Vacuum Structure of Two-Dimensional Gauge Theories for Arbitrary Lie Groups

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Abstract

Using the well established machinery of Wilson loop calculations we investigate the multiple vacua of two dimensional Yang-Mills theories with infinitely massive adjoint matter. In particular, via group theoretical techniques we calculate string tensions between charges and find the number of vacua for each compact Lie symmetry group. The counting of vacua is in agreement with the standard classification based on the topology of the effective gauge group $\pi_1(G/Z)$ when one considers arbitrary numbers of adjoint charges in the system. For systems with limited numbers of charges we find additional ”meta-stable” vacuum states. Finally we discuss t’Hooft’s disorder operators in this setting as number operators for the multiple vacua.

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1 Introduction and Motivation

After the invention of instantons [1] it became clear that the vacuum structure of gauge theories is non-trivial and has definite effects on the physics of the theory. The existence of large gauge transformations and consequent non-contractable paths in the space of fields leads to a parameterization of distinct vacuum states by the periodic vacuum angle $\theta$ ([2]-[4]). Unfortunately, this simple and apparently universal picture is incomplete as shown by the well known four-dimensional examples of gluodynamics with large ($N_c$) numbers of colours, supersymmetric QCD with arbitrary numbers of ($N_f$) flavours and colours and QCD with massless quarks, in addition to others. In all these cases we find that the parameter $\theta$ does not appear simply as the argument of a periodic function as a standard classification based on integer topological numbers would predict. Rather, $\theta$ dependence in the theory comes about through the ratio $\theta/N$ which seems to suggest a new classification principle. Indeed, in the chiral ($m = 0$) limit of four-dimensional QCD, Ward identities [5] imply $\theta$ dependence appears only through the ratio $\theta/N_f$ and, for a fixed value of $\theta$, there are still $N_f$ different vacuum states. Similarly in gluodynamics [6], $\theta$ dependence appears as $\theta/N_c^4$. Consequently we can retain $2\pi$ periodicity in the variable $\theta$ only if the vacuum is $N_f$ or $N_c$-fold degenerate, respectively for each value of $\theta$. Labeling these degenerate vacua implies an additional superselection rule over and above the original $\theta$ parameterization. In an important test of this picture of vacuum structure, we will investigate the nature of this superselection rule in a solvable model.

In addition to gaining direct results about the details of vacuum structure, we will be considering a solvable model in part due to inconsistencies amongst different techniques for investigating the vacuum structure of gauge groups. For example, consideration of the Witten index and other aspects of supersymmetric theories in four dimensions gives contradictory results in these cases. More precisely, in four dimensional supersymmetric field theories with arbitrary gauge group, the exactly calculable gluino condensate $\langle \lambda \lambda \rangle \sim e^{i\theta/N}$ is not vanishing [7]. This result is unexpected since the number of fermionic zero modes in the tunneling transition associated with the standard instanton is larger than two for any gauge group. Consequently, this analysis predicts that the number of vacuum states is equal to the quadratic Casimir of adjoint representation $C_2(Ad)$. In general this is in conflict with an analysis of the Witten index [8] for supersymmetric QCD. For a gauge group of rank $r$ the Witten index predicts the number of vacua to be $r + 1$ (independent of space-time dimension) which differs from the quadratic Casimir except in the cases of $SU(N)$ and $Sp(N)$.

Additionally, contradictory results about vacuum structure are not limited to four dimensions as shown in the analysis of two dimensional QCD with dynamical fermions in adjoint representation, ([9]-[13]). In particular, there is the following discrepancy: Standard bosonization techniques (and large $N$ arguments [13]) predict the existence of a fermion condensate

\[4\text{We recall that the } \theta \text{ dependence of physics is linked to the } U(1) \text{ problem. Indeed, if we believe that the resolution of } U(1) \text{ problem appears within the framework of the papers [6], we must assume that the topological susceptibility}

\[K = i \int d^4 x \langle 0 | T \{ Q(x) , Q(0) \} | 0 \rangle \sim \frac{d \langle 0 | Q | 0 \rangle}{d\theta} \sim \frac{1}{N_c}
\]

in pure gluodynamics is not zero and $K \sim \frac{1}{N_c}$. Here $Q = \frac{1}{32\pi^2} G_{\mu\nu} G_{\mu\nu}$ is topological density. It demonstrates one more time that the dependence on $\theta$ comes through $\theta/N_c$. In particular, it is expected that $\langle Q \rangle \sim \sin(\frac{\theta}{N})$.
for an arbitrary gauge group. At the same time, the standard topological classification of vacuum states with the corresponding counting of zero modes, does not seem to support such a condensation. Such a discrepancy in the two dimensional theory is very similar to what we discussed above in four dimensional supersymmetric models where there existence of a gluino condensate does not appear to be supported by vacuum transitions with the standard properties.

In short, it is clear that current understanding of the classification of vacuum states is not complete. More sophisticated methods of classification may be necessary in order to match well known (but indirect) results with the analysis of admissible large gauge transformation in gauge theories. Before increasing the complexity of the analysis though, it would be very useful to understand the problems formulated above on a more detailed level in toy models. Thus our main goal is an analysis of the multiplicity of vacuum states in two dimensional gluodynamics with arbitrary (compact Lie) gauge group. Recently the problem of classification of vacuum states for the model based on an $SU(N)$ gauge group has been discussed [14] (see also [15] for a classification in the finite temperature case). There it was explicitly shown that the model exhibits exactly $N$ different vacuum states. In addition, the string tension and vacuum energy in each given vacuum state were found using two independent approaches- the standard Hamiltonian approach and Wilson loop calculations. Using the same machinery here we will see that all alternative vacua corresponding to the extra superselection rule follow from a topological classification of the large gauge transformations. We emphasize that our results are strictly from group theoretic calculations to determine the number of stable ground states and do not involve direct input from topology.

The paper is organized as follows. In the next section we will outline the topological classification of vacua in gauge theories and the connection to boundary conditions. Once one has particular boundary conditions then the classification problem can be recast in terms of Wilson loops and, for two-dimensional gauge theories, group theory. In discussing some of the details of the group theoretical calculations, it will become clear that counting true vacua with non-unitary gauge groups is somewhat more complicated than the previously dealt with unitary case. In particular we will demonstrate the existence of ‘meta-stable’ vacua in the (unphysical) limit when the charges in the system are infinitely massive. Finally, we will consider the problem of enumerating vacua in two-dimensional gauge theories from a dual perspective in terms of disorder operators.

2 Topology and Boundary Conditions

The standard method of classifying the multiplicity of vacua in a particular gauge theory with adjoint matter hinges on identifying the effective gauge group. Here, since gauge transformations operate by adjoint action on all fields, the true gauge group is the quotient of the gauge group and its center. This quotient is multiply connected. For simply connected semi-simple gauge group $G$ with center $Z$,

$$\pi_1(G/Z) = \pi_0(Z) = Z$$

(1)
This gives a classification of gauge fields which are constrained to be flat connections at infinity. In that case
\[
\lim_{|x| \to \infty} A_\mu(x) = ig^\dagger(x)\nabla g(x)
\]  
(2)

Where \( g(x) \) is a mapping of the circle at infinity to the gauge group \( G/Z \). Since \( G/Z \) is a symmetry of the Hamiltonian, we expect that all physical states carry a representation of \( \pi_1(G/Z) \). In the case where the center of the group is Abelian all of its irreducible representations are one dimensional and further, when \( Z \sim \mathbb{Z}_{n_1} \times \cdots \times \mathbb{Z}_{n_j} \), we are lead to a classification of all physical states in terms of \( j \) generators of \( Z \), \( \{ z_1, \cdots, z_j \} \). If \( Z \) is a unitary realization of \( Z \) and \( |\psi> \) is a physical state we have
\[
Z|\psi> = e^{i(z_1 + \cdots + z_j)}|\psi>
\]  
(3)

A more direct way to classify vacua in the case of SU(N) gauge group, without resorting to topological arguments, was developed in [14] following the example of multiple vacua in the massive Schwinger model ([16], [17]). These methods were previously introduced by Witten [9] to identify the existence of \( \theta \)-vacua in two dimensional non-Abelian field theories. To review, in the Schwinger model multiple vacua are considered as generalized boundary conditions on the model where static charges reside at either end of the spatial dimension of the world. In the non-Abelian case we have the same situation except that the c-number charges are replaced by static Lie algebra valued colour charges \( T_R \) and \( T_{\bar{R}} \) at either end of the world. Here the \( T \)’s are the generators of the colour group in the representation \( R \) and its conjugate, respectively and form a discrete choice of boundary conditions as opposed to the continuum in the Schwinger model.

The key to utilizing such a picture of multiple vacua as generalized boundary conditions hinges on the fact that static charges in a gauge theory have a natural interpretation in terms of Wilson loops as can be seen directly in the Hamiltonian formulation of the problem [14]. If \( \text{tr}_R \) is the trace in the representation \( R \) of the gauge group then we can impose boundary conditions directly in the action by including a Wilson loop
\[
Z \to \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \exp (-\int d^2x \mathcal{L}) \text{tr}_R P \exp (i \oint_{C \to \infty} dx^\mu A_\mu(x))
\]  
(4)

The number of distinct choices of representation in which to take the trace of the boundary loop is directly related to the transformation properties of the loops under \( Z \), the center of the gauge group due to the presence of only adjoint charges in the system. Since different boundary charges may be equivalent up to decays via adjoint charges, there are only a small number of stable vacua in any theory with dynamical charges since all decays are possible in this case. What is interesting though is that the number of stable vacuum states for a theory with static charges depends on the number of charges in the system and is in general greater or equal to the number of stable vacuum states in the dynamical theory. This contrasts the case of SU(N) which was considered previously where there are \( N \) stable vacuum states for both static and dynamical matter. Investigating such differences was a main motivation for the present work since, as mentioned earlier, in four dimensional theories a similar discrepancy is observed for the gauge groups other than SU(N).
First of all let us recall why the calculation of nested Wilson loops is an appropriate method for analysis of the vacuum structure of the theory. As is known, different vacuum states are classified by different boundary conditions which can be expressed by formula (4) with a Wilson loop inserted at infinity. At the same time if we are interested in the string tension for a very heavy charge-anti-charge pair of a particular representation R in a given vacuum state, this is nothing but the calculation of a loop in representation R in the background of the external loop and similarly for additional charges. The vacuum expectation value of corresponding system can be calculated by using the standard technique ([18]-[22]) and the string tension can be easily extracted. What is important for us is the observation that a stable vacuum state corresponds to the situation when string tension for all states is a non-negative number. Negative string tension between a pair of charges corresponds to the unstable situation where it is energetically favourable for the charges to be infinitely separated.

Let us review the previous calculation [14]. The configuration we are interested in here is that of a pair of adjoint charges in a fundamental background with the Wilson correlator of a single loop in the adjoint representation of SU(N) with boundary $C_2$ enclosing total area $S_2$ nested within a loop $C_1$. The contour $C_1$, which encloses an area $S_1 + S_2$, is taken in the k-fold anti-symmetric (fundamental) representation ($k = 0 \ldots N - 1$). The expectation value $\langle W(C_1, C_2) \rangle$ of such a configuration can be calculated explicitly with the result

$$\langle W(C_1, C_2) \rangle = \frac{N!}{(N-k)!k!} e^{-g^2 S_1/(N-k)} \left[ 1 + \frac{(N-k)(N+1)}{k+1} e^{-g^2 S_2 (N-k)} \right]$$

Where we will take the leading factor, which is just the contribution of a k-fundamental loop of total area $S_1 + S_2$, to be normalization which physically corresponds to a constant colour-electric field throughout the one dimensional world. The relevant part of the formula, in square brackets, describes different states with string tensions $0$, $g^2(N-k)$, $g^2k$ and $g^2(N+1)$. Since for each ($k = 0 \ldots N - 1$) these are non-negative we conclude that for a theory based on an SU(N) gauge group the number of vacuum states is equal to N. If we had considered different (non-fundamental) representations for the external loop which labels vacua, we would have found negative string tensions for some states-those corresponding to unstable charge configurations. Of course, this result was anticipated from more general considerations [9], but we believe that the explicit demonstration of instability is an important check of our formalism.

With this example in mind we are ready precisely formulate our approach to enumerating the vacuum states of 1+1 dimensional Yang-Mills theories with heavy adjoint matter. As in the unitary case, we will calculate the vacuum expectation value of the system of adjoint Wilson loops bounded by a single loop taken in different representations. Only those boundary representations which lead to non-negative string tensions count as vacuum states otherwise the configuration is unstable and should be excluded from the consideration. Looking at systems with more than one pair of adjoint charges (more than one adjoint loop) it will become clear that stability is a more complex issue for a general gauge group than for SU(N). These complications will lead us to consider generalizations of the notion of N-ality for arbitrary Lie and a complete classification of vacua.
In this section we present some explicit calculations of the enumeration of vacuum states for arbitrary compact Lie groups. We should note that these calculations are completely independent of the topological considerations of Section 2 as we will use group theoretic methods to calculate string tensions directly in different vacua. Of course, anticipating our results, we note that the calculations of this section are in general agreement with the topological classification. In terms of group theory this enumeration involves calculation of the Kronecker products of arbitrary representations with that of the adjoint and decomposing the result into a sum of irreducible representations. While it is difficult to show in general that a non-fundamental representation as a boundary charge leads to an unstable configuration, it will become clear later that we need only consider fundamental external charges and hence we will only consider these. To begin we will consider an example of the symplectic groups $\text{Sp}(2N)$ since they are similar in complexity to the unitary group and they show the basic characteristics of the considerations we must make for general simple Lie groups.

In $\text{Sp}(8)$ each of the representations is given by a Young table, a tensor representation in contrast to the spinor representations which arise in the orthogonal groups. As $\text{Sp}(8)$ is 36 dimensional we can easily identify the adjoint representation with the symmetric combination of two 8 dimensional fundamental representations; $(2000)$ in Young table notation as detailed in the appendix. The other three fundamentals are given by anti-symmetric combinations of the 8 dimensional $(1000)$ representation: $(1100)$, $(1110)$ and $(1111)$ which are 27, 48 and 42 dimensional respectively. From tables of branching rules [23] or directly from a table of products [24] one can easily find the product of each of the fundamentals with an adjoint representation. Using the expressions in the appendix for the quadratic Casimirs of arbitrary tensor representation for the classical Lie groups and subtracting the Casimir of the external fundamental we have the string tensions for states occurring in each loop configuration

$$36 \otimes 8 : 0, \frac{9}{2}g^2, \frac{11}{2}g^2 \quad 36 \otimes 27 : 0, g^2, 5g^2, 6g^2$$

$$36 \otimes 48 : 0, \frac{3}{2}g^2, \frac{7}{2}g^2, 6g^2 \quad 36 \otimes 42 : 0, 3g^2, 6g^2$$

It appears that each fundamental vacuum is stable under interaction with a single adjoint. Explicit calculations for other symplectic groups suggest that this is the general case with the number of stable vacua for $\text{Sp}(2N)$ equal to $N+1$. It is interesting to compare this result with naive expectations from four dimensional calculations mentioned in the introduction. Here, $N+1$ different vacuum states agrees with a classification based on the adjoint quadratic Casimir as in supersymmetric theories [7], and calculations of the Witten index but clearly differs from general topological considerations where the vacua should be a representation of $\pi_1(G/Z) \sim \mathbb{Z}_2$ with only two elements.

The other classical Lie groups we have yet to consider are the orthogonal groups. Typically these groups are more complicated to deal with since they have spinor representations which cannot be characterized by a Young tableaux. For our purposes though the presence of spinor representations simplifies matters in the following way. Since the adjoint representation of an orthogonal group $\text{SO}(N)$ is given by the anti-symmetric combination of two $N$-dimensional vector representations, the adjoint is a tensor representation and cannot interact with a spinor to produce an unstable configuration. In short we need only consider the stability of tensor
fundamentals interacting with adjoint charges. Since we are considering only pure tensor representations, the arguments for SO(2N+1) also apply to SO(2N). With this in mind we consider in detail only the case of SO(9).

For SO(9) the adjoint is 36 dimensional and represented by a Young table (1100). Since it also a fundamental there are only two tensor fundamentals to consider in this case. First is the 9 dimensional vector representation (1000). The normalized string tensions for the configuration of a single adjoint loop with this external charge are \(0, 5g^2, 16g^2\) and we see that all these configurations are stable. The other tensor fundamental is 84 dimensional with Young table (1110). Here we begin to see some differences with the case of SU(N) in that this fundamental does not form a stable configuration with a single adjoint loop. This is clear from the normalized string tensions \(-5g^2, 0, g^2, 3g^2, 6g^2\) and \(9g^2\). Consequently we find for the case of SO(9) there are three different stable vacuum states corresponding to the trivial vacuum, one with a single spinor representation as a boundary charge and one with the 9 dimensional vector representation as a boundary charge. This situation is in fact generic. For the orthogonal groups one can show using branching rules ([25], [24]) that all tensor fundamentals consisting of anti-symmetric combinations of more than two vector representations are unstable when combined with adjoint charges. Hence for SO(2N+1) we have three stable vacuum states and for SO(2N) there are four due to the second spinor representation. Remarkably these results disagree with naive expectations of supersymmetric calculations, the Witten index and topological arguments.

Next we will investigate the five exceptional Lie groups. We will consider only the simplest, \(G_2\) of rank two in any detail. Consulting standard group theory references [26] we find that the first fundamental is 7 dimensional and the second, which is also the adjoint, is 14 dimensional. Consequently there are only two different possible vacuum states here: the trivial and the one given by a 7 dimensional fundamental boundary charge which calculation [24] shows to be stable acting with an adjoint. Again we find that this result agrees with the supersymmetric calculation where the number of vacua is given by the quadratic Casimir of the adjoint representation but disagrees with the Witten index calculation and topology. By homotopy arguments since \(G_2\) is simply connected and it has a trivial center we have trivial \(\pi_1(G/Z)\) and expect only a single, trivial vacuum.

In a similar fashion one can investigate the stability of fundamental boundary charges for the other exceptional Lie groups \(F_4, E_6, E_7\) and \(E_8\). Using Kronecker product tables from [27] one finds that for \(F_4\) only the 26 dimensional fundamental gives a stable configuration when combined with a single adjoint. For \(E_6\) there are two degenerate 27 dimensional fundamentals each of which are stable and \(E_7\) has only a single 56 dimensional fundamental which results in a stable configuration. The last of the exceptional groups, \(E_8\) has no fundamental representations leading to stable configurations. This exhausts all of the possible cases for simple compact Lie groups and the results that we have obtained are listed in Table 1. The results of this section are suggestively place in the row labeled meta-stable vacua and comparing with the group theory and topological information tabulated there it is clear that none of the classification schemes we have been considering are accurate. Naively, one would think that all the vacua we have demonstrated here are perfectly stable with respect to the creation of a pair of adjoint charges and should be taken into account. At least, in case of \(SU(N)\) such a procedure reproduced all of the know results for enumerating the multiple vacua of the theory. Clearly an understanding of what is going on for \(SU(N)\) theory is not
sufficient to understand the general situation. This is obvious if one notes that the different approaches we have been considering: gluino condensates, the Witten index and the topology of the configuration space all give the same number of vacua for SU(N) - N. The essential details of the situation are not clear from the unitary case and we will need to investigate the general case more carefully.

Up to this point we have been considering the stability problem only with respect to creation of a pair of adjoint particles. This approach is perfectly general in Abelian gauge theories where the interactions have the same character for arbitrary numbers of interacting charges. However there are complications in the case of non-Abelian theories where interaction between different representations become quite involved. Before analyzing this more general case we return momentarily to the example of SU(N). Consider multiple adjoint loops in the standard background of a fundamental loop which corresponds to the possibility of creation of arbitrary numbers of particles. In SU(N) the adjoint by definition transforms trivially under the center (has no N-ality) and each of the fundamentals transforms with a different phase (has a different N-ality) so each is stable [9] under decays via adjoint charges into each other or the trivial representation. What is important for us is that this statement does not depend on the number of internal loops. The crucial element of this observation is the existence of a conserved quantum number: N-ality. Identifying the analogous feature of a general Lie group will lead us to the stable vacuum states.

Consequently we need to check if a boundary charge is stable against decay via multiple adjoint charges. For SU(N) this was not necessary due to the different N-ality of the fundamentals but in the general case we will use stability to determine the analog of N-ality for a general Lie group. As a concrete example let us return to the case of Sp(8) for which we have seen four different, apparently stable vacua (6). Instead of a single adjoint loop we now consider the system containing two adjoint Wilson loops nested within a single fundamental loop. Consider for instance the interaction of two adjoint representations with a single 27 dimensional fundamental: $36 \otimes 36 \otimes 27$. In the process of calculating (6) one finds the decomposition of the Kronecker product

$$36 \otimes 27 = 27 \oplus 36 \oplus 315 \oplus 594$$

and hence adding a second adjoint representation to the product produces a term $36 \otimes 36$ which certainly contains the trivial representation and will correspond to states with negative string tension once one renormalizes by the 27 dimensional boundary loop. By considering a similar process with more adjoint factors, the 42 and 48 dimensional representations can be shown to be unstable as well. Consequently we find that while all of the fundamental charges of Sp(8) are stable against decay via one adjoint charge, only the 8 dimensional one appears stable against an arbitrary number. While one cannot test the general situation for arbitrary groups via the method of loop calculations there is a different approach which will lead us to the correct generalization of N-ality for general Lie groups.

By now it is clear that the stable fundamentals which define different vacua for a given group are exactly the fundamental charges which cannot decay into each other nor the trivial representation via interactions with adjoint charges. In the physical picture these mark the end of decay chains and are the minimum energy representatives of their respective classes. Mathematically this idea is formalized in the concept of minimal representations which have been utilized before in the physics literature especially by Goddard and Olive [28] in discussions of monopole stability. As noted there, if we have the lattice of weights $\Lambda(G)$ for
a simply connected compact Lie group $G$ then the lattice of roots (weights of the adjoint representation) $\Lambda_{Ad}(G)$ is a subgroup of it. Consequently we can form the cosets $\Lambda/\Lambda_{Ad}$ by identifying weights which differ by integral linear combinations of roots and in this way we identify precisely which representations lie in like decay chains. These cosets form a finite dimensional Abelian group which is isomorphic to the center $Z$ of the group $G$ and now the importance of the center in classifying vacuum states becomes obvious. Clearly the notion of N-ality for the unitary groups is generalized here where to each coset one can assign a (conserved) element of an Abelian group- typically, but not always, an integer.

Once we have determined the classes to which different representations belong the question remains: Is there a representative of each class which is stable? The answer to this question is yes. First we deal with the identity coset where the representation with minimum quadratic Casimir (energy) is just the trivial representation. Clearly this coset includes the adjoint representation and all other representations which transform trivially under the center of the group. As for the other cosets we can equivalently define the minimal weights $\theta$ as those which lie closest to the origin of the weight diagram or have the least Casimir or where all weights $\{\theta\}$ of a representation lie on the same orbit under the action of the Weyl group. The positive (dominant) elements of the minimal weights then identify the minimal representations of the group and identify the complete set of stable charges which label non-trivial vacuum states for any Lie group.

It is now an easy task to enumerate the minimal representations for all simple, compact Lie groups. In the case of the special unitary group SU(N) these are of course the N-1 fundamental representations. For the symplectic group Sp(2N) only the 2N dimensional fundamental is stable in interactions with adjoint charges. The other classical Lie group is the orthogonal group SO(N) and here we have two different cases. For SO(2N+1) only the $2^N$ dimensional spinor representation is minimal but for SO(2N) things are more interesting where both of the spinor representations are stable in addition to the N dimensional vector representation. As for the exceptional groups, both 27 dimensional fundamentals of $E_6$ are minimal as is the 56 dimensional fundamental of $E_7$. The remaining compact Lie groups $E_8$, $F_4$ and $G_2$ have no non-trivial minimal representations, that is all representations can decay to the trivial one via adjoint charges. Now that we have identified all possible boundary charges which give rise to non-trivial vacua we see that all candidates are fundamental representations justifying a restriction we placed on ourselves earlier. We now tabulate this information in the last two rows of Table 1. The number of stable vacua is just the number of minimal representations in addition to the trivial representation which labels the standard perturbative vacuum.

Some comments are in order. We have shown that when enumerating the stable vacuum states of Yang-Mills theories with heavy adjoint matter the center symmetry of the group is crucial. Neither supersymmetric nor Witten index calculations lead to consistent results but topological considerations, namely the connectivity of the effective gauge group $G/Z$ does. In particular, the group structure of the center gives the structure of the vacuum states and in particular if representations $R_1$ and $R_2$ label different stable vacua then the multiplication table of the center determines the character of the vacuum with label $R_1 \otimes R_2$.

The second remark is much more speculative: We found that in the limit of very heavy quark mass, the meta-stable vacuum states will be stable with respect to decays by a single pair of adjoint charges but not in more complicated scenarios. For instance, in the chiral limit when an arbitrary number of massless quarks may appear, those vacuum states are certainly
unstable. Since the number of stable and meta-stable vacua are different for a given theory one could expect a phase transition with variation of the quark mass for those $\theta$ vacuum states which are meta-stable, rather than stable. If this indeed true, such a transition might have some relation to the problem of chiral condensation in $QCD_2$ with adjoint matter we discussed in the introduction.

5 Disorder Dual Operator

Now we turn to a discussion of confinement phenomena in two-dimensional gluodynamics from a point of view dual to the one we have been using to enumerate vacuum states. Of course, as we have shown in the previous sections, all physics is determined by the effective gauge group $G/Z$ so a dual picture has little new to offer. Regardless, we feel that a different point of view on the same phenomenon is desirable and in this section we investigate these confinement properties in terms of a so-called disorder operator $M$. As we will show, $M$ leads to algebraic structures analogous to number and ladder operators for multiple vacua in a gauge theory.

The importance of disorder operator in gauge theories has been emphasized by ’t Hooft[30] (see also [29]), who argued that rather than instantons it is the field configurations with non-trivial $\mathbb{Z}_N$ topological properties (for $SU(N)$ gauge group) that should be considered responsible for long range interactions. Analogous disorder variables have been used in different fields of physics ([31]-[34]) and most of the ideas described here have been extracted from the classical papers ([30]-[34]). In what follows we deal with the specific case of an $SU(N)$ gauge group however generalization to any compact Lie group follows by replacing $\mathbb{Z}_N$ factors by elements from the center of the relevant gauge group.

Now we are in a position to define our disorder operators $\{M(x)\}$. $M(x_0)$ is an operator of large gauge transformation which acts on fields with a gauge transformation $U(x)$ which is singular at the point $x_0$. Consequently this transformation has the property that as $x$ encircles $x_0$, $U$ does not return to its original value but acquires a $\mathbb{Z}_N$ phase:

$$U(\phi = 2\pi) = e^{-i2\pi k/N}U(\phi = 0) \quad (8)$$

From this definition we can easily find the expression for $\langle M \rangle$ in our case. Under a large gauge transformation the expression for the partition function of the gauge theory in the non-trivial vacuum with label $k$ (4) acquires a phase

$$\langle k|M|k \rangle = \frac{1}{2} \int D\psi D\bar{\psi} DA \exp \left(-\int d^2x \mathcal{L} \right) \text{tr}_R P \exp \left(i \int_{C \to \infty} dx^\mu A_\mu(x) \right) \text{tr}_R P \exp \left(i \int_{C \to \infty} dx^\mu U^\dagger \partial_\mu U \right) = e^{-i2\pi k/N} \quad , \quad k = 0, 1, \ldots N - 1 \quad (9)$$

where $U$ is the matrix of the large gauge transformation (8). Therefore, the vacuum expectation value of the disorder operator is a constant - a condensate of $M$ in some sense - and its phase is a label of the specific vacuum state in which we are. Notice that this situation is very similar to what we have in a two dimensional Abelian $\theta$ vacuum with the only difference being that the label which marks the vacuum state in our case is a discrete rather than a continuous variable ($\theta \in [0, 2\pi]$). Hence we see that the disorder operator $M$ is a number
operator for the vacua, returning the associated phase when acting on a particular vacuum state.

Two remarks are in order. First of all, operator $M$ as it is defined is not gauge invariant. It can be easily seen from the formula (8) there is an explicit dependence on the position of singularity $x_0$. This phenomenon is similar to the analysis of the Dirac string attached to a magnetic monopole where the position of the string depends on the gauge but the fact of its existence does not. In our calculation of the vacuum expectation value of $M$, we integrate over all of 1+1 dimensional space-time and thus the contribution of the string is recorded. Therefore $\langle M \rangle$ is a gauge invariant characteristic even though the operator $M$ itself is not.

Our second remark concerns the formula (9). As we discussed earlier, the external Wilson loop is in a fundamental representation therefore the gauge transformation defined by the formula (8) leads to the nontrivial phase in the calculation of $\langle M \rangle$ (9). If we had considered an adjoint external loop instead we would have got the trivial result $\langle M \rangle = 1$. Such a result certainly means that any adjoint external loop does not describe a new vacuum state, but rather, is equivalent to the trivial one - a fact which we already knew. However, what is useful from this point of view is we see the importance of the center $Z$ of the gauge group directly. Only elements nontrivial with respect to the center may lead to the nontrivial phases in (9).

Because (9) is so important we would like to explain this result in another way. From an analysis of the 't Hooft commutation relations [30] it follows

$$W(C)M(x) = M(x)W(C) \exp\left(\frac{2i\pi n}{N}\right)$$  \hspace{1cm} (10)

Here $W(C)$ is Wilson loop operator and $n$ counts the number of times that $C$ winds around $x$. It is easy to see from the original expression (4) that an extra Wilson loop insertion (one above another) generates the following transformation of the vacuum label $k$: $k \rightarrow k + 1$. So, the Wilson operator acts on $k$-variable as a ladder operator and takes us from the $k$ vacuum to the $k + 1$ vacuum state. Of course we have all seen this before in the standard commutation relations of quantum mechanics: If the operator $P = -i \frac{d}{dx}$ is momentum operator, then the unitary operator for a finite translation $a$ is $\exp(iaP)$. Taking into account the commutation relation $[P, x] = -i$, we have the identity

$$e^{iP} e^{ix} = e^{ia} \cdot e^{ix} e^{iP},$$  \hspace{1cm} (11)

which is analogous to the 't Hooft relation (10) with the obvious substitution:

$$e^{iP} \rightarrow W; \hspace{1cm} e^{ix} \rightarrow M; \hspace{1cm} e^{ia} \rightarrow e^{i\frac{2\pi}{N}}.$$  \hspace{1cm} (12)

From this analogy we see one more time that $\langle M \rangle$ should have the exponential form (9) with a phase determined by the amount of finite translation.

Finally we note that such a behaviour of vacuum expectation value (9) is in a perfect agreement with 't Hooft’s conjecture about properties of the vacuum condensation of the dual variable in confinement phase. In our case we can go even further though because the absence of propagating degrees of freedom makes possible the calculation of not only $\langle M \rangle$ itself, but the correlation functions like $\langle M(x_1), M^\dagger(x_2) \rangle = \langle M \rangle \cdot \langle M^\dagger \rangle \sim const$ as well. They simply factorize. In general case, when physical degrees of freedom are present the similar behaviour remains the same only in the limit $(x_1 - x_2)^2 \rightarrow \infty$ as a manifestation of the standard properties of cluster decomposition.
We have explicitly calculated string tensions between heavy adjoint charges and consequently found the number of vacuum states in two dimensional gluodynamics for each compact Lie symmetry group. It was found that string tension, and consequently the spectrum of the theory, depends on which vacua one is in. Additionally, it was shown that the number of vacuum states is in agreement with topological classification and it is given by $\pi_1$ of the corresponding effective gauge group as documented in Table 1. We believe that the number of absolutely stable vacua will remain the same even if one introduces the physical degrees of freedom so long as the system respects the center symmetry. The most important observation which can be derived from this study is the existence of a new superselection rule which must be imposed in order to count these new vacuum states. One could always say that an introduction of fundamental fields will destroy this classification: therefore there is no reason to study configurations which can not survive in the presence of fundamental matter. It is well known, however, why this argument is flawed. It is believed that the main configurations in $QCD_4$ responsible for the confinement are pure gluon (not quark) configurations and, at least in the large $N_c$ limit this statement is certainly correct. Therefore, one could expect that an introduction of fundamental fermions might change some qualitative characteristics, but not a quantitative picture derived in pure gluodynamics.

We believe that this is also true in $QCD_2$ where we essentially introduced and studied this new superselection rule. We hope that the future investigation of these, fundamentally new superselection rules and vacuum states may shed the light on the many issues raised in the Introduction regarding supersymmetric models, chiral condensates and behaviour in the large $N_c$ limit. We feel that adopting a point of view about classifying vacuum states as we have outlined here may lead to progress in resolving current dilemmas.

Finally, it might happen that a useful tool to formulate (and answer) questions about vacuum structure is a disorder operator in the dual description. Rather than the Wilson loop operator it is the disorder operator which plays the fundamental role of enumerating the vacua in the theory. At least recent developments [35] in supersymmetric models are based on the dual picture. In this approach the operator of the creation of a point monopole can be regarded as a large gauge transformation with the corresponding properties. This operator is in a sense analogous to our disorder operator (9) and, as is known, the monopole condensation (which is formally described as a nonzero condensate of the monopole creation operator) is the fundamental property of the theory with confinement. Of course, our operator (9) is in some sense trivial because it does not create any physical degrees of freedom. However, we believe that the main topological properties of this operator remain the same even in the presence of physical degrees of freedom.

Acknowledgment

We thank Sasha Polyakov for discussions and for the historical remarks regarding the disorder operator in the dual description. This work is supported in part by the Natural Sciences and Engineering Research Council of Canada. L.P. is supported in part by a University of British Columbia Graduate Fellowship.
Here we explain some of the notations and details of the group theory used in our analysis. First we need to define the Young tableaux notation we use. Since any tensor representation of a rank n compact Lie group may be denoted by a Young table we will deal with convenient characterizations of these tables. Let $l_i$ be the number of boxes in the $i$th row of the Young table for some representation $R$ then then we have non-increasing sequence $l_1 \geq l_2 \geq \ldots \geq l_n$. In the text this sequence is denoted $(l_1 l_2 \ldots l_n)$.

Expressions for the quadratic Casimirs of tensor representations for the classical are given in terms of simple formulae [36] of the table variables $l_i$. If we denote the total number of boxes in the Young table for representation $R$ by $l = \sum l_i$ then

$$SU(N) : \quad C_2(R) = \sum_{i=1}^{N} l_i (N + l_i + 1 - 2i - l/N) \quad (13)$$

$$Sp(2N) : \quad C_2(R) = \frac{1}{2} \sum_{i=1}^{N} l_i (2N + l_i + 2 - 2i) \quad (14)$$

$$SO(N) : \quad C_2(R) = \sum_{i=1}^{N} l_i (N + l_i - 2i) \quad (15)$$

Of particular interest for us are the details of adjoint representations for different groups. These are tensor representations for every compact Lie group and, hence, can be stated in terms of table variables $l_i$. For the classical Lie groups we have the following table configurations for adjoint representations

$$SU(N) : \quad l_1 = 2, l_2 \ldots l_{N-1} = 1 \quad (16)$$

$$Sp(2N) : \quad l_1 = 2, l_2 \ldots l_N = 0 \quad (17)$$

$$SO(2N + 1) : \quad l_1 = l_2 = 1, l_3 \ldots l_N = 0 \quad (18)$$

$$SO(2N) : \quad l_1 = l_2 = 1, l_3 \ldots l_N = 0 \quad (19)$$

For representations of the exceptional groups and spinor representations of the orthogonal groups one may calculate quadratic Casimirs directly from knowledge of the weights and the Weyl formula but for our purposes it was sufficient to use existing tables [23] and [27].
References


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13


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Table 1: Group theoretic and topological information important for discussion of multiple vacua in Yang-Mills theories. Details on the minimal (dominant) weights for each group can be found in the text.

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