On the black-hole kink

Pedro F. González-Díaz*
Centro de Física "Miguel Catalán",
Instituto de Matemáticas y Física Fundamental,
Consejo Superior de Investigaciones Científicas,
Serrano 121, 28006 Madrid (SPAIN)

May 16, 1996

Abstract

By allowing the light cones to tip over on hypersurfaces according to the conservation laws of an one-kink in static, Schwarzschild black hole metric, we show that in the quantum regime there also exist instantons whose finite imaginary action gives the probability of occurrence of the kink metric corresponding to single chargeless, nonrotating black holes taking place in pairs, the holes of each pair being joined on an interior surface, beyond the horizon.

PACS number(s): 04.70.Bw, 04.70.Dy, 04.60.Gw

*E-mail: iodpf21@cc.csic.es
1 Introduction

The idea that we shall explore in this paper is based on looking at the spacetime metric of a (D-1)-wormhole as the metric that results on the constant-time hypersurfaces corresponding to purely future directed or purely past directed light-cone orientations of a D-dimensional black-hole spacetime where we allow all possible light cone orientations compatible with the existence of a gravitational kink\(^1\).

We shall restrict to the physically most interesting example with D=4, which associated a Schwarzschild black hole to a three-dimensional wormhole.

We first briefly review the general topological concept of a kink and its associated topological charge. Let \((M, g_{ab})\) be a given D-dimensional spacetime, with \(g_{ab}\) a Lorentz metric on it. One can always regard \(g_{ab}\) as a map from any connected D-1 submanifold \(\Sigma \subset M\) into a set of timelike directions in \(M\) \(^2\). Metric homotopy can then be classified by the degree of this map. This is seen by introducing a unit line field \(\{n, -n\}\), normal to \(\Sigma\), and a global framing \(u_i: i=1,2,...,D-1\), of \(\Sigma\). A timelike vector \(v\) can then be written in terms of the resulting tetrad framing \((n, u_i)\) as \(v = v^0n + v^iu_i\), such that \(\sum_{i}^{D-1}(v^i)^2 = 1\). Restricting to time orientable manifolds \(M\), \(v\) then determines a map

\[ K: \Sigma \rightarrow S^{D-1} \]

by assigning to each point of \(\Sigma\) the direction that \(v\) points to at that point. This mapping allows a general definition of kink and kink number. Respect to hypersurface \(\Sigma\), the kink number (or topological charge) of the Lorentz metric \(g_{ab}\) is defined by\(^2\)

\[ \text{kink}(\Sigma; g_{ab}) = \text{deg}(K), \]

so this topological charge measures how many times the light cones rotate all the
way around as one moves along $\Sigma^3$.

In the case of an asymptotically flat spacetime the pair $(\Sigma, g)$ will describe an asymptotically flat kink if $\text{kink}(\Sigma; g) \neq 0$. All of the topological charge of the kink in the metric $g$ is in this case confined to some finite compact region. Outside that region all hypersurfaces $\Sigma$ are everywhere spacelike. For the case of a spherically symmetric kink, to asymptotic observers, the compact region containing all of the topological charge coincides with the interior of either a black hole when the light cones rotate away from the observers (positive topological charge), or a white hole when the asymptotic observers "see" light cones rotating in the opposite direction, toward them (negative topological charge).

Topology changes, such as handles or wormholes, can occur in the compact region supporting the kink, but not outside it. All topologies are actually allowed to happen in such a region. Therefore, in the case of spherically symmetric kinks, the supporting region should be viewed as an essentially quantum-spacetime construct. This is the view we shall assume throughout this paper.

## 2 The Schwarzschild Kink

We can take for the static, spherically symmetric metric of a three-dimensional wormhole

$$ds^2 = (1 - \frac{2M}{r})^{-1} dr^2 + r^2 d\Omega_2^2,$$

(2.1)

where $d\Omega_2^2$ is the metric on the unit two-sphere. Metric (2.1) describes a spacetime which (i) is free from any curvature singularity at $r = 0$, and (ii) possesses an apparent (horizon) singularity at $r = 2M$ that is removable by a suitable
coordinate transformation. The re-definition

\[ r = \frac{M}{2} \left( \frac{u}{\mu} + \frac{\mu}{u} \right)^2, \]  
(2.2)

where \( \mu \) is an arbitrary scale, transforms metric (2.1) into

\[ ds^2 = \frac{M^2}{4} \left( \frac{u}{\mu} + \frac{\mu}{u} \right)^4 \left( \frac{4du^2}{u^2} + d\Omega^2_2 \right). \]  
(2.3)

Along the complete \( u \)-interval, \((\infty, \mu)\), metric (2.3) varies from an asymptotic region at \( u = \infty \) to a minimum throat at \( u = \mu \) (i.e. at \( r = 2M \)).

On the other hand, metric (2.1) is in fact a constant time section, \( T = t_0 \), of a Schwarzschild black hole,

\[ ds^2 = -(1 - \frac{2M}{r})dT^2 + (1 - \frac{2M}{r})^{-1}dr^2 + r^2d\Omega^2_2. \]  
(2.4)

Metric (2.1) can likewise be regarded as being described by constant Euclidean time \( \tau = -iT \) sections of the Gibbons-Hawking instanton\(^4\) associated to (2.4). Each of such three-wormholes would correspond to a given Einstein-Rosen bridge\(^5\) on this instanton, so that one of the two halves of the wormhole should then be described in the unphysical\(^4\) exterior region created in the Kruskal extension of metric (2.4). In order to avoid the need of using such an unphysical region to describe a complete wormhole, we shall consider that metric (2.1) corresponds to a given fixed value of time \( T \) in the kink extension of (2.4).

We take for the metric that describes a spherically symmetric one-kink in four dimensions\(^6\),

\[ ds^2 = -\cos 2\alpha(dt^2 - dr^2) \pm 2\sin 2\alpha dtdr + r^2d\Omega^2_2, \]  
(2.5)

where \( \alpha \) is the angle of tilt of the light cones, and the choice of sign in the second term depends on whether a positive (upper sign) or negative (lower sign)
topological charge is being considered. An one-kink is ensured to exist if \( \alpha \) is allowed to monotonously increase from 0 to \( \pi \), starting with \( \alpha(0) = 0 \). Then metric (2.5) converts into (2.4) if we use the substitution

\[
\sin \alpha = \sqrt{\frac{M}{r}}
\]  

(2.6)

and introduce a change of time variable \( t + g(r) = T \), with

\[
\frac{dg(r)}{dr} = \tan 2\alpha.
\]  

(2.7)

Now, since \( \sin \alpha \) cannot exceed unity, it follows that \( \infty \geq r \geq M \), so that \( \alpha \) varies only from 0 to \( \frac{\pi}{2} \). In order to have a complete one-kink gravitational defect, we need therefore a second coordinate patch to describe the other half of the \( \alpha \) interval, \( \frac{\pi}{2} \leq \alpha \leq \pi \).

The kink metric (2.5), which is defined by coordinates \( t, r, \theta, \phi \) and satisfy (2.6) and (2.7), restricts the Schwarzschild solution to cover only the region \( \infty \geq r \geq M \). If one wants to extend such a metric to describe the region beyond \( r = M \) as well, two procedures can in principle be followed: (i) if the compact support of the kink is assumed to be classical, then one lets \( \alpha \) continue to increase as \( r \) decreases from \( r = M \) until \( \alpha = \pi \) at \( r = 0 \) to produce a manifold which has a homotopically nontrivial light cone field and one kink. This procedure makes metric (2.5) and definitions (2.6) and (2.7) to hold asymptotically only, and since, classically, one should assume a continuous distribution of matter in the kink support, the momentum-energy tensor can be chosen to satisfy reasonable physical conditions such as the weak energy condition\(^\text{6} \). (ii) The second procedure can apply when one assumes the black hole interior (i.e. the supporting compact region of the kink) to be governed by quantum mechanics. The simplest quantum condition to be satisfied by the interior region supporting a black- or white-kink
arises from imposing metric (2.5) to hold along the radial coordinate interval
of the kink, i.e.: $\infty \geq r \geq M$, rather than asymptotically only. Actually, in
this case, the kink geometry should hold in the two coordinate patches which
we need to describe the complete one-kink gravitational defect. The need for a
second coordinate patch can most clearly be seen by introducing the new time
coordinate

$$\bar{t} = t + h(r),$$  \hspace{1cm} (2.8)

which transforms metric (2.5) into the standard metric

$$ds^2 = -\cos 2\alpha \bar{t}^2 \mp 2k\bar{t} dr + r^2 d\Omega^2, \hspace{1cm} (2.9)$$

provided

$$\frac{dh(r)}{dr} = \frac{dg(r)}{dr} - \frac{k}{\cos 2\alpha}, \hspace{1cm} (2.10)$$

with $k = \pm 1$ and the choice of sign in the second term of (2.9) again depending
on whether a positive (upper sign) or negative (lower sign) topological charge is
considered. The choice of sign in (2.10) is adopted for the following reason. The
zeros of the denominator of $dh/dr = (\sin 2\alpha \mp 1)/\cos 2\alpha$ correspond to the two
horizons where $r = 2M$, one per patch. For the first patch, the horizon occurs at
$\alpha = \frac{\pi}{4}$ and therefore the upper sign is selected so that both $dh/dr$ and $h$
remain well defined and hence the kink is not lost in the transformation from (2.5) to
(2.9). For the second patch the horizon occurs at $\alpha = \frac{3\pi}{4}$ and therefore the lower
sign in (2.10) is selected. $k = +1$ will then correspond to the first coordinate
patch and $k = -1$ to the second one.

Metric (2.9) can be transformed directly into the Schwarzschild metric (2.4) if
we use (2.6) and the new coordinate transformation

$$\bar{t} = T - f(r), \hspace{1cm} (2.11)$$
where
\[ \frac{df(r)}{dr} = \frac{k}{\cos 2\alpha}. \] (2.12)

We impose then the standard kink metric (2.9) to hold along \( \infty \geq r \geq M \) on the two patches \( k = \pm 1 \), similarly to as it has been made in the de Sitter kink\(^7\). It can be seen that this condition respects conservation of energy-momentum tensor only if we assume an energy spectrum \( \frac{kn}{M} \), \( n = 0, 1, 2, \ldots \) to hold in the compact, internal region supporting the kink\(^3\): along the radial coordinate interval \( 2M \geq r > M \) of the first patch there would be no spherical surface with nonzero energy and therefore this interval does not contribute the stress tensor \( T_{\mu\nu} \); as one gets at \( r = M \) on the first patch it would appear a "delta-function-like" concentration of positive energy on that surface corresponding to the quantum level \( n = 1 \). This would at first glance blatantly violate energy-momentum conservation. However, the continuity of the angle of tilt \( \alpha \) at \( \frac{\pi}{2} \) implies that the two coordinate patches are identified at exactly the surfaces \( r = M \). Thus, since there would be an identical "delta-function-like" concentration of negative energy-momentum at \( n = 1 \) on \( r = M \) in the second patch, the total stress tensor \( T_{\mu\nu} \) will also be zero at the minimal surface \( r = M \). Thus, although the Birkhoff’s theorem ensures\(^8\) the usual Schwarzschild metric as the unique spherically symmetric solution to the four-dimensional vacuum Einstein equation, the violation of this classical result the way we have shown above implies an allowed quantized extension from it because this extension entails no violation of energy-momentum conservation at any interior spacelike hypersurface.

This result should be interpreted as follows. All what is left at length scales equal or smaller than the minimum size of the bridge (i.e. for \( n \geq 1, r \leq M \) is
some sort of quantized “closed” baby universe with maximum size $M$, whose zero total energy may be regarded as the sum of the opposite-sign eigenenergies of two otherwise identical harmonic oscillators with the zero-point energy substracted. The positive energy oscillator would play the role of the matter field part of a constrined Hamiltonian, $H = 0$, and the negative energy oscillator would behave like though it were the gravitational part of this Hamiltonian constraint. On the other hand, to an asymptotic observer in either patch, the above quantized kink geometry would look like that of a black hole if the topological charge is positive, and like that of a white hole if the topological charge is negative. In the latter case, to the asymptotic observer there would actually be a topological change by which an asymptotically flat space converts into asymptotically flat space plus a baby universe being branched off from it. Now, since from a quantum-mechanical standpoint white and black holes with the same mass are physically indistinguishable$^9$, it follows that to asymptotic observers the asymptotically flat space of black holes is physically indistinguishable from asymptotically flat space plus a baby universe, with such a baby universe living outside the realm of the two coordinate patches where the kink is defined, in the inaccessible region between $r = M$ and $r = 0$.

Metric (2.9) contains still the geodesic incompleteness at $r = 2M$ of metric (2.4). This incompleteness can be removed by the use of Kruskal technique. Thus, introducing the metric

$$ds^2 = -F(U, V)dUdV + r^2dΩ^2_2,$$ (2.13)

in which

$$F = \frac{4M \cos 2\alpha}{\beta} \exp \left( -2\beta k \int_{\infty/M}^2 \frac{dr}{\cos 2\alpha} \right),$$ (2.14)
\[ U = \mp e^{\beta \bar{t}} \exp \left( 2\beta k \int_{\infty/M}^{r} \frac{dr}{\cos 2\alpha} \right), \quad (2.15) \]
\[ V = \mp \frac{1}{2\beta M} e^{-\beta \bar{t}}, \quad (2.16) \]

where \( \beta \) is an adjustable parameter which will be chosen so that the unphysical singularity at \( r = 2M \) is removed, and the lower integration limit \( \infty/M \) refers to the choices \( r = \infty \) and \( r = M \), depending on whether the first or second patch is being considered. Using (2.6) we obtain from (2.14)

\[ F = 4M \left( \frac{1 - 2M/r}{\beta} \right) \left( \frac{r}{M} \left( \frac{2M}{r} - 1 \right) \right)^{-4\beta kr}. \]

This expression would actually have some constant term coming from the lower integration limits \( \infty/M \). We have omitted at the moment such a term because it is canceled by the similar constant term which appears in the Kruskal coordinate \( U \) when forming the Kruskal metric from (2.13)-(2.16).

Unphysical singularities are then avoided if we choose

\[ \beta = \frac{1}{4kM}. \quad (2.17) \]

Whence

\[ F = \frac{16kM^3}{r} e^{-\frac{r}{2M}}, \quad (2.18) \]
\[ U = \mp e^{\frac{\bar{t}}{4M}} \exp \left( \frac{2M - r}{M} \right), \quad V = \mp \frac{k}{2} e^{-\frac{\bar{t}}{4M}}, \quad (2.19) \]

where\(^1\)

\[ \bar{t} = t_0 - k \int_{\infty/M}^{r} \frac{dr}{\cos 2\alpha} \]
\[ = \bar{t}_0 - k \left( r - 2M \ln \left( \frac{M}{2M - r} \right) \right), \quad (2.20) \]

with the constant \( \bar{t}_0 \) being obtained from \( t_0 \) after absorbing the term arising from the lower integration limit \( \infty \) or \( M \), depending on whether the first or
second patch is being considered. We finally obtain for the Kruskal metric of the Schwarzschild kink

\[ ds^2 = -\frac{32kM^3}{r}e^{-\frac{r}{2M}}dUdV + r^2d\Omega^2. \]  

(2.21)

Except for the sign parameter \( k \), this metric is the same as the Schwarzschild-Kruskal metric.

Because of continuity of the angle of tilt \( \alpha \) at \( \frac{\pi}{2} \), the two coordinate patches can be identified to each other only on the surfaces at \( r = M \). Such an identification should occur both on the original and the new regions created by the Kruskal extension, and represents a bridge that connects asymptotically flat regions of the two coordinate patches. Any \( T = \text{const.} \) section of this spacetime construct will then describe halves of a three-dimensional wormhole whose neck is now at \( r = M \), rather than \( r = 2M \). One can then describe the two halves of a complete wormhole just in the physical original regions of either patch \( k = +1 \) or patch \( k = -1 \).

The causal structure of the considered geometry could at first glance be thought of as being unstable due to mass-inflation caused by the unavoidable presence of a Cauchy horizon\(^{11} \): because quanta that enter the future event horizon at arbitrary late time suffers an arbitrarily large blue shift while propagating parallel to the Cauchy horizon, there will be in general a mass-inflation singularity along a part of the horizon in one patch caused by small fluctuations in the other. However, using the spherical shell approach in the lightlike limit\(^{12} \) where a mass shell is allowed to move toward \( r = 0 \) in the field of an interior mass distribution, it can be shown that our kink model with quantized support prevents the occurrence of any mass-inflation singularity. In fact, any interior energy fluctuation in
one patch is necessarily sign-reversed to the energy of the imploding shell in the other. Therefore, a mass increase must now occur in the expanding fluctuation shell, rather than in the imploding shell, and the mass variation of these two shells is nonsingular everywhere for \( r \geq M \), even at the collision radius where one would expect the mass singularity to occur. We actually expect that, at that radius, imploding and expanding gravitational masses are both finite with half and twice their respective asymptotic values.

### 3 Euclidean formalism

The Euclidean section of the Schwarzschild solution is asymptotically flat and nonsingular because it does not contain any points with \( r < 2M \). Thus, the curvature singularity does not lie on the Euclidean section. Here I shall consider the instantons that can be associated with the black hole kinks, and show that their Euclidean sections can be extended beyond the horizon down to the surface \( r = M \).

The Euclidean continuation of the metrics which contain one kink should be obtained by putting

\[
\tilde{t} = i\tilde{\tau}.
\]

Using (2.6) and (2.7) we then have

\[
d\tilde{\tau} = -idT + \frac{ik}{\cos 2\alpha}dr.
\]

This Euclidean continuation would give rise to metrics which are positive definite if we choose either the usual continuation \( T = i\tau \), for \( r \geq 2M \), or the new Euclidean continuation \( r = -i\rho \), \( M = -i\mu \), for \( r < 2M \), where \( r \) becomes timelike, and we transform a space coordinate into a time coordinate. In the first
In order for the new continuation \( r = -i\rho, M = -i\mu \) to give rise to a metric which is positive definite for \( r < 2M \), we would also continue the angular polar coordinates such that \( \theta = -i\Theta, \phi = \phi \). With this choice we then had for the orthogonal coordinates

\[
x = -X = -\rho \sinh \Theta \cos \phi, \quad y = -Y = -\rho \sinh \Theta \sin \phi,
\]

\[
z = -iZ = -i\rho \cosh \Theta,
\]

so that \( r = \sqrt{(x^2 + y^2 + z^2)} = \pm i\rho \) and \( |Z| \geq \rho \). Therefore, we in fact have \( \phi = \arctan \frac{y}{x} = \arctan \frac{Y}{X} \), and \( \theta = \arccos \frac{z}{r} = \pm i\Theta \), with \( \Theta = \cosh^{-1} \frac{Z}{\rho} \). Hence, the metric on the unit two-sphere \( d\Omega_2^2 \) should transform as

\[
d\Omega_2 = \pm i d\omega_2 = i(d\Theta^2 + \sinh^2 \Theta d\phi^2)^{\frac{1}{2}}.
\]

The choice of the minus sign for the Euclidean continuation of both the radial coordinate \( r \) and the polar angle \( \theta \) would allow us to have the same action continuation as that corresponding to the continuation \( T = i\tau \); i.e. \( S = iI \), where
$S$ and $I$ are the Lorentzian and Euclidean action, respectively, since the scalar curvature transforms as $R(r) = -R(\rho)$ under continuation $r = -i\rho$.

Thus, for the continuation $r = -i\rho$, $M = -i\mu$, $d\Omega_2 = \pm i d\omega_2$ for $r < 2M$, metric (2.9) becomes

$$ds^2 = \cos 2\alpha d\bar{\tau}^2 \pm 2kd\bar{\tau}d\rho + \rho^2 d\omega_2^2,$$  

(3.7)

which corresponds to the new Euclidean subsection $2M > r > M$, with positive definite metric

$$ds^2 = \left(\frac{2\mu}{\rho} - 1\right) dT^2 + \left(\frac{2\mu}{\rho} - 1\right)^{-1} d\rho^2 + \rho^2 d\omega_2^2,$$  

(3.8)

and can be maximally-extended to the Kruskal metric

$$ds^2 = \frac{32\mu^3 k}{\rho} e^{-\frac{\rho}{2\mu}} d\hat{U}d\hat{V} + \rho^2 d\omega_2^2,$$  

(3.9)

where in this case

$$\hat{U} = \mp e^{\frac{\rho}{4k\mu}} e^{\frac{\rho}{2\mu}} \left(\frac{2\mu - \rho}{\mu}\right), \quad \hat{V} = \mp \frac{k}{2} e^{-\frac{\rho}{4k\mu}}. \quad (3.10)$$

Using a positive definite metric such as (3.8) leads however to the problem that the azimuthal angle $\theta$ is a periodic variable only outside the horizon. In this case the transverse two-manifold, which is a two-sphere of positive scalar curvature outside the Euclidean horizon, becomes a hyperbolic plane of negative scalar curvature inside the horizon and any boundary at finite geodesic distance inside the horizon is no longer compact. In particular, the boundary at $r = M$ would then have the noncompact topology $S^1 \times R^2$, with $S^1$ corresponding to time $T$. One cannot identify this geometry at $\rho = 2\mu$ with the geometry at $r = 2M$ corresponding to the compact topology $S^1 \times S^2$ of the Gibbons-Hawking instanton$^4$. Nevertheless, avoidance of the spacetime singularities in the
calculation of the black hole action does not actually require having a positive
definite metric in our spacetime kinks. Indeed, the “tachyonic continuation” of
the signature \(+ + - -\) (which is \(- - + +\)) that corresponds to a real azimuthal
periodic variable \(\theta\) also inside the horizon and implies the same metrics as (3.7),
(3.8) and (3.9) but with the sign for the polar coordinate terms reversed, can not
only avoid singularities but erase them even at \(r = 0\). In order to see this, let us
consider the new variables \(y + z = U\) and \(y - z = V\) in the Kruskal metric (2.21)
which then becomes
\[
ds^2 = -\frac{32kM^3}{r}e^{-\frac{r}{2M}}(dy^2 - dz^2) + r^2d\Omega_2^2,
\]
with
\[
y^2 - z^2 = ke^{\frac{r}{2M}}\left(1 - \frac{r}{2M}\right),
\]
\[
\frac{y + z}{y - z} = ke^{\frac{i}{2M}}e^{\frac{r}{2M}}\left(\frac{r}{2M} - 1\right).
\]
The singularity at \(r = 0\) lies on the surfaces \(y^2 - z^2 = k\). This singularity can be
avoided by defining either a new coordinate \(\zeta = iy\) or a new coordinate \(\xi = iz\).
For the first choice the metric takes the Euclidean form
\[
ds^2 = \frac{32kM^3}{r}e^{-\frac{r}{2M}}(dz^2 + d\zeta^2) + r^2d\Omega_2^2,
\]
which is positive definite in the patch \(k = +1\) and has in fact signature \(- - + +\)
in the patch \(k = -1\). The radial coordinate is then defined by
\[
z^2 + \zeta^2 = ke^{\frac{r}{2M}}\left(\frac{r}{2M} - 1\right).
\]
On the section on which \(z\) and \(\zeta\) are both real (the usual Euclidean section for
patch \(k = +1\)) \(\frac{r}{2M}\) will be real and greater or equal to 1 on patch \(k = +1\), and
\(\frac{1}{2} \leq \frac{r}{2M} \leq 1\) on patch \(k = -1\), the lower limit \(\frac{1}{2}\) being imposed by the continuity
of the kink at $\alpha = \frac{\pi}{2}$. Define the imaginary time by $T = i\tau$. This continuation leaves invariant the form of the metric (3.14) and is therefore compatible with the coordinate transformation $\zeta = iy$. Then, from (2.20) and (3.13) we obtain

$$z - i\zeta = \pm \left(z^2 + \zeta^2\right)^{\frac{1}{2}} e^{i\frac{\tau}{4kM}}. \tag{3.16}$$

It follows that for this time continuation $\tau$ is periodic with period $8\pi kM$. On this nonsingular Euclidean section, $\tau$ has then the character of an angular coordinate which rotates about the "axis" $r = 2M$ clockwise in patch $k = +1$, and anticlockwise about the "axis" $r = 0$ in patch $k = -1$. Any boundary $\partial M_k$ in this Euclidean section has topology $S^1 \times S^2$ and so is compact in both coordinate patches. Since the scalar curvature $R$ vanishes, the action can be written only in terms of the surface integrals corresponding to the fixed boundaries. This action can be written

$$I_k = \frac{1}{8\pi} \int_{\partial M_k} d^3x K_k, \tag{3.17}$$

where $K_k = K - \frac{1}{2}(1+k)K^0$, $K$ being the trace of the second fundamental form of the boundary, and $K^0$ the trace of the second fundamental form of the boundary imbedded in flat space. This action was evaluated in the case of the positive definite metric which corresponds to $k = +1$. It is $I_{+1}(M) = 4\pi iM^2$. In the case $k = -1$, fixing the boundary at the surface $r = A = M$, we also have

$$I_{-1}(M) = \frac{1}{8\pi} \int_{\partial M_{-A}} K d\Sigma$$

$$= -4\pi i(2r - 3M) \bigg|_{r=M} = 4\pi iM^2.$$  

For the second choice of coordinates, $\xi = iz$, metric (3.11) takes the form

$$ds^2 = -\frac{32kM^3}{r} e^{-\frac{r}{4M}} (dy^2 + d\xi^2) + r^2 d\Omega_2^2, \tag{3.18}$$

15
which is positive definite in patch \( k = -1 \) and has again signature \(- + + +\) in patch \( k = +1 \). The radial coordinate is now defined by

\[
y^2 + \xi^2 = ke^{\frac{r}{2M}} \left( 1 - \frac{r}{2M} \right),
\]

so that on the section on which \( y \) and \( \xi \) are both real (the usual Euclidean section for patch \( k = -1 \)) \( \frac{r}{2M} \) will be in the interval \( \frac{1}{2} \leq \frac{r}{2M} \leq 1 \) on patch \( k = +1 \), and greater or equal to 1 on patch \( k = -1 \). We define now the imaginary \( r \) and \( M \) by \( r = -i\rho \) and \( M = -i\mu \), keeping \( T \) and the azimuthal coordinate \( \theta \) real. In order for this definition to be compatible with the coordinate transformation \( \xi = iz \), it should leave metric (3.18) formally unchanged. For this to be accomplished one must also continue the line element \( ds \) itself, namely \( ds = -i d\sigma \), instead of the azimuthal angle \( \theta \). This requirement becomes most natural if we recall that the interval \( ds \) has the same physical dimension as that of \( r \) and \( M \), and that the ”tachyonic” mass \( \mu \) should be associated with an imaginary relativistic interval. Then, from (2.20) and (3.13) we obtain

\[
y - i\xi = \pm (y^2 + \xi^2)^{\frac{1}{2}} e^{i\frac{r}{4\mu}}.
\]

It is now the Lorentzian time \( T \) which becomes periodic with period \( 8\pi k\mu \). On this new nonsingular Euclidean section, \( T \) would have the character of an angular coordinate which rotates about the ”axis” \( \rho = 0 \) clockwise in the patch \( k = +1 \), and anticlockwise about the ”axis” \( \rho = 2\mu \) in the patch \( k = -1 \). In such a new section, the action is given by (3.17), where now \( K_k = K - \frac{1}{2}(1 - k)K^0 \). On the patch \( k = +1 \), we have

\[
I_{+1}(\mu) = \frac{1}{8\pi} \int_{\partial M_{+1}^4} K d\Sigma
= 4\pi i M (2r - 3M) \bigg|_{r=M} = -4\pi i M^2 = 4\pi i\mu^2.
\]
In the patch $k = -1$, taking $K^0 = \frac{2}{r}$ and following Gibbons and Hawking\textsuperscript{4}, we obtain the action $I_{-1}(\mu)$ which turns out to be the same as $I_{+1}(\mu)$.

Thus, on the coordinate patch $k = +1$, the Euclidean continuation (3.1) of the time coordinate $\bar{t}$ of the kink metric contains both the continuation for time $T$, $T = i\tau$, where the apparent singularity at $r = 2M$ is like the irrelevant singularity at the origin of the polar coordinates provided that $\frac{T}{4M}$ is regarded as an angular variable and is identified with period $2\pi$\textsuperscript{4}, and a new continuation $r = -i\rho$, which also implies ”tachyonic” continuations $M = -i\mu$ and $ds = -i d\sigma$, where the curvature singularity at $\rho = 0$ becomes again like a harmless polar-coordinate singularity provided that $\frac{T}{4\mu}$ is regarded as an angular variable and is identified with period $2\pi$. The transverse two-manifold is now a compact two-sphere both outside and inside the Euclidean horizon and any boundaries have compact topology $S^1 \times S^2$, with $S^1$ corresponding to $\tau$ outside the horizon and to $T$ inside the horizon. Since these topological products are compact, have the same Euler characteristic and are both orientable, they are homeomorphic to each other with a continuous mapping between them. Therefore, one can identify the two corresponding geometries at the Euclidean horizons ($r = 2M$ and $\rho = 2\mu$) which, respectively, $\tau$ rotates about at zero geodesic distance and is the geodesic distance at which $T$ rotates about $\rho = 0$. The Gibbons-Hawking instanton can then be extended beyond the Euclidean horizon down to just the boundary surface at $r = M$ ($\rho = \mu$) where the first and second patches must be somehow joined onto each other. The resulting Euclidean section does not contain any points with $r < M$ and therefore the curvature singularity is still avoided, as it also is in the baby universe sector ($\rho < \mu$) due to the periodic nature of the instantonic time $T$. The spacetime of the extended instanton covers the entire domain of
the coordinate patch $k = +1$ and that of the baby universe is outside the two coordinate patches.

Euclidean continuation (3.1) on coordinate patch $k = -1$ leads to the same instantonic sections as for patch $k = +1$, but now $T = i\tau$ corresponds to the section inside the horizon $r = 2M$ up to $r = M$, and $r = -i\rho$, $M = -i\mu$, $ds = -id\sigma$ define the section outside the horizon $\rho = 2\mu$, with $\tau$ and $T$ respectively rotating about $r = 0$ and $2\mu$, anticlockwise in both cases. Since the boundaries at constant radial coordinates on both sides of the Euclidean horizon have compact topology $S^1 \times S^2$, the geometries at the Euclidean horizon ($r = 2M$ and $\rho = 2\mu$) can also be identified, leading to an instanton which covers the entire coordinate patch $k = -1$.

On any $\tau-r$ plane in the coordinate patch $k = +1$ we can define the amplitude $\langle \tau_2 \mid \tau_1 \rangle$ to go from the surface $\tau_1$ to the surface $\tau_2$ which is dominated by the action $I_1(M) = 4\pi iM^2$, corresponding to the circular sector limited by the times $\tau_1$ and $\tau_2$ on a circle centered at $r = 2M$ with large radius $r_0 \gg 2M$. Similarly, on the $t-\rho$ plane in the patch $k = +1$ the amplitude $\langle t_2 \mid t_1 \rangle$ to go from the surface $t_1$ to the surface $t_2$ is dominated by the action $I_2(\mu) = 4\pi i\mu^2$ that corresponds to the sector limited by times $t_1$ and $t_2$ from $\rho = \mu$ to $\rho = 2\mu$ on a circle centered at $\rho = 0$. An asymptotic observer in patch $k = +1$ would interpret these results as providing the probability of the occurrence in the vacuum state of, respectively, a black hole with mass $M$ or a white hole with mass $\mu$. In the coordinate patch $k = -1$, the same observer would reach the same interpretation but for a black hole with mass $\mu$ or a white hole with mass $M$.

Any constant time section of these instantons would represent the half of a three-dimensional wormhole either in the first or the second coordinate patch.
The connection of one such wormhole halves in the first patch to other wormhole half in the second patch would take place on an equatorial surface $r = A$ and produce a complete wormhole with two original asymptotic regions, one in patch $k = +1$ and the other in patch $k = -1$.

Acknowledgements

This work was supported by DGICYT under Research Projects No. PB94-0107-A.
References


3 A. Chamblin, *Kinks and singularities*, DAMTP preprint R95/44.


