Integrability, Duality and Strings

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1 Introduction.

After the two years ago work of Seiberg and Witten [1, 2], the Pandora box of string theory has been opened once more. The magic opening word has been duality. A web of exciting interrelated results it contains has appeared during the last months: string-string duality (U duality) [3]; the physical interpretation of the conifold singularity [4]; heterotic-type II dual pairs [3, 5, 6, 7, 8]; R-R states and Dirichlet-branes [9, 10]; derivation from M-theory of dualities in string theory; and non-perturbative enhancement of symmetries [11, 12, 13].

String-string duality was first pointed out between heterotic strings compactified on $T^4$, and type II$_A$ strings on K3 [5]. The first check of this duality relation between these two different approaches to string theory is of course trying to understand the equivalence, for the type II$_A$ string on K3, of the well known enhancement of symmetry for the heterotic string on $T^4$ at certain points in the moduli space. This problem was beatifully solved in [5], where the enhanced non abelian gauge symmetries appear associated to the orbifold singularities of K3. These singularities are of A-D-E type, and its combinaton corresponds to a group of total rank $\leq 20$. Geometrically, these singularities arise from the collapse of a set of 2-cycles; the number of collapsing cycles equals the rank of the gauge group, and the intersection matrix is given by the Dynkin diagram of the singularity. This geometry can be directly connected with Strominger’s suggestion for the interpretation, in Calabi-Yau threefolds, of conifold singularities, where a 3-cycle collapses to a point; in this case, and for the type II$_B$, a massless soliton with R-R charge can be interpreted in terms of an appropriated 3-brane wrapping around the 3-cycle. For the case of K3, the enhancement of symmetry is interpreted in terms of 2-branes wrapping around 2-cycles that collapse at the orbifold point.

String-string duality in six dimensions can now be used to produce dual heterotic-type II pairs in four dimensions, by compactifying on a 2-torus. This is the point where string-string duality and type II T-duality produces the desired S duality for the heterotic string [14].

The dual pairs become then the string analog of the celebrated Seiberg-Witten solution of $N = 2$ gauge theories. In fact, in the simpler case of supersymmetric gauge theories, the quantum moduli of a particular theory with some gauge group and matter content is characterized by the classical special geometry of the moduli of a curve, while in the string dual pairs context the quantum moduli of a certain heterotic string compactified on some Calabi-Yau manifold $X$ is given by the classical special geometry of the complex structures moduli space of some other Calabi-Yau manifold $Y$, in which the type II string theory is compactified.

Taking now into account that field theory is the point particle limit of string theory, it was very natural to expect that the Seiberg-Witten quantum moduli for $N = 2$ gauge theories should be naturally derived as the point particle limit of heterotic-type II dual pairs, provided the low energy theory defined by the heterotic string corresponds to the
desired $N=2$ gauge theory [15, 16]. This is in fact what happens when a singular point in
the moduli space of the type II theory is blown up; this point is in the weak coupling and
point particle limit region of the moduli, and is a non abelian enhancement of symmetry
point, with the non abelian symmetry that of the $N=2$ gauge theory under study.

Already at this level of the discussion an interesting subtlety, connecting a non trivial
way global symmetries of field theory and string theory global symmetries, appears. In
fact, the point particle limit of the string defines a coordinate for the field theory quantum
moduli which involves both the string tension and the heterotic dilaton, both taken in
a double scaling limit. The importance of this fact is that for the corresponding moduli
of the dual type II string we can define global stringy transformations, associated to
symmetries of the Calabi-Yau threefold $Y$, which induce transformations on the field
theory quantum moduli parameter. These transformations have, at the field theory level,
the interpretation of global $R$-symmetries that we can not quotient by.

The above described mechanism, for the enhancement of non abelian symmetry in type
II$_A$ string on $K3$, allows to think of phenomena of enhancement of symmetry at regions
in the moduli space of type II theories that are at very strong coupling. In fact, from the
analysis in [17], topology changing transitions must be taken into account as a present
mechanism in the singular loci associated to vanishing of the discriminant. Some of these
transitions are geometrically identical to the process of regularization of singularities,
again characterized by a Dynkin diagram and a set of collapsing 2-cycles. This comment
unifies two apparently unrelated facts: topology changing amplitudes, and singularities
associated to enhanced non abelian symmetries. Even when these phenomena take place
in very stringy regions of the moduli of type II theory, it is possible to use a “gauge field
theory auxiliary model” to describe them [11]. The interplay between perturbative and
non perturbative enhancement of non abelian gauge symmetries has been started to be
understood on the context of compactifications of $F$-theory [18] on elliptic fibrations [19],
a question also related to the problem of finding heterotic-heterotic dual pairs.

In this notes we will try to attack some of the exciting developments taking place
around $M$-theory and $F$-theory from a more abstract, but at the same time simpler,
point of view, that arising from the integrable model introduced by Donagi and Witten
[20]. The essence of this integrable model is a two dimensional gauge theory with a Higgs
field living in the adjoint representation defined on a reference Riemann surface; in what
follows we will think of this surface as the genus one elliptic curve of the $N=4$ theory
determined by Seiberg and Witten, $E_\tau$. The integrable model, when defined on $E_\tau$, can
be used to derive the Seiberg-Witten solution of $N=2$ supersymmetric gauge theory.

The relevant part of the integrable model we will extensively make use of all over this
notes is the role played by the $\tau$ moduli on the reference surface: it defines the “scale”
with respect to which we “measure” the two dimensional gauge invariant $\text{tr} \phi^k$ quantities.

In these notes evidence will be presented for interpreting the moduli of Donagi-Witten
theory as the stringy moduli of a Calabi-Yau manifold [21]. A reason for such an inter-
pretation is that in both cases $N=2$ field theory is obtained by the same type of blow up procedure. What in the Calabi-Yau framework is purely stringy, within the Donagi-Witten approach is interpreted in terms of changes in the moduli of the “reference” Riemann surface $E_{\tau}$. The most interesting part of this construction is that in the integrable model context, the roles played by $u$ (the value of $tr\phi^2$ relative to the quadratic differential of $E_{\tau}$) and $\tau$ are very symmetric. We can change $u$ or, on the same footing, modify the scale defined by $\tau$.

It is therefore tempting to consider the duality arising in Donagi-Witten picture as the same sort of exchange taking place in $F$-theory elliptic fibrations between the two base $\mathbb{P}^1$, an exchange at the root of heterotic-heterotic duality and non perturbative enhancement of non abelian symmetry.

We feel that if $M$-theory and $F$-theory approaches are deep and stringy in spirit, the approach based on integrable models can be complementary replacing the climbing up in dimensions for the abstract set up of integrability.

2 Seiberg-Witten Theory.

For simplicity, and because on this lecture we will only consider the case of $SU(2)$, we reduce this brief introduction on Seiberg-Witten theory to the simplest case, that of $N=2$ supersymmetric gauge theory with gauge group of rank equal one.

The potential for the scalar superpartner $\phi$ is given by

$$V(\phi) = \frac{1}{g^2} tr[\phi, \phi^\dagger]^2. \quad (2.1)$$

Vanishing of the potential leads to a flat direction defined by

$$\phi = \frac{1}{2} a \sigma^3, \quad (2.2)$$

where $a$ is a complex parameter, and $\sigma^3$ is the diagonal Pauli matrix. As vacuum states corresponding to values $a$ and $-a$ are equivalent, since they are related by the action of the Weyl subgroup, a gauge invariant parameterization of the moduli space is defined in terms of the expectation value of the Casimir operator,

$$u = <tr\phi^2> = \frac{1}{2} a^2. \quad (2.3)$$

For non vanishing $u$, the low energy effective field theory contains only one abelian $U(1)$ gauge field; this gauge field, besides the photon, can be described, up to higher than two derivative terms, by a holomorphic prepotential $F(a)$. The dual variable is defined through

$$a_D \equiv \frac{\partial F}{\partial a}. \quad (2.4)$$
The Seiberg-Witten solution is obtained when an elliptic curve $\Sigma_u$ is associated to each value of $u$ in the moduli space; the periods of this curve are required to satisfy

$$
a(u) = \oint_A \lambda,
\quad a_D(u) = \oint_B \lambda,
$$
with $A$ and $B$ cycles on the homology basis, and $\lambda$ a meromorphic 1-form satisfying

$$
\frac{d\lambda}{du} = \frac{dx}{y},
$$

with $\frac{dx}{y}$ is the abelian differential of the curve $\Sigma_u$.

From the BPS mass formula for the particle spectrum,

$$
M_{BPS} = |a n_e + a_D n_m|,
$$

and equation (2.5), we observe that massless particles will appear whenever the homology cycles of $\Sigma_u$ contract to a point.

Classically, solutions $a(u)$ and $a_D(u)$ are given by the Higgs expressions

$$
a(u) = \sqrt{2u},
\quad a_D(u) = \tau_0 \sqrt{2u},
$$

with $\tau_0 = \frac{\theta}{2\pi} + i\frac{4\pi}{g^2}$ the bare coupling constant. The one loop renormalization of the coupling constant leads, in the Higgs phase, to

$$
a(u) = \sqrt{2u},
\quad a_D(u) = 2i\frac{a}{\pi} \log \left( \frac{a}{\Lambda} \right) + \frac{ia}{\pi},
$$

Solution (2.9) is locally correct as a consequence of the holomorphy of the prepotential. The global solution, consistent with holomorphy, is given by the cycles of the elliptic curve

$$
y^2 = (x-u)(x-\Lambda^2)(x+\Lambda^2),
$$

which leads to the existence of a dual phase where $a(u)$ behaves logarithmically, as a consequence of the appearance in the spectrum of a charged massless monopole.

The $Sl(2,\mathbb{Z})$ duality transformations, acting on the vector $(a, a_D)$, became a non perturbative symmetry only if they are part of the monodromy around the singularities. The monodromy subgroup of $Sl(2,\mathbb{Z})$ for the $SU(2)$ case is given by the group $\Gamma_2$ of unimodular matrices, congruent to the identity up to $\mathbb{Z}_2$.

Some points must be stressed, concerning Seiberg-Witten solution:
C-1 Under a soft supersymmetry breaking term, the massless particles appearing at the singularities condense, moving the theory into the confinement (in the case that the condensing particles are monopoles), or the oblique confinement phase (if they are dyons).

C-2 The different phases of the theory, corresponding as mentioned in the above comment to the kind of state giving rise to the singularity, are interchanged by the action of the part of the duality group that is not in the monodromy group,

$$Sl(2, \mathbb{Z})/\Gamma_2.$$  \hfill (2.11)

C-3 The elliptic modulus $\tau_u$ of the curve $\Sigma_u$ is given by

$$\tau = \frac{da_D}{da}.$$  \hfill (2.12)

This modulus becomes 0, 1 or $\infty$ at the singularities.

C-4 The point $u = 0$, corresponding to (classical) enhancement of symmetry, is not a singular point, and therefore no enhancement of symmetry does take place dynamically.

C-5 There exits a global $\mathbb{Z}_2$ R-symmetry acting on the moduli space by

$$u \rightarrow -u.$$  \hfill (2.13)

This transformation is part of $Sl(2, \mathbb{Z})/\Gamma_2$, and it interchanges the monopole and dyon singularities. We can not quotient the moduli space by (2.13).

C-6 The dynamically generated scale $\Lambda$ of the theory is not a free parameter that can be changed at will. If that was the case, we would be able to consider a family of $SU(2)$ theories with different values of $\Lambda$, with the $\Lambda = 0$ theory possessing an enhancement of symmetry singular point. Nevertheless, it is possible to think that once our theory is embedded into string theory, the dilaton can effectively change the value of $\Lambda$, modifying the picture of dynamical enhancement of symmetries. This particular issue will be considered in the rest of this notes.

### 3 $N = 4$ to $N = 2$ Flow.

Let us start considering the algebraic geometrical description of $N = 4$ $SU(2)$ super Yang-Mills [2]. For this theory we can define a complexified coupling constant,

$$\tau = \frac{\theta}{2\pi} + i \frac{4\pi}{g^2},$$  \hfill (3.1)
on which the duality group $SL(2, \mathbb{Z})$ acts in the standard way through
\[
\tau \rightarrow \frac{a \tau + b}{c \tau + d}. \tag{3.2}
\]
As the $\beta$-function for the $N=4$ theory vanishes, we can take the classical solution
\[
a(u) = \sqrt{2u}, \\
a_D(u) = \tau \sqrt{2u}, \tag{3.3}
\]
as exact. The algebraic geometrical description of this solution requires finding, for given values of $\tau$ and $u$, an elliptic curve $E_{(\tau, u)}$ such that
\[
a(u) = \oint_A \lambda, \\
a_D(u) = \oint_B \lambda, \tag{3.4}
\]
where $A$ and $B$ are the two homology cycles, and
\[
\frac{d\lambda}{du} = \frac{dx}{y} = w, \tag{3.5}
\]
with $w$ the abelian differential of the elliptic curve $E_{(\tau, u)}$ (as in the $N=2$ theory of the previous section). This problem can be easily solved using the modular properties of Weierstrass $P$-function. In a factorized form, the solution is given by [2]
\[
y^2 = (x - e_1(\tau)u)(x - e_2(\tau)u)(x - e_3(\tau)u), \tag{3.6}
\]
in terms of the Weierstrass invariants $e_i(\tau)$:
\[
e_1(\tau) = \frac{1}{3}(\theta_2^4(0, \tau) + \theta_3(0, \tau)) = \frac{2}{3} + 16q + 16q^2 + \cdots, \\
e_2(\tau) = -\frac{1}{3}(\theta_2^3(0, \tau) + \theta_3^3(0, \tau)) = -\frac{1}{3} - 8q^{1/2} - 8q - 32q^{3/2} - 8q^2 + \cdots, \\
e_3(\tau) = \frac{1}{3}(\theta_1^4(0, \tau) - \theta_3^2(0, \tau)) = -\frac{1}{3} + 8q^{1/2} - 8q + 32q^{3/2} - 8q^2 + \cdots, \tag{3.7}
\]
where we have used Jacobi’s $\theta$-functions, and $q \equiv e^{2\pi i \tau}$. By an affine transformation, the curve (3.6) can be rewritten as
\[
y^2 = x(x - 1)(x - \lambda(u, \tau)), \tag{3.8}
\]
with
\[
\lambda(u, \tau) = \frac{u(e_3 - e_1)}{u(e_2 - e_1)}. \tag{3.9}
\]
From equation (3.9), we observe that the elliptic modulus of the curve $E_{(\tau, u)}$ is independent of the value of $u = \langle tr\phi^2 \rangle$. Interpreting $\tau$ as the “bare” coupling constant, and the
elliptic modulus of $E_{(\tau,u)}$ as the effective wilsonian coupling constant, the previous fact simply reflects the vanishing of the $\beta$-function.

Using relations (3.7), we can now work the weak coupling limit ($q \to 0$) of (3.6):

$$y^2 = (x - \frac{2}{3}u)(x + \frac{1}{3}u)^2.$$  \hspace{1cm} (3.10)

Redefining $x$ as $(x + \frac{1}{3}u)$, we get

$$y^2 = (x - u)x^2,$$  \hspace{1cm} (3.11)

which is precisely the $\Lambda \to 0$ limit of the Seiberg-Witten curve (2.10).

There already exists strong evidence on the Montonen-Olive duality invariance [22] of the $N = 4$ gauge theory. Two facts contribute to the existence of this symmetry: i) The vanishing of the $\beta$-function, and ii) the correct spin content for electrically and magnetically charged BPS-particles. This second fact depends on the multiplicities of $N = 4$ vector multiplets$^1$. One thing that prevents the existence of exact Montonen-Olive duality in $N = 2$ is the fact that the $W^{\pm}$'s are in $N = 2$ vector multiplets, while the magnetic monopoles are in $N = 2$ hypermultiplets. We can easily make the theory $N = 4$ invariant by simply adding a massless hypermultiplet in the adjoint representation. Naively, we can now have the following physical picture in mind: when a soft breaking term is added for the hypermultiplet in the adjoint, $N = 4$ is broken down to $N = 2$, but still preserving the number of degrees of freedom which are necessary for having duality, namely the number of degrees of freedom of the $N = 4$ theory. If the soft breaking mass term goes to $\infty$, we obtain at low energy the pure $N = 2$ theory. It looks like if in order to have duality for this $N = 2$ theory we should be faced to deal with a set of very massive states which have “a priori” no interpretation in the context of the pure $N = 2$ theory$^2$.

Motivated by the previous discussion, we will work out in some detail the soft breaking $N = 4$ to $N = 2$ flow for the $SU(2)$ gauge theory.

The curve describing the massive case was derived in reference [2], and is given by

$$y^2 = (x - e_1(\tau)\tilde{u} - e_1^2(\tau)f)(x - e_2(\tau)\tilde{u} - e_2^2(\tau)f)(x - e_3(\tau)\tilde{u} - e_3^2(\tau)f),$$  \hspace{1cm} (3.12)

where $f = \frac{1}{4}m^2$, and $\tilde{u}$ is related to $u = <tr\phi^2>$ by the renormalization

$$\tilde{u} = u - \frac{1}{2}e_1(\tau)f.$$  \hspace{1cm} (3.13)

$^1$The multiplicities for massless irreducible representations are:

\begin{itemize}
  \item $N = 2$: (1,2,2) for spin 1 (vector),
  \item (2,4) for spin 1/2 (hypermultiplet);
  \item $N = 4$: (1,4,6) for spin 1.
\end{itemize}

$^2$At this point we can offer an heuristic interpretation in terms of a similar phenomenon. It is well known that, in a theory which is anomaly free, when some chiral fermion, that contributes to the anomaly cancellation, becomes decoupled some extra state appears in the spectrum to maintain the anomaly cancellation right. In our context the anomaly condition is replaced by the duality symmetry, and anomaly matching conditions by some sort of “duality” matching conditions.
The singularities of (3.12) are located at
\[ \tilde{u} = e_1(\tau)f, e_2(\tau)f, e_3(\tau)f. \] (3.14)

### 3.1 Extended Moduli.

In order to present the singularities of the massive curve, it is convenient to introduce an “extended” moduli, parameterized by \( \tilde{u} \) and \( \tau \). In this extended moduli the singularity loci (3.14) have the look depicted in Figure 1.

Figure 1: The singularity loci (3.14); the point A corresponds to \( \tilde{u} = -\frac{1}{3}f \) (\( u = 0 \)).

The figure is only qualitative, pretending mainly to stress the fact that for \( \tau = i\infty \) the singularities \( e_2(\tau)f \) and \( e_3(\tau)f \) coincide, while for \( \tau \to 0 \) the coalescing singularities are the ones defined by the Weierstrass invariants \( e_1(\tau) \) and \( e_2(\tau) \). Notice also that in the massless limit \( f \to 0 \) the curve (3.12) becomes the \( N=4 \) curve (3.6). In contrast to what happens for the \( N=4 \) case, the moduli of the curve (3.12) depends on \( \tau \) and \( u \), a fact that reflects the renormalization of the bare coupling constant \( \tau \). To see the dependence on \( u \) and \( \tau \) of the moduli of (3.12) it is convenient to write the curve in the standard form
\[ y^2 = x(x-1)(x - \lambda(\tilde{u}, \tau, f)), \] (3.15)
with
\[ \lambda(\tilde{u}, \tau, f) = \frac{(e_3(\tau) - e_1(\tau))(\tilde{u} + f(e_3(\tau) + e_1(\tau)))}{(e_2(\tau) - e_1(\tau))(\tilde{u} + f(e_2(\tau) + e_1(\tau)))}. \] (3.16)

It is easy to check that the singularity loci (I,II,III) appearing in Figure 1 correspond respectively to \( \lambda(\tilde{u}, \tau, f) = 1, \infty, 0 \).

A simple minded approach to Figure 1 is to compare it with a fictitious extended moduli for the pure \( N=2 \) theory, parameterized by \( u \) and the dynamically generated
scale $\Lambda$. In this interpretation, the singular loci II and III of Figure 1 will represent the split of the “classical” singularity represented by the point $A$, which corresponds to the point $u = 0$ of enhancement of $SU(2)$ symmetry. However, this interpretation is too naive, and is not taking into account that the pure $N = 2$ theory is only recovered in the $m \to \infty$ limit, with $m$ the mass of the extra hypermultiplet in the adjoint representation.

### 3.2 Extended Moduli and Integrability.

The possibility to encode exact information for $N = 2$ systems in terms of algebraic geometry opens the way to a connection with integrable systems [23, 24, 20]. In particular, the extended moduli space parameterized by $\tilde{u}$ and $\tau$ can also be approached in terms of an integrable system, introduced by Donagi and Witten [20]. Let us briefly review this work.

The integrable model associated with an $N = 2$ gauge theory with gauge group $G$ is defined by a bundle

$$X \to \mathcal{U},$$

with $\dim \mathcal{U} = r = \text{rank} G$, and the fiber $X_{\mathcal{U}}$ the jacobian $\text{Jac}(\Sigma_{\pi})$, where $\Sigma_{\pi}$ is the hyper-elliptic curve of genus $g = r$ solving the model in the Seiberg-Witten sense [25] ($\bar{u} = (u_1, \ldots, u_r)$). A Poisson bracket structure can be defined on $X$ with respect to which the $u_i$ are the set of commuting hamiltonians. Following Hitchin’s work [26], we can model the integrable system (3.17) in terms of a gauge theory with gauge group $G$ and a Higgs field $\Phi$ in the adjoint representation of $G$, defined on a Riemann surface $\Sigma$. In very synthetic terms, given the data $(\Sigma, G, \Phi)$, we define $X$ as the cotangent bundle $T^* \mathcal{M}_{G}^{\Sigma}$, with $\mathcal{M}_{G}^{\Sigma}$ the moduli of $G$-connections defined on the Riemann surface $\Sigma$. The set of commuting hamiltonians has now a very nice geometrical interpretation. For $G = SU(n)$, the gauge invariants $\text{tr}(\phi)^k$ define holomorphic $k$-differentials on the Riemann surface $\Sigma$. Now, we can expand these gauge invariant Casimirs in terms of a basis of $k$-differentials of $\Sigma$. The coefficients will define the set of commuting hamiltonians. There is, in addition, a simple geometrical way, using the concept of ramified coverings (spectral curves), to define on a given Riemann surface $\Sigma$ the gauge theory used above to define the integrable model; the receipt is the following: on a Riemann surface $\Sigma$ we define a holomorphic 1-differential $\Phi$, that will play the role of the Higgs field, valued in the adjoint representation of the gauge group $G$. Given $\Phi$, the spectral cover $C$ of $\Sigma$ is then

$$\det(t - \Phi) = 0.$$ 

(3.18)

This relation can be written (in the $SU(n)$ case) as

$$t^n + t^{n-2}W_2(\phi) + \cdots + W_n(\phi) = 0,$$

(3.19)

with $W_k(\phi)$ gauge invariant holomorphic $k$-differentials on $\Sigma$. As has already been mentioned, the commuting hamiltonians are given by the coefficients of $W_k(\phi)$ with respect
to the basis of holomorphic $k$-differentials of $\Sigma$. Now, the definition on $\Sigma$ of an $SU(n)$ gauge bundle $V$ is rather simple: for any point $w \in \Sigma$, from (3.19) we get $n$ different points $v_i(w)$ in $C$; each of these points is associated to a one dimensional eigenspace of $\Phi(w)$ that we will denote by $L_{v_i(w)}$. As the point $v_i$ in $C$ is moved, it defines a line bundle $L$ on the spectral cover $C$, and a vector bundle $V$ on the Riemann surface $\Sigma$:

$$V = \bigoplus_{i=1}^{n} L_{v_i(w)}. \quad (3.20)$$

However, this vector bundle does not yet define an $SU(n)$ gauge theory on $\Sigma$; in order to do so, an extra condition must be imposed: the line bundle $\det V$ on $\Sigma$ has to be a trivial bundle. It turns out that this condition is equivalent to defining the $X_\pi$ fiber of the corresponding integrable system (3.17) as the kernel of the map from $\text{Jac}(C) \to \text{Jac}(\Sigma)$, defined by the map $L \to N(L)$, with $N(L)$ given by

$$N(L) = \bigotimes_{i=1}^{n} L_{v_i(w)}, \quad (3.21)$$

a line bundle on $\Sigma$ (recall that $L$ was defined as a line bundle on the spectral cover $C$).

For the $SU(2)$ example, the previous discussion can be materialized in simple terms. As reference surface $\Sigma$, we take the $N=4$ solution

$$E : \quad y^2 = (x - e_1(\tau))(x - e_2(\tau))(x - e_3(\tau)). \quad (3.22)$$

Then, the spectral cover is defined by (3.22) and

$$0 = t^2 - x + \tilde{u}. \quad (3.23)$$

This spectral cover is a genus two Riemann surface that we are parameterizing by the $\tau$ moduli of (3.22), and the $\tilde{u}$ parameter in (3.23), i.e., by the extended moduli we are considering in this notes. The $SU(2)$ curve can now be obtained projecting out in $\text{Jac}(C)$ the part coming from $\text{Jac}(E)$, using the $\mathbb{Z}_2 \times \mathbb{Z}_2$ automorphisms of the system defined by (3.22) and (3.23): $\alpha : t \to -t$, $\beta : y \to -y$. As the abelian differential of $E$ is given by $\frac{dx}{y}$, the part of $\text{Jac}(C)$ not coming from $E$ is precisely the $\alpha\beta \mathbb{Z}_2 \times \mathbb{Z}_2$-invariant part.

After this brief review of Donagi-Witten theory we would like to address the attention of the reader to a very simple physical interpretation of the role played by $\tau$ (the $N=4$ moduli). What we are going to compare to the vacuum expectation values parameterizing the Coulomb phase, the coordinates of a point $\varpi$, are the coefficients defined in the expansion of the $k$-differentials $\text{tr} \phi^k$ (with $\phi$ the Higgs field of the auxiliary gauge theory) in the basis of the $k$-differentials of the reference Riemann surface $\Sigma$. For $SU(2)$, and with $\Sigma = E$, we get a relation of the type

$$\text{tr} \phi^2 = u\Phi_\tau^{(2)}, \quad (3.24)$$
with $\Phi^{(2)}_\tau$ the quadratic differential on $E$ for the moduli value $\tau$. The physical way to understand (3.24) is that we are measuring the value of $\text{tr}\phi^2$ with respect to the “scale” defined in terms of the moduli $\tau$ of the $E$ curve. This comment will be important later on, and will be at the basis of the use of the Donagi-Witten framework to define a stringy representation of the extended moduli. In fact, if we think of $\tau$ as defining the scale we use to measure $\text{tr}\phi^2$, it is natural to think on some relation between $\tau$ and the dilaton field.

### 3.3 The Soft Supersymmetry Breaking Scale.

Let us now use $f = \frac{1}{4} m^2$ as the unit scale to measure $u = \langle \text{tr}\phi^2 \rangle$. In order to do so, we define the dimensionless quantity

$$\hat{u} \equiv \frac{u}{f}. \quad (3.25)$$

The first thing to be noticed is that in the pure $N=2$ decoupling limit, $f \to \infty$, the whole Seiberg-Witten plane $u$ is “blown down” to the point $\hat{u} = 0$, i.e., to the point of enhancement of symmetry. Moreover, using again (3.25) we can consider that the point at $\infty$ of the Seiberg-Witten plane is “blown up” into the line $\hat{u}$ minus the origin.

Moreover, the scale $\Lambda^2$ is given by $2q^{1/2}m^2$, and therefore in the decoupling limit $m \to \infty$ we can only get a finite scale by means of a double scaling limit:

$$\Lambda^2 = \lim_{q \to 0} \lim_{m \to \infty} 2q^{1/2}m^2, \quad (3.26)$$

i.e., in the weak coupling limit $q \to 0$. Thus, if we work in the $(\hat{u}, \tau)$-plane, the whole Seiberg-Witten $N=2$ physics is condensed at the singular (enhancement of symmetry) point $(\hat{u} = 0, \tau = i\infty)$. However, it is important to stress that we can think of the $(\hat{u}, \tau)$-plane without imposing any restriction on the value of $f$. In this plane it is only the singular point $(\hat{u} = 0, \tau = i\infty)$ that admits a pure $N=2 \text{SU}(2)$ gauge theory interpretation; all other points can only be interpreted in terms of the richer theory containing the hypermultiplet in the adjoint representation. At this point of the discussion we can face two different questions:

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3Even though it was not explicitly mentioned, the mass term breaking $N=4$ to $N=2$ already enters Donagi-Witten construction. In fact, the Higgs field $\Phi$ has a pole with residue given by $\left( \begin{array}{c} 1 \\ 1 \\ \vdots \\ -(n-1) \end{array} \right)$ for $\text{SU}(n)$.

4Morally speaking, when we measure in units of $f$ and we take $f \to \infty$ the theory becomes effectively scale invariant.
i) How to get the $N = 2$ Seiberg-Witten solution from just the singular point $(\hat{u} = 0, \tau = i\infty)$ of the $(\hat{u}, \tau)$-plane.

ii) How to interpret in terms of duality the rest of the $(\hat{u}, \tau)$-plane.

Let us start by considering for the time being, and in very qualitative terms, the second of the above interrogants. The action of duality on the $(\hat{u}, \tau)$-plane can be properly defined by the conditions on the moduli

$$ S : 1 - \lambda(\hat{u}, \tau) = \lambda(\hat{u}', -\frac{1}{\tau}), $$

$$ T : \frac{1}{\lambda(\hat{u}, \tau)} = \lambda(\hat{u}'', \tau + 1), $$

(3.27)

defining the $S$ and $T$ action on the $(\hat{u}, \tau)$-plane respectively by $S : (\hat{u}, \tau) \rightarrow (\hat{u}', -\frac{1}{\tau})$, $T : (\hat{u}, \tau) \rightarrow (\hat{u}'', \tau + 1)$. It is easy to verify from (3.16), (3.25), and the redefinition $\tilde{u} = u - \frac{1}{2}e_1(\tau)f$ that

$$ \hat{u}' = \tau^2(\hat{u} + \frac{1}{2}(e_2 - e_1)f), $$

$$ \hat{u}'' = \hat{u}. $$

(3.28)

To derive (3.28) we have only used the modular properties of the Weierstrass invariants $e_i(\tau)$, and the fact that

$$ 1 - \lambda(\tilde{u}, \tau) = \lambda(\tilde{u}^M, -\frac{1}{\tau}), $$

$$ \tilde{u}^M = \tau^2\tilde{u}, $$

(3.29)

that is, $\tilde{u}$ is a modular form of weight two. Once we have implemented the action of the duality group $SL(2, Z)$ on the $(\hat{u}, \tau)$-plane we observe that the $S$-duality transformations move the pure $N=2$ $SU(2)$ point $A$ into a point with no pure $N=2$ interpretation. This is reminiscent to what happens in string theory, where $R - R$ states with no world sheet interpretation are necessary to implement duality, and where these states are light only in the strong coupling regime, in this case defined by the dilaton. In summary, and as a temptative answer to the second question posed above, we can say that the $(\hat{u}, \tau)$-plane beyond the $N = 2$ point $(\hat{u} = 0, \tau = i\infty)$ is necessary to implement duality. But before entering into a more concrete discussion of the previous argument, let us go first to discuss the first question.

### 3.4 The Blow-up Microscope.

In the previous section we have concentrated the whole $N = 2$ Seiberg-Witten solution into the singular point $A (\hat{u} = 0, \tau = i\infty)$ of the $(\hat{u}, \tau)$-plane. Now, we face the problem to
unravel, just from one point, the whole $N=2$ quantum moduli. By the previous discussion
we know that the $\hat{u} \neq 0$ has no interpretation in terms of pure $N=2 SU(2)$ gauge theory;
therefore, to recover Seiberg-Witten quantum moduli we will need to add some extra
direction parameterized by the pure $N=2$ Seiberg-Witten moduli $u = < tr \phi^2 >$. Nicely
enough, we have a natural way to create this “extra” dimension, using simply the fact
that the point $A$ in the $(\hat{u}, \tau)$-plane is a singular point: this singularity can be blown-up.
To see how this can be done, and get a certain familiarity with the blow-up procedure, let
us first present some simple examples, referring to [27] for a more exhaustive treatment.

i) Blowing up a crossing.

Let us consider the crossing of the lines ($y = x$) and ($y = 0$). To resolve (blow up)
this crossing we define a new variable $v$ as $v \equiv y/x$, obtaining
\[ v = 1. \tag{3.30} \]

The coordinate $v$ is a coordinate parameterizing an exceptional divisor $E$ which is intro-
duced through the blow up. The result can be represented as in Figure 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{fig2.png}
\caption{The (single) blow up of the crossing between $y = x$ and $y = 0$.}
\end{figure}

ii) Blowing up a tangency.

We consider now the tangency point $(0,0)$ between the parabola $y = x^2$ and the line
$y = 0$. To blow this tangency up will require a procedure in two steps. In the first, the
variable $v$ is introduced by the definition
\[ v \equiv y/x, \tag{3.31} \]

\footnote{Let’s recall that the parameter $\lambda(\hat{u}, \tau)$ is ill defined at $A$.}
and we obtain a crossing in the \((v, x)\)-plane,

\[ v = x. \]  

(3.32)

To blow up this crossing we repeat the discussion in the above paragraph, introducing a new variable \(w\),

\[ w = v/x, \]  

(3.33)

so that the original parabola turns into \(w = 1\). This two stepped process has introduced two exceptional divisors \(E_1\) and \(E_2\), parameterized respectively by \(v\) and \(w\); the pictorial representation is that depicted in Figure 3.

![Figure 3: The double blow up of the tangency point between the parabola \(y = x^2\) and \(y = 0\).](image)

Armed with this artillery we can now blow up the point \(A\) in the \((\hat{u}, \tau)\)-plane. In the neighborhood of this point, the loci II and III are given by

\[
\begin{align*}
II & : \quad \hat{u}(\tau) = 8q^{1/2}, \\
III & : \quad \hat{u}(\tau) = -8q^{1/2}.
\end{align*}
\]  

(3.34)

We can now consider, instead of the \(\tau\)-coordinate, a coordinate \(\epsilon\) defined by

\[ \epsilon \equiv 8q^{1/2}. \]  

(3.35)

Using now the blow up of a crossing, as described in i) above, we obtain an exceptional divisor \(E\) and the picture of Figure 2.

Now we can try, following the reasoning at the beginning of this section, to identify the exceptional divisor \(E\) with the Seiberg-Witten quantum moduli for \(SU(2)\). This is now rather simple using the relation

\[ \Lambda^2 = 8q^{1/2}f, \]  

(3.36)
Figure 4: The double blow up of the tangency point $A$ in the $(\hat{u}, \tau)$-plane.

with $f = \frac{1}{4} m^2$. In this way, we obtain that the divisor $E$ is parameterized by

$$\Lambda^2 / u,$$

where now $u$ can be identified with the Seiberg-Witten quantum moduli parameter, with the points $C$ and $B$ in Figure 4 corresponding to the two singularities at $u = \pm \Lambda^2$, and the point $A$ to $u = \infty$.

### 3.5 Double Covering and $R$-Symmetry.

The action of the $Sl(2, \mathbb{Z})$ modular group on the $(\hat{u}, \tau)$-plane was defined by equations (3.27) and (3.28). Let us now consider the effect of the generator $T$,

$$T : (\hat{u}, \tau) \rightarrow (\hat{u}, \tau + 1)$$

on the coordinate $\epsilon / \hat{u}$ of the exceptional divisor $E$ introduced by the blow up; obviously, the action of $T$ induces on $E$ the global $\mathbb{Z}_2$ transformation

$$\epsilon / \hat{u} \rightarrow -\epsilon / \hat{u},$$

as $\epsilon = 8q^{3/2} = 8e^{\pi i \tau}$. If we now identify, as was proposed above, the $E$ divisor with the Seiberg-Witten quantum moduli space for the $N=2 SU(2)$ theory, then the transformation (3.39) will become the $\mathbb{Z}_2$ global $R$-symmetry $u \rightarrow -u$.

To quotient by transformation (3.38) is equivalent to using coordinates $(\epsilon^2, \hat{u})$, instead of $(\epsilon, \hat{u})$. In these new coordinates, the loci II and III of Figure 1 can be described, in the neighborhood of the singular point $(\hat{u} = 0, \tau = i \infty)$, by the parabola $\hat{u}^2 = \epsilon^2$. The tangency can now be blown up through the two stepped process described above. The two exceptional divisors and coordinates are shown in Figure 5.
Figure 5: Exceptional divisors and coordinates arising from the blow up of the tangency point of the parabola $\hat{u}^2 = \epsilon^2$ with the line $\epsilon = 0$.

Using again (3.36) we get on $E_2$ the dimensionless coordinate

$$\Lambda^4/u^2.$$  \hfill (3.40)

In Figure 5 the point $B$ corresponds to $\Lambda^4 = 1$, the point $A$ to $u = 0$, and the point $C$ to $u = \infty$. The extra divisor at the origin, $E_1$, appears because we have quotiented by transformation (3.39). The important point to be stressed here is that the quotient by the global $\mathbb{Z}_2$ $R$-symmetry $u \rightarrow -u$ is not allowed in the pure $N=2$ $SU(2)$ gauge theory. Hence, at this level, the quotient by the action of $T$ is only a formal manipulation. Its real physical meaning will depend on our way to understand the $T$ transformation (3.38) as a real symmetry of the theory.

### 3.6 Singular Loci.

The singular loci, in the $(\hat{u}, \tau)$-plane, for the curve (3.12) are given by

\begin{align}
\hat{C}_\infty & \equiv \{ \tau = i\infty / \epsilon = 0 \}, \\
\hat{C}_0 & \equiv \{ \hat{u}(\tau) = \frac{3}{2}e_1(\tau) \}, \\
\hat{C}_C^{(1)} & \equiv \{ \hat{u}(\tau) = e_3(\tau) + \frac{1}{2}e_1(\tau) \}, \\
\hat{C}_C^{(2)} & \equiv \{ \hat{u}(\tau) = e_2(\tau) + \frac{1}{2}e_1(\tau) \}, \\
\hat{C}_1^+ & \equiv \{ \tau = 0 / \epsilon = 1 \}, \\
\hat{C}_1^- & \equiv \{ \tau = 1 / \epsilon = -1 \},
\end{align}  \hfill (3.41)

In the above set, two different types of singular loci can be distinguished: $\hat{C}_0$, $\hat{C}_C^{(1)}$ and $\hat{C}_C^{(2)}$, describing singularities of the massive curve of the $N=2$ theory with $N=4$ matter content, and the loci $\hat{C}_\infty$, $\hat{C}_1^+$ and $\hat{C}_1^-$, related only to the value of $\tau$.  

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Some sort of “duality” between this two types of loci can already be noticed through the $T$-transformation (3.38), as (3.38) interchanges $\hat{C}_1^+$ with $\hat{C}_1^-$, and $\hat{C}_C^{(1)}$ with $\hat{C}_C^{(2)}$. Moreover, by (3.17) $\hat{C}_0$ and $\hat{C}_\infty$ are kept fixed.

4 String Theory Framework.

4.1 The Moduli of Complex Structures.

For Calabi-Yau practitioners the formal similarities between the loci (3.41) and the singular loci of complex structures of $\mathbb{P}_{11226}$ or of $\mathbb{P}_{11222}$ would become immediately clear [28, 29]. We will concentrate our discussion in $\mathbb{P}_{11226}$.

The mirror of this weighted projective Calabi-Yau space $\mathbb{P}_{11226}$ is defined by the polynomial

$$ p = z_1^{12} + z_2^{12} + z_3^6 + z_4^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^2 z_2^2. \tag{4.1} $$

The moduli of complex structures, parameterized by $\psi$ and $\phi$, presents a $\mathbb{Z}_{12}$ global symmetry $A$:

$$
\begin{align*}
\psi & \rightarrow \alpha \psi, \\
\phi & \rightarrow -\phi,
\end{align*}
\tag{4.2}
$$

where $\alpha$ is such that $\alpha^{12} = 1$. The singular loci in the compactified moduli space are given by

$$
\begin{align*}
\mathcal{C}_{con} & \equiv \{864\psi^6 + \phi = \pm 1\}, \\
\mathcal{C}_1 & \equiv \{\phi = \pm 1\}, \\
\mathcal{C}_\infty & \equiv \{\phi, \psi = \infty\}, \\
\mathcal{C}_0 & \equiv \{\psi = 0\}. \tag{4.3}
\end{align*}
$$

The locus $\mathcal{C}_0$ appears as the result of quotienting by the $\mathbb{Z}_{12}$ symmetry (4.2). In fact, $g^2$ fixes the whole line $\psi = 0$. Defining the $A$-invariant quantities

$$
\xi \equiv \psi^{12}, \quad \eta \equiv \psi^6 \phi, \quad \zeta \equiv \phi^2, \tag{4.4}
$$

the moduli space can be defined as the affine cone in $\mathbb{C}^3$ given by the relation

$$
\xi \zeta = \eta^2. \tag{4.5}
$$

To compactify this space we can embed $\mathbb{C}^3$ into $\mathbb{P}^3$, with homogeneous coordinates $[\hat{\xi}, \hat{\eta}, \hat{\zeta}, \hat{\tau}]$, in such a way that $\xi = \hat{\xi}/\hat{\tau}$, $\eta = \hat{\eta}/\hat{\tau}$, $\zeta = \hat{\zeta}/\hat{\tau}$. Now, the compactified moduli is defined by the projective cone $\mathcal{Q}$, defined as

$$
\hat{\xi} \hat{\zeta} = \hat{\eta}^2, \tag{4.6}
$$

**Strictly, the mirror manifold is defined by $\{p = 0\}/G$, where $G$ is the group of reparametrization symmetries of $p$ [28].**
and the loci (4.3) become

\[ C_{\text{con}} \equiv \mathcal{Q} \cap \{(864)^2 \hat{\xi} + \hat{\zeta} + 1728 \hat{\eta} - \hat{\tau} = 0\}, \]
\[ C_1 \equiv \mathcal{Q} \cap \{\hat{\eta} - \hat{\tau} = 0\}, \]
\[ C_{\infty} \equiv \mathcal{Q} \cap \{\hat{\tau} = 0\}, \]
\[ C_0 \equiv \mathcal{Q} \cap \{\hat{\xi} = \hat{\eta} = 0\}. \]  

(4.7)

Notice for instance that in the compactified moduli the loci \( C_1 \) and \( C_{\infty} \) meet at the point \([\hat{\xi} = 1, \hat{\eta} = 0, \hat{\zeta} = 0, \hat{\tau} = 0]\) of \( \mathbb{P}^3 \).

The toric representation (see Appendix) of the projective cone \( \mathcal{Q} \) is given in Figure 6.

![Toric diagram of the projective cone \( \mathcal{Q} \), containing the three different coordinate patches.](image)

For the coordinates in chart I we introduce the variables

\[ \frac{1}{x} = -\frac{864 \psi^6}{\phi}, \quad y = \frac{1}{\phi^2}. \]  

(4.8)

In this chart the four singularity loci are those depicted in Figure 7.

When passing to the coordinate chart II, parametrized by coordinates \((x, x^2 y)\), the four singularity loci will appear in a different fashion, as shown in Figure 8.

One should notice that in this chart the locus \( C_1 \) is tangent, at the origin, to the locus \( C_{\infty} \). This tangency can therefore be blown up in the usual way (see Figure 9).

It is clarifying to point out that the blow up of Figure 9 can be constructed in a toric way, giving rise to the toric vectors shown in Figure 10.

4.2 Back to the \((\hat{u}, \tau)\)-plane.

Inspired by the previous discussion, let us come back to the \((\hat{u}, \tau)\)-plane. We will introduce a new variable \( \check{u} \) defined by

\[ \check{u} \equiv \hat{u}/\epsilon. \]  

(4.9)
Figure 7: Singular loci of the compactified moduli space as seen in coordinate chart I.

Figure 8: Singular loci of the compactified moduli space as seen in coordinate chart II.
Figure 9: Blow up of the tangency point of $C_1$ to $C_\infty$.

Figure 10: Toric diagram of the blow up shown in Figure 9. The divisors $E_1$ and $E_2$ correspond, respectively, to the toric diagram vectors $(-1, -1)$ and $(0, -1)$. 
Now, let us quotient by the global transformation

\[
\tilde{u} \rightarrow -\tilde{u}, \\
\epsilon \rightarrow -\epsilon,
\]

which is nothing but the $T$ transformation defined in (3.38). Following the same steps as for the Calabi-Yau moduli space, we introduce the variables

\[
\xi \equiv \tilde{u}^2, \quad \eta \equiv \frac{\tilde{u}}{\epsilon}, \quad \zeta \equiv \frac{1}{\epsilon^2},
\]

satisfying again just the same sort of relation,

\[
\xi \zeta = \eta^2.
\]

Now, we embed the cone (4.12) into $\mathbb{P}^3$, using homogeneous coordinates $[\hat{\xi}, \hat{\eta}, \hat{\zeta}, \hat{\tau}]$. It is now obvious that after this compactification the loci $C_\infty = \{\tau = i\infty\}$ and $C_1 = \{\tau = 0\}$ become

\[
C_1 \equiv Q\cap\{\hat{\eta} - \hat{\tau} = 0\}, \\
C_\infty \equiv Q\cap\{\hat{\tau} = 0\},
\]

with $Q$ the projective cone defined by $\hat{\xi} \hat{\zeta} = \hat{\eta}^2$. In this approach, coordinates in chart I are $\frac{1}{\zeta} = \epsilon^2$, and $\frac{\eta}{\zeta} = \hat{u}$. In chart II, the coordinates will be $(\xi = \frac{1}{\hat{u}}, \frac{\zeta}{\hat{\eta}} = \epsilon^2)$; obviously, in this chart we notice that $C_1 = \{\epsilon^2 = 1\}$ is tangent to $C_\infty$ at the origin, i.e., for $\{\hat{u} = \infty\}$. If we blow this tangency up we get two exceptional divisor $E_1$ and $E_2$ parameterized respectively by $\epsilon^2$ and $\epsilon^2/\hat{u}$.

Notice now that the extended moduli space we started with, namely the $(\hat{u}, \epsilon)$-plane, is a double covering of the chart I of the compactified moduli space defined by the projective cone $Q : \hat{\xi} \hat{\zeta} = \hat{\eta}^2$ for $\xi$, $\eta$ and $\zeta$ as given in equation (4.11).

If we now consider the double cover of chart II, as defined by coordinates $(1/\hat{u}, \epsilon/\hat{u})$, we will get a crossing between $C_1^\pm$ and $C_\infty$. By blowing this crossing up, we get only one exceptional divisor $E$, this time parameterized by $\epsilon$. In the $(\hat{u}, \epsilon)$-plane, this exceptional divisor can be identified with the line $\hat{u} = \infty$.

At this point, we should come back to our discussion in section 3.2. The Donagi-Witten integrable model framework is a natural set up for interpreting the $\tilde{u}$ variable introduced in (4.9). In fact, when we change the value of the $\tau$ moduli of $E$ in equation (3.24), we are effectively changing the value of $\text{tr}\phi^2$ as it is measured in the new unit.
obtained by varying the $\tau$ moduli; this is the same as the phenomena we have in equation (4.9) when we change the value of $\epsilon$ with fixed value of $\hat{u}$. In fact, in the $(\hat{u}, \epsilon)$ variables used in the compactification of the moduli, the lines of fixed $\hat{u}$ in the $(\hat{u}, \epsilon)$-plane are parameterized by $\hat{u}$. Using $\hat{u}$ as the dimensionless parameter, we have two candidates for Seiberg-Witten moduli: the divisor obtained from the blow up of the tangency point at $\hat{u} = 0$, $\tau = i\infty$, and any line of fixed $\hat{u}$, in which $\hat{u}$ changes depending on the value of $\tau$ ($\epsilon$); are both lines candidates to $SU(2)$ theory, and, if so, what is the (different) physics they are representing? In the last part of these notes, we will argue that both lines describe an $SU(2)$ theory. However, before that discussion we will proceed to compare the two moduli spaces more carefully.

4.3 Comparing Blow Ups.

Let us now compare the blow up in Figure 5 and the blow up of the tangency point $(x = 1, y = 0)$ between $C_{\text{con}}$ and $C_{\infty}$. In the second case the parabola is defined by

$$y = \left(\frac{1-x}{x}\right)^2,$$

and the blow up is that in Figure 11,

\[
\begin{array}{c}
\frac{yx^2}{(1-x)^2} \\
\frac{yx}{(1-x)} \\
\frac{yx^2}{(1-x)^2} = 1 \\
(1-x)/x
\end{array}
\]

Figure 11: Blow up of the tangency point of $C_{\text{con}}$ with $C_{\infty}$.

Identifying the two blow ups, we get the following relation:

$$\frac{yx^2}{(1-x)^2} = \frac{\epsilon^2}{\hat{u}^2}.$$  \hfill (4.15)

From (4.15), we get the identifications

$$y = \epsilon^2$$ \hfill (4.16) \\
$$\hat{u} = \pm \frac{1-x}{x}.$$ \hfill (4.17)
As it can be seen from equation (4.15) the blow up of the conifold singularity produces an exceptional divisor parameterized by $1/\tilde{u}^2$, which can only be related to the Seiberg-Witten moduli plane if we first perform the quotient by the $\mathbb{Z}_2$ $R$-symmetry

$$\tilde{u} \rightarrow -\tilde{u}. \quad (4.18)$$

When we considered the same problem from the point of view of the $(\hat{u}, \epsilon)$-plane, we notice that transformation (4.18) was induced by the $T$ action $(\hat{u}, \tau) \rightarrow (\hat{u}, \tau + 1)$, which does not modify the vacuum expectation value $\hat{u}$. We then conclude that the $T$ transformation $(\hat{u}, \epsilon) \rightarrow (\hat{u}, -\epsilon)$, having nothing to do with a $\mathbb{Z}_2$ $R$-symmetry, induces, on the $\tilde{u}$ variable obtained from the blow up, a $\mathbb{Z}_2$ $R$-symmetry (which is the global transformation remaining after taking into account instantonic effects on the $SU(2)$ theory). In a simple scheme,

<table>
<thead>
<tr>
<th>$(\hat{u}, \tau)$</th>
<th>$\rightarrow$</th>
<th>$\hat{u}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\hat{u}, \tau + 1)$</td>
<td><strong>blow up</strong></td>
<td>$\mathbb{Z}_2$ $R$-symmetry</td>
</tr>
</tbody>
</table>

| variable | $\rightarrow$ | $-\tilde{u}$ |

It must also be noticed that the blow up in chart II of the tangency between $C_1$ and $C_\infty$ introduces the exceptional divisor $E_2$ (see Figure 9), parameterized by $y$, i.e., $E_2 = \{x = 0\}$. From (4.17) we see that $x = 0$ corresponds to $\hat{u} = \infty$, in agreement with our discussion in section 4.2, concerning the tangency between $\hat{C}_1$ and $\hat{C}_\infty$ in the $(\hat{u}, \epsilon)$-plane compactified by quotienting by (4.10). In order to get some insight on the physical meaning of these identifications, we will first need to comment on the concept of heterotic-type II dual pairs.

### 4.4 Heterotic-Type II Dual Pairs.

In what follows, a very succinct definition of heterotic-type II dual pairs will be presented, concentrating in the part of the string moduli space associated to vector excitations.

A heterotic-type II dual pair is given by a pair $(X,Y)$ of Calabi-Yau manifolds, such that the quantum moduli of an heterotic string compactified on $X$ coincides with the classical moduli of Kähler structures of $Y$. The necessary condition for having a dual pair is

$$b_{1,1}(Y) = b_{2,1}(Y^*) = V,$$  \hspace{1cm} (4.19)

where $b_{1,1}(Y)$ and $b_{2,1}(Y^*)$ denote Hodge numbers of the manifold $Y$ and its mirror\(^7\) $Y^*$ respectively, and where $V$ stands for the number of massless vectors for the heterotic

\(^7\)Two Calabi-Yau spaces $Y$ and $Y^*$ are said to constitute a mirror pair if they correspond to the same conformal field theory, i.e., when taken as target spaces for two dimensional nonlinear $\sigma$-models they give rise to isomorphic $N = 2$ superconformal field theories; this isomorphism maps the complex structure moduli of $Y$ to the Kähler moduli of $Y^*$, and vice versa. A necessary condition for $Y$ and $Y^*$ to be a mirror pair is that their Hodge numbers satisfy $b_{p,q}^Y = b_{d-p,q}^{Y^*}$. 

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string compactified on $X^8$. We are not taking into account in (4.19) the graviphoton as an extra massless vector, while the vector superpartner of the dilaton its included.

In more precise terms, the idea of heterotic-type II dual pairs can be introduced using prepotentials. Let us denote by $F_{\text{het}}(T_i, S)$ the prepotential at tree level of the heterotic string compactified on the manifold $X$, with $T_i$, $i = 1, \ldots, 4$, representing the scalar component of the massless vector fields, and $S$ the dilaton. In the same way as for the rigid case, presented in section 2, this prepotential will become quantum mechanically corrected by one loop perturbative effects, and by non perturbative corrections:

$$F_{\text{het}}(T_i, S) = F_{\text{het}}^0(T_i, S) + F_{\text{het}}^{\text{one loop}}(T_i) + F_{\text{het}}^{\text{non-perturbative}}(T_i, S). \quad (4.20)$$

Let us denote by $F_{\text{II}}(t_j)$, $(j = 1, \ldots, n+1)$, the prepotential characterizing the special Kähler geometry of the moduli space of complex structures of some Calabi-Yau space $Y$. The pair $(X, Y)$ of Calabi-Yau manifolds will define a heterotic-type II dual pair if there exits a map $(T_i, S) \rightarrow (t_j)$ such that $F_{\text{II}}(t_j) = F_{\text{het}}(T_i, S)$.

The main difficulty in finding heterotic-type II dual pairs or, in other words, finding the heterotic dual version of a type II string compactified on some Calabi-Yau manifold $Y$, is to discover which variable $t_j$ corresponds to the heterotic dilaton $S$. A partial solution to this question comes from requiring $Y$ to be a $K_3$-fibration.

Let us assume we have a type II_A string compactified on $Y$ (type II_B on its mirror image $Y^\ast$). Then, in the large radius limit we can characterize the corresponding point in the moduli $\mathcal{M}$ of Kähler structures (dim $\mathcal{M} = b_{1.1}(Y) = b_{2,1}(Y^\ast)$) by a Kähler form in $H_2(Y)$:

$$B + iJ = \sum (B + iJ)_j e_j \equiv \sum t_j e_j. \quad (4.21)$$

(On this paragraph we are following presentation in reference [30]). To each element $e_j$ in $H_2(Y)$ we can associate an element in the dual $H_4(Y)$, i.e., a divisor $D_j$ of complex codimension one. The holomorphic prepotential for the type II_A theory is then given by

$$F_{\text{II}} = -\frac{i}{6} \sum_{j_1, j_2, j_3} (D_{j_1} \wedge D_{j_2} \wedge D_{j_3}) t_{j_1} t_{j_2} t_{j_3} + \cdots, \quad (4.22)$$

where the dots stand for instanton corrections, and $(D_{j_1} \wedge D_{j_2} \wedge D_{j_3})$ is the intersection product, as homology classes, of these divisors. Now, we can try to discover which particular divisor $D_S$ corresponds to the heterotic dilaton. From Peccei-Quinn symmetry we can fix the tree level part of the heterotic prepotential,

$$F_{\text{het}}^0(T_i, S) = \sum SC_{jk} T_j T_k. \quad (4.23)$$

Comparing (4.22) and (4.23) we get the following constraints on the “dilaton” divisor $D_S$:

$$D_S \wedge D_S \wedge D_S = 0,$$

$$D_S \wedge D_S \wedge D_j = 0, \forall j. \quad (4.24)$$

\[8\]An analogous condition to (4.19) should be satisfied for the part of the moduli space coming from massless neutral hypermultiplets.
From the second of the above constraints, we notice that
\[ D_S \wedge D_S = 0. \] (4.25)

Two more extra constraints can be derived for \( D_S \):
\[ e_S \wedge C \geq 0, \] (4.26)

for any algebraic curve \( C \), and
\[ D_S.c_2(X) = 24. \] (4.27)

Condition (4.26) implies that if the Kähler form \( J \) is in the Kähler cone, then \( J' = J + \lambda e_S \) is also in the Kähler cone. Condition (4.27) is more involved, and takes into account higher derivative terms.

The previous set of conditions on \( D_S \) implies that \( Y \) should be a \( K_3 \)-fibration with base space \( \mathbb{P}^1 \), and fiber a \( K_3 \)-surface. In particular, it also means that the heterotic dilaton has a geometrical interpretation as the size of the fiber \( \mathbb{P}^1 \). Once we have discovered the coordinate \( t_j \) for the dilaton, we can use the inverse mirror map to define the relation between the heterotic dilaton and the complex structure moduli of the mirror manifold.

Let us consider in particular the example proposed by Kachru and Vafa [8]. The Calabi-Yau space is the quintic in \( \mathbb{P}^{11226} \), and its mirror \( Y^* \) is the one defined by the polynomial (4.1). Using the results in Appendix A.5 (see equation (A.40)), we get the relation
\[ S = \frac{1}{2\pi i} \log \left( \frac{2 - z - 2\sqrt{1 - z}}{z} \right), \] (4.28)
with \( z \) defined in (A.37),
\[ z = \frac{1}{\phi^2}. \] (4.29)

Expanding (4.28) around \( z = 0 \), i. e., in the weak coupling limit \( \phi \to \infty \), we get
\[ S = \frac{1}{2\pi i} \log \left( \frac{z}{4} \right) \] (4.30)
which, in terms of the variable \( y \) defined in (4.8), becomes the Kachru-Vafa relation [8]
\[ y \simeq e^{-2\pi iS}. \] (4.31)

Combining now this equation with that from (4.17), obtained by comparing the blow up around the conifold point, we get the stringy interpretation of the \( N = 4 \) parameter \( \tau \) in terms of the dual heterotic dilaton.

---

9More precisely, from conditions (4.24), (4.26) and (4.27) it is possible to conclude that \( D_S \), as an element in \( H_4 \), is given by the \( K_3 \)-surface itself, and thus the corresponding cohomology is given by the size of the basis \( \mathbb{P}^1 \).
This identification sends some light on our previous discussion on Donagi-Witten theory. In fact, what we identify with the heterotic dilaton is the moduli of the reference curve $E_\tau$, in terms of which we were “measuring” the two dimensional Higgs gauge invariant quantity $\text{tr}\phi^2$ (see equation (3.24)).

Once we have identified the moduli corresponding to the heterotic dilaton, we can, in the two moduli case, identify the heterotic $T$ with the second generator of $H_2(Y)$. Using again the inverse mirror map we get, for $Y = \mathbb{P}_{\{11226\}},$

$$x = \frac{1728}{j(T)} + \cdots$$

in the large $S$ limit. From (4.32), we can identify the point $(x = 1, y = 0)$, where $C_{con}$ is tangent to $C_\infty$, as the corresponding to the enhancement of symmetry at $T = i$: $U(1) \to SU(2)$ for the heterotic string compactified on $T^2 \times K_3$. In our comparison between the $(x, y)$-plane and the $(\hat{u}, \tau)$-plane of the $N=2$ theory with $N=4$ matter content, this point of enhancement of symmetry corresponds to $(\tau = i\infty, \hat{u} = 0)$. Moreover, the quantum moduli space for $N=2$ $SU(2)$ Yang-Mills theory can be exactly recovered in the point particle limit of the string by blowing up the tangency point $(x = 1, y = 0)$ (see Figure 11) and identifying the second divisor introduced by the blow up $E_2$ with the Seiberg-Witten moduli space according to (4.15) [15].

4.5 Non Perturbative Enhancement of Gauge Symmetry.

A natural question we can immediately make ourselves is whether there exists any other point in the $(x, y)$-plane, not necessarily in the perturbative region $y \to 0$, which can be interpreted in terms of an enhancement of symmetry. Following the general philosophy we have learned in the Seiberg-Witten analysis of the rigid theory, we should expect this thing to take place whenever an electrically charged particle becomes massless. In the case of the special geometry of the $(x, y)$ moduli, there are two special coordinates $t_1$ and $t_2$. They are associated with the two $U(1)$ vector fields corresponding to $b_{1,1}(\mathbb{P}_{\{11226\}}) = 2$. Moreover, from the BPS mass formula, we can get massless electrically charged particles with respect to each of these $U(1)$’s at points where some of these variables vanish. In particular, for $(x = 0, y = 1)$ we have, from (4.32), that $t_2 = \frac{T-i}{T+i}$ becomes zero. Now, a question arises: what about points where $t_1$ (the variable we have associated to the dilaton) vanishes?

However, before trying to answer this question, it is worth to point out that on the locus $C_1 = \{\phi^2 = 1\}$ we have, from (4.28),

$$t_1 = 0,$$

so the candidate to look for an enhancement of gauge symmetry is the locus $C_1$. We stress the fact that it is the $U(1)$ factor corresponding to the dilaton the one that becomes $SU(2)$.
Using the results stated in A.4 and A.6, we can qualitatively understand this enhancement of symmetry. To do so, we must first recall that the discriminant locus $\varphi^2 = 1$ is a singular point for the Picard-Fuchs differential equation associated to the perestroikha corresponding to the resolution of the $A_1$ singularity. As explained in Appendix A.4, the preimage, in the resolution of this singularity, of the singular point is a 2-cycle (there are $n - 1$ 2-cycles for a generic $A_{n-1}$ singularity). Now, we can consider 2-branes wrapped around this 2-cycle. The mass of these 2-branes will become zero whenever the size of the 2-cycle becomes zero. From equation (4.28) we observe that this is what precisely happens at the discriminant locus $\varphi^2 = 1$ provided we measure the size of the 2-cycle in terms of the corresponding Kähler form. At this point a subtlety arises that we would like to mention: in terms of the variable $z$, the one dimensional perestroikha defined by the resolution of the $A_1$ singularity is defined by going from $z = 0$ to $z = \infty$ [31]. The discriminant locus corresponds to the singularity at $z = 1$. Now, we parameterize this path in terms of $S$. From $z = 0$ to $z = 1$ we are moving along the the imaginary axis, taking from $S = i\infty$ to $S = 0$; however, from $z = 1$ to $z = \infty$, using (4.28), we get to the strange point $S = i0 - \frac{1}{2}$.

This discussion leads to a natural question: we must now try to find out whether this non perturbative enhancement of symmetry has an equivalent when working at the level of the $(\hat{u}, \epsilon)$-plane.

But before discussing this general issue let us first present in more concrete terms the Calabi-Yau interpretation of the $(\hat{u}, \epsilon)$-plane. This interpretation is based on the following set of similarities:

i) The two conifold branches and the loci II and III play the same roles.

ii) The locus $\mathcal{C}_1 = \{\epsilon^2 = 1\}$, and the locus $\varphi^2 = 1$.

iii) Identification of the blow up of the weak coupling point limit point $(\hat{u} = 0, \tau = i\infty)$, and the blow up of the conifold singularity $(x = 1, y = 0)$ leads to the relation

\[
\frac{1}{\hat{u}^2} = \frac{y x^2}{(1-x)^2}.
\]

iv) The $T$ transformation $(\hat{u}, \tau) \to (\hat{u}, \tau + 1)$ and the symmetry transformation $A : \psi \to \alpha \psi, \phi \to -\phi$, with $\alpha^{12} = 1$.

v) $A$ interchanges the conifold branches and the $\mathcal{C}_1$ branches in the very same way as $T$.

vi) The locus $\mathcal{C}_0 = \{\psi = 0\}$ in the $(x, \sqrt{y})$ transforms under $A$ as shown in Figure 12, which is, as Figure 13 clarifies, exactly what happens to the locus I in the $(\hat{u}, \epsilon)$-plane.

vii) In the Calabi-Yau case, $\phi \to -\phi$ can be associated to the Weyl symmetry of an $A_1$ singularity.
Figure 12: *The A transformation.*

Figure 13: *The action of T in the $(\hat{u}, \epsilon)$-plane.*
All this similarities between the singular loci in the $(\hat{u}, \tau)$-plane, and the moduli of complex structures of $\mathbb{P}_{11226}$, can be connected through a one to one map between the two moduli spaces [21],

$$
x = \frac{3/2e_1(\tau)}{3/2e_1(\tau) - \hat{u}}, \quad \sqrt{y} = -\frac{e_2(\tau) - e_3(\tau)}{3e_1(\tau)},
$$

whose weak coupling limit reproduces relations (4.17),

$$
x = \frac{1}{1 \pm \hat{u}} + \cdots \quad y = \epsilon^2 + \cdots
$$

Let us now finish with some general comments on the question raised above concerning the non perturbative enhancement of symmetry. The picture we have in mind can be summarized as follows: we start from the Donagi-Witten framework; the main point on that construction is the appearance of the $N=4$ moduli, $\tau$, used to measure the value of $\text{tr}\phi^2$ (for the $SU(2)$ case), and therefore to define the quantum moduli parameters of the $N=2$ gauge theory. Some sort of duality naturally appears in this context, between $\hat{u}$ and $\epsilon$: we can change the two dimensional Higgs field by varying the value of $\hat{u}$ for fixed $\epsilon$, or move the scale defined by the reference surface $E_\tau$ by moving its moduli, $\tau$. In both cases, and in the $(\hat{u}, \epsilon)$ variables used to make explicit the correspondence with the Calabi-Yau moduli space, the two descriptions are parameterized in terms of $\hat{u}$. The Calabi-Yau interpretation we have proposed, relates the $N=4 \tau$ with the heterotic dilaton; in this way, the interchange between $\hat{u}$ and $\epsilon$ seems to be the reflection, at the level of the Donagi-Witten approach, of a duality symmetry between $S$ and $T$. In this spirit, the singularity loci defined by $\tau = 0, 1$ will appear as singularities of the $SU(2)$ theory described by fixing $\hat{u}$ and moving the moduli of the reference curve $E_\tau$. It is natural to conjecture that the richness of the Donagi-Witten model already contains the essence of heterotic-heterotic duality, codified in the dual role played in this scheme by $\hat{u}$ and $\tau$. This issue will be presented elsewhere.

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A Appendix.

A.1 Calabi-Yau Manifolds and Weighted Projective Spaces.

The weighted projective space \( \mathbb{P}^{d+1}_{\{k_0\ldots k_{d+1}\}} \) with homogeneous coordinates \([z_0, \ldots, z_{d+1}]\), is defined by the equivalence relation

\[
[z_0, \ldots, z_{d+1}] \sim [\lambda^{k_0} z_0, \ldots, \lambda^{k_{d+1}} z_{d+1}] \tag{A.1}
\]

A Calabi-Yau manifold of complex dimension \(d\) can be defined as the vanishing locus of a homogeneous polynomial \(W\), of degree \(\sum_i k_i = k\):

\[
W = \sum a_{i_0\ldots i_{d+1}} z_0^{i_0} \ldots z_{d+1}^{i_{d+1}}, \quad \sum_{l=0}^{d+1} i_l k_l = k. \tag{A.2}
\]

Examples of vanishing polynomials are the following:

i) \( \mathbb{P}^4_{\{11222\}} \), \( \sum k_i = 8 \).

\[
W = z_1^8 + z_2^8 + z_3^4 + z_4^4 + z_5^4. \tag{A.3}
\]

ii) \( \mathbb{P}^4_{\{11226\}} \), \( \sum k_i = 12 \).

\[
W = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2. \tag{A.4}
\]

For some values of \(a_{i_0\ldots i_{d+1}}\) in (A.2), for which \(\frac{\partial W}{\partial z_i} = 0\) has solution other than \(z_i = 0\) for all \(z_i\), the manifold defined by \(W\) develops singularities (see A.5 for more details on this point). For the above examples,

i) Singularity: \( z_1 = z_2 = 0, z_3^4 + z_4^4 + z_5^4 = 0 \).

ii) Singularity: \( z_1 = z_2 = 0, z_3^6 + z_4^6 + z_5^2 = 0 \).

The values of \(a_{i_0\ldots i_{d+1}}\) for which the defined manifold is not smooth define the discriminant locus of the Calabi-Yau manifold; the discriminant locus is complex codimension one.

As an example, the discriminant locus of

\[
W = z_1^{12} + z_2^{12} + z_3^6 + z_4^6 + z_5^2 - 12\psi z_1 z_2 z_3 z_4 z_5 - 2\phi z_1^6 z_2^6 \tag{A.5}
\]

is given by

\[
\Delta = ((864\psi^6 + \phi)^2 - 1)(\phi^2 - 1). \tag{A.6}
\]
A.2 Toric Construction.

Let $\mathbb{Z}^n$ be the $n$-dimensional lattice of integer numbers. A fan $\Delta \subset \mathbb{Z}^n$ is defined as a set of cones $\sigma_i$ in the real vector space $\mathbb{N}_\mathbb{R} = \mathbb{N} \otimes \mathbb{R}$, such that the face of any cone $\sigma_i$ in $\Delta$ is also in $\Delta$. The idea of toric geometry consists of associating to each cone a patch of coordinates, and to use the fan combinatorics to define the transition functions between different patches. In order to do so, some rules are needed:

R-1 Given a cone $\sigma_i$, the dual cone $\bar{\sigma}_i \subset M_\mathbb{R}$, with $M$ the dual lattice to $\mathbb{N}$, is defined through

$$\bar{\sigma}_i \cap M = Z_{\geq 0}m_{i,1} + \cdots + Z_{\geq 0}m_{i,d_i} \quad (A.7)$$

R-2 The set (of vectors expanding the cones) $m_i,l$ satisfies relationships of the form

$$\sum_p p_i(l)m_{i,l} = 0. \quad (A.8)$$

R-3 To each $\sigma_i$ the patch of coordinates $U_{\sigma_i}$ is associated through

$$u_{\sigma_i} = \{(u_{i,1}, \ldots, u_{i,d_i}) : \prod_l u_{i,l}^{p_i(l)} = 1\}. \quad (A.9)$$

R-4 Given the patches $U_{\sigma_i}, U_{\sigma_j}$, in order to find the coordinate change leading from one of them to the other, we must find a relation of the form

$$\sum q_i^{(i)}m_{i,l} + \sum q_m^{(j)}m_{j,m} = 0 \quad (A.10)$$

R-5 The transition functions are defined by

$$\prod_l u_{i,l}^{q_i^{(i)}} \prod_m u_{j,m}^{q_m^{(j)}} = 1. \quad (A.11)$$

As a simple example let us consider the case of a fan with only one cone, as that depicted in Figure 14. For the picture in (a), the dual cone $\bar{\sigma}$ is defined by vectors $(1,0)$ and $(-1,2)$, so that

$$\bar{\sigma} \cap M = Z_{\geq 0}(1,0) + Z_{\geq 0}(-1,2) + Z_{\geq 0}(0,1). \quad (A.12)$$

Using (A.9), we get

$$u_{(1,0)}^2 = u_{(-1,2)}u_{(0,1)}. \quad (A.13)$$

Defining $z \equiv u_{(1,0)}, x \equiv u_{(-1,2)}$ and $y \equiv u_{(0,1)}$, equation (A.13) defines the manifold $\mathbb{C}^2 / \mathbb{Z}_2$: $xy = z^2$. In just the same way, the cone in (b) describes $\mathbb{C}^2 / \mathbb{Z}_3$; in general, as in the diagram in (c), we obtain the orbit space $\mathbb{C}^2 / \mathbb{Z}_n$. 

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Figure 14: Toric cones giving rise to singularities of different orders.

Figure 15: Toric construction of the projective space $\mathbb{P}_{(112)}$. 
A Dynkin diagram $A_{n-1}$ can be associated to these spaces, defined as the set of internal lattice points on the edge $[(1,0), (n,1)]$.

A more detailed example would be working out the manifold associated with the fan in Figure 15.

The dual cones $\tilde{\sigma}_i$ are given by

\[
\begin{align*}
\tilde{\sigma}_1 &= [(1,0), (0,1)], \\
\tilde{\sigma}_2 &= [(0,-1), (1,-2)], \\
\tilde{\sigma}_3 &= [(-1,0), (-1,2)].
\end{align*}
\]

Using the rules given above, coordinates in the different patches are given by

\[
\begin{align*}
U_{\sigma_1} &= (u,v), \\
U_{\sigma_2} &= (v^{-1}, uv^{-2}), \\
U_{\sigma_3} &= (u^{-1}, u^{-1}v^2, u^{-1}v).
\end{align*}
\]

It must be stressed that the chart $U_{\sigma_3}$ is again isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$:

\[
u^{-1}(u^{-1}v^2) = (u^{-1}v)^2.
\]

This is already clear from the cone $\sigma_3$, in the fan of Figure 15: the edge $[(0,1), (-2, -1)]$ contains one interior lattice point, $(-1,0)$. The space defined by (A.15) is the weighted projective space $\mathbb{P}_{\{112\}}$.

### A.3 Toric Blow up.

The space $\mathbb{C}^2/\mathbb{Z}_n$ is an example of a singularity of type $A_{n-1}$, attending to Arnold’s classification (see next paragraph). The resolution of this singularity consists in finding a smooth manifold $\mathcal{V}$, and a proper map $\Pi: \mathcal{V} \to \mathbb{C}^2/\mathbb{Z}_n$, which is biholomorphic outside the preimage $\Pi^{-1}(0)$ of the singular point in $\mathbb{C}^2/\mathbb{Z}_n$. A toric approach to $\mathcal{V}$ is the construction of a new fan, obtained by adjoining vectors to the old one in such a way that all lattice points in each cone of the new fan are generated by the vectors defining the cone.

As an example, consider the fan associated to $\mathbb{C}^2/\mathbb{Z}_2$ given in Figure 14.(a). It is clear that in order to generate all lattice points in $\sigma$ we need to include the vector $(1,1)$. The new fan, defined by vectors $(0,1)$, $(1,1)$ and $(2,1)$, contains two patches, and defines the resolution of the singularity $\mathbb{C}^2/\mathbb{Z}_2$.

From the above, it is clear that the blow up of the $A_{n-1}$ spaces $\mathbb{C}^2/\mathbb{Z}_n$ requires including as many extra vectors as points are there in the corresponding Dynkin diagram.
A.4 Brief Review of Singularity Theory.

A.4.1 Singularities and Platonic Bodies.

Let \( \Gamma \) be a discrete subgroup of \( SO(3) \), and let us denote by \( \Gamma^* \) the preimage of \( \Gamma \) by the covering map \( SU(2) \to SO(3) \) (the classical notation for the cyclic, dihedral, tetrahedron, octahedron and icosahedron groups is, respectively, \( \mathbb{Z}_n, D_{2n}, T, O \) and \( I \)). The action of \( \Gamma^* \) on \( \mathbb{C}^2 \) will be defined through the algebra \( A_{\Gamma^*} \) of polynomials in two complex variables invariant under the action of \( \Gamma^* \). The generators of \( A_{\Gamma^*} \), denoted by \( x, y \) and \( z \), satisfy a relation \( R_{\Gamma^*}(x, y, z) = 0 \), that defines a hypersurface \( V \) in \( \mathbb{C}^3 \).

\[
\begin{array}{|c|c|c|}
\hline
\Gamma^* & R_{\Gamma^*}(x, y, z) = 0 & \text{Dynkin diagram} \\
\hline
\mathbb{Z}_n & xy = z^n & A_{n-1} \\
D_{2n} & xy^2 - x^{n+1} + z^2 = 0 & D_{n+2} \\
T & x^4 + y^3 + z^2 = 0 & E_6 \\
O & x^3 + yy^3 + z^2 = 0 & E_7 \\
I & x^5 + y^3 + z^2 = 0 & E_8 \\
\hline
\end{array}
\]

The hypersurface \( V \) defined by \( R_{\Gamma^*}(x, y, z) = 0 \) is isomorphic to the orbit space \( \mathbb{C}^2/\Gamma^* \). All these surfaces have a unique singular point \( O \). The resolution of the singularity takes place through a map \( \Pi : \tilde{V} \to V \), where \( \tilde{V} \) is a smooth manifold; the preimage \( \Pi^{-1}(O) \) of the singular point is the union of 2-cycles \( \Gamma_i \),

\[
\Pi^{-1}(O) = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_\mu,
\]

with \( \mu \) the number of points of the corresponding Dynkin diagram. Furthermore, the intersection between these 2-cycles is determined by the intersection matrix of the Dynkin diagram. Thus, we associate 2-cycles \( \Gamma_i \) with the points of the diagram, and a link \((i, j)\) whenever the corresponding 2-cycles \( \Gamma_i, \Gamma_j \) intersect.

A.4.2 Dynkin Diagrams and Picard-Lefschetz Theory.

Let \( f : \mathbb{C}^n \to \mathbb{C} \) be a Morse function\(^{10}\), i.e., a function such that all its critical points are non degenerate; the corresponding critical values will be denoted by \( \alpha_1, \ldots, \alpha_\mu \). The level manifold \( V_{\alpha_i} \), associated to each critical value, is defined as follows:

\[
V_{\alpha_i} = \{(z_1, \ldots, z_n), \ f(z_1, \ldots, z_n) = \alpha_i\} \equiv f^{-1}(\alpha_i).
\]

Notice that \( V_{\alpha_i} \) are singular hypersurfaces of dimension \( n - 1 \). The singular point \( a_i \in V_{\alpha_i} \) is the critical point with critical value \( \alpha_i \).

\(^{10}\)In the case considered in the text, that of Calabi-Yau threefolds, \( n = 3 \).
Let us now consider a point \( \alpha \in C \), which corresponds to a regular value of \( f \), so that the level manifold \( V_\alpha \) is non singular. From the regular point \( \alpha \) to the critical values \( \alpha_i \), a set of paths \( \varphi_i \) can be defined, through

\[
\varphi_i(\tau) = \begin{cases} 
\alpha & \tau = 0 \\
\alpha_i & \tau = 1,
\end{cases}
\]  

(A.19)

and such that \( \varphi_i(\tau) \) is a regular value of \( f \) for \( \tau \neq 1 \).

Given the homology \( H_{n-1}(V_\alpha) \), a vanishing cycle \( \Delta_i \in H_{n-1}(V_\alpha) \) will be an \((n-1)\)-cycle that, when “transported” by the path \( \varphi_i \), will contract to the critical point \( a_i \), i.e., to the singular point of the level manifold \( V_{\alpha_i}; \) therefore, a vanishing cycle \( \Delta_i \) can be associated to each path \( \varphi_i \). They define a basis of \( H_{n-1}(V_\alpha) \).

For each path \( \varphi_i \) we can define a loop \( \gamma_i \) starting at the point \( \alpha \), and going around the critical value \( \alpha_i \). Picard-Lefschetz theory associates to each loop \( \gamma_i \) a monodromy matrix \( h_i \),

\[
h_i : H_{n-1}(V_\alpha) \to H_{n-1}(V_\alpha) \]  

(A.20)

As an example, consider the Landau-Ginzburg potential

\[
W = z^n + t_1 z .
\]  

(A.21)

In the \( w \)-plane we have \((n-1)\)-critical values \( \alpha_i \). Let us denote by \( a_i \) the corresponding critical points, i.e., the different vacua of \( W \). The level manifold \( V_\alpha \) at a regular point \( \alpha \) consist of the set of \( n \) points

\[
V_\alpha = \{ z^n + t_1 z = \alpha \} .
\]  

(A.22)

Calling these points \( z_1, \ldots, z_n \), the homology \( H_0(V_\alpha) \) is generated by the \( n-1 \) 0-cycles \( \Delta_i = [z_i, z_{i+1}], \) \( i = 1, \ldots, n-1 \). now, we define the set of paths \( \varphi_i \) in the \( w \)-plane (see Figure 16). When we transport the cycle \( \Delta_i \) from \( \alpha \) to \( \alpha_i \), we immediately observe that this 0-cycle contracts to the critical point \( a_i \). Therefore, we have \( n-1 \) vanishing cycles in \( H_0(V_\alpha) \). It is then immediate to see that they define the Dynkin diagram \( A_{n-1} \).

### A.5 Discriminants and Perestroïkas.

Let us consider the Calabi-Yau manifold \( X \) in \( \mathbb{P}_{\{11226\}} \) defined by

\[
x_1^{12} + x_2^{12} + x_3^6 + x_4^6 + x_5^2 = 0.
\]  

(A.23)

To study the Kähler classes of (A.23), we can consider the complex deformations of its mirror manifold \( Y \) defined by

\[
x_1^{12} + x_2^{12} + x_3^6 + x_4^6 + x_5^2 + \bar{a}_1 a_1 x_1^6 x_2^6 + \bar{a}_2 a_2 x_1 x_2 x_3 x_4 x_5 = 0.
\]  

(A.24)
When noticing that all terms in (A.24) are of the form $x_1^{n_1} x_2^{n_2} x_3^{n_3} x_4^{n_4} x_5^{n_5}$, a point in $\mathbb{R}^5$ can be associated to each monomial: $(n_1, n_2, n_3, n_4, n_5)$. Therefore, equation (A.24) is represented by a collection of (seven) points in $\mathbb{R}^5$. Following notation from reference [31] we will denote this set by $\mathcal{A}$:

$$\mathcal{A} \equiv \{(12, 0, 0, 0, 0), (0, 12, 0, 0, 0), (0, 0, 6, 0, 0), (0, 0, 0, 6, 0), (0, 0, 0, 0, 2), (6, 6, 0, 0, 0), (1, 1, 1, 1, 1)\}. \quad (A.25)$$

The set $\mathcal{A}$ lies in a four dimensional polytope $P$, called Newton polytope, whose corners are determined by the vectors associated to the monomials in (A.23).

Defining polynomials for faces $\Gamma$ in the Newton polytope can be easily done by just considering the set of monomials associated to points in that face. The 1-dimensional face spanned by vectors $(12, 0, 0, 0, 0), (0, 12, 0, 0, 0)$ is the polytope corresponding to

$$W_\Gamma = x_1^{12} + x_2^{12} + \bar{a}_1x_1^6 x_2^6. \quad (A.26)$$

This decomposition into faces of the Newton polytope also provides a powerful tool for the characterization of the singularities arising in the moduli space. The number of points in $\mathcal{A}$ that lie in the interior of the face determined by the monomials $x_1^{12}$ and $x_2^{12}$, namely one point: $(6, 6, 0, 0, 0)$, implies the existence of a singularity of $A_1$ type [32].

Let us consider now a triangulation of the set $\mathcal{A}$. This triangulation will define a fan of cones, in terms of which we can give a toric representation of the manifold (A.24). Part of the idea of topology changing amplitudes is related to the existence of different triangulations for the same set $\mathcal{A}$ of points.

Before entering the question of topology change, we will concentrate on the discriminant $\Delta$ of (A.24). We consider a homogeneous polynomial

$$W = \sum a_i m_i \quad (A.27)$$
defined in terms of the set of monomials \( m_i = \prod x_i^{n_i} \), where we have introduced parameters \( a_i \) for each monomial. Then, the discriminant is the condition obtained on the coefficients \( a_i \) by requiring that there exist solutions to

\[
\frac{\partial W}{\partial x_i} = 0,
\]

(A.28)

for \( x_i \) not equal zero for all \( i \). Partial discriminants (those coming from the corresponding Landau-Ginzburg potential \( W_\Gamma \)) \( \Delta_\Gamma(a_i) \) are naturally associated to faces \( \Gamma \) of the Newton polytope. In this way, the face \( \Gamma \) determined by vectors \((12, 0, 0, 0, 0), (0, 12, 0, 0, 0)\), whose defining polynomial is that given in (A.26), leads to the discriminant

\[
\Delta_\Gamma(\tilde{a}_1) = (1 - \frac{4}{\tilde{a}_1^2}).
\]

(A.29)

Vanishing of the discriminant defines the discriminant locus, the point in moduli space (parameterized by \( a_i \)) where the manifold defined as the locus \( W = 0 \) becomes singular.

Now we can try to understand the discriminant locus, in string language, from the type II A or II B point of view. In order to do so, we will make use of mirror symmetry to interpret the quantities \( a_i \) parameterizing the complex structures of the manifold \( Y \), defined by \( W \), in terms of Kähler classes of its mirror \( X \).

In example (A.24), the moduli space of complex structures, parameterized by \( \tilde{a}_1 \) and \( \tilde{a}_2 \), is isomorphic to \((\mathbb{C}^*)^2 \) \(^{11}\). Let us now define new coordinates \( u_i \) by

\[
u_i = -\frac{1}{2\pi} \log |\tilde{a}_i| \quad \text{(A.30)}
\]

in \( \mathbb{R}^2 \). These coordinates can be interpreted in the spirit of the monomial divisor map [33]. In fact, the Kähler classes in \( H_2(X) \) can be parameterized in terms of quantities \( t_i \) related to the complex deformation variables \( \tilde{a}_i \) by \( \tilde{a}_i = e^{2\pi i t_i} \). Asymptotically \( (\tilde{a}_i \to \infty) \), the imaginary part of \( t_i \), given by (A.30), is the (real) Kähler form \( J_i \) in (4.21).

The structure of Kähler cones on \( \mathbb{R}^2 \) can be now determined by using the properties of the discriminant: when moving to large values of \( \tilde{a}_i \), we notice that the discriminant is dominated by some particular monomial; we will refer to this monomial by \( r_{\Delta_i} \). Therefore, \( \mathbb{R}^2 \) is divided into sectors where a particular monomial \( r_{\Delta_i} \) is dominating, as indicated in Figure 17.

The discriminant locus now appears concentrated on the walls separating different regions.

\(^{11}\)By rescaling \( x_i \to \lambda_i x_i \), five of the possible deformations \( a_i \) have been set to one. The (reduced) moduli space is

\[
\mathcal{M} \simeq \frac{(\mathbb{C}^*)^7}{(\mathbb{C}^*)^5} \simeq (\mathbb{C}^*)^2.
\]

See [31] for a detailed discussion on this point.
A beautiful result in toric geometry allows us to associate with each monomial $r_{\Delta_i}$ a particular triangulation of the set of points $\mathcal{A}$ determining the Newton polytope: given a triangulation $\Delta_i$ (just as the one the monomials dominating in the discriminant along different regions introduce) of $\mathcal{A}$, then

$$r_{\Delta_i} = \prod a_k \sum_{\sigma \in \Delta_i, \alpha_k \in \sigma} \text{vol}(\sigma), \quad (A.31)$$

where the sum is over all triangles with $\alpha_k$ (the vector defined by the monomial in $W$ with $a_k$ coefficient) as a vertex.

With the above geometry, whenever a wall separating different Kähler cones is transversed, a transition between different triangulations takes place; this process is called a \textit{perestroïka}. This transition represents a change in topology\footnote{The topology change does not modify the Hodge numbers, but the rational curves lying on the cones the wall bounds.}, so it is a candidate to topology changing amplitudes. In the particular case given by (A.24), we can expect transitions on the wall determined by vanishing of the discriminant locus $\Delta_f(\tilde{a}_1)$, defined in (A.26).

\textbf{A.5.1 Picard-Fuchs Equations.}

To each point in the moduli space of complex structures, parameterized by $a_i$, we can associate a fiber defined by the $H_2$ cohomology of its mirror manifold $X$; elements in $H_2$ of the mirror will be denoted by $\Phi(a_i)$. Armed with this fibration, we can understand what happens as we move around the discriminant locus (This is exactly the same problem we have delt with in previous section, in connection with singularity theory; in that case, we had the homology $H_{n-1}(V_n)$ for the map $f : \mathbb{C}^n \to \mathbb{C}$, and we can study...
using Picard-Lefschetz theory the monodromy matrix arising in $H_{n-1}(Y_\alpha)$ when moving around critical values, which are just the equivalent notion, within this context, to the discriminant locus.

The first thing we need to do is defining coordinates associated to each particular triangulation of the set $A$; then, a differential Picard-Fuchs equation will be associated to transitions between different triangulations. In order to interpret the complete moduli space of complex structures of $Y$ in terms of Kähler classes, it is necessary to include triangulations that can omit points in $A$ while always containing the Newton polytope $P$.

In the example we are dealing with a simple perestroïka takes place. Defining

$$
\alpha_1 = (6, 6, 0, 0, 0),
\alpha_2 = (12, 0, 0, 0, 0),
\alpha_3 = (0, 12, 0, 0, 0),
$$

(A.32)

the only two possible triangulations are those depicted in Figure 18.

\begin{figure}[ht]
\centering
\begin{subfigure}{0.4\textwidth}
\centering
\begin{tikzpicture}
\draw (0,0) -- (0,4);
\node at (0,4.5) {$\alpha_2$};
\node at (0,3.5) {$\alpha_1$};
\node at (0,2.5) {$\alpha_3$};
\end{tikzpicture}
\caption{(a)}
\end{subfigure}
\begin{subfigure}{0.4\textwidth}
\centering
\begin{tikzpicture}
\draw (0,0) -- (0,4);
\node at (0,4.5) {$\alpha_2$};
\node at (0,3.5) {$\alpha_1$};
\node at (0,2.5) {$\alpha_3$};
\end{tikzpicture}
\caption{(b)}
\end{subfigure}
\caption{The only possible Perestroïka in one dimension.}
\end{figure}

The triangulation Figure 18.(b) corresponds to resolution of the $A_1$ singularity, as we have included the vertex $\alpha_1$; in this situation, we have a cone in the fan of the manifold with one edge going from the origin to the point $\alpha_1$. On the other hand, the triangulation in (a) does not see the point $\alpha_1$, so that it produces a singular space, $\mathbb{C}^2/\mathbb{Z}_2$.

Following [31], a recipe can be given to build up the differential equation associated to a simple perestroïka between sets of points where only two triangulations are possible: Perestroïkas are associated with sets of $N + 2$ points in $\mathbb{R}^N$ that are not contained in an $\mathbb{R}^{N-1}$ hyperplane. Denoting by $\alpha_l$ these points, we must find relations of the form

$$
\sum m_l \alpha_l = 0
$$

(A.33)
in order to define the invariant coordinate

\[ z \equiv \prod a_i^{m_i}. \quad \text{(A.34)} \]

The differential operator is defined as

\[ \Box = \prod_{m_i>0} \left( \frac{\partial}{\partial a_i} \right)^{m_i} - \prod_{m_i<0} \left( \frac{\partial}{\partial a_i} \right)^{-m_i}, \quad \text{(A.35)} \]

where \( a_i \) stands for the coefficients of the monomials in the polynomial \( W \).

For the example in Figure 18, we get

\[ 2\alpha_1 = \alpha_2 + \alpha_3, \quad \text{(A.36)} \]

so that

\[ z = 4 \frac{a_2 a_3}{a_1^2}. \quad \text{(A.37)} \]

As in (A.24) we have chosen \( a_2 = a_3 = 1 \), \( z \) becomes \( z = \frac{4}{a_1^2} \), and the operator (A.35) is

\[ \Box = \left( z \frac{d}{dz} \right)^2 - z \left( z \frac{d}{dz} \right) \left( z \frac{d}{dz} + \frac{1}{2} \right). \quad \text{(A.38)} \]

The invariant \( z \) parameterizes the transition between the two triangulations.

From (A.29) we notice that the discriminant is given by \( z = 1 \). Moreover, the solution obtained from \( \Box f = 0 \),

\[ f = C_1 + C_2 \log \left( \frac{2 - z - 2\sqrt{1-z}}{z} \right), \quad \text{(A.39)} \]

has non trivial monodromy around \( z = 1 \) (namely, \(-\Pi\)).

Through the monomial divisor map, equation (A.39) provides us with the coordinate in \( H_2(X) \) corresponding to \( z \):

\[ t = \frac{1}{2\pi i} \log \left( \frac{2 - z - 2\sqrt{1-z}}{z} \right). \quad \text{(A.40)} \]

The following facts, arising from (A.40), must be stressed:

i) \( t = 0 \) when \( z \) is in the discriminant locus \( (z = 1) \).

ii) As we move around the discriminant locus, we pick up a monodromy for \( t \) which is the Weyl group of the \( A_1 \) singularity (A.36).

\[ ^{13}\text{For convenience, the factor } 4 \text{ (appearing in expression (A.29) for the discriminant } \Delta_{\Gamma(a_1)} \text{) has been introduced.} \]
iii) Moving from $z = 1$ to $z = 0$ the singularity in $\mathbb{C}^2/\mathbb{Z}_2$ has been blown up to infinite size.

iv) An interesting point is the size of the 2-cycle used to blow up the $A_1$ singularity: the point $z = \infty$ is expected to correspond to a singular space and, certainly, $J = 0$; however, $B$ is not zero at that point ($B = -\frac{1}{2}$). On the other hand, at the discriminant locus $z = 1$, we have $B + iJ = 0$, the difference being the value of the $B$ field.
References


