we can neglect contributions of $O(a_t^2)$ and higher. The fermionic part becomes in this limit

$$\hat{H}_F = \sum_{\vec{x}} \left\{ \frac{1}{2} P_e(\vec{x}) \left[ \hat{\alpha}^\dagger(\vec{x}) \hat{\eta}_k(\vec{x}) \hat{U}_k(\vec{x}) \hat{\alpha}(\vec{x} + \hat{k}) + \hat{\alpha}(\vec{x} + \hat{k}) \hat{\eta}_k(\vec{x}) \hat{U}_k^\dagger(\vec{x}) \hat{\alpha}(\vec{x}) \right. \right. \\
+ \left. \left. \beta^\dagger(\vec{x} + \hat{k}) \hat{\eta}_k(\vec{x}) \hat{U}_k^\dagger(\vec{x}) \beta(\vec{x}) + \hat{\beta}(\vec{x}) \hat{\eta}_k(\vec{x}) \hat{U}_k(\vec{x}) \hat{\beta}(\vec{x} + \hat{k}) \right] \right\}.$$ \tag{B.13}

We can bring this Hamiltonian to a more conventional form with the canonical transformation

$$\hat{\chi}(\vec{x}) = \frac{1}{\sqrt{2}} P_e(\vec{x}) \left( \hat{\beta}^\dagger(\vec{x}) - \hat{\alpha}(\vec{x}) \right) + \frac{1}{\sqrt{2}} P_0(\vec{x}) \left( \hat{\beta}(\vec{x}) - \hat{\alpha}^\dagger(\vec{x}) \right)$$

$$\hat{\psi}(\vec{x}) = \frac{1}{\sqrt{2}} P_e(\vec{x}) \left( \hat{\beta}^\dagger(\vec{x}) + \hat{\alpha}(\vec{x}) \right) + \frac{1}{\sqrt{2}} P_0(\vec{x}) \left( \hat{\beta}(\vec{x}) + \hat{\alpha}^\dagger(\vec{x}) \right). \tag{B.14}$$

The operators $\hat{\chi}$ and $\hat{\psi}$ and their adjoint’s $\hat{\chi}^\dagger$ and $\hat{\psi}^\dagger$ satisfy anticommutation relations like eq. (B.5). Expressed in terms of these operators the Hamiltonian (B.14) reads

$$\hat{H}_F = \frac{1}{2} \sum_{\vec{x},\vec{k}} \left[ \hat{\psi}^\dagger(\vec{x}) \hat{\eta}_k(\vec{x}) \hat{U}_k(\vec{x}) \hat{\psi}(\vec{x} + \hat{k}) + \hat{\psi}(\vec{x} + \hat{k}) \hat{\eta}_k(\vec{x}) \hat{U}_k^\dagger(\vec{x}) \hat{\psi}(\vec{x}) \right. \\
+ \hat{\chi}(\vec{x}) \hat{\eta}_k(\vec{x}) \hat{U}_k(\vec{x}) \hat{\chi}(\vec{x} + \hat{k}) + \hat{\chi}^\dagger(\vec{x} + \hat{k}) \hat{\eta}_k(\vec{x}) \hat{U}_k^\dagger(\vec{x}) \hat{\chi}^\dagger(\vec{x}) \right] \tag{B.15}$$

$$+ m \sum_{\vec{x}} (-1)^{|\vec{x}|} \left[ \hat{\psi}^\dagger(\vec{x}) \hat{\psi}(\vec{x}) - \hat{\chi}^\dagger(\vec{x}) \hat{\chi}(\vec{x}) \right].$$

We recognize (B.16) as the Susskind Hamiltonian [22] for two independent staggered fermion fields $\psi$ and $\chi$.

References


23


