Ray-Singer Torsion for a Hyperbolic 3-Manifold and
Asymptotics of Chern-Simons-Witten Invariant

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Abstract: The Ray-Singer torsion for a compact smooth hyperbolic 3-dimensional manifold \( \mathcal{H}^3 \) is expressed in terms of Selberg zeta-functions, making use of the associated Selberg trace formulae. Applications to the evaluation of the semiclassical asymptotics of the Witten’s invariant for the Chern-Simons theory with gauge group \( SU(2) \) as well as to the sum over topologies in 3-dimensional quantum gravity are presented.

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1 Introduction

Recently, the topology of manifolds has been studied with the help of quantum field theory methods. In this approach the partition functions of quadratic functionals play an important role. It has been shown that the Ray-Singer analytic torsion (a topological invariant) \([1]\) can be obtained within quantum field theory as the partition function of a certain quadratic functional \([2, 3]\).

Furthermore, new invariants related to 3-manifolds within the framework of Chern-Simons gauge theory have been constructed in Ref. \([4]\). These invariants (well-defined topological invariants) have been specified in terms of the axioms of topological quantum field theory \([5]\). The equivalent derivation of 3-manifold invariants has also been given combinatorially in Ref. \([6, 7]\), where modular Hopf algebras associated to quantum groups have been used. A considerable interest in topological quantum field theory has been stimulated by the introduction of the Witten’s invariant. \(\text{iThis invariant has been explicitly calculated for a number of 3-manifolds and gauge groups [8–15].}\)

It has been observed that the semiclassical approximation can be associated with the asymptotics \( k \rightarrow \infty \) of Witten’s invariant \( Z_W(k) \) (the level \( k \in \mathbb{Z} \)) of a 3-manifold \( M \) and gauge group \( G \). Typically this expression is a sum over partition functions of quadratic functionals. For

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a number of 3-manifolds and $G = SU(2)$ and more general groups the large-$k$ limit has been explicitly calculated in Refs. [4, 10, 16, 11, 13–15, 17].

The method considered in Ref. [17] for evaluating the partition function (and the partition function appearing in the semiclassical approximation) for a class of quadratic functionals is an extension and refinement of the method proposed in [2, 3]. This has led to the formulae for the partition function as a topological invariant for a wider class of quadratic functionals. In addition, the Ray-Singer torsion as a function of the cohomology [18] has been obtained as the partition function of a quadratic functional.

These methods have been applied in order to derive the usual Ray-Singer torsion associated with a flat connection. When the cohomology of the connection is non-vanishing the torsion is metric-dependent. However, in some cases, the metric-dependence factors out in a simple way as a power of the volume of the manifold to give a topological invariant [17].

In this paper, we compute the Ray-Singer torsion for a compact smooth 3-dimensional hyperbolic manifold $H^3 \equiv H^3/\Gamma$, $H^3$ being the Lobachevsky space and $\Gamma$ is a co-compact discrete group of isometries (for more detail see [19]). This result will be used for the evaluation of the semiclassical asymptotics of the Witten’s invariant related to the manifold $H^3$ and the gauge group $SU(2)$ as well as for an expression of the one-loop 3-dimensional Euclidean partition function in the case of negative cosmological constant.

The contents of the paper are the following. In Sect. 2 we review the relevant information on the semiclassical approximation in the Chern-Simons theory, involving partition functions of quadratic functionals. The analytic properties of zeta and eta functions associated with elliptic operator acting on $p$-form are discussed in Sect. 3. The partition function related to the elliptic resolvent and the trace formula for the Laplacian on $p$-forms are investigated respectively in Sects. 4 and 5. The new theoretical result of this paper, the explicit computation of the Ray-Singer torsion and asymptotics of Witten’s $H^3$ invariant, is presented in Sect. 6. We end with some conclusions in Sect. 7. Finally the Appendices A and B contain a summary of the Plancherel measure, zeta functions, and the Selberg trace formula for transverse vector fields.

2 The Partition Function and Semiclassical Approximation

In this Section, we briefly summarize the formalism we shall use in the paper. To start with, we recall that the partition function of a quadratic functional can be formally rewritten as follows

$$Z(\beta) = \int_\mathcal{G} D\omega e^{-\beta S(\omega)},$$

(2.1)

where $S(\omega) = F(\omega,\omega)$ is a real-valued quadratic functional on the space $\mathcal{G}$ of sections $\omega$ (in a vectorbundle over a manifold $M$) and $\beta$ is a complex-valued scaling parameter.

Let $\langle \cdot, \cdot \rangle$ b the inner product in $\mathcal{G}$ which determines an integration measure $D\omega$. One can write $S(\omega) = \langle \omega, T\omega \rangle$, where $T$ is a uniquely determined self-adjoint operator.

Generally speaking, Eq. (2.1) is mathematically ill-definite and the partition function is divergent, but in our cases, since formally it depends on functional determinants of elliptic operators, it may be regularised via zeta-function regularisation [20]. If $S(\omega)$ is non-degenerate, $\ker(T) = 0$ then one can obtain a finite expression for the partition function, which depends on the choice of inner product $\langle \cdot, \cdot \rangle$ in $\mathcal{G}$. If $S(\omega)$ is degenerate, the partition function diverges also due to the divergent volume $V(\ker(T))$, but it can be formally expressed in terms of $\zeta \equiv \dim\mathcal{G} - \dim(\ker(T))$ and the (divergent) volume $V(\ker(T))$. A method for evaluating the partition function in the degenerate case has been proposed in [2, 3] and requires the functional $S(\omega)$ to have an additional structure associated with it, namely a resolvent.

Let $(\mathcal{G}_p, T_p)$ be a complex, i.e. a sequence of vector space $\mathcal{G}_p$ and linear operators $T_p$ acting from space $\mathcal{G}_p$ to the space $\mathcal{G}_{p+1}$, $(\mathcal{G}_0 = \mathcal{G}_{N+1})$ and satisfying $T_{p+1}T_p = 0$ for all $p = 0, 1, ... N$. 

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Let us define the adjoint operators $T_p^* : \mathcal{G}_{p+1} \mapsto \mathcal{G}_p$ by $\langle a, T_p b \rangle_{p+1} = \langle T_p^* a, b \rangle_p$. A resolvent $R(S)$ of the functional $S$ (a chain of linear maps) and the cohomology spaces, has the form

$$0 \mapsto \mathcal{G}_N \xrightarrow{T_N} \mathcal{G}_{N-1} \xrightarrow{T_{N-1}} \ldots \xrightarrow{T_1} \ker(T) \mapsto 0,$$  \hspace{1cm} (2.2)

$$H^p(R(S)) = \ker(T_p)[\text{Im}(T_{p+1})]^{-1}. \hspace{1cm} (2.3)$$

A generalisation of the Faddeev-Popov method given in Ref. [3] requires the cohomology of the resolvent to vanish. As a result the volume $V(\ker(T))$ can be evaluated from resolvent (2.2) in terms of the divergent volumes $V(\mathcal{G}_p)$.

If the self-adjoint operators $\triangle_p : \mathcal{G}_p \mapsto \mathcal{G}_p$,

$$\triangle_p = T_p^* T_p + T_{p-1} T_{p-1}^*, \hspace{1cm} (2.4)$$

are elliptic differential operators, then the complex $(\mathcal{G}_p, T_p)$ is an elliptic complex. As anticipated, for the elliptic resolvent the determinants associated with partition function can be regularised by standard zeta-function regularization techniques. Thus a finite expression for the partition function will depend on the choice of resolvent (2.2) and inner products $\langle \cdot, \cdot \rangle_p$ in the $\mathcal{G}_p$. For the elliptic complex the spaces $\mathcal{G}_p$ are the spaces of smooth sections in vector bundles over the manifold $M$, the inner products in the $\mathcal{G}_p$ can be constructed from Hermitian structures in the bundles and a metric on $M$. It has been shown [3] that when the compact closed manifold $M$ has odd dimension the partition function is invariant under variation of the inner products in the $\mathcal{G}_p$ and under a certain variation of the maps $T_p$ in the resolvent. In particular when the definition of the functional $S$ and resolvent $R(S)$ does not require choices of Hermitian structures or metric on $M$ the partition function is a topological invariant. This results was generalised in Ref. [17] to the case where the cohomology of the resolvent is non-vanishing.

We conclude this Sect. introducing the Witten’s invariant, since the method of evaluating the partition function of a quadratic functional can be used in its the semiclassical approximation. The invariant is defined by the partition function associated with a Chern-Simons gauge theory, i.e.

$$Z_W(k) = \int DA e^{ikCS(A)}, \hspace{1cm} k \in \mathbb{Z}, \hspace{1cm} (2.5)$$

and

$$CS(A) = \frac{1}{4\pi} \int_M \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right). \hspace{1cm} (2.6)$$

Since $CS(A)$ does not contain the metric on $M$, the quantity $Z_W(k)$ is expected to be metric independent, namely to be a (well-defined) topological invariant of $M$. Indeed, this fact has been proved in Refs. [6,7]. The formal integration in (2.5) is over the gauge fields $A$ in a trivial bundle, i.e. 1-forms on the 3-dimensional manifold $M$ with values in Lie algebra $\mathfrak{g}$ of the gauge group $G$. In the formula (1.6) the group $G = SU(N)$ identified with its fundamental representation, while the correct expression in the general case can be found in Ref. [21].

In the limit $k \mapsto \infty$, Eq. (2.5) is given by its semiclassical approximation, involving only partition functions of quadratic functionals [4]

$$\sum_{[A_f]} e^{ikCS(A_f)} \int D\omega e^{ik \int_M \text{Tr}(\omega \wedge d_{A_f} \omega)}. \hspace{1cm} (2.7)$$

In above equation the sum is taken over representatives $A_f$ for each point $[A_f]$ in the moduli-space of flat gauge fields on $M$. In addition the $\omega$ are Lie-algebra-valued 1-forms and $d_{A_f}$ is the covariant derivative determined by $A_f$, namely

$$d_{A_f} \omega = d\omega + [A_f, \omega]. \hspace{1cm} (2.8)$$
We shall use the method which enables the partition functions in Eq. (2.7) to be evaluated in complete generality (for more detail, see Ref. [17]), namely the cohomology of \( d_{A_f} \) is not required to vanish. The general form of an action for each partition function of the form (2.7) is
\[
S(\omega) = -\int_M \lambda_g \text{Tr}(\omega \wedge d_{A_f}\omega), \quad \beta = \frac{ik}{4\pi\lambda_g},
\]  
(2.9)
where \( \lambda_g \) is an arbitrary parameter. The inner products in the space \( \Omega^q(M, g) \) of \( g \)-valued \( q \)-forms naturally can be chosen as \( S(\omega) = (\omega, *d_{A_f(1)}\omega)_0 \), \( T \equiv *d_{A_f(1)} \). The canonical elliptic resolvent for quadratic functional (2.9) is given by
\[
0 \longrightarrow \Omega^0(M, g) \xrightarrow{d_{A_f(0)}} \ker(d_{A_f(1)}) \hookrightarrow 0,
\]  
(2.10)
and the resolvent has cohomology spaces \( H^0(R(S)) = H^1(d_{A_f}), \ H^1(R(S)) = H^0(d_{A_f}). \)

3 Zeta and Eta Functions

Let \( \{\mu_j^{(p)}\}_{j=0}^\infty \) denote the non-zero eigenvalues (appearing the same number of times as its multiplicity) of positive, selfadjoint Laplace operators \( \triangle_p \) and \( \mu_j^{(p)} \leq \mu_{j+1}^{(p)} \) for all \( j \). The zeta function associated with operators \( \triangle_p \),
\[
\zeta(s|\triangle_p) = \sum_j (\mu_j^{(p)})^{-s},
\]  
(3.1)
is well-defined analytic function for \( \text{Re}\, s > 0, \ s \in \mathbb{C} \) and it can be analytically continued to a meromorphic function on complex plane \( \mathbb{C} \), regular at \( s = 0 \). The set \( \{\mu_j^{(p)}\}_{j=0}^\infty \) is the union of the non-zero eigenvalues of \( T_p^*T_p \) and \( T_{p-1}^*T_{p-1} \). It can be shown that
\[
\{\mu_j^{(p)}\} = \{\lambda_j^{(p)}\} \cup \{\lambda_m^{(p-1)}\},
\]  
(3.2)
where \( \{\lambda_j^{(p)}\} \) and \( \{\lambda_m^{(p-1)}\} \) are the non-zero eigenvalues of \( T_p^*T_p \) and \( T_{p-1}^*T_{p-1} \) respectively. In accordance with the Ref. [17], the zeta functions \( \zeta(s|T_p^*T_p) \) \( p = 0, 1, ..., N \) are well-defined and analytic for \( \text{Re}(s) > 0 \) and can be analytically continued to meromorphic functions on \( \mathbb{C} \), regular at \( s = 0 \) and satisfy the formula
\[
\zeta(s|\triangle_p) = \zeta(s|T_p^*T_p) + \zeta(s|T_{p-1}^*T_{p-1}).
\]  
(3.3)
One can define the heat kernel of elliptic operator
\[
\text{Tr} \left( e^{-t\triangle_p} \right) = \frac{-1}{2\pi i} \text{Tr} \int_\gamma e^{-z(t - \triangle_p)^{-1}} \, dz,
\]  
(3.4)
where \( \gamma \) is an arc on complex plane \( \mathbb{C} \). By the standard result in operator theory there exists \( \epsilon, \delta > 0 \) such that for \( 0 < t < \delta \) the heat kernel expansion holds
\[
\text{Tr} \left( e^{-t\triangle_p} \right) = \sum_{0 \leq l \leq l_0} a_l(\triangle_p)t^{-l} + O(t^\epsilon).
\]  
(3.5)
Starting with the formula [17]
\[
\zeta(0|\triangle_p) = a_0(\triangle_p) - \dim(\ker(\triangle_p)) = a_0(\triangle_p) - \dim H^p(R(S)),
\]  
(3.6)
one can shown that the zeta function $\zeta(s||T|)$ ($|T| = \sqrt{T^2}$ is defined via spectral theory) is well-defined and analytic for $\text{Re}(s) > 0$ and can be continued to a meromorphic function on $\mathbb{C}$, regular at $s = 0$ and satisfies the formula
\[
\zeta(0||T|) = \sum_{p=0}^{N} (-1)^p(a_0(\triangle_p) - \dim H^p(R(S))). \tag{3.7}
\]

Finally the eta function
\[
\eta(s|\triangle_p) = \sum_j \text{sign}(\mu_j^{(p)})|\mu_j^{(p)}|^{-s}, \tag{3.8}
\]
is well-defined and analytic function for $\text{Re}s > 0$ and it can be analytically continued to a meromorphic function on $\mathbb{C}$, regular at $s = 0$.

## 4 Quadratic Functional with Elliptic Resolvent

Let $M$ be a compact oriented Riemannian manifold without boundary, and $n = 2m + 1 = \dim M$ is the dimension of the manifold. Let the quadratic functionals be defined on the space $G = \mathcal{G}(M,\xi)$ of smooth sections in a real Hermitian vectorbundle $\xi$ over $M$.

Let $\chi : \pi_1(M) \longrightarrow O(V,\langle \cdot,\cdot \rangle_V)$ be a representation of $\pi_1(M)$ on real vectorspace $V$. The mapping $\chi$ determines (on a basis of standard construction in differential geometry) a real flat vectorbundle $\xi$ over $M$ and a flat connection map $D$ on the space $\Omega(M,\xi)$ of differential forms on $M$ with values in $\xi$. One can say that $\chi$ determines the space of smooth sections in the vectorbundle $\Lambda(TM)^* \otimes \xi$.

Let $D_q$ denote the restriction of $D$ to the space $\Omega^q(M,\xi)$ of $q$-forms and
\[
H^q(D) = \ker(D_q)/\text{Im}(D_{q-1}), \tag{4.1}
\]
are the cohomology spaces. A canonical Hermitian structure of the bundle $\chi$ which $D$ is compatible with associated to $\langle \cdot,\cdot \rangle_V$. The above mentioned Hermitian structure determines for each $x \in M$ a linear map $\langle \cdot,\cdot \rangle_x : \Omega_x \otimes \Omega_x \longrightarrow \mathbb{R}$, and the diagram for linear maps hold (see Ref. [17] for details)
\[
(\Lambda^p(T_xM)^* \otimes \Omega_x) \otimes (\Lambda^q(T_xM)^* \otimes \Omega_x) \xrightarrow{\Delta^*} \Lambda^{p+q}(T_xM)^* \otimes (\Omega_x \otimes \Omega_x) \xrightarrow{(\cdot,\cdot)_x} \Lambda^{p+q}(T_xM)^*, \tag{4.2}
\]
where the image of $\omega_x \otimes \tau_x$ under this map has been denote by $\langle \omega_x,\tau_x \rangle_x$.

For odd $m$ we define the quadratic functional $S_D$ on $\Omega^m(M,\xi)$ by
\[
S_D(\omega) = \int_M \langle \omega(x) \wedge (D_m\omega)(x) \rangle_x, \tag{4.3}
\]
where $D_m$ is the restriction of $D$ to $\Omega^m(M,\xi)$. One can construct from the metric on $M$ and Hermitian structure in $\xi$ a Hermitian structure in $\Lambda(T_xM)^* \otimes \xi$ and the inner products $\langle \cdot,\cdot \rangle_q$ in the space $\Omega^q(M,\xi)$. Thus
\[
S_D(\omega) = \langle \omega, T\omega \rangle_m, \quad T = *D_m, \tag{4.4}
\]
where $*$ is the Hodge-star map. Remind that the map $T$ is formally selfadjoint with the property $T^2 = D_m^*D_m$. For the functional (4.3) there is a canonical topological elliptic resolvent $R(S_D)$,
\[
0 \xrightarrow{0} \Omega^0(M,\xi) \xrightarrow{D_0} \ldots \xrightarrow{D_m^{-2}} \Omega^{m-1}(M,\xi) \xrightarrow{D_m^{-1}} \ker(S_D) \xrightarrow{0} 0. \tag{4.5}
\]
From Eqs. (2.2) and (4.5) it follows that for the resolvent \( R(S_D) \) we have \( N = m, \mathcal{G}_p = \Omega^{m-p}(M, \xi), T_p = D_{m-p} \) and \( H^p(R(S_D)) = H^{m-p}(D) \). Note that \( S \geq 0 \) and therefore \( \ker(S_D) \equiv \ker(T) = \ker(D_m) \).

Let us choose an inner product \( \langle \cdot, \cdot \rangle_{H^p} \) in each space \( H^p(R(S_D)) \). The partition function of \( S_D \) with the resolvent (4.5) can be written in the form (see Ref. [17])

\[
Z(\beta) = Z(\beta; R(S_D), \langle \cdot, \cdot \rangle_H, \langle \cdot, \cdot \rangle) = \pi^{\zeta/2} e^{-\frac{\pi i}{4}} (\zeta^{\pm n}) |\beta|^{-\zeta/2} \tau(M, \chi, \langle \cdot, \cdot \rangle_H)^{1/2}, \tag{4.6}
\]

where for \( \beta = i\lambda, \lambda \in \mathbb{R} \), we have \( \theta = \pm \pi/2 \). The function \( \zeta \) appearing in the partition function above can be expressed in terms of the dimensions of the cohomology spaces of \( D \).

First of all, if the dimension of \( M \) is odd then for all \( p = 0, 1, \ldots, N, a_0 = 0 \) in the asymptotic expansion (3.5). Since \( H^p(R(S_D)) = H^{m-p}(D) \) (the Poincaré duality) for the resolvent (4.5) it follows from Eq. (3.7) that for \( N = m, \)

\[
\zeta \equiv \zeta(0|T|) = -\sum_{p=0}^{N} (-1)^p \dim H^p(R(S)) = (-1)^{m+1} \sum_{q=0}^{m} (-1)^q \dim H^q(D). \tag{4.7}
\]

The factor \( \tau(M, \chi, \langle \cdot, \cdot \rangle_H) \) is independent of the choice of metric \( g \) on \( M \) [17]. In fact this quantity is associated with the Ray-Singer torsion [1] of the representation \( \chi \) of \( \pi_1(M) \) constructed using the metric \( g \). Thus \( \tau(M, \chi, \langle \cdot, \cdot \rangle_H) \) is a version of the Ray-Singer torsion as a function of the cohomology defined and shown to be metric-independent in Ref. [18]. If \( H^0(D) \neq 0 \) and \( H^q(D) = 0 \) for \( q = 1, \ldots, m, n = 2m+1 \) is the dimension of \( M \), then the product

\[
\tau(M, \chi, \langle \cdot, \cdot \rangle_H) = \tilde{\tau}(M, \chi, g) \cdot V(M)^{-\dim H^0(D)}, \tag{4.8}
\]

is independent of the choice of metric \( g \), i.e. the metric dependence of the Ray-Singer torsion \( \tilde{\tau}(M, \chi, g) \) factors out as \( V(M)^{-\dim H^0(D)} \).

The dependence of \( \eta = \eta(0|T_D) \) on the connection map \( D \) can be expressed with the help of formulae for the index of the twisted signature operator for a certain vectorbundle over \( M \otimes [0, 1] \) [22]. It can be shown that [17]

\[
\eta(s|B^{(l)}) = 2\eta(s|T_{D^{(l)}}), \tag{4.9}
\]

where the \( B^{(l)} \) are elliptic selfadjoint maps on \( \Omega(M, \xi) \) defined on \( q \)-forms by

\[
B_q^{(l)} = (-i)^{\lambda(q)} (\ast D^{(l)} + (-1)^{q+1} D^{(l)} \ast), \tag{4.10}
\]

In this formula \( \lambda(q) = (q+1)(q+2) + m + 1 \) and for the Hodge star-map we have used \( \ast \alpha \wedge \beta = \langle \alpha, \beta \rangle_{vol} \). From the Hodge theory

\[
\dim \ker B^{(l)} = \sum_{q=0}^{n} \dim H^q(D^{(l)}). \tag{4.11}
\]

The metric-dependence of \( \eta \) enters through \( L^j(TM) \) and \( \eta(0|T_{D^{(l)}}) \), where \( L^j(TM) \) is the \( j \)'th term in Hirzbruch \( L \)-polynomial (see for detail Ref. [22]) and \( D^{(l)} \) is an arbitrary flat connection map on \( \Omega(M, \xi) \). For \( n = 3 \) the only contribution of the \( L \)-polynomial is \( L_0 = 1 \) and the metric-dependence of \( \eta \) is determined alone by \( \eta(0|T_{D^{(l)}}) \).
5 Laplacian on Forms and Trace Formula

In the application we shall consider a smooth compact 3-dimensional hyperbolic manifold. The gauge group to be $G = SU(N)$ (its Lie algebra are identified with their fundamental representations). Let us define an inner product in the Lie algebra $\mathfrak{g}$ by $\langle a, b \rangle_\mathfrak{g} = -\lambda_g \text{Tr}(ab)$ with scaling parameter $\lambda_g > 0$. In the semiclassical approximation, the partition functions are the partition functions of functionals of the form $i\lambda S_{A_i}$, with $S_{A_i} = S_D$ given by Eq. (2.9), $\xi = M \times \mathfrak{g}$ with Hermitian structure determined by $\langle \cdot, \cdot \rangle_\mathfrak{g}$ and $D = dA_f = d + ad(A_f)$ (where $A_f$ is a flat gauge field and $ad: \mathfrak{g} \to \text{End}(\mathfrak{g})$ is the adjoint representation, so the Eq. (2.8) holds).

In the following, we recall some results in the Hodge theory. Let $\delta$ be the Hodge operator acting on $p$-forms defined on a smooth 3-dimensional manifold and $\Delta = \delta \delta + \delta d$ is the Laplace operator. The following facts are well known.

A transverse vector field is a co-closed one-form, i.e. $\delta A = 0$. The Hodge decomposition of such a vector field is $A = \delta \omega + dJ + H$ for some 2-form $\omega$, where $H$ is a harmonic and transverse vector field. Then the condition $\delta A = 0$ would imply $\Delta J = 0$, which in turn is equivalent to $J$ being a constant. Hence, modulo harmonic vector fields, each transverse vector is actually co-exact.

If $\{\lambda^{(p)}_l\}_{l=0}^\infty$ are the eigenvalues of the operator $\delta d$ restricted on $p$-forms and $\{\nu^{(p)}_l\}_{l=0}^\infty$ are the eigenvalues of $d\delta$ restricted on $p$-forms, then $\lambda^{(p)}_l = \nu^{(p+1)}_l$ with equal multiplicity. It follows that if $A$ is transverse and $J$ is a closed two-form such that $J = dA$ locally, then the eigenvalue problem $\Delta A \equiv d\delta A = \lambda A$ gives all the eigenvalues of the problem $\Delta J \equiv d\delta J = \lambda J$.

Let $\chi(\gamma)$ be a character of $\Gamma$, i.e. an homomorphism $\chi(\gamma) : \Gamma \to S^1$. Then a twisted $p$-form, is one such that $\gamma^* A(x) = \chi(\gamma) A(x)$ for any $\gamma \in \Gamma$, and we denote by $b_p$ the number of twisted harmonic $p$-forms (twisted Betti numbers), i.e. the number of twisted zero modes.

In order to obtain a trace formula for the Laplace-type operator acting on transverse vector fields (starting from a trace formula for general vector fields) we need to isolate the contribution of the scalar longitudinal mode. More exactly, let $\Delta^\perp_1$ and $\Delta_0$ be the Laplacians acting on transverse vector and scalar fields respectively. Then for $A = A^\perp + dJ$, $\delta A^\perp = 0$, we have $\Delta A = \Delta^\perp_1 A^\perp + d\Delta_0 J$ and the same decomposition holds for polynomials (or possibly for a class of well behaved functions). One can choose a basis $\{A_l\}$ for the space of 1-forms and a set $A_l = A^\perp_l + dJ_l + H_l$, where $A^\perp_l$ is transverse and $H_l$ is harmonic. This is the orthogonal Hodge decomposition as applied to 1-forms, in view of the fact noted above that transverse vectors are actually co-exacts.

Denoting $\Delta_1$ the operator acting on vector fields (1-form), we have

$$\text{Tr } F(\Delta^\perp_1) = \sum_l (A_l, F(\Delta) A_l) = b_1 F(0) + \sum_l (A^\perp_l, F(\Delta^\perp_1) A^\perp_l) + \sum_l (dJ_l, F(\Delta) dJ_l)$$

$$= \text{Tr } F(\Delta^\perp_1) + \sum_l (dJ_l, F(\Delta) dJ_l), \quad (5.1)$$

where $F$ is a suitable function and the crossed terms were zero by orthogonality. Then the eigenvalues of operator $\Delta$ on exact 1-forms are equal to the eigenvalues of $\Delta_0$ acting on the scalars which are the divergence of a vector fields, i.e the co-exact 0-forms with equal multiplicity. Hence the last term in Eq. (5.1) is the trace of $F(\Delta_0)$ on co-exact scalars. From the Hodge decomposition it follows that each scalar is of the form $\phi = \delta \omega_1 + h$, where $h$ is harmonic. Working as we did in Eq. (5.1) it follows easily that $\text{Tr } F(\Delta_0) = \text{Tr } F(\Delta_0)|_{(\text{co-exact})} + b_0 F(0)$, where the first trace is over all scalars and all non-vanishing eigenvalues. Hence we get

$$\text{Tr } F(\Delta^\perp_1) + (b_1 - b_0) F(0) = \text{Tr } F(\Delta_1) - \text{Tr } F(\Delta_0). \quad (5.2)$$

Thus $\text{Tr } F(\Delta_1)$ can be rewrite as the sum of $\text{Tr } F(\Delta^\perp_1)$ and $\text{Tr } F(\Delta_0)$ plus the "number of zero modes" of $\Delta^\perp_1$, namely the sum associated with the orthogonal decomposition of the vector
representation of $SO(N)$ into irreducible summands [23]. Both traces in the right hand side of
the Eq. (5.2) are known in terms of the geometric data of the manifold.

Let us generalize the above result to a trace formula for transverse p-form defined on a
smooth compact manifold. More generally our goal is to derive the link between the trace
of an arbitrary function $F(\Delta^\bot)$, computed by using constrained eigenfunctions of $\Delta^\bot$, and the
 corresponding unconstrained quantity. We shall consider completely antisymmetric tensors of
order $p$, that is $p$-forms $\omega_p$. From the Hodge theory we have the orthogonal decomposition
\[
\omega_p = \delta\omega_{p+1} + d\omega_{p-1} + h_p ,
\]
where $h_p$ being a harmonic $p$-form, and the two equivalent eigenvalues problems
\[
\Delta_p \omega_p = \lambda \omega_p \iff \begin{cases} 
\Delta_{p+1} d\omega_p = \lambda d\omega_p \\
\Delta_{p-1} \delta\omega_p = \lambda \delta\omega_p .
\end{cases}
\]

This means that the spectra of the Hodge Laplacian acting on exact $p$-forms and on co-exact
$(p - 1)$-forms are the same. The transverse part of the antisymmetric tensor is represented by
the co-exact $p$-form $\omega^\bot_p = \delta\omega_{p+1}$, which trivially satisfies $\delta\omega^\bot_p = 0$, and we denote by $\Delta^\bot_p = \delta d$
the restriction of the Laplacian on the co-exact $p$-form.

Choosing a basis $\{\omega^l_p\}$ of $p$-forms (eigenfunctions of the Laplacian) we get
\[
\sum_l \langle \omega^l_p, F(\Delta_p)\omega^l_p \rangle = \sum_l \langle \omega^\bot_l_p, F(\Delta_p)\omega^\bot_l_p \rangle + \sum_l \langle d\omega^l_{p-1}, F(\Delta_p) d\omega^l_{p-1} \rangle + b_p F(0) .
\]

Using the previous properties of $p$-forms and the Hodge Laplacian one can obtain
\[
\sum_l \langle d\omega^l_{p-1}, F(\Delta_p) d\omega^l_{p-1} \rangle = \sum_l \langle \delta d\omega^l_p, F(\Delta_{p-1}) \delta d\omega^l_p \rangle
= \sum_l \langle \omega^l_{p-1}, F(\Delta_{p-1}) \omega^l_{p-1} \rangle - \sum_l \langle d\omega^l_{p-2}, F(\Delta_{p-1}) d\omega^l_{p-2} \rangle - b_{p-1} F(0) .
\]

In this way we get
\[
\text{Tr } F(\Delta^\bot_p) = \sum_{j=0}^p (-1)^j \left[ \text{Tr } F(\Delta_{p-j}) \right] - \tilde{b}_p F(0) ,
\]
where we have put
\[
\tilde{b}_p = \sum_{j=0}^p (-1)^j b_{p-j} .
\]

Separating the $j = 0$ term and using Eq. (5.7), one has the trace formula
\[
\text{Tr } F(\Delta_p) = \text{Tr } F(\Delta_p^\bot) + \text{Tr } F(\Delta_{p-1}^\bot) + (\tilde{b}_p - \tilde{b}_0) F(0) ,
\]
which is the generalization of Eq. (5.2).

Using the zeta function $\zeta(s|\Delta_p)$ (see the Eqs. (3.1) and (3.3)) associated with operators $\Delta_p$
and making the choice $F(x) = x^{-s}$, with the convention $F(0) = 0$, one can rewrite the Eq. (5.9)
in the form
\[
\zeta(s|\Delta_p) = \zeta(s|\Delta_p^\bot) + \zeta(s|\Delta_{p-1}^\bot) .
\]

Furthermore the definition of zeta-function regularized determinant, namely
\[
\ln \det \Delta_p = -\zeta'(0|\Delta_p) ,
\]
gives
\[ \det \Delta_p = \det \Delta^+_p \det \Delta^+_{p-1}. \quad (5.12) \]
Since \( *\Delta = \Delta * \) the Hodge duality leads also to formula
\[ \det \Delta_p = \det \Delta_{d-p}. \quad (5.13) \]
As a consequence
\[ \det \Delta^+_p = \det \Delta^+_{d-p-1}, \quad (5.14) \]
and one can consider the spectral properties of the transverse Laplace operator only.

### 6 Ray-Singer Torsion and Asymptotics of Witten’s \( H^3 \) Invariant

Recall the classification of all vector bundles over \( S^1 \) for the vector space \( \mathcal{J} \) over fields \( \mathbb{R}, \mathbb{C} \) and a body \( \mathcal{H} \). For \( \mathcal{J} = \mathbb{R} \) there are trivial and 1-dimensional vector bundles. Let \( (\mathcal{E}, p, S^1) \) be a bundle associated with 1-dimensional bundle. Then space \( \mathcal{E} \) is a circle and the map \( p : \mathcal{E} \rightarrow S^1 \) is a double covering. But for \( \mathcal{J} = \mathbb{C}, \mathcal{H} \) any vector bundle over base \( S^1 \) is a trivial bundle.

The partition function (we consider vector line bundles over the manifold \( S^1 \otimes \mathcal{H}^3 \)) is
\[ (2\pi \sqrt{\lambda_g})^{\zeta(4)} e^{-\frac{i\pi}{4} \eta(A_f, g) k^{-\frac{1}{2}} \tau (M, A_f, <.,.>_{H(A_f)})^{1/2}}, \quad (6.1) \]
where
\[ \zeta(A_f) = \dim H^0(A_f) - \dim H^1(A_f), \quad (6.2) \]
\[ \eta(A_f) = \eta(0| T_{A_f}), \quad (6.3) \]
the quantity \( \zeta(A_f) \) are formally given by \( \zeta(0| |T|) \) and \( \lambda_g \equiv k(4\pi \lambda g)^{-1}, i\lambda \equiv \beta \). Let the gauge group be \( G = SU(2) \), and up to gauge equivalence \( A_f = 0 \) is the only flat gauge field on \( \mathcal{H}^3 \). Since \( H^0(A_f) = su(2) \) and \( H^1(A_f) = 0 \), we get
\[ (2\pi)^3 \lambda_g^{3/2} k^{-3/2} \tau (\mathcal{H}^3, A_f = 0, <.,.>_{H(0,A_f=0)})^{1/2} V(SU(2))^{-1}, \quad (6.4) \]
where we take the inner product in \( g = su(2) \) to be \( \langle a, b \rangle = -\lambda_g Tr(ab) \),
\[ \tau (\mathcal{H}^3, A_f = 0, <.,.>_{H^0(A_f=0)})^{1/2} = \tilde{\tau}(\mathcal{H}^3, \chi, g)^{3/2} V(\mathcal{F})^{-3/2}, \quad (6.5) \]
where \( \tilde{\tau}(\mathcal{H}^3, \chi, g) \) is the Ray-Singer torsion of \( \mathcal{H}^3 \), and \( V(\mathcal{F}) \) is its volume.

The Ray-Singer torsion can be expressed in terms of the Selberg zeta functions \( Z_p(s) \) introduced in the Appendix B. To begin with, let us introduce the Ruelle’s zeta function in three dimension
\[ \mathcal{R}(s) = \prod_{p=0}^2 Z_p(p+s)(-1)^p = \frac{Z_0(s) Z_2(2+s)}{Z_1(1+s)}, \quad (6.6) \]
which can be extended to the entire complex plane as a meromorphic function [24]. The importance of Ruelle’s zeta-function lies in the two theorems, first proved in [23] for any \( N \)-dimensional compact hyperbolic manifold. For the sake of completeness we shall briefly discuss these aspects for the 3-dimensional case.

First let us introduce the Ray-Singer torsion of the 3-manifold, associated with a character \( \chi \), by the formula
\[ \tilde{\tau}(\mathcal{H}^3, \chi, g) = \frac{(\det \Delta_1)^{1/2}(\det \Delta_3)^{3/2}}{\det \Delta_2}, \quad (6.7) \]
where $\Delta_p$ is the laplacian restricted on $p$-forms and the determinants are defined by means of zeta-function regularization. If the zero modes exist, the determinants are to be defined by omitting the zero modes from the Dirichelet series which are relevant to zeta functions. By the Hodge duality analysis, presented in Sect. 5, we may also rewrite Eq. (6.7) in the form

$$\tilde{\tau}(H^3, \chi, g) = \frac{(\det \Delta_0)^{3/2}}{(\det \Delta_1)^{1/2}} = \frac{(\det \Delta_0)}{(\det \Delta_1^{1/2})^{1/2}}. \quad (6.8)$$

The Selberg trace formulae of Appendix B allow one to evaluate the analytic continuations of the related zeta functions $\zeta(s|\Delta_p)$. If there are no zero modes then \[19\]

$$\det \Delta_0 = Z_0(2) \exp \left(-\frac{V(F)}{6\pi}\right). \quad (6.9)$$

For the transverse 1-form one may use again the corresponding Selberg trace formula. In addition, an intermediate regularization for the identity element contribution should be performed, since the Plancherel measure in 3-dimension case has no gap term (see for detail the Appendix A). Therefore let us consider the Selberg trace formula (B.1) of the Appendix B and related to the operator $\Delta_1(m^2) \equiv \Delta_1 + m^2$. In the final formulae the limit $m \to 0$ have to be taken. A straightforward computation leads to

$$\zeta(s|\Delta_1(m^2)) = \frac{V(F)}{(4\pi)^3/2\Gamma(s)} \left[(m)^{3-2s}\Gamma(s-3/2) + 2(m)^{1-2s}\Gamma(s-1/2)\right] + \mathcal{I}_1(s, m) \Gamma(s), \quad (6.10)$$

where

$$\mathcal{I}_1(s, m) = \frac{1}{\Gamma(1-s)} \int_0^\infty dt (2tm + t^2)^{-s} \frac{Z'_1}{Z_1}(1 + t + m). \quad (6.11)$$

Zeta-function regularization of the determinant yields

$$\ln \det \Delta_1(m^2) = -\zeta'(0|\Delta_1(m^2)) = \frac{mV(F)}{2\pi} \left[\frac{m^2}{3} - 1\right] + \ln Z_1(1 + m). \quad (6.12)$$

For the vanishing Betti number $\tilde{b}_1$ the limit $m \to 0$ gives

$$\det \Delta_1^{1/2} = Z_1(1). \quad (6.13)$$

As a consequence, we have the Ray-Singer torsion in the form

$$\tilde{\tau}^2(H^3, \chi, g) = \frac{Z_0(2)^2}{Z_1(1)} \exp \left(-\frac{V(F)}{3\pi}\right). \quad (6.14)$$

In this form the dependence on the volume has been extracted. In fact, if one computes the Ray-Singer torsion for $H^3$, one naively obtains $\tilde{\tau}^2(H^3) \simeq \exp \left(-\frac{V_3}{3\pi}\right)$, with $V_3$ very large, and the result is zero in the limit $V_3 \to \infty$.

It should be noted that we can rewrite the torsion as

$$\tilde{\tau}^2(H^3, \chi, g) = \frac{Z_0(0)^2}{Z_1(1)} \exp \left(\frac{V(F)}{3\pi}\right) = \mathcal{R}(0), \quad (6.15)$$

where the use of functional equation (B.7) has been made, namely

$$Z_0(2) = Z_0(0) \exp \left(\frac{V(F)}{3\pi}\right). \quad (6.16)$$
This result is a particular case of Fried’s first theorem [23] in three dimensions: when the twisted Betti’s numbers all vanish, one has \( \tilde{\tau}(\mathcal{H}^3, \chi, g) = |R(0)|^{1/2} \).

The second theorem (in three dimensions) states that in the presence of non-vanishing Betti numbers, the leading term in the Laurent expansion of \( R(s) \) around \( s = 0 \) is

\[
(-4)^{b_0} \tilde{\tau}^2(\mathcal{H}^3, \chi, g) s^{4b_0 - 2b_1}.
\] (6.17)

So the analytic torsion can still be identified as the leading term of the Ruelle zeta-function at \( s = 0 \). Then since

\[
\det \Delta_0 = \frac{1}{b_0!} Z_0^{(b_0)}(2) \exp \left( -\frac{V(F)}{6\pi} \right),
\] (6.17)

and for \( \tilde{b}_1 = b_1 - b_0 > 0 \),

\[
\det \Delta_1^{\frac{1}{b_1}} = \frac{1}{b_1!} Z_1^{(b_1)}(1),
\] (6.18)

one may also write

\[
\tilde{\tau}^2(\mathcal{H}^3, \chi, g) = \frac{\tilde{b}_1! (Z_0^{(b_0)}(2))^2}{(b_0!)^2 Z_1^{(b_1)}(1)} \exp \left( -\frac{V(F)}{3\pi} \right).
\] (6.19)

We conclude this Sect. with some remarks. It is an useful fact that in three dimensions one has \( Z_2(s) = Z_0(s) \), essentially because \( \Lambda^2 \mathbb{C}^2 \) is isomorphic to \( \Lambda^0 \mathbb{C}^2 \). In the second case (non-vanishing Betti numbers) the functional equation seems useless. Besides one may note that for non-trivial characters \( b_0 = 0 \) and possibly also \( b_1 = 0 \), so one falls in previous case. For trivial character, on the other hand, one has \( b_0 = 1 \) (for any closed manifold) and \( b_1 = 0 \) for an infinite number of \( \mathcal{H}^3 \), so that \( R(s) \) has a zero at \( s = 0 \) of order 4. However, it is known that there exist a class of compact hyperbolic manifolds which admits arbitrarily large value of \( b_1 \), the so called Haken class.

As a consequence, the asymptotics of the Witten’s invariant for a smooth compact hyperbolic 3-manifold has been expressed as a function of the Selberg zeta-functions \( Z_p \).

7 Conclusions

In this paper the Ray-Singer torsion for a 3-dimensional compact hyperbolic manifold has been evaluated as a function of the Selberg zeta-functions and the volume of the fundamental domain. As first application, we have derived explicit formulae for the Chern-Simons-Witten invariant related to this manifold for arbitrary values of the level \( k \) making use of quantum field theory methods. Our results have been obtained for the continuous group \( G = SU(2) \), even though they may be extended to more general groups. The final formulae are given in a form where the behaviour as \( k \rightarrow \infty \) is obvious. In this connection we have explicitly exhibited the first term in the level \( k \) asymptotic expansion for compact hyperbolic families of 3-manifolds. This paper has shown the validity of the asymptotic expansion for a wide class of hyperbolic 3-manifolds, and we hope that this analysis may be extended to a larger class of examples.

With regard to possible physical applications, we would like to mention that the evaluation of the Ray-Singer torsion presented in this paper for the compact 3-hyperbolic manifold may be useful within the Euclidean path-integral approach to 3-dimensional quantum gravity, where the partition function is evaluated by summing contributions from all possible topologies [25]. In fact, for negative cosmological constant \( \Lambda \), the classical extrema of the Euclidean action are hyperbolic manifolds. In particular, we may consider a compact hyperbolic manifold. It has been shown that 3-dimensional gravity can be rewritten as a Chern-Simons theory for a suitable gauge group [26]. Therefore in the one-loop partition function the quantum prefactor turns out to be dependent only on the Ray-Singer torsion of a compact hyperbolic manifold. The result
of Sect. 6 leads to the conclusion that the dependence on the volume of the Ray-Singer torsion is exponentially decreasing, making a contribution to the one-loop Euclidean partition function of the same nature of the one corresponding to the classical action. This fact stems from the Eq. (6.14) and the result of ref. [25], namely we have

\[ Z_{H^3} \equiv \tilde{\tau} \frac{1}{2} (H^3)^{1/2} \exp \left( -\frac{V(\mathcal{F})}{4\pi G\sqrt{|\Lambda|}} \right). \]  

(7.1)

As a consequence, the one-loop Euclidean partition function, including only one extremum with \( \Lambda < 0 \) and in absence of zero modes, reads

\[ Z_{H^3} \equiv \left( \frac{Z_{1}(2)}{Z_{1}(1)} \right)^{1/4} \exp \left( -\frac{V(\mathcal{F})}{4\pi G} \left[ \frac{1}{G\sqrt{|\Lambda|}} + \frac{1}{3} \right] \right), \] 

(7.2)

where the second term in the exponential is the first quantum correction. Thus, in this case the volume dependence on the partition function has been completely extracted and the exponential suppression, due to the large volume dependence, confirmed.

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A The Plancherel Measure Associated with Transverse Vector Fields on \( H^3 \)

Let \( A = A_a dx^a \) be a 1-form in \( H^3 \), while the transverse gauge is \( \nabla^a A_a = 0 \). Then it may always be written in the form \( \nabla^a \omega_{ab} \), for some 2-form \( \omega_{ab} \) [27]. The eigenvalue equation for the Laplace operator takes the form

\[ g^{ab} \nabla_a \nabla_c A_c - R^a A_a = -\lambda A_c. \]  

(A.1)

This operator is self-adjoint in the Hodge scalar product

\[ (A, B) = \int_{H^3} A \wedge * B = \int_{H^3} g^{ab} A_a B_b \sqrt{g} d^3 x. \] 

(A.2)

The spectrum of the operator presented in Eq. (A.1) is continuous and it contains a gap, since \( \lambda \geq (\rho_N - 1)^2 \), where \( \rho_N = (N - 1)/2 \). The exception is the case \( N = 3 \), where there are harmonic and square integrable one-forms [28]. Because of the gap, we use the parameter \( r^2 = \lambda - (\rho_N - 1)^2 \geq 0 \) to label the spectrum. We define the density of states (the Plancherel measure) \( \mu_N(r) \), so that the zeta function per unit volume [29, 19]

\[ \zeta_N(s|\Delta_1) = \text{Tr}[g_{ab} \Delta - R_{ab}]^{-s} = \int_0^{\infty} [r^2 + (\rho_N - 1)^2]^{-s} \mu_N(r) dr, \] 

(A.3)

has a residue at its most right pole as demanded by the general theory. Using the asymptotic behaviour of the solutions, one finds the formula

\[ \mu_N(r) = \frac{2(N - 1)}{2^N \pi^{N/2} \Gamma(N/2)} \left| \frac{\Gamma(ir + \rho_N + 1)}{\Gamma(ir)} \right|^2. \]  

(A.4)
For example, in the case of $N = 2$ the measure contains a contribution of a discrete spectrum and it can be written (perhaps improperly) as

$$
\mu_2(r) = \frac{r}{2\pi} \tanh \pi r + \frac{1}{2\pi} \delta(r - i/2).
$$

(A.5)

Therefore the trace of the vector heat kernel in two dimensions is

$$
K(t) \equiv \text{Tr} \left( e^{-t\Delta_1} \right) = \frac{1}{2\pi} \int_0^\infty e^{-t(r^2+1/4)}r \tanh(\pi r)dr + \frac{1}{2\pi}.
$$

(A.6)

For $N = 3$ the zeta function in Eq. (A.3) generally speaking do not exist and a mass term is needed to achieve convergence. Doing this we obtain

$$
\zeta_3(s|\Delta_1) = \frac{2\varrho^{2s-3}}{(4\pi)^{3/2}\Gamma(s)} \left[ (m\varrho)^{3-2s}\Gamma(s - 3/2) + 2(m\varrho)^{1-2s}\Gamma(s - 1/2) \right],
$$

(A.7)

where the curvature radius $\varrho$ has been reinserted. Note that the zero mass limit is not uniform in $s$, i.e. it depends on $s$.

For $N = 4$ no mass term is needed and we get

$$
\zeta_4(s|\Delta_1) = \frac{3\varrho^{2s-4}}{16\pi^2} \left[ \frac{4^{s-2}}{(s-2)(s-1)} + \frac{4^{s-1}}{s-1} \right] - \frac{3\varrho^{2s-4}}{4\pi^2} \int_0^\infty \left( r^2 + \frac{9}{4} \right) \left( r^2 + \frac{1}{4} \right)^{-s} \frac{1}{1 + e^{2\pi r}} dr.
$$

(A.8)

It can be shown explicity that the integral in the last equation is an analytic function of $s$.

**B  The Selberg Trace Formula and Zeta Functions**

We now specialize the results of Sect. 3 to 3-manifold with topology $\mathcal{H}^3$. First we remind shortly some geometric information necessary for the trace formula.

Let $\gamma$ be a closed geodesic in $\mathcal{H}^3$ and $l_\gamma$ its length. Let us consider a parallel translation of any vector along $\gamma$. After the journey, the vector will be rotated in the two-space orthogonal to the tangent vector at $\gamma$. Let $R_\gamma$ be a corresponding $SO(2)$ matrix of rotation. Thus every closed geodesic is associated with a certain element of $SO(2)$ and a certain number $l_\gamma$. If $n$ is the winding number of $\gamma$ then $R_\gamma = R_\delta^n$ for some rotation $R_\delta$, and $R_\delta$ being not a power of any other rotation. In this case $\gamma = \delta^n$ and we call $\delta$ a primitive geodesic. The set of all primitive geodesics will be denoted by $\mathcal{P}$. For any $\gamma$ we also define the factor $S(n, l_\gamma) = \det |1 - N_\gamma^n R_\delta^n|$, where $N_\gamma = \exp l_\gamma$. Finally, let $\mathcal{F}$ being the fundamental domain for the group $\Gamma$ (relative to the invariant Riemannian measure) and $V(\mathcal{F})$ is its volume. Thus one can now state the trace formula [23, 24].

**Proposition 1** Let $h(r)$ be a function even and holomorphic in a strip larger than 1 about the real axis and such that $h(r) = O(r^{-3-\varepsilon})$ uniformly in the strip as $r \to \infty$. Then

$$
\sum_j h(r_j) + (b_1 - b_0)h(0) = V(\mathcal{F}) \int_0^\infty h(r)\mu_3(r)dr
$$

$$
+ \sum_{\langle \gamma \rangle \in \mathcal{P}} \sum_{n=1}^{\infty} \text{Tr}[R_\gamma^n] \frac{\lambda^n(\gamma)l_\gamma N_\gamma^n}{S(n; l_\gamma)} \hat{h}(nl_\gamma),
$$

(B.1)

where each number $r_j$ is the positive root of $r_j^2 = \lambda_j$, $\lambda_j$ are the eigenvalues of the Laplace operator acting on transverse vector fields and the summation over $j$ includes all the non-vanishing eigenvalues counted along with their degeneracy.
In addition $\hat{h}(p) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ipr} h(r) dr$ and the Plancherel measure has the form $\mu_3(r) = (r^2 + 1)\pi^{-2}$.

**Proposition 2** The Selberg trace formula for twisted scalar fields is similar and reads [30, 19]

\[
\sum_j h(r_j) + b_0 h(i) = V(\mathcal{F}) \int_0^\infty h(r) \Phi_3(r) dr \\
+ \sum_{l} \sum_{\gamma} \frac{\chi^2(\gamma)}{S(n; l)} \hat{h}(nl_\gamma),
\]

where now each number $r_j$ is the root of $r^2 = \lambda_j - 1$ in the upper half complex plane, $\lambda_j$ are the eigenvalues of the Laplace operator acting on twisted scalar fields and the sum is over all the non vanishing eigenvalues, including degeneracy.

Furthermore, the measure is $\Phi_3(r) = r^2(2\pi^2)^{-1}$. The Selberg zeta functions we are going to introduce are important in the following.

The $\Xi$ function for 1-forms (for 0-forms the definition is the same, but the factor $\text{Tr}[R^n_\gamma]$ in the equations below is put equal to one) can be written as

\[
\Xi_1(s) = \sum_{\gamma} \sum_{n=1}^\infty \text{Tr}[R^n_\gamma] \frac{\chi^n(\gamma)}{S(n; l_\gamma)} \exp[-(s-2)nl_\gamma],
\]

while the Selberg zeta function is given by

\[
\mathcal{Z}_1(s) = \exp \left\{ - \sum_{\gamma} \sum_{n=1}^\infty \frac{\chi^n(\gamma)}{S(n; l_\gamma)} \exp[-(s-2)nl_\gamma] \right\},
\]

and therefore $\Xi_1(s) = \mathcal{Z}_1^*(s)/\mathcal{Z}_1(s)$. The presence of the exponential makes it easy to expect analyticity for $\Xi_1(s)$ in the strip $\text{Re } s > 2$. Indeed this the case, although it is far from trivial, due to exponential proliferation of closed geodesics with increasing lengths [31]. It can also be shown that the analytically continued function $\Xi_1(s)$ has simple poles at $s_j = \pm ir_j$. If there are zero modes, then a pole also exists at $s = 1$, of residue $b_1 - b_0$.

**Remark 1** The $\Xi_1(s)$ function satisfies the functional equation

\[
\Xi_1(s+1) + \Xi_1(-s+1) = \frac{2V(\mathcal{F})}{\pi} (s^2 - 1).
\]

The poles of $\Xi_1(s)$ become poles or zeroes of $\mathcal{Z}_1(s)$, depending on the sign of the residues and the functional equation for $\mathcal{Z}_1(s)$ has the form

\[
\mathcal{Z}_1(-s+1) = \mathcal{Z}_1(s+1) \exp \left[ \frac{2V(\mathcal{F})}{\pi} s(1 - \frac{s^2}{3}) \right].
\]

In particular, $\mathcal{Z}_1(s)$ has a pole or a zero at $s = 1$ of order $b_1 - b_0$ depending on whether this number is negative or positive, namely $\mathcal{Z}_1(s) = s^{(b_1 - b_0)} G_1(s)$ with finite number $G_1(0)$.

Generally speaking zeta functions $\mathcal{Z}_p(s)$ can actually be defined not only for 1- or 0-forms, but also for $p$-forms. In the case of 0-forms, the defining equations are those for $\mathcal{Z}_1(s)$ in which the factor $\text{Tr}[R^n_\gamma]$ is replaced with one.

**Remark 2** The functional equation for the $\mathcal{Z}_0(s)$ is

\[
\mathcal{Z}_0(-s+1) = \mathcal{Z}_0(s+1) \exp \left[ - \frac{V(\mathcal{F})}{3\pi} s^3 \right].
\]

The $\mathcal{Z}_0(s)$ is the entire function of order three, and all poles of $\Xi(s)$ are zeroes of the $\mathcal{Z}_0(s)$, a fact which holds true in any odd dimension. In particular, it has a zero of order $b_0$ at $s = 0$, namely $\mathcal{Z}_0(s) = s^{b_0} G_0(s)$ with finite number $G_0(0)$. Finally for $p$-forms one replaces $R_\gamma \in SO(2)$, acting on $\mathbb{C}^2$, with its representation acting on the exterior algebra $\Lambda^p \mathbb{C}^2$. 

\[14\]
References


