Cylindrical versus Spherical Resonant Antennas for Gravitational Wave Detection

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Abstract

The principles and detection of gravitational waves by resonant antennas are briefly discussed. But the main purpose of this short note is to compare the two geometries of resonant antennas, the well-known cylindrical to the spherical type. Some features of a two sphere observatory are also discussed.

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I. INTRODUCTION

The demonstration by Hertz of the existence of electromagnetic [EM] waves predicted by the Maxwell’s equations was indeed a remarkable discovery. Einstein’s equations of general relativity written some eighty years ago also lead us to a wave equation for gravitational waves [GW]. Despite bold pioneering efforts of eminent scientist Weber [1] and several ongoing international collaborations of excellent scientists [2] we still await the direct detection of GW waves. The reasons are well-known

1. The gravitational coupling constant is approximately $10^{-38}$ times weaker than the electromagnetic coupling.

2. The elementary mode of vibration induced by gravitational wave is quadrupole whereas an elementary mode excited by EM wave is dipole. A simple way to see why dipole radiation does not exist for gravity is because there is no negative mass and hence no gravitational dipole.

In the weak field approximation to Einstein’s equations we can treat GW waves as gravitons [spin two massless particle] propagating on almost flat space [i.e. very close to Minkowski space]. However the existence of gravitons should not be confused with the weak field limit. Their existence is implied by the radiative solutions of the Einstein’s equations. Like EM waves GW are transverse and have two states of polarization. GW waves are more complicated than EM waves because they unlike the latter self interact [or in other words contribute to their own source].

The main goal is to first detect the GW waves directly. Once this is done then one can use the gravitational radiation as a powerful new probe and address questions, such as,

1. Examining interesting astrophysical systems like coalescing neutron-star binaries, black holes, supernovae, pulsars all of which are sources of gravitational radiation.
2. GW from the early universe is expected to hold clues about inflation. In particular GW excited during inflation as quantum mechanical fluctuations is a key test of inflation and also allows one to learn about specifics of the inflationary model [3].

3. Experimental information connected with string theory [which has the promise of providing a unified theory of particles and their interactions] may be obtained. GW detection would help test general relativity against other competing theories of gravitation such as scalar-tensor theories [4].

Detectors for GW waves fall into two broad categories, i: Resonant Detectors and ii: laser interferometric devices. We are concerned here with Resonant Detectors in particular comparing the two geometries cylindrical [bar] with spherical. The practical cylindrical bar antenna has been with for about 30 years whereas the suggestion for the practical implementation of spherical geometry is recent [5,6]. Making a spherical antenna requires more sophisticated technology in several areas such as casting, cooling, suspension, and transducer attachment. However it has some added advantages over its cylindrical counterparts for it provides:

1. Enhanced sensitivity, an order of magnitude better than the corresponding cylindrical geometry.

2. Omni directionality.

3. Multi mode measurement capabilities.

The suggestion for using a spherical detector for gravitational wave was made as early as 1971 by Forward [7] where it was indicated that by a suitable positioning of a set of transducers on the sphere one could determine the direction, the amplitude, and the polarization of the gravitational wave. A free sphere has five degenerate quadrupole modes of vibration
that will interact strongly with a gravitational wave. Each free mode can act as a separate antenna, oriented towards a separate polarization or direction. Wagoner and Paik [8] showed that the angle-averaged energy absorption cross section of a spherical antenna is much larger [by a factor of 60] than a cylindrical bar [length to diameter ratio of 4.2] with the same quadrupole mode frequency. So the question arises why these results were ignored? One [main] reason is that a simple spherical detector is not a practical detector. For one requirement of practicality is a set of secondary mechanical resonators, which act as mechanical impedance transformers between the primary vibrational modes of the antenna and the actual motion sensors, producing the essential increase in the electromechanical coupling. All successful cryogenic bar-type detectors have such resonators. This practical consideration led Johnson and Merkowitz [9] to propose a method of positioning six radial transducers on a truncated icosahedron to construct a nearly spherical detector. They showed that a spherical detector cooled to ultralow temperature can have sensitivity comparable or even better than the first generation Laser Interferometric Gravitational Wave Observatory [LIGO] detectors in the frequency range around 1 kHz. A network of six cylindrical detectors with appropriate orientation can cover the whole sky isotropically [like a spherical detector] and have source direction resolution [10,11]. Such a network of six colocated cylindrical detectors have has a sensitivity $\frac{1}{7}$ that of a single spherical detector made of the same material and with the same resonant frequency [11]

**II. RESONANT ANTENNAS-SPHERICAL AND CYLINDRICAL**

In the weak field approximation we can write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$  \hspace{1cm} (1)
where \( h_{\mu\nu} \) \( \ll 1 \) is the GW perturbation to the flat space time metric \( \eta_{\mu\nu} \). As is known and discussed in previous section GW waves have two states of polarizations \([h_+ \text{ and } h_\times]\) with quadrupole patterns. The perturbation \( h \) will cause a change \( \Delta L \) of a length \( L \) between two free masses, such that

\[
\Delta L = \frac{hL}{2}
\]  

(2)

assuming the wavelength of GW waves is much larger than the mass separation \( L \).

It is well-known that the metric perturbation \( h \) caused at the detector location by a GW burst generated due to conversion of mass \( M_{gw} \) into gravitational radiation at a distance \( R \) from the detector position is,

\[
h = \frac{1}{R\omega_0} \sqrt{\frac{8G_N M_{gw}}{c\tau_{gw}}},
\]  

(3)

where \( \tau_{gw} \) is the duration of the GW burst, \( G_N \) is Newton’s constant and \( \omega_0 \) is the angular resonance frequency of the antenna. It is instructive to write this in the form

\[
h \approx 1.76 \times 10^{-17} \times \frac{1}{\omega_0} \times \sqrt{\frac{1}{10\tau_{gw}}} \times \frac{10 \text{Mpc}}{R} \times \sqrt{\frac{M_{gw}}{M_s}},
\]  

(4)

Assuming that \( M_{gw} \sim 10^{-2} M_s, \tau_{gw} \sim 10^{-3} \) and \( \omega_0 = 2\pi \times 1000 \text{ rad/s} \) in the above we find \( h \approx 3 \times 10^{-18} \) [for an event at the center of our galaxy, \( R = 8.5 \text{ kpc} \)] and \( h \approx 2.6 \times 10^{-21} \) [for a source of GW waves located in the virgo cluster].

The energy absorbed [captured] by the bar from GW burst is related to the absorption cross section and the incident flux [by the general definition of cross section]

\[
\sigma_{abs} = \frac{\Delta E_{abs}(\omega)}{\Phi(\omega)}
\]  

(5)

here \( \Delta E_{abs}(\omega) \) is the energy absorbed by the detector at frequency \( \omega \) and \( \Phi(\omega) \) is the incident flux measured in \( \text{W m}^{-2}\text{Hz} \). For a bar antenna [at the first longitudinal resonance mode]
\[ \sigma_{\text{abs}} = \frac{8}{\pi} \frac{G_N M v_s^2}{c^3} \sin^4 \delta \cos^2 2\Psi \]  

(6)

where \( v_s \) is the sound velocity, \( M \) is mass of the antenna, \( \delta \) is the angle between the bar axis and direction of source and \( \Psi \) is the angle between the plane [formed by the bar axis and direction of source] and the polarization plane. The cross section increase with both \( M \) and the square of the sound velocity in medium \( v_s \). In the case of a spherical detector one finds for the absorption cross section is

\[ \sigma_{\text{abs}} = F_{\text{ln}} \frac{G_N M v_s^2}{c^3} \frac{\Gamma_{\text{ln}}}{(\omega - \omega_{\text{ln}})^2 + \Gamma_{\text{ln}}^2/4} \]  

(7)

\( F_{\text{ln}} \) is a dimensionless coefficient which is characteristic of each mode. Assuming general relativity \( F_{\text{ln}} \) is zero unless \( l = 2 \). \( \omega_{\text{ln}} \) and \( \Gamma_{\text{ln}} \) are respectively, the mode resonance frequency and linewidth.

To facilitate comparison between cylindrical and spherical geometries we note \( F_{21} \approx 3 \) which is about 15% better than the cylinder’s \( 8/\pi \) [Eq. 6]. If we average over polarizations and directions the sphere’s cross section is a factor 4.4 better than the cylinder [assuming same masses for both] this result is in agreement with [8,11,5] An aluminum [Al5056] cylinder [optimal orientation] of length=3 m, and diameter = 0.6 m has mass of 2.3 tons, the first longitudinal mode has angular frequency of \( 2\pi \times 910 \) rad/s and the corresponding absorption cross section is \( \sigma_{\text{abs}} = 4.3 \times 10^{-21} \) cm\(^2\) Hz. A sphere [omnidirectional] of same material having a diameter 3.1 m will have a mass of 42 tons. The sphere has the same fundamental frequency. The absorption cross section is \( \sigma_{\text{abs}} = 9.2 \times 10^{-20} \) cm\(^2\) Hz. Similar numbers are noted in [11,5]. We note that although the length of the cylinder and the diameter of the sphere are comparable the latter is much more heavier, representing an advantage. A conservative estimate of the sensitivity of this sphere is \( h \approx 7 \times 10^{-22} \). This could be compared to a sensitivity of \( h \approx 4 \times 10^{-22} \) [in the same frequency range i.e. \( \sim 2\pi 1000 \) Hz] for the best interferometers and that is if GW arrives at optimum direction and polarization.
Higher mode cross section values can be found by computing $F_{22}, F_{23}, F_{24}, F_{25},...$ [5]. One finds that we obtain larger cross sections at higher harmonics for a spherical detector with respect to its cylindrical counterpart. This suggests that by using spherical detector as a xylophone we can scan the frequency range 1 – 5 kHz thus allowing us to study the stochastic background of gravitational waves. We note that a single sphere is not sufficient to identify a GW event, for at least two antennas are necessary for minimum coincidence analysis. A two sphere GW observatory has one important advantage over a network of several directional antennas with different orientations. The coincidence analysis is much simpler since the same amount of energy is absorbed by each detector while for an array each member will absorb according to its orientation. A coincidence analysis between spherical detectors is given in [11].

**Present Detectors-Experimental Considerations**

The vibrations in the bar are converted to electrical signals [by electromechanical transducers], the electrical signals are amplified and pass through filters to optimize the signal to noise [SNR] ratio. The minimum vibration energy which can be detected [SNR=1] is [from simple analysis]

$$E_{\text{min}} = k_B T_{\text{eff}} = \frac{k_B T}{\xi Q} + 2k_B T_n.$$  \tag{8}$$

$T$ is the bar’s temperature, $T_n$ is electronic noise temperature, $\xi$ is the fractional part of energy available to the amplifier and $Q$ is the quality factor of all the apparatus. $T_{\text{eff}}$ is referred to as the effective noise temperature. It is clear that to compete against thermal noise $[k_B T]$ in the bar $\xi Q$ must be as large as possible. Clearly $T$ and $T_n$ must be small. SQUID amplifiers can go down to the quantum limits $T_n \sim 6 \times 10^{-8}$ K but the difficulty lies in matching them to the transducers. For a FET amplifier the best value of $T_n$ is more
like $T_n \approx 0.1$ K. Let us see what kind of numbers are involved if we assume a value of $h \sim 3 \times 10^{-21}$ which we obtained for the Virgo Cluster, Eq. 4. Now $h$ is related to the mass of detector $M$, its length $L$, $\tau_{gw}$ and speed of sound in bar material by [6]

$$h = \frac{1}{\tau_{gw}} \times \sqrt{\frac{L}{v_s^2}} \times \sqrt{\frac{E}{M}}. \quad (9)$$

Using $h \sim 2.6 \times 10^{-21}$ in the above equation we find $E \sim 1.5 \times 10^{-30}$ joule. Choosing this value of $E$ as our $E_{\text{min}}$ in 8 we find that the set $T \sim 40$ mK, $\xi Q \sim 10^6$ and $T_n \sim .75 \times 10^{-7}$ satisfies equation 8. Meeting these requirements in an experiment [which is conducted over long time] is quite demanding. From these numbers we can easily appreciate the challenging task of the experimentalists trying a direct detection of gravitational waves.

Let us now look at the characteristics of some the resonant antennas [bar type] in world. We use the following notation: Name of Experiment(Location, Detector mass in kg, Temperature in Kelvin, Amplifier used, data taking, sensitivity $h$). Following this notation we have

- **CRAB** (Tokyo, 1200, 4.2, parametric, 1991, $2 \times 10^{-22} \text{[monochromatic]}$)
- **EXPLORER** (Rome, 2300, 2.5, SQUID, July 1990(1986), $7 \times 10^{-19}$)
- **NAUTILUS** (Rome, 2300, 0.1, SQUID, 1994-95, $3 \times 10^{-18}$)
- **ALTAIR** (Rome, 390, 4.2, SQUID, ——, ——)
- **AURIGA** (Legnaro, 2300, 0.1, SQUID, 1995, ——)
- **ALLEGRO** (Louisiana, 2300, 4.2, SQUID, June 1991, $7 \times 10^{-19}$)
- **NIOBE** (Perth, 1500, 4.2, parametric, June 1993, $7 \times 10^{-19}$)
- Moscow University (Moscow, 1500, 290, tunnel, 1993, $7 \times 10^{-17}$).

For ALTAIR and AURIGA the sensitivity has not been listed for they are yet to be fully tested. Sensitivity refers to a GW burst lasting 1 ms, with the exception of the Tokyo group which is searching for the continuous GW from the CRAB pulsar. The Rome antenna EXPLORER has reached sustained strain sensitivity $h = 6 \times 10^{-19}$ for millisecond bursts
over several consecutive month period [12]. ALLEGRO and the EXPLORER were also operated in coincidence. They were aligned parallel to each other and operated for 180 days [June 24th 1991- December 16th 1991] so that the same gravitational wave burst would have produced in the antennas signals with the same amplitude. Preliminary analysis of the data gives a negative result [6,2]. In particular no coincident events were found above 200 mK [2].

III. CONCLUSIONS

In conclusion we have examined the advantages of the spherical detector over its cylindrical counterpart and find that our preliminary study confirms the optimistic results reported in [9,11,5].
REFERENCES


