DIMENSIONALLY CONTINUED OPPENHEIMER-SNYDER
GRAVITATIONAL COLLAPSE. I – SOLUTIONS IN EVEN DIMENSIONS

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Abstract

The extension of the general relativity theory to higher dimensions, so that the field equations for the metric remain of second order, is done through the Lovelock action. This action can also be interpreted as the dimensionally continued Euler characteristics of lower dimensions. The theory has many constant coefficients apparently without any physical meaning. However, it is possible, in a natural way, to reduce to two (the cosmological and Newton’s constant) these several arbitrary coefficients, yielding a restricted Lovelock gravity. In this process one separates theories in even dimensions from theories in odd dimensions. These theories have static black hole solutions. In general relativity, black holes appear as the final state of gravitational collapse. In this work, gravitational collapse of a regular dust fluid in even dimensional restricted Lovelock gravity is studied. It is found that black holes emerge as the final state for these regular initial conditions.

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I. INTRODUCTION

The theory of general relativity has predicted the existence of black holes in our universe. The relevance of black holes comes from their unavoidable emergence in complete gravitational collapse of astrophysical objects, as was first shown by Oppenheimer and Snyder [1].

General relativity was formulated in a spacetime with four dimensions, of course. However, there are theoretical hints that we might live in a world with more dimensions. Kaluza-Klein theories, with the help of extra compact dimensions, try to unify the gravitational field to the other known gauge fields. String theory, a theory that embraces gravity and other interactions in a unified picture, lives, in its heterotic form, in ten dimensions. Turning-off all other fields, the low-energy limit of string theory yields Einstein-Hilbert action plus terms involving quadratic powers in the curvature. For the quantum version of the theory, these quadratic corrections should be proportional to the Gauss-Bonnet term in order to give ghost-free meaningful interactions.

On the other hand, one can ask what is the most natural generalization of Einstein gravity to other dimensions while keeping the same degrees of freedom. Although pure general relativity can be formulated in other dimensions, when one goes to dimensions higher than four it is not anymore unique. The natural generalization is given by the Lovelock action [2] so that the field equations for the metric remain of second order. The theory can also be considered as the topological extension of Einstein-Hilbert action [3]. In this theory new terms make their appearance by taking into the action the Euler densities of the spaces with dimensions lower than the space in consideration. The Euler density of the space in consideration yields a topological term only, with no pure dynamical content. In four dimensions one has to take in consideration two Euler densities. The Euler density of the 0-dimensional space which is proportional to $\sqrt{-g}$, and the Euler density of
the 2-dimensional space, proportional to $\sqrt{-g}R$, where $g$ is the determinant of the metric and $R$ the Ricci curvature scalar. Thus Lovelock gravity in four dimensions reduces to Einstein gravity, with action $\frac{1}{16\pi G} \int d^4x \sqrt{-g}(-2\Lambda + R)$, where $\Lambda$ and $G$ are the cosmological and Newton’s constant, respectively. A similar construction and action is obtained for three dimensions. In six dimensions one has still to add the Euler characteristic of four dimensional space, i.e. the Gauss-Bonnet term, to have the Lanczos action, given by,

$$\frac{1}{16\pi G} \int d^6x \sqrt{-g} \left(-2\Lambda + R + \alpha_2(R_{\alpha\beta\gamma\sigma}R^{\alpha\beta\gamma\sigma} - 4R_{\alpha\beta}R^{\alpha\beta} + R^2)\right),$$

where $\alpha_2$ is a new constant.

For each two new dimensions there exists a new constant $\alpha_p$. These constants do not seem to have a direct physical meaning. In order to find a meaningful set of constants in any dimension $D$, it was proposed in [4] a method which restricts drastically the number of independent constants to two, $G$ and $\Lambda$, thus yielding a restricted Lovelock gravity. This method separates, in a natural manner, theories in even dimensions ($D = 2n$, with $n = 1, 2, ..$) from theories in odd dimensions ($D = 2n + 1$).

Several static and cosmological solutions within Lovelock gravity have been found [5]. In the restricted setting of [4], where the independent constants are reduced to two, wormhole [6] and black hole solutions have also been found [4] both in even and odd dimensions. Higher dimensional black holes may shed some light on the understanding of non-perturbative effects in quantum gravity. They can also expose which of the features of the usual four-dimensional black hole solutions remain in higher dimensions. Since in general relativity black holes appear as the final state of gravitational collapse it is important to know if the black hole solutions found in Lovelock gravity can, in an analogous manner, form from gravitational collapse. We will show that, indeed, black holes form from regular initial data. A possible scenario for the occurrence of this collapse in $D$ dimensions, would be in the very early universe, before the $D - 4$ extra dimensions have been compactified. In turn, these newly formed higher dimensional black holes could play a role in the compactification process.
In this work we study gravitational collapse of a dust fluid in Lovelock gravity within the context of the restricted coefficients found in [4]. We will analyse the even dimensional case only. For the odd dimensional case see [7]. We thus generalize the Oppenheimer-Snyder collapse. In section II the Lovelock gravity for restricted coefficients is presented. In section III we display the static solutions in even dimensions found in [4]. In section IV we find some cosmological solutions for perfect fluids. In section V we match the solutions found in section IV to the solutions of section III. Finally, in section VI we show that black holes can form through gravitational collapse in Lovelock gravity. Section VI presents some conclusions. In the rest of the paper we usually do \( G = c = 1 \).

II. THE LOVELOCK THEORY

The most general action in \( D \geq 3 \) spacetime dimensions that yields the same degrees of freedom of Einstein’s theory is the so called Lovelock action, given by [2,3]

\[
S = \int \mathcal{L}_D = \kappa \sum_{p=0}^{\lfloor(D-1)/2\rfloor} \alpha_p \int_M \epsilon_{a_1 \ldots a_D} R^{a_1 a_2} \wedge \ldots \wedge R^{a_{2p-1} a_{2p}} \wedge e^{a_{2p+1}} \wedge \ldots \wedge e^{a_D} + S_m,
\]

(2.1)

where \( R^{ab} = d\omega^{ab} + \omega^a_c \wedge \omega^{cb} \) is the curvature two-form, \( e^a \) is the local frame one-form, and \( \omega^{ab} \) is the spin connection, with \( a_i = 0, 1, \ldots, D - 1 \). The symbol \( \lfloor \rfloor \) over the summation symbol means one should take the integer part of \((D - 1)/2\). \( S_m \) is a phenomenological action which describes the macroscopic matter sources.

In general, the constant coefficients \( \alpha_p \) are arbitrary. However, it is shown in [4] that taking certain special choices one is able to get simple meaningful solutions. Following [4] one first considers embedding the Lorentz group \( SO(D - 1, 1) \) into de anti-de Sitter group \( SO(D - 1, 2) \), and then separates into two distinct classes of Lagrangians: Lagrangians for even dimensions and Lagrangians for odd dimensions.
For even dimensions, $\mathcal{D} = 2n$, $(n = 2, 3, \ldots)$, one chooses the following Lagrangian

$$L_{2n} = \kappa \hat{R}^{A_1 A_2} \wedge \hat{R}^{A_3 A_4} \wedge \cdots \wedge \hat{R}^{A_{\mathcal{D}-1} A_{\mathcal{D}}} Q_{A_1 A_2 \cdots A_{\mathcal{D}}},$$

(2.2)

with $A_1, A_2 = 0, 1, \ldots, \mathcal{D}$ being anti-de Sitter indices. $\hat{R}^{A_1 A_2}$ is the anti-de Sitter curvature two-form constructed with the $SO(\mathcal{D} - 1, 2)$ connection, $W_{A_1 A_2}$. In order to yield a non-trivial action, the tensor $Q$ is chosen to be an invariant tensor under the Lorentz group only, i.e., $Q_{A_1 A_2 \cdots A_{\mathcal{D}}} = \epsilon_{a_1 a_2 \cdots a_{\mathcal{D}}}$ for $A_i = a_i$ $(i = 0, \ldots, \mathcal{D} - 1)$ and zero otherwise. Decomposing the connection $W^{AB}$ into the connection under $\mathcal{D}$ rotations, $\omega^{ab}$, and inner translations, $e^a$, one finds the anti-de Sitter curvature $\hat{R}$ in terms of the Lorentz curvature $R$

$$\hat{R}^{ab} = R^{ab} + \frac{1}{l^2} e^a \wedge e^b,$$

(2.3)

where $l$ is a scale factor which is to be related to the cosmological constant $l^2 = -\frac{1}{\Lambda}$. Using (2.3) one finds that the Lagrangian (2.2) can be put in the form

$$L_{2n} = \kappa \left( R^{a_1 a_2} + \frac{1}{l^2} e^{a_1} \wedge e^{a_2} \right) \wedge \cdots \wedge \left( R^{a_{\mathcal{D}-1} a_{\mathcal{D}}} + \frac{1}{l^2} e^{a_{\mathcal{D}-1}} \wedge e^{a_{\mathcal{D}}} \right) \epsilon_{a_1 a_2 \cdots a_{\mathcal{D}}}. $$

(2.4)

This Lagrangian gives the Born-Infeld gravity [8]. Comparison of (2.4) and (2.1) gives the coefficients $\alpha_p$,

$$\alpha_p = \kappa \binom{\mathcal{D}}{p} l^{-\mathcal{D}+2p}, \quad \mathcal{D} = 2n$$

(2.5)

where, for convenience one can choose $\kappa$ as

$$\kappa = \frac{l^{\mathcal{D}-2}}{32 \pi G n}, \quad \mathcal{D} = 2n$$

(2.6)

For odd dimensions, $\mathcal{D} = 2n - 1$, one can find a construction similar to the Chern-Simons action construction in three dimensions. One starts with the Euler density in one dimension above $\mathcal{D}$, $E_{2n}$, which is an exact form, and can be written as an exterior derivative of a Lagrangian in $2n - 1$ dimensions, i.e., $E_{2n} = dL_{2n-1}$, see [4]. Since gravitational collapse
in odd dimensions has different features from collapse in even dimensions we study odd dimensional collapse in another work [7].

Given the action (2.1), the field equations are obtained by the variation with respect to the one-forms $e^a$. Under the assumption of zero torsion, the variation with respect to the spin connection $\omega^{ab}$ vanishes identically. Although the equations have powers in the curvatures, they remain by construction second order in the metric. The field equations are given by

$$-\kappa \frac{[(D-1)/2]}{\sum_{p=0}^{n-1} \alpha_p (D - 2p) \epsilon_{a_1...a_D} R^{a_1a_2} \wedge .. \wedge R^{a_{2p-1}a_{2p}} \wedge e^{a_{2p+1}} \wedge .. \wedge e^{a_{D-1}} = Q_{a_D}, \quad (2.7)$$

where $Q_{a_D}$ is a $(D - 1)$-form associated with the energy momentum tensor $T^a_b$ through the following expression

$$Q_i = \frac{1}{(D-1)!} T_i^{a_1} \epsilon_{a_1...a_D} e^{a_2} \wedge ... \wedge e^{a_D}. \quad (2.8)$$

### III. EXTERIOR VACUUM SOLUTIONS

In the vacuum all components of the energy-momentum tensor vanish, so that the field equations (2.7) are given by

$$-\kappa \sum_{p=0}^{n} \alpha_p (D - 2p) \epsilon_{a_1...a_D} R^{a_1a_2} \wedge .. \wedge R^{a_{2p-1}a_{2p}} \wedge e^{a_{2p+1}} \wedge .. \wedge e^{a_{D-1}} = 0. \quad (3.1)$$

Inserting the coefficients $\alpha_p$ and the constant $\kappa$ given in (2.5) and (2.6) in equation (3.1), one gets for even dimensions $(D = 2n)$,

$$(R^{a_1a_2} + \ell^{-2} e^{a_1} \wedge e^{a_2}) \wedge .. \wedge (R^{a_{2n-3}a_{2n-2}} + \ell^{-2} e^{a_{2n-3}} \wedge e^{a_{2n-2}}) \wedge e^{a_{2n-1}} \epsilon_{a_1a_2...a_{2n}} = 0. \quad (3.2)$$

We consider now a static, spherical symmetric spacetime. One can write the metric in the following form,
\[
    ds_+^2 = -g^2(r_+)\, dt_+^2 + g^{-2}(r_+)\, dr_+^2 + r_+^2\, d\Omega_{D-2}^2, \tag{3.3}
\]

where \( t \) and \( r \) are the time and radial coordinates and \( d\Omega_{D-2}^2 \) is the arc-element of a unit \((D - 2)\)-sphere. The subscript \( + \) reminds that (3.3) is to be viewed as an exterior solution.

With metric (3.3) and equations (3.1) and (3.2), Bañados, Teitelboim and Zanelli found the following exact solution for \( D = 2n \) [4],

\[
    ds_+^2 = -\left[1 - (2M/r_+)\, \frac{1}{n-1} + (r_+/l)^2\right]\, dt_+^2 + \frac{dr_+^2}{1 - (2M/r_+)\, \frac{1}{n-1} + (r_+/l)^2} + r_+^2\, d\Omega_{D-2}^2. \tag{3.4}
\]

These solutions describe black holes. We will show that they also represent the exterior vacuum solution to a collapsing (or expanding) dust cloud in Lovelock’s theory.

### IV. INTERIOR MATTER SOLUTIONS

The interior spacetime is modeled by a homogeneous collapsing (or expanding) dust cloud, whose metric is described by the Friedmann-Robertson-Walker in \( D \) dimensions

\[
    ds^2 = -dt^2 + a^2(t) \left[ \frac{dr^2}{1 - kr^2} + r^2\, d\Omega_{D-2}^2 \right]. \tag{4.1}
\]

The coordinates \( t \) and \( r \) are comoving coordinates (we omit throughout the subscript \( - \) to indicate an interior solution). The constant \( k \) can take the values, \( k = 0, \pm 1 \). The energy-momentum tensor for a perfect fluid is given by

\[
    T_{\alpha\beta} = (\rho + p)\, u_{\alpha}\, u_{\beta} + pg_{\alpha\beta}, \tag{4.2}
\]

where \( \rho \) is the energy-density, \( p \) the pressure, and \( u^\alpha \) is the \( D \)-velocity of the fluid. From (4.1)-(4.2) and Lovelock equations (2.7) we obtain

\[
    (D - 1)! \sum_p \alpha_p (D - 2p) \left( \frac{k + \dot{a}^2}{a^2} \right)^p = \rho + p \tag{4.3}
\]

and
\[(D - 2)! \sum_p (D - 2p) \left( \frac{k + \dot{a}^2}{a^2} \right)^{p-1} \left( 2p \frac{\ddot{a}}{a} - (D - 2p - 1) \frac{k + \dot{a}^2}{a^2} \right) = 0, \quad (4.4)\]

where the coefficients \(\alpha_p\) are given in (2.5), and \(\kappa\) is given in (2.6). Rearranging (4.3)-(4.4) we obtain the following system of equations

\[-B \frac{d}{d\tau} \left( \frac{\dot{a}}{a} \right) + \frac{k}{a^2} = \rho + p \quad (4.5)\]

\[(D - 1) B \left( \frac{\dot{a}}{a} \right) \left[ -\frac{k}{a^2} + \frac{d}{d\tau} \frac{\dot{a}}{a} \right] = \dot{\rho} \quad (4.6)\]

where,

\[B \equiv (D - 2)! \sum_p \alpha_p 2p (D - 2p) \left( \frac{\dot{a}^2 + k}{a^2} \right)^{p-1}. \quad (4.7)\]

Equations (4.5)-(4.6) have a first integral given by

\[\dot{a}^2 = -k - \left( a \frac{\dot{a}}{l} \right)^2 + \left( a \frac{\dot{a}}{l} \right)^2 \left[ 16 \pi l^2 \rho_0 \left( \frac{a_0}{a} \right)^{D-1} \right]^{2/(D-2)}, \quad (4.8)\]

where \(\rho_0\) and \(a_0\) are constants.

The Ricci quadratic scalar and the Kretschmann scalar are given by

\[R^{ab} R_{ab} = -(D - 1)^2 \left( \frac{\ddot{a}}{a} \right)^2 + (D - 1) \left[ \frac{\ddot{a}}{a} + (D - 2) \frac{\dot{a}^2 + k}{a^2} \right]^2, \quad (4.9)\]

\[R^{abcd} R_{abcd} = (D - 1) \left[ \left( \frac{\ddot{a}}{a} \right)^2 + \left( \frac{\dot{a}^2 + k}{a^2} \right)^2 \right] \quad (4.10)\]

respectively.

Taking (4.5) and (4.6) yields

\[\dot{\rho} + (D - 1) (\rho + p) \frac{\dot{a}}{a} = 0. \quad (4.11)\]

We now assume a dust fluid, \(p = 0\). For such an equation of state we can integrate (4.11) to give
\[ \rho = \rho_0 \left( \frac{a_0}{a} \right)^{D-1} \]  

(4.12)

where \( \rho_0 \) and \( a_0 \) are the constants defined above.

In general it is not possible to obtain an exact analytical solution of (4.8) for \( k = \pm 1 \). However, restricting to \( D = 4 \), one of course obtains the Lemaître models, of which the closed and open Friedmann universes are the particular cases found for \( l \to \infty \). For \( k = -1 \) and \( D \neq 4 \) there is a special solution with zero matter content, taken in the limit \( l \to \infty \) and given by

\[ a(t) = \pm t \]  

(4.13)

where for the \( - \) sign one takes \(-\infty < t < 0\) and for the \( + \) sign \( 0 < t < \infty \). There are no singularities in these solutions since, as one can show, the curvature scalars are null. This solution indicates that the higher order terms in the curvature appearing in the Lovelock gravity act, in some sense, as matter terms [9].

The marginally bound case, \( k = 0 \), allows a second integral of (4.8) given by

\[ a = a_0 \left\{ \frac{16}{(D-1)!} \rho_0 l^2 \sin^{D-2} \left[ -\frac{D-1}{D-2} \left( \frac{t-t_0}{l} \right) \right] \right\}^{\frac{1}{D-1}} \]  

(4.14)

where \( t_0 \) gives the time for which \( a = 0 \), and without loss of generality, one can put \( t_0 = 0 \). We take \(-\pi < t < 0\). For \(-\pi < t < \pi/2\) the cloud is expanding. For \(-\pi/2 < t < 0\) the cloud is collapsing. And \( t = \pi/2 \) is a moment of time-symmetry.

Inserting (4.14) in (4.12) we obtain the evolution of the density in the \( k = 0 \) dust model,

\[ \rho(t) = \left( \frac{D-1}{16 \pi} \right)^2 \sin^{D-2} \left[ -\frac{D-1}{D-2} \left( \frac{t}{l} \right) \right] \]  

(4.15)

The curvature scalars (4.9)-(4.10) and the density (4.15) diverge, at \( t = -\pi \) (representing the appearance of a singularity), and \( t = 0 \) (denoting the formation of a singularity).

We can now take the limit of zero cosmological constant. Indeed, expanding (4.14) in powers of \( 1/l \) yields
\[
\left( \frac{a}{a_0} \right)^{D-1} \approx \alpha \left[ (\bar{t})^{D-2} \left( \frac{1}{l} \right)^{D-4} - \frac{D-2}{3} (\bar{t})^D \left( \frac{1}{l} \right)^{D-2} + \mathcal{O}^D \left( \frac{1}{l} \right) \right]. \tag{4.16}
\]

where \( \alpha \equiv \frac{16 \pi}{(D-1)!} \rho_0 \) and, \( \bar{t} \equiv -\frac{D-1}{D-2} t \).

When \( D \neq 4 \) and \( l \to \infty \) one gets \( a = 0 \), i.e., no physical solution. For \( D = 4 \) and \( l \to \infty \) one recovers the usual Friedmann \( k = 0 \) case,

\[
\left( \frac{a}{a_0} \right)^3 \approx \left[ \frac{3}{2} \sqrt{\frac{8 \pi}{3} \rho_0} t \right]^2, \tag{4.17}
\]

whereas the density goes like \( \rho \sim t^{-2} \).

V. JUNCTION CONDITIONS

Now we match the exterior and interior spacetimes found in sections III and IV, respectively, across an interface of separation \( \Sigma \). The junctions conditions are [10]

\[
ds^2_+|_{\Sigma} = ds^2_-|_{\Sigma} \tag{5.1}
\]

\[
K_{\alpha\beta}^+|_{\Sigma} = K_{\alpha\beta}^-|_{\Sigma} \tag{5.2}
\]

where \( K_{\alpha\beta} \) is the extrinsic curvature,

\[
K_{\alpha\beta}^\pm = -n^\pm_\epsilon \frac{\partial^2 x^\pm_\epsilon}{\partial \xi^\alpha \partial \xi^\beta} - n^\pm_\epsilon \Gamma^\gamma_\epsilon_{\gamma\delta} \frac{\partial x^\gamma_\pm}{\partial \xi^\alpha} \frac{\partial x^\delta_\pm}{\partial \xi^\beta} \tag{5.3}
\]

and \( n^\pm_\epsilon \) are the components of the unit normal vector to \( \Sigma \) in the coordinates \( x^\pm_\epsilon \), and \( \xi \) represents the intrinsic coordinates in \( \Sigma \). The subscripts \( \pm \) represent the quantities taken in the exterior and interior spacetimes. Both the metrics and the extrinsic curvatures in (5.1)-(5.2) are evaluated at \( \Sigma \). The metric intrinsic to \( \Sigma \) is written as

\[
ds^2_\Sigma = -d\tau^2 + R^2(\tau) d\Omega^2_{D-2}. \tag{5.4}
\]

Where \( \tau \) is the proper time on \( \Sigma \) and \( d\Omega^2_{D-2} \) denotes the line element on a \( D-2 \) dimensional sphere.
Using the junction condition (5.1), metric (5.4) and the exterior metric (3.4) we obtain
\[ r_+ = R(\tau), \quad (5.5) \]
and
\[ \left[ 1 - (2M/r_+)^{\frac{1}{n-1}} + (r_+/l)^2 \right] \dot{r}_+^2 - \left[ 1 - (2M/r_+)^{\frac{1}{n-1}} + (r_+/l)^2 \right]^{-1} \dot{r}_+^2 = 1, \quad (5.6) \]
where \( \dot{\cdot} \equiv \frac{d}{d\tau} \), and both equations are evaluated at \( \Sigma \). From now on, we will usually omit the subscript \( \Sigma \) to denote evaluation at the interface. Using (5.5) in (5.6) we find
\[ \frac{dt_+}{d\tau} = \sqrt{\left[ 1 - (2M/R)^{\frac{1}{n-1}} + (R/l)^2 \right] + \dot{R}^2} \left[ 1 - (2M/R)^{\frac{1}{n-1}} + (R/l)^2 \right]. \quad (5.7) \]
The unit normal to \( \Sigma \) in the exterior spacetime is
\[ n^+ = \left( -\frac{dr_+}{d\tau}, \frac{dt_+}{d\tau}, 0, \ldots, 0 \right). \quad (5.8) \]
From (5.3) we then get
\[ K^+_{\theta\theta} = R \left( \frac{1}{n-1} - 1 \right)^{\frac{1}{2}} \left( \frac{R}{l} \right)^2 + \dot{R}^2. \quad (5.9) \]
In what follows the other components of \( K^+_{ab} \) is not needed.

The unit normal to \( \Sigma \) in the interior spacetime is
\[ n^- = \left( 0, \frac{a}{\sqrt{1 - kr^2}}, 0, \ldots, 0 \right) \quad (5.10) \]
and from (5.3) we have
\[ K^-_{\theta\theta} = R(\tau) \sqrt{1 - k (r_\Sigma)^2}. \quad (5.11) \]
From the condition \( K^+_{\theta\theta} = K^-_{\theta\theta}, \) (5.9) and (5.11) we obtain
\[ \dot{R}^2 + \left( \frac{R}{l} \right)^2 + k (r_\Sigma)^2 = \left( \frac{2M}{R} \right)^{2/(D-2)}. \quad (5.12) \]
Multiplying equation (4.8) by \((r_\Sigma)^2\) we get
\[
\dot{R}^2 + \left(\frac{R}{\ell}\right)^2 + k (r_\Sigma)^2 = \left(\frac{R}{\ell}\right)^2 \left[\frac{16\pi l^2 \rho_0}{(D - 1)!} \left(\frac{R_0}{R}\right)^{D-1}\right]^{2/(D-2)}.
\] (5.13)

Comparing equation (5.12) and (5.13) we have
\[
M = \left(\frac{1}{l}\right)^{D-4} \frac{8\pi}{(D - 1)!} \rho_0 R_0^{D-1},
\] (5.14)

which is the mass of the cloud expressed in terms of the constants given in the problem. This expression is valid for any value of \(k\), \(k = 0, \pm 1\).

VI. BLACK HOLE FORMATION

In order to study black hole formation in this theory we work with the \(k = 0\) model solution found in (4.14). The interior metric is then
\[
ds^2 = -dt^2 + a^2(t) \left(dr^2 + r^2 d\Omega_{D-2}^2\right).
\] (6.1)

where for convenience we rewrite (4.14) as
\[
a = \left(\frac{2M}{r_\Sigma^{D-1} t^{D-2}} \sin^{D-2} \left[-\frac{D - 1}{D - 2} \left(\frac{t}{\ell}\right)\right]\right)^{1/(D-1)},
\] (6.2)

and we have used equation (5.14) and \(R_0 = a_0 r_\Sigma\).

The exterior metric is given in (3.4) and as we have shown in section V, it is possible to make a smooth junction between both spacetimes.

We assume that gravitational collapse occurs for \(-\frac{\pi}{2} \leq t \leq 0\). The time \(t = -\frac{\pi}{2}\) marks the onset of collapse. At this moment there are no singularities in spacetime, as the curvature scalars (4.9)-(4.10) and the density (4.15) indicate. In fact, the singularity appears only at \(t = 0\), where all these quantities blow up.

To know whether a black hole as formed or not, one has to search for the appearance of an apparent horizon and an event horizon. The apparent horizon is defined in [11] to be the
boundary of the region of trapped two-spheres in spacetime. To find this boundary on the interior spacetime one looks for two spheres \( Y \equiv a(t) r \) = constant whose outward normals are null, i.e.,

\[
\nabla Y \cdot \nabla Y = 0.
\]

(6.3)

Using metric (6.1) in (6.3) yields,

\[
\left( \frac{da(t)}{dt} \right)^2 = \frac{1}{r^2}.
\]

(6.4)

Using (6.2) in (6.4) gives the evolution of the apparent horizon in comoving coordinates,

\[
r = r_\Sigma \left( \frac{1}{2m} \right)^{1/D-1} \sin^{1/D-1} \left[ \frac{-D-1}{D-2} \left( \frac{t}{l} \right) \right] \cos \left[ \frac{-D-1}{D-2} \left( \frac{t}{l} \right) \right].
\]

(6.5)

where \( m \equiv \frac{M}{l} \). For \( D = 4 \) and \( l \to \infty \) this expression reduces to the usual expression for the apparent horizon in the Friedmann metric,

\[
t = -\frac{2}{3} \left( \frac{2M}{r_\Sigma^3} \right) r^3.
\]

(6.6)

Now, the apparent horizon first forms at the surface \( r_\Sigma \). Then, for \( r = r_\Sigma \), equation (6.5) gives the time \( t \) at which the apparent horizon first forms. On the other hand, one should also be able to find the formation time of the apparent horizon on the surface \( \Sigma \) through an equation on \( \Sigma \), equation (5.12). Indeed, at the junction one has \( R = a(t) r_\Sigma \). Then from junction condition (5.12) and equation (6.4) we have that the apparent horizon first forms when

\[
R \left[ 1 + \left( \frac{R}{l} \right)^2 \right]^{(D-2)/2} = 2M.
\]

(6.7)

Using (5.5) this also gives \( r_+ \left[ 1 + \left( \frac{r_+}{l} \right)^2 \right]^{(D-2)/2} = 2M \). For Friedmann \( (l \to \infty \) and \( D = 4 \) the above expression reduces to \( r_+ = 2M \), as expected. Dividing equation (6.7) by \( l \) and defining \( x \equiv \frac{R}{l} \) we get

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\[ x \left[ 1 + x^2 \right]^{(D-2)/2} = 2m, \]  
\hfill (6.8)

where \( m \equiv \frac{M}{l} \) as above. Now, the time of formation of the apparent horizon can be found through equation

\[ R_{AH} = a(t_{AH}) \, r_\Sigma \]
\[ = \left\{ 2M l^{D-2} \sin^{D-2} \left[ -\frac{D - 1}{D - 2} \left( \frac{t_{AH}}{l} \right) \right] \right\}^{1/(D-1)}. \]  
\hfill (6.9)

In terms of \( x \) and \( m \) (6.9) reads

\[ x_{AH} = \left\{ 2m \sin^{D-2} \left[ -\frac{D - 1}{D - 2} \left( \frac{t_{AH}}{l} \right) \right] \right\}^{1/(D-1)}. \]  
\hfill (6.10)

Given a dimension \( D \) and an \( m \) one can obtain \( x \) through equation (6.8). Then equation (6.10) gives implicitly \( t_{AH} \), the time of the formation of the apparent horizon on the surface \( \Sigma \). For instance, for \( D = 6 \) and \( m = 1 \) we find \( t_{AH} = -0.53l \). Putting this value back in equation (6.5) we verify that everything checks.

The event horizon, being a null spherical surface, is determined through the null outgoing lines of metric (6.1), i.e.,

\[ \frac{dt}{dr} = a(t). \]  
\hfill (6.11)

Equation (6.11) can be put in the following integral form,

\[ \frac{r}{r_\Sigma} = -\frac{D - 2}{D - 1} \left( \frac{1}{2m} \right)^{1/(D-1)} \int_{u_0}^{u_1} \frac{du}{\sin^{(D-2)/(D-1)}(u)}, \]  
\hfill (6.12)

\( u \equiv -\frac{D-1}{D-2} \frac{t}{l} \) and \( m \) has been defined above. Now, the time \( u_1 \) is precisely equal to the formation time of the apparent horizon, since in vacuum both horizons coincide [12]. One has then to integrate (6.12) to find the time \( u_0 \) at which the event horizon first forms, at \( r = 0 \).

This can be done numerically. For \( D = 6 \) and \( m = 1 \) we obtain \( t_0 = -\frac{4}{5} u_0 l = -1.57l \). A plot in comoving coordinates \((t, r)\) shows the evolution of the apparent and event horizons. We do this for \( D = 4, 6, 10, 26 \) (see figures 1,2,3,4). Making a matching to the vacuum exterior spacetime one finds the usual Penrose diagram for gravitational collapse and formation of a black hole in an anti-de Sitter background, see figure 5.
VII. CONCLUSIONS

We have analysed gravitational collapse in Lovelock gravity which is a natural extension of Einstein’s general relativity to higher dimensions. It was shown that within a restricted set of Lovelock coefficients, gravitational collapse of a regular initial non-rotating dust cloud proceeds, to form event and apparent horizons, and terminates at a spacelike curvature singularity, in much the same way as the Oppenheimer-Snyder collapse. As in the case of the wormhole solutions found in [6] and the black hole solutions found in [4] the collapsing solutions studied here show that some important features of classical general relativity are preserved and carried into Lovelock gravity.

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REFERENCES


Figure Captions

Figure 1. Oppenheimer-Snyder collapse in D=4 dimensions in an asymptotically anti-de Sitter spacetime. The interior dust cloud in comoving coordinates $(t, r)$ fills the whole diagram. The left side represents the center of the cloud $r = 0$, the right side the surface of the cloud $\frac{r}{r_0} = 1$. The evolution of the event horizon (dashed line) and apparent horizon (full line) are drawn. The singularity occurs at $t = 0$.

Figure 2. Dimensionally continued Oppenheimer-Snyder collapse in D=6 dimensions in an asymptotically anti-de Sitter spacetime. See subtitle of figure 1 for more detailed explanation.

Figure 3. Dimensionally continued Oppenheimer-Snyder collapse in D=10 dimensions in an asymptotically anti-de Sitter spacetime. See subtitle of figure 1 for more detailed explanation.

Figure 4. Dimensionally continued Oppenheimer-Snyder collapse in D=26 dimensions in an asymptotically anti-de Sitter spacetime. See subtitle of figure 1 for more detailed explanation.

Figure 5. Penrose diagram for the collapse of a dust cloud in an asymptotically anti-de Sitter spacetime. Each point in the diagram represents a $D - 2$ sphere. (eh=event horizon, ah=apparent horizon).