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SUPERSYMMETRY AND DUALITIES *

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ABSTRACT

Duality transformations with respect to rotational isometries relate supersymmetric with non-supersymmetric backgrounds in string theory. We find that non-local world-sheet effects have to be taken into account in order to restore supersymmetry at the string level. The underlying superconformal algebra remains the same, but in this case T-duality relates local with non-local realizations of the algebra in terms of parafermions. This is another example where stringy effects resolve paradoxes of the effective field theory.

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1 Outline and summary

These notes, based on two lectures given at the Trieste conference on “S-Duality and Mirror Symmetry”, summarize some of our recent results on supersymmetry and duality [1, 2, 3]. The problem that arises in this context concerns the supersymmetric properties of the lowest order effective theory and their behaviour under T-duality (and more generally U-duality) transformations (see also [4]). We will describe the geometric conditions for the Killing vector fields that lead to the preservation or the loss of manifest space-time supersymmetry under duality, and classify them as translational versus rotational respectively. Analyzing this problem in detail we find that there is no contradiction at the level of string theory. This is part of the standard lore that stringy effects manifest as paradoxes of the effective field theory, and supersymmetry is no exception to it.

In toroidal compactifications of superstring theory T-duality is a symmetry that interchanges momentum with winding modes, and it manifests geometrically as a small-large radius equivalence in the effective theory of the background massless fields. This equivalence has been generalized to arbitrary string backgrounds with Abelian isometries following Buscher’s formula that was originally derived for the β -function equations of the metric $G_{\mu\nu}$, antisymmetric tensor field $B_{\mu\nu}$ and dilaton Φ to lowest order in the σ -model coupling constant (inverse string tension) α' (see for instance [5], and references therein). One might naively expect that if a background is supersymmetric (eg a bosonic solution of effective supergravity), the dual background will also share the same number of supersymmetries. This is a reasonable expectation provided that there are no non-local world-sheet effects, so that the dual faces of the theory provide a trustworthy low energy approximation to string dynamics in the α' -expansion. There are circumstances, however, when the dual geometry has strong curvature singularities and the effective field theory description breaks down in their vicinity. This is a typical property of gravitational backgrounds having Killing isometries with fixed points, in which case the T-dual background exhibits curvature singularities. Then, T-duality appears to be incompatible with space-time supersymmetry, and non-local world-sheet effects have to be taken into account for resolving this issue consistently. After all, it is a well-known fact that symmetries not commuting with a given Killing vector field are not manifest in the dual background of the effective field theory description.

In the first part most of our discussion will concentrate on 4-dim space-time backgrounds, thinking of superstring vacua as arising from ten dimensions by compactification on a 6-dim internal space K . It is much simpler to expose the main ideas in this case and employ the notion of self-duality (when it is appropriate) to refine the distinction we want to make. In four dimensions, and in the presence of some Killing isometries, it is also possible to intertwine T with S-duality, thus forming a much larger symmetry group known as U-duality. We will present the essential features of this idea at the level of the effective theory by considering 4-dim backgrounds with (at least) one isometry, in which case the U-duality corresponds to $O(2, 2)$ transformations. The supersymmetric properties of the effective theory will also be considered under the action of such generalized

symmetries, where the effect on supersymmetry can be even more severe.

In the second part we will investigate more elaborately the issue of supersymmetry versus duality in heterotic σ -models, where we analyze possible anomalies and find some modifications of Buscher's rules. We consider the heterotic string in 10-dim flat Minkowski space with $SO(32)$ gauge group (though similar arguments apply to the $E_8 \times E_8$ string) and study the effect of T-duality with respect to rotations in a 2-dim plane, using a manifest $ISO(1,7) \times SO(30) \times SO(2)$ symmetric formalism. We prove that the number of space-time fermionic symmetries remains unchanged. The conformal field theory analysis suggests that the emission vertices of low energy particles in the dual theory are represented by non-local operators, which do not admit a straightforward α' -expansion. In fact the underlying conformal theory provides a natural explanation of this, where local realizations of the superconformal algebra become non-local in terms of parafermions. In view of these circumstances, certain theorems relating world-sheet with manifest space-time supersymmetry should be revised. Also, the powerful constraints imposed by supersymmetry on the geometry of the target space are not valid for non-local realizations.

Some of the topics of this work may also be relevant for superstring phenomenology, where we mainly work in terms of the lowest order effective theory. The issue of space-time supersymmetry versus duality, which arises to lowest order in α' , demonstrates explicitly that an apparently non-supersymmetric background can qualify as a vacuum solution of superstring theory, in contrary to the "standard wisdom" that has been considered so far. So, whether supersymmetry is broken or not in various phenomenological applications cannot be decided, unless one knows how to incorporate the appropriate non-local world-sheet effects that might lead to its restoration at the string level. Also, various gravitational solutions, like black holes, might enjoy some supersymmetric properties in the string context. This could also provide a better understanding of the way that string theory, through its world-sheet effects, can resolve the fundamental problems of the quantum theory of black holes. Finally, from the low energy point of view the fact that T-duality relates supersymmetric with non-supersymmetric backgrounds could provide examples of the mechanism advocated by Witten to shed new light in the cosmological constant problem [6]. Hence, we view T-duality at this moment as a method for probing some of these possibilities, and leave to future investigation the applicability of these ideas to the physical problems of black holes and cosmology within superstring theory.

2 Supersymmetry and dualities: a first clash

Consider the class of 4-dim backgrounds with one isometry generated by a Killing vector field $K = \partial/\partial\tau$. The signature is taken to be Euclidean, but most of the results generalize to the physical Minkowskian signature. The metric can be written locally in the form

$$ds^2 = V(d\tau + \omega_i dx^i)^2 + V^{-1}\gamma_{ij}dx^i dx^j \ , \quad (2.1)$$

where $x^1 = x$, $x^2 = y$, $x^3 = z$ are coordinates on the space of non-trivial orbits of $\partial/\partial\tau$, and V , ω_i , γ_{ij} are all independent of τ . Of course, ω_i are not unique, since they are defined up to a gauge transformation $\omega_i \rightarrow \omega_i - \partial_i\lambda$ that amounts to the coordinate shift $\tau \rightarrow \tau + \lambda(x^i)$. These backgrounds provide gravitational solutions of the β -function equations with constant dilaton and antisymmetric tensor fields to lowest order in α' . Although this class is rather restrictive, it is a good starting point for exploring the clash of supersymmetry with dualities in the effective theory.

Performing a T-duality transformation to the gravitational background (2.1) we obtain the following configuration in the σ -model frame,

$$G_{\mu\nu} = \frac{1}{V} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \gamma_{ij} & \\ 0 & & & \end{pmatrix} \quad (2.2)$$

with non-trivial dilaton and antisymmetric tensor fields

$$\Phi = -\frac{1}{2} \log V, \quad B_{\tau i} = \omega_i. \quad (2.3)$$

A necessary condition for having manifest space-time supersymmetry in the new background is given by the dilatino variation $\delta\lambda = 0$,

$$\frac{1}{2} H_{\mu\nu\rho} \pm \sqrt{\det G} \epsilon_{\mu\nu\rho}{}^\sigma \partial_\sigma \Phi = 0, \quad (2.4)$$

where $H_{\mu\nu\rho}$ is the field strength of $B_{\mu\nu}$. Substituting the expressions (2.2) and (2.3) we arrive at the differential equations

$$\partial_i \omega_j - \partial_j \omega_i \pm \frac{\sqrt{\det \gamma}}{2V^3} \epsilon_{\tau ij}{}^k \partial_k V = 0, \quad (2.5)$$

which select only a subclass of solutions (2.1) of the vacuum Einstein equations.

It is useful at this point to introduce a non-local quantity b ,

$$\partial_i b = \frac{1}{2} V^2 \sqrt{\det \gamma} \epsilon_i{}^{jk} (\partial_j \omega_k - \partial_k \omega_j), \quad (2.6)$$

where ϵ_{ijk} is the fully antisymmetric tensor with respect to the metric γ with $\epsilon_{123} = 1$. It is then clear that the dilatino variation of the dual background is zero provided that $b \pm V$ is constant. This constraint can be reformulated as an axionic instanton condition on the dual background, because $V = e^{-2\Phi}$, and b coincides with the axion field associated with $B_{\tau i} = \omega_i$. Another useful interpretation of b in the initial frame (2.1) is given by the nut potential of the original metric [7]. We note that b exists only on-shell. Furthermore, the identification of the nut potential of the original metric with the axion field of the T-dual background is important for describing not only the necessary condition for having manifest space-time supersymmetry, but also later for understanding the concept of U-duality at work.

The original background being purely gravitational trivially satisfies the dilatino equation. Manifest space-time supersymmetry requires the existence of a Killing spinor η , which in turns implies the existence of a complex structure as a bilinear form in η and $\bar{\eta}$ in the usual way. But a Kahlerian Ricci-flat metric is also hyper-Kahler, and hence self-dual (or anti-self-dual),

$$R_{\mu\nu\rho\sigma} = \pm \frac{1}{2} \sqrt{\det G} \epsilon_{\rho\sigma}{}^{\kappa\lambda} R_{\mu\nu\kappa\lambda} . \quad (2.7)$$

Consequently, exploring the behaviour of space-time supersymmetry for T-duality transformations of purely gravitational backgrounds forces us to consider self-dual solutions of the lowest order effective theory. In this case the original backgrounds exhibit $N = 4$ extended superconformal symmetry associated with the three underlying complex structures. According to the standard rules of manifest supersymmetry it is natural to expect that the dual background (2.2) and (2.3) will also exhibit $N = 4$ world-sheet supersymmetry, and its metric will be conformally equivalent to a hyper-Kahler metric $G'_{\mu\nu}$ (in the presence of non-trivial $B_{\mu\nu}$ and Φ) so that

$$G = \Omega G' , \quad \square' \Omega = 0 . \quad (2.8)$$

It will be demonstrated later that this is possible only for $b \pm V = 0$ (up to a constant). Otherwise manifest space-time supersymmetry is lost, and non-local realizations of the $N = 4$ superconformal algebra have to be introduced for resolving this paradox of the effective theory at the string level. In this case the dual background admits no Killing spinors, thus providing a vacuum solution of superstring theory that is apparently non-supersymmetric.

A potential trouble with supersymmetry has already been spotted in the adapted coordinate system (2.1) for the Killing isometry $\partial/\partial\tau$. The covariant description of the criterion for having manifest space-time supersymmetry after duality can be formulated as follows: we divide the Killing vector fields K_μ into two classes, the translational and the rotational. The first class consists of Killing vector fields with self-dual (respectively anti-self-dual) covariant derivatives

$$\nabla_\mu K_\nu = \pm \frac{1}{2} \sqrt{\det G} \epsilon_{\mu\nu}{}^{\rho\sigma} \nabla_\rho K_\sigma , \quad (2.9)$$

while the second class encompasses all the rest. Consider now the conjugate fields $S_\pm = b \pm V$ and introduce the quadratic quantity

$$\Delta S_\pm = \gamma^{ij} \partial_i S_\pm \partial_j S_\pm , \quad (2.10)$$

which is clearly ≥ 0 . It is a well-known theorem that $\Delta S_\pm = 0$ for translational isometries (in which case S_\pm is constant), while $\Delta S_\pm > 0$ for rotational isometries [8]. In the latter case it is always possible to choose the coordinates x , y and z so that $S_\pm = z$. This analysis provides the covariant distinction between those isometries that preserve manifest space-time supersymmetry after duality and those that do not. The result can

be easily extended from purely gravitational backgrounds to more general solutions with $N = 4$ superconformal symmetry having antisymmetric tensor and dilaton fields. Indeed, if the original background has a metric which is hyper-Kähler up to a conformal factor, as in (2.8), the T-dual background will also be manifestly supersymmetric if the Killing vector field is translational with respect to G' .

Next, we investigate in detail the specific form of hyper-Kähler metrics with one isometry, and use it to determine the dependence of the corresponding complex structures and the Killing spinors on the Killing coordinate τ . It turns out that for rotational isometries these quantities depend explicitly on τ , which is the key for having non-local realizations of supersymmetry after T-duality. This problem does not arise for translational isometries as nothing depends explicitly on τ ; for this reason translational isometries are also known as tri-holomorphic.

Translational isometries : In this case the adapted coordinate system (2.1) can be chosen so that

$$\gamma_{ij} = \delta_{ij} , \quad \partial_i(V^{-1}) = \pm \frac{1}{2} \epsilon_{ijk} (\partial_j \omega_k - \partial_k \omega_j) , \quad (2.11)$$

and so self-dual metrics with translational isometry are determined by solutions of the 3-dim Laplace equation

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)V^{-1} = 0 . \quad (2.12)$$

The localized solutions are of the general form [7, 9]

$$V^{-1} = \epsilon + \sum_{i=1}^n \frac{m_i}{|\vec{x} - \vec{x}_{0i}|} , \quad (2.13)$$

where ϵ is a constant that determines the asymptotic behaviour of the metric and m_i , \vec{x}_{0i} are moduli parameters. The apparent singularities of the metric are removable if all $m_i = M$ and τ is taken to be periodic with range $0 \leq \tau \leq 4\pi M/n$. For $\epsilon = 0$ the resulting metrics are the multi-center Gibbons-Hawking metrics, with $n = 2$ being the simplest non-trivial example known as the Eguchi-Hanson instanton. For $\epsilon \neq 0$ (in which case its value is normalized to 1) one has the multi-Taub-NUT metrics, with $n = 1$ being the ordinary self-dual Taub-NUT metric.

The three independent complex structures are known to be τ -independent. Choosing $\omega_3 = 0$ for convenience, we have the following result for the corresponding Kähler forms in the special frame (2.11) [10]:

$$\begin{aligned} F_1 &= (d\tau + \omega_2 dy) \wedge dx - V^{-1} dy \wedge dz , \\ F_2 &= (d\tau + \omega_1 dx) \wedge dy + V^{-1} dx \wedge dz , \\ F_3 &= (d\tau + \omega_1 dx + \omega_2 dy) \wedge dz - V^{-1} dx \wedge dy . \end{aligned} \quad (2.14)$$

As for the Killing spinors of the background (2.1), (2.11) one can easily check that they are the constant, independent of any coordinates.

Rotational isometries : In this case the self-duality condition reduces to a non-linear equation in three dimensions involving a function $\Psi(x, y, z)$. The metric (2.1) can always

be chosen so that [8]

$$\begin{aligned}\omega_1 &= \mp \partial_y \Psi, & \omega_2 &= \pm \partial_x \Psi, & \omega_3 &= 0, \\ V^{-1} &= \partial_z \Psi, & \gamma_{ij} &= \text{diag}(e^\Psi, e^\Psi, 1),\end{aligned}\tag{2.15}$$

where Ψ satisfies the continual Toda equation

$$(\partial_x^2 + \partial_y^2)\Psi + \partial_z^2 e^\Psi = 0.\tag{2.16}$$

It can be verified directly that $\Delta S_\pm = 1$, and hence $S_\pm = z$ indicating the anomalous behaviour of manifest space-time supersymmetry under rotational T-duality transformations.

The Kahler forms that describe the three independent complex structures in this case are not all τ -independent; one of them is a $SO(2)$ -singlet, while the other two form a $SO(2)$ -doublet. We have explicitly the following result [2] for the doublet,

$$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = e^{\frac{1}{2}\Psi} \begin{pmatrix} \cos \frac{\tau}{2} & \sin \frac{\tau}{2} \\ \sin \frac{\tau}{2} & -\cos \frac{\tau}{2} \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},\tag{2.17}$$

where

$$\begin{aligned}f_1 &= (d\tau + \omega_2 dy) \wedge dx - V^{-1} dz \wedge dy, \\ f_2 &= (d\tau + \omega_1 dx) \wedge dy + V^{-1} dz \wedge dx\end{aligned}\tag{2.18}$$

and for the singlet,

$$F_3 = (d\tau + \omega_1 dx + \omega_2 dy) \wedge dz + V^{-1} e^\Psi dx \wedge dy.\tag{2.19}$$

It is straightforward to verify that they are covariantly constant on-shell and satisfy the $SU(2)$ Clifford algebra, as required. As for the Killing spinors in this case one finds that they are constant spinors up to an overall phase $e^{\pm i\tau/4}$ that depends explicitly on τ .

Rotational isometries are more rare than translational isometries in 4-dim hyper-Kahler geometry. The only example known to this date with only rotational isometries is the Atiyah-Hitchin metric, whereas other metrics like the Eguchi-Hanson and Taub-NUT exhibit both (see for instance [10]). The flat space also exhibits both type of isometries. Choosing $\Psi = \log z$ as the simplest solution of the continual Toda equation we obtain the metric

$$ds^2 = z d\tau^2 + dx^2 + dy^2 + \frac{1}{z} dz^2,\tag{2.20}$$

which is the flat space metric written in terms of the coordinates $2\sqrt{z} \cos(\tau/2)$ and $2\sqrt{z} \sin(\tau/2)$. Introducing $r^2 = 4z$ and $\theta = \tau/2$ we obtain the metric in standard polar coordinates,

$$ds^2 = dx^2 + dy^2 + dr^2 + r^2 d\theta^2.\tag{2.21}$$

This example is rather instructive because it provides a good approximation of target space metrics around the fixed points (located at $r = 0$) of a Killing isometry. It is clear

that the dual background will have a curvature singularity at $r = 0$, where the σ -model formulation of strings is expected to break down. If one insists on exploring the dual metric from the low energy point of view by determining the corresponding dilaton field to lowest order in α' , space-time supersymmetry will appear to be lost. Clearly more powerful techniques should be used in order to understand the behaviour of strings close to this point.

One might think that any isometry with fixed points could lead to anomalous behaviour of space-time supersymmetry under duality, as it is seen from the effective theory viewpoint. This is however not true in general. The relevant analysis of this issue is rather simple for 4-dim spaces M . At a fixed point p the action of K gives rise to an isometry $T_p(M) \rightarrow T_p(M)$ on the tangent space, which is generated by the antisymmetric 4×4 matrix $\nabla_\mu K_\nu$. Any such matrix can have rank 2 or 4, provided that the Killing vector field K is not zero everywhere. For rank 2 the fixed points form a 2-dim subspace called bolt. The metric close to a bolt can be approximated by (2.21) and the apparent singularity at $r = 0$ is nothing but a coordinate singularity in the flat polar coordinate system on R^2 . For rank 4, p is an isolated fixed point called nut after the fixed point at the center of the self-dual Taub-NUT metric. In this case the metric has a removable singularity and it is approximated using the flat polar coordinate system on R^4 centered at the nut. Consider now for general M (not necessarily hyper-Kahler) the decomposition

$$\nabla_\mu K_\nu = K_{\mu\nu}^+ + K_{\mu\nu}^- \quad (2.22)$$

into self-dual and anti-self-dual parts. A nut is said to be self-dual (or anti-self-dual) if $K_{\mu\nu}^\pm$ is zero (ie., K is translational) at p . If M is hyper-Kahler, as it is the case of interest here, then K will be translational everywhere. Therefore we can have a Killing isometry with a fixed point (self-dual nut) which preserves manifest space-time supersymmetry under T-duality. For bolts, however, we always have

$$K_{\mu\nu}^+ K^{+\mu\nu} = K_{\mu\nu}^- K^{-\mu\nu} , \quad (2.23)$$

and so K_μ cannot be translational since otherwise both $K_{\mu\nu}^\pm = 0$ (ie., K_μ will be constant everywhere if M is hyper-Kahler). Only in this case manifest supersymmetry behaves anomalously under duality. For this reason we may consider the bolt-type metric (2.21) as a characteristic example of rotational isometries. In the next section we will study the 10-dim analogue of this in heterotic string theory and perform duality transformations with respect to rotations in a 2-dim plane.

We return now to the general situation where the T-duality transformation is performed on a purely gravitational background (2.1). The standard description of T-duality as a canonical transformation in τ and its conjugate momentum [11] amounts to

$$\tau \rightarrow \int (V^{-1} \partial \tau - \omega_1 \partial x - \omega_2 \partial y) dz - (V^{-1} \bar{\partial} \tau + \omega_1 \bar{\partial} x + \omega_2 \bar{\partial} y) d\bar{z} , \quad (2.24)$$

where we have set $\omega_3 = 0$. Applying this formula to the frame with translational isometries we find that the complex structures (2.14) become in the dual background [2]

$$\tilde{F}_i^\pm = V^{-1} (\pm d\tau \wedge dx^i - \frac{1}{2} \epsilon_{ijk} dx^j \wedge dx^k) . \quad (2.25)$$

Here \pm refers to the right or left structures, which are not the same because the duality generates non-trivial torsion (2.3). We also note in this case that the dual metric (2.2) is conformally flat (and hence hyper-Kahler) with a conformal factor $\Omega = V^{-1}$ that satisfies the Laplace equation (2.12) in agreement with (2.8). Hence, the dual background exhibits $N = 4$ superconformal symmetry which is locally realized with the aid of the dual complex structures, and space-time supersymmetry is manifest.

For rotational isometries the duality transformation (2.24) acts differently on the complex structures and yields the dual forms

$$\tilde{F}_3^\pm = V^{-1}(\pm d\tau \wedge dz + e^\Psi dx \wedge dy) , \quad (2.26)$$

and

$$\begin{aligned} \tilde{f}_1^\pm &= V^{-1}(\pm d\tau \wedge dx - dz \wedge dy) , \\ \tilde{f}_2^\pm &= V^{-1}(\pm d\tau \wedge dy + dz \wedge dx) . \end{aligned} \quad (2.27)$$

Then, we see explicitly that $\tilde{F}_{1,2}^\pm$ become non-locally realized in terms of the target space fields [2]. These non-local variables can be used explicitly to construct a new realization of the $N = 4$ superconformal algebra, which is preserved under duality. Indeed only the realization changes form, since otherwise T-duality would not be a string symmetry. Hence, we conclude that non-local world-sheet effects have to be taken into account in order to restore the space-time supersymmetry that is apparently lost in this case. If one takes seriously the low energy effective theory will face the paradox that the dual metric (2.2) is not conformally hyper-Kahler anymore. In fact, even if we have one local complex structure (provided in this example by \tilde{F}_3) we do not seem to have manifest $N = 1$ space-time supersymmetry, although the converse is always true. Further details on the resolution of this issue will be presented in the next section.

The non-local realization of the $N = 4$ superconformal algebra that arises in this context is reminiscent of the parafermionic realizations in conformal field theory. Unfortunately there is no exact conformal field theory description of the backgrounds (2.2), (2.3) and so to strengthen this analogy we appeal to another example that a similar problem arises after T-duality. Namely, consider the pair of coset models $SU(2) \times U(1)$ and $(SU(2)/U(1)) \times U(1)^2$, which are related by T-duality with respect to a rotational isometry. Both models have $N = 4$ world-sheet supersymmetry, but in the latter it is non-locally realized in terms of the $SU(2)/U(1)$ parafermions [12]. The relevant analysis for the transformation of the underlying complex structures has been performed in this case [2], in exact analogy with the above discussion, but we omit the details here. The background that arises to lowest order in α' for the second coset turns out not to be manifestly space-time supersymmetric.

A note of general value is that Killing spinors with explicit dependence on the Killing coordinate τ are not maintained after duality.

Finally, we discuss briefly the behaviour of supersymmetry under U-duality transformations. For 4-dim string backgrounds with one isometry it is possible to intertwine T

with S-duality non-trivially and produce new symmetry generators. The key point in this investigation is provided by the Ehlers transform, which for pure gravity is a continuous $SL(2)$ symmetry acting on the space of vacuum solutions [13, 7]. It acts non-locally, as its formulation requires the notion of the nut potential b . More precisely, the dimensional reduction of 4-dim gravity for the metrics (2.1) reads as follows,

$$\int d^4x \sqrt{\det G} R^{(4)} \rightarrow \int d^3x \sqrt{\det \gamma} \left(R^{(3)} - \frac{1}{2} V^{-2} (\partial_i V \partial^i V - \partial_i b \partial^i b) \right), \quad (2.28)$$

and so $b \pm V$ form a conjugate pair of $SL(2)/U(1)$ σ -model variables. The celebrated Ehlers transform is

$$\begin{aligned} V' &= \frac{V}{(Cb + D)^2 - C^2 V^2}, \\ b' &= \frac{(AD + BC)b + AC(b^2 - V^2) + BD}{(Cb + D)^2 - C^2 V^2}, \end{aligned} \quad (2.29)$$

where $AD - BC = 1$. Recall now the observation that the nut potential coincides with the axion field of the T-dual background (2.2), (2.3). Therefore the Ehlers transform behaves as an S-duality transformation, where one starts from a purely gravitational solution and reaches the new one (2.29) via a sequence of T-S-T transformation within the context of the string effective theory (switching on and then off again non-trivial torsion and dilaton fields). Straightforward generalization of this argument to the full massless sector leads to an enlarged symmetry of the β -function equations that is called U-duality (given in its continuous form). For 4-dim backgrounds S and T-S-T are two $SL(2)$ symmetries that combine into an $O(2, 2)$ group [1], while for the 10-dim heterotic string compactified on a 7-dim torus this procedure yields the bigger group $O(8, 24)$ [14].

U-duality transformations have a more severe effect on supersymmetries, provided that the intertwining of S with T is performed using rotational isometries. It is rather instructive for this purpose to consider the class of self-dual metrics and ask whether the Ehlers transform always preserves the self-duality. The answer is yes for translational isometries and no for rotational. A simple example to demonstrate this is provided by the $SL(2)$ action (2.29) on the flat space metric (2.20) written in polar coordinates. Choosing $A = D = 1$ and $B = 0$ we may verify that the new metric fails to be self-dual [1]. On the contrary, the same group element acts in the translational frame by a simple shift

$$V^{-1} \rightarrow V^{-1} + 2C, \quad (2.30)$$

and while it preserves the self-duality it has a non-trivial effect on the boundary conditions; starting from (2.13) with $\epsilon = 0$, the parameter C generates solutions with $\epsilon \neq 0$. The main point we wish to make here is that rotational isometries, from the space-time point of view in the T-S-T dual face, apparently destroy all three complex structures; otherwise Ricci-flatness would imply self-duality for the transformed metric (2.29).

It will be interesting to explore the possibility to have non-local realizations of supersymmetry in this case by reformulating the Ehlers symmetry as a non-local transformation on the target space coordinates. This way we hope to extend the results of

our investigation on “supersymmetry versus duality” to the most general situation for U-duality symmetries that arise by compactification to three or even two space-time dimensions [15].

3 Duality in heterotic string theory

We begin this section by working out the T-duality transformation for a general $(1, 0)$ heterotic σ -model with arbitrary connection and background gauge field. We find that if one does not want to have a non-local dual world-sheet action, due to anomalies which appear when implementing the duality transformation, one has to transform under the isometry the right-moving fermions. This yields a non-trivial transformation of the background gauge field under T-duality [3]. We also find that if in the original model the gauge and the spin connections match, so that there is anomaly cancellation in the effective theory, the change in the gauge field under T-duality ensures the same matching condition in the dual theory. Furthermore, if the original theory had $(2, 0)$ or $(2, 2)$ superconformal invariance, the dual theory also has these properties.

We consider a manifold M with metric G_{ij} , antisymmetric tensor field B_{ij} and a background gauge connection V_{iAB} associated to a gauge group $G \in O(32)$ (for simplicity we consider the $O(32)$ heterotic string). We introduce the $(1, 0)$ superfields

$$X^i(\sigma, \theta) = x^i + \theta \lambda^i, \quad \Psi^A(\sigma, \theta) = \psi^A + \theta F^A, \quad (3.1)$$

where $x^i(\sigma)$ are the fields embedding the world-sheet in the target space, F^A are auxiliary fields, and the fermions λ and ψ have opposite world-sheet chirality. Using light-cone coordinates on the world-sheet, $\sigma^\pm = (\sigma^0 \pm \sigma^1)/\sqrt{2}$, and defining the operator

$$D = \frac{\partial}{\partial \theta} + i\theta \frac{\partial}{\partial \sigma^+}, \quad D^2 = i\partial_+, \quad (3.2)$$

and

$$\mathcal{D}\Psi^A = D\Psi^A + V_i{}^A{}_B(X)DX^i\Psi^B, \quad (3.3)$$

we consider the Lagrangian density

$$L = \int d\theta \left(-i(G_{ij} + B_{ij})DX^i\partial_-X^j - \delta_{AB}\Psi^A\mathcal{D}\Psi^B \right). \quad (3.4)$$

Eliminating the auxiliary fields one obtains [16]

$$L = (G_{ij} + B_{ij})\partial_+x^i\partial_-x^j + iG_{ij}\lambda^iD_- \lambda^j + i\psi^AD_+\psi^A + \frac{1}{2}F_{ijAB}\lambda^i\lambda^j\psi^A\psi^B, \quad (3.5)$$

with world-sheet supercurrent of type $(1, 0)$,

$$\mathcal{G}_+ = (2G_{ij} + B_{ij})\partial_+x^i\lambda^j - \frac{i}{2}H_{ijk}\lambda^i\lambda^j\lambda^k. \quad (3.6)$$

We will carry out a duality transformation with respect to an isometry of the metric that leaves (3.4) and (3.5) invariant. Following the procedure outlined in [17], we gauge

the isometry, with some gauge fields A_{\pm} , and add an extra term with a Lagrange multiplier making the gauge superfield strength vanish. If we integrate out the Lagrange multiplier, the gauge superfields become pure gauge. Using the invariance of the action we can change variables to remove all presence of gauge fields and recover the original action. If instead we integrate first over A_{\pm} and then fix the gauge, we obtain the dual theory. In our case we will perform these steps in a manifestly $(1,0)$ -invariant formalism. Recall that the necessary condition for gauging an isometry generated by a Killing vector field K^i of the metric is [18]

$$K^i H_{ijk} = \partial_j U_k - \partial_k U_j , \quad (3.7)$$

and

$$\delta_K B_{ij} = \partial_i (K^l B_{lj} + U_j) - \partial_j (K^l B_{li}) , \quad (3.8)$$

for some vector U . Then, the conserved $(1,0)$ -supercurrent for the first term of (3.4) is

$$\mathcal{J}_- = (K_i - U_i) \partial_- X^i, \quad \mathcal{J}_+ = (K_i + U_i) D X^i , \quad (3.9)$$

so that $D\mathcal{J}_- + \partial_- \mathcal{J}_+ = 0$.

We now introduce $(1,0)$ gauge fields $\mathcal{A}_- = A_- + \theta \chi_-$ and $\mathcal{A} = \chi + i\theta A_+$ of bosonic and fermionic character respectively. If $\epsilon(\sigma, \theta)$ is the gauge parameter, we can take $\delta_\epsilon \mathcal{A}_- = -\partial_- \epsilon$ and $\delta_\epsilon \mathcal{A} = -D\epsilon$, and with the variation $\delta_\epsilon X^i = \epsilon K^i(X)$ the gauge invariant Lagrangian (assuming that $K^i U_i$ is constant) is

$$(G_{ij} + B_{ij}) D X^i \partial_- X^j + \mathcal{J}_+ \mathcal{A}_- + \mathcal{J}_- \mathcal{A} + K^2 \mathcal{A}_- \mathcal{A} . \quad (3.10)$$

The left-moving part $\Psi(D\Psi + V_i D X^i)\Psi$ is invariant under this global transformation when the isometry variation can be compensated by a gauge transformation

$$\begin{aligned} \delta X^i &= \epsilon K^i(X) , & \delta \Psi &= -\kappa \Psi , \\ \delta_K V_i &= \mathcal{D}_i \kappa = \partial_i \kappa + [V_i, \kappa] , \end{aligned} \quad (3.11)$$

which implies

$$K^i F_{ij} = D_j \mu ; \quad \mu = \kappa - K^i V_i . \quad (3.12)$$

Making ϵ a function of $(1,0)$ superspace one obtains after some algebra $\delta_\epsilon(\Psi^T \mathcal{D}\Psi) = D\epsilon \Psi^T \mu \Psi$, and hence, adding the coupling $\mathcal{A} \Psi^T \mu \Psi$ we achieve gauge invariance. The full gauge invariant Lagrangian reads

$$L = -i \left((G_{ij} + B_{ij}) D X^i \partial_- X^j + \mathcal{J}_+ \mathcal{A}_- + \mathcal{J}_- \mathcal{A} + K^2 \mathcal{A}_- \mathcal{A} \right) - (\Psi^T \mathcal{D}\Psi + \mathcal{A} \Psi^T \mu \Psi) . \quad (3.13)$$

Add now the Lagrange multiplier superfield term $i\Lambda(D\mathcal{A}_- - \partial\mathcal{A})$ and integrate out \mathcal{A} and \mathcal{A}_- to obtain the dual Lagrangian

$$\tilde{L}_{cl} = -i \left((\tilde{G}_{ij} + \tilde{B}_{ij}) D X^i \partial_- X^j + (\mathcal{J}_+ + D\Lambda) \frac{1}{K^2} (\partial_- \Lambda + i\Psi^T \mu \Psi - \mathcal{J}_-) \right) - \Psi^T \mathcal{D}\Psi . \quad (3.14)$$

In coordinates adapted to the Killing vector the dual values for \tilde{G}_{ij} , \tilde{B}_{ij} , \tilde{V}_i are

$$\begin{aligned}
\tilde{G}_{00} &= \frac{1}{K^2}, & \tilde{G}_{0\alpha} &= \frac{1}{K^2}U_\alpha, & \tilde{G}_{\alpha\beta} &= G_{\alpha\beta} - \frac{K_\alpha K_\beta - U_\alpha U_\beta}{K^2}, \\
\tilde{B}_{0\alpha} &= -\frac{1}{K^2}K_\alpha, & \tilde{B}_{\alpha\beta} &= B_{\alpha\beta} + \frac{K_\alpha B_{0\beta} - K_\beta B_{0\alpha}}{K^2}, \\
\tilde{V}_{0AB} &= -\frac{1}{K^2}\mu_{AB}, & \tilde{V}_{\alpha AB} &= V_{\alpha AB} - \frac{1}{K^2}(K_\alpha + U_\alpha)\mu_{AB}.
\end{aligned} \tag{3.15}$$

These results are equivalent to Buscher's formulae (see for instance [5]), but in this case we find a change in the background gauge field as well.

The preceding formulae were obtained using only classical manipulations. In general, however, there will be anomalies and the dual action may not have the same properties as the original one. Depending on the choice for μ and the gauge group $G \in O(32)$, the dual theory may be afflicted with anomalies, in which case the two theories are not equivalent. Equivalence would follow provided we include some Wess-Zumino-Witten terms generated by the quantum measure. If we want the two local Lagrangians L and \tilde{L} to be equivalent, we must find the conditions on G , B , V and μ in order to cancel the anomalies. Using standard techniques in anomaly computations, the variation of the effective action is expressible in the form (see [3] for details)

$$\begin{aligned}
\delta\Gamma_{eff} &= -\frac{1}{4\pi} \int Tr(\omega_i \partial_- X^i - \mathcal{A}_- \Omega) D(\epsilon(K^j \omega_j + \Omega)) d^3 Z \\
&\quad - \frac{1}{4\pi} \int Tr(V_i D X^i - \mathcal{A} \Omega) \partial_- (\epsilon(K^j V_j + \mu)) d^3 Z,
\end{aligned} \tag{3.16}$$

where $d^3 Z = d^2 \sigma d\theta$, and we have introduced the effective gauge field

$$V_-^a{}_b = \omega_-^a{}_b - A_- \Omega^a{}_b. \tag{3.17}$$

Note that the dual theory will contain non-local contributions unless we cancel the anomaly, which is a mixture between $U(1)$ and σ -model anomalies [19]. The simplest way to cancel it is to assume that the spin and gauge connection match in the original theory, a condition that also makes the space-time anomalies to cancel (see for instance [20]). In this case if $\mu = \Omega$ with matching quadratic Casimirs, the anomalous variation $\delta\Gamma_{eff}$ can be cancelled by a local counterterm. A test of the validity of the generalized duality transformation is that if we start with a theory with matching spin and gauge connection $\omega = V$, the dual theory is also guaranteed to have $\tilde{\omega} = \tilde{V}$. It is straightforward to verify this condition in our case. This is also important for the consistency of the model with respect to global world-sheet and target-space anomalies and it implies that if the original theory is conformally invariant to $O(\alpha')$, so is the dual theory.

There is yet one more possible source of anomalies under duality if the original model is $(2, 0)$ or $(2, 2)$ -superconformal invariant. In the $(2, 2)$ case for instance we have a $U(1)_L \times U(1)_R$ current algebra. The manifold has a covariantly constant complex structure,

$\nabla_k J^i_j = 0$, $J^i_k J^k_j = -\delta^i_j$. The R-symmetry is generated by the rotation $\delta\lambda^i = \epsilon J^i_j \lambda^j$ with current $\mathcal{J}_+ = iJ_{ij}\lambda^i\lambda^j$. This current has an anomaly

$$\partial_- \mathcal{J}_+ = -\frac{1}{4\pi} R_{ijk}{}^l J_l{}^k \partial_+ x^i \partial_- x^j, \quad (3.18)$$

which can be removed only if the right-hand side is cohomologically trivial. From [17] we know that T-duality preserves $N = 2$ global supersymmetry, hence, we should be able to improve the dual R-current so that the $U(1)_L \times U(1)_R$ current algebra is preserved, as needed for the application of the theorem in [21]. Since generically a T-duality transformation generates a non-constant dilaton, the energy-momentum tensor of the dual theory contains an improvement term due to the dilaton Φ of the form $\partial_+^2 \Phi$. As a consequence of $N = 2$ global supersymmetry there should also be an improvement term in the fermionic currents and in the $U(1)$ currents. Since the one-loop β -function implies (in complex coordinates) $R_{\alpha\bar{\beta}} \sim \partial_\alpha \partial_{\bar{\beta}} \Phi$, we can improve the $U(1)_L \times U(1)_R$ currents so that they are chirally conserved. In the (2,2) case the improvements are

$$\begin{aligned} \Delta \mathcal{J}_+ &= \partial_\alpha \Phi \partial_+ Z^\alpha - \partial_{\bar{\alpha}} \Phi \partial_+ Z^{\bar{\alpha}}, \\ \Delta \mathcal{J}_- &= -(\partial_\alpha \Phi \partial_- Z^\alpha - \partial_{\bar{\alpha}} \Phi \partial_- Z^{\bar{\alpha}}). \end{aligned} \quad (3.19)$$

With these improvements the currents are chirally conserved to order α' (and presumably to all orders, since the higher loop counterterms are cohomologically trivial for a (2,2) supersymmetric σ -model); hence we conclude that under duality the (2,0) or (2,2) superconformal algebra is preserved, as was also pointed out earlier. We meet the conditions to apply the Banks et al theorem [21] implying that the theory is space-time supersymmetric.

In the following we give a concrete answer to the puzzles raised in section 2 for space-time supersymmetry in the context of 10-dim heterotic string theory. We consider the motion in flat Minkowski space

$$ds^2 = dr^2 + r^2 d\theta^2 + (dx^i)^2 - (dx^0)^2, \quad i = 1, \dots, 7 \quad (3.20)$$

and revisit the problem of “space-time supersymmetry versus duality” first from the effective action point of view, and then within the framework of conformal field theory. The frame (3.20) provides the natural generalization of the bolt-type coordinates (2.21). Before duality we have the full $ISO(1,9)$ Lorentz invariance and $O(32)$ gauge symmetry. Since we perform duality in (3.20) with respect to rotations in a 2-dim plane, only the subgroup of $ISO(1,9)$ commuting with them will be a manifest local symmetry of the effective action. Similarly if we preserve manifest (1,0) supersymmetry on the world-sheet and avoid anomalies, we embed the isometry group $SO(2) \subset G \equiv SO(32)$. The subgroup of G commuting with $SO(2)$ is $SO(30) \times SO(2)$ and once again this will be a manifest symmetry in the low energy theory. It is well known that under T-duality, symmetries not commuting with the ones generating duality are generally realized non-locally. Hence although the dual background still contains all the original symmetries from the CFT point of view, the low energy theory does not seem to exhibit them. The theory will be

explicitly symmetric under $ISO(1,7) \times SO(30) \times SO(2)$ only. We want to make sure nevertheless that the original space-time supersymmetry is preserved, although not in a manifest $O(1,9)$ -covariant formalism. For this we can consider the variation of the fermionic degrees of freedom in a formalism adapted to the $ISO(1,7) \times SO(30) \times SO(2)$ symmetry, and look for which combination of the $O(1,9)$ fermions are annihilated by supersymmetry.

The low energy approximation to the heterotic string is given by $N = 1$ supergravity coupled to $N = 1$ super Yang-Mills in $d = 10$. In ten dimensions we can impose simultaneously the Majorana and Weyl conditions [22]; then, in terms of $SO(1,7)$, a Majorana-Weyl spinor of $SO(1,9)$ becomes a Weyl spinor. Write the Dirac algebra (in an orthonormal frame) as

$$\begin{aligned}\Gamma_\mu &= \tau_3 \otimes \gamma_\mu ; \quad \mu = 0, 1, \dots, 7 , \\ \Gamma_{7+i} &= i\tau_i \otimes 1 ; \quad i = 1, 2 , \\ \bar{\Gamma} &= \tau_3 \otimes \gamma_9 ,\end{aligned}\tag{3.21}$$

where γ_9 is the analogous of the 4-dim γ_5 in eight dimensions. Ten dimensional indices will be hatted. The supersymmetric variation of the ten-dimensional fermions is given by

$$\begin{aligned}\delta\hat{\Psi}_{\hat{\mu}} &= (\partial_{\hat{\mu}} - \frac{1}{4}\omega_{\hat{\mu}\hat{a}\hat{b}}\Gamma^{\hat{a}\hat{b}})\hat{\epsilon} , \\ \delta\hat{\lambda} &= (\Gamma^{\hat{\mu}}\partial_{\hat{\mu}}\Phi - \frac{1}{6}H_{\hat{\mu}\hat{\nu}\hat{\rho}}\Gamma^{\hat{\mu}\hat{\nu}\hat{\rho}})\hat{\epsilon} , \\ \delta\chi^A &= -\frac{1}{4}F^A_{\hat{\mu}\hat{\nu}}\Gamma^{\hat{\mu}\hat{\nu}}\hat{\epsilon}\end{aligned}\tag{3.22}$$

for the gravitino, dilatino and gluino, respectively.

The dual background is

$$\begin{aligned}d\tilde{s}^2 &= dr^2 + \frac{1}{r^2}d\tilde{\theta}^2 - (dx^0)^2 + (dx^i)^2 ; \quad i = 1, \dots, 7, \\ \Phi &= -\log r , \quad V_\mu dx^\mu = \frac{1}{r^2}d\tilde{\theta}M ,\end{aligned}\tag{3.23}$$

where M is the matrix describing the embedding of the spin connection in the gauge group, which we take to be the standard one acting only on two of the right-moving fermions. The background gauge field strength is

$$F = -\frac{2}{r^3}dr \wedge d\tilde{\theta} ,\tag{3.24}$$

and so decomposing the above variations with respect to $SO(1,7)$ we find: for the gravitino

$$\begin{aligned}\delta\Psi_\mu &= \partial_\mu\epsilon , \quad \delta\Psi_{\{r\}} = \partial_{\{r\}}\epsilon , \\ \delta\Psi_{\{\hat{\theta}\}} &= (\partial_\theta + \frac{i}{4r^2}\tau_3 \otimes 1)\epsilon ,\end{aligned}\tag{3.25}$$

for the dilatino

$$\delta\lambda = -\frac{i}{r}(\tau_1 \otimes 1)\epsilon, \quad (3.26)$$

and for the gluino

$$\delta\chi^A = 0, \quad \delta\chi = -\frac{i}{r^2}(\tau_3 \otimes 1)\epsilon \quad (3.27)$$

for $A \in SO(32)$ and along the embedded $SO(2)$ respectively. In the preceding formulas ϵ is an $SO(1,7)$ Weyl spinor with the same number of independent components as a 10-dim Majorana-Weyl spinor.

If we define now

$$\begin{aligned} \tilde{\Psi}_\mu &= \Psi_\mu, & \tilde{\Psi}_{\{r\}} &= \Psi_{\{r\}}, \\ \tilde{\Psi}_{\{\tilde{\theta}\}} &= \Psi_{\{\tilde{\theta}\}} + \frac{i}{4}e^\Phi(\tau_2 \otimes 1)\lambda, \end{aligned} \quad (3.28)$$

it is easy to see that the new fields transform as $\delta\tilde{\Psi}_\mu = \partial_\mu\epsilon$. Similarly,

$$\begin{aligned} \tilde{\lambda} &= \lambda + ie^{-\Phi}(\tau_2 \otimes 1)\chi, \\ \tilde{\chi} &= \chi - ie^\Phi(\tau_2 \otimes 1)\lambda \end{aligned} \quad (3.29)$$

have vanishing variation under space-time supersymmetry. Furthermore, they have the correct chiralities as dictated by the ten-dimensional multiplet. Thus if we use a formalism covariant only under the explicit $SO(1,7) \times SO(30) \times SO(2)$ symmetry of the dual background we recover the full number of supersymmetric charges. From the low-energy effective action this is the most we could expect since at the level of the world-sheet CFT the full symmetry $SO(1,9) \times SO(32)$ is only realized non-locally. If we want to consider the complete symmetry and the complete massless spectrum in the dual theory it seems that the only reasonable thing to do is to go back to the two-dimensional point of view.

In the remaining part we will see how to obtain in principle the vertex operators for the full massless spectrum in the dual theory by investigating in detail the way the full symmetry is realized from the conformal field theory point of view. The main point is to show explicitly that there are indeed world-sheet operators in the dual theory associated to the space-time supersymmetry charges, although some world-sheet non-locality is generated. We will find at this end an interesting interplay between the picture-changing operator and T-duality.

We are considering the effect of rotational duality in a 2-dim plane, and so the relevant part of the free heterotic Lagrangian is

$$L = \partial_+\vec{x} \cdot \partial_-\vec{x} + i\vec{\lambda} \cdot \partial_-\vec{\lambda} + i\psi^A \partial_+\psi^A + \dots, \quad (3.30)$$

where the vector quantities are two-dimensional. The isometry is $\vec{x} \rightarrow \exp(i\alpha\sigma_2)\vec{x}$, but for the time being we shall work in an unadapted frame. Hence for (3.30) we can perform duality only in the bosonic sector. The world-sheet supercurrent is

$$\mathcal{G}_+ = \vec{\lambda} \cdot \partial_+\vec{x} = \vec{\lambda} \cdot \vec{P}_+, \quad (3.31)$$

where \vec{P}_+ is a chiral current generating translations in the target space. It is convenient to work in canonical pictures [23] ($-\frac{1}{2}$ for fermion vertices, -1 for boson vertices). The space-time supersymmetry charge is

$$Q_\alpha^{(-\frac{1}{2})} = \oint e^{-\frac{\phi}{2}} S_\alpha , \quad (3.32)$$

where ϕ is the scalar which bosonizes the superconformal ghost current and S_α is the spin-field associated to the λ -fermions. The translation operator in the -1 picture is

$$P_\mu^{(-1)} = \oint e^{-\phi} \lambda_\mu . \quad (3.33)$$

Note that in these definitions only the space-time fermion and the (β, γ) -ghosts appear. Hence

$$\{Q_\alpha^{(-\frac{1}{2})}, Q_\beta^{(-\frac{1}{2})}\} = \Gamma^\mu P_\mu^{(-1)} \quad (3.34)$$

is satisfied, and if we choose to perform duality for the bosonic part of the Lagrangian only, the same relationships should still hold.

From this point of view there is clearly no problem with space-time supersymmetry. However, in constructing scattering amplitudes we need to use vertex operators in different pictures. Hence any problem should come from the interplay with the picture changing operator \mathcal{P} . The picture changing operator acting on a vertex operator $V_q(z)$ in the q -picture can be represented as [23]

$$\mathcal{P}V_q(z) = \lim_{w \rightarrow z} e^{\phi(w)} \mathcal{G}_+(w) V_q(z) . \quad (3.35)$$

The only possible difficulties may appear in anomalies in the world-sheet supercurrent under duality. Since \mathcal{G}_+ does not commute with purely bosonic rotations, after duality \mathcal{G}_+ will become non-local in the world-sheet. To guarantee that there are no problems with \mathcal{G}_+ we want to make sure that the dual world-sheet supercurrent still has the form $\vec{\lambda} \cdot \vec{\tilde{P}}_+$, where $\vec{\tilde{P}}_+$ is the representation of the translation current in the dual theory, and it is here that the non-locality resides. In fact the full theory in (3.30) can be constructed out of the knowledge that P_+^i ($i = 1, 2$) is chirally conserved and that its operator product expansion (OPE) is $P^i(z)P^j(w) \sim \frac{\delta^{ij}}{(z-w)^2}$. It is hard to believe that the existence of the chiral currents is going to be lost under duality. To make sure that this is not the case, the simplest thing to do is to include sources for these currents and then follow their transformation under duality.

Following [17] we gauge the symmetry $\vec{x} \rightarrow \exp(i\alpha\sigma_2)\vec{x}$ and concentrate only on the bosonic part of (3.30), the only one relevant due to the previous arguments. Thus our starting point is

$$L = D_+ x^T D_- x + \Lambda F_{+-} , \quad (3.36)$$

where $D_\pm x = \partial_\pm x + i\sigma_2 x A_\pm$, $F_{+-} = \partial_+ A_- - \partial_- A_+$ and Λ is a Lagrange multiplier. Here and in the following we drop for convenience the vectorial notation for \vec{x} and $\vec{\lambda}$, and introduce $\epsilon = i\sigma_2$. Using the Λ -equation of motion, $A_\pm = \partial_\pm \alpha$, $D_\pm x = e^{-\alpha\epsilon} \partial_\pm (e^{\alpha\epsilon} x)$,

and changing variables $x \rightarrow e^{-\alpha\epsilon}x$, the original theory is recovered. It proves convenient to parametrize locally

$$A_+ = \partial_+ \alpha_L, \quad A_- = \partial_- \alpha_R. \quad (3.37)$$

Then (3.36) has the symmetries

$$\delta x = e^{-\epsilon\alpha_R} a_R, \quad \delta \lambda = -x^T \epsilon e^{-\epsilon\alpha_R} a_R, \quad (3.38)$$

and

$$\delta x = e^{-\epsilon\alpha_L} a_L, \quad \delta \lambda = x^T \epsilon e^{-\epsilon\alpha_L} a_L, \quad (3.39)$$

yielding respectively the conserved currents

$$\partial_-(e^{\epsilon\alpha_R} D_+ x) = 0, \quad \partial_+(e^{\epsilon\alpha_L} D_- x) = 0. \quad (3.40)$$

When $\alpha_R = \alpha_L = \alpha$, we recover the original currents $\partial_\pm(e^{\epsilon\alpha}x)$. Acting on x , these symmetries commute.

The sources to be added to (3.36) should be gauge invariant,

$$J_- e^{\epsilon\phi_R} D_+ x + J_+ e^{\epsilon\phi_L} D_- . \quad (3.41)$$

The exponents are non-local in A_\pm , and they make the coupling gauge invariant. This also guarantees that the currents in (3.41) satisfy the OPE of the original theory as expected. In our example, let us take for simplicity $J_+ = 0$. The most straightforward way to integrate out the gauge fields is to work in the light-cone gauge $A_- = 0$. Then the integral over A_+ becomes a δ -function which can be solved in two ways. If we choose to solve it in order to write the Lagrange multiplier Λ as a function of the other fields, we recover the original theory.

On the other hand if we choose to solve the adapted coordinate to the isometry in terms of Λ and the other variables, we obtain the dual theory. Furthermore we also obtain a determinant, which when properly evaluated ([5] and references therein) yields the transformation of the dilaton. Using polar coordinates r, θ for the 2-dim plane we obtain the equation

$$\partial_- w = w \frac{i}{r^2} \partial_- \Lambda - \frac{r}{2i} (J_- - J_-^* w^2), \quad (3.42)$$

where $w = e^{i\theta}$ and J_-^* is the complex conjugate of J_- . This is a Riccati equation, which can be solved order by order in J_- . Restoring powers of α' we have

$$\partial_- w = iw \frac{\alpha'}{r^2} \partial_- \Lambda - \frac{r}{2i\alpha'} (J_- - J_-^* w^2). \quad (3.43)$$

The lowest order solution is

$$w = \exp i \int \frac{\alpha'}{r^2} \partial_- \Lambda d\sigma^-, \quad (3.44)$$

and to this order the current looks like

$$\partial_+ \left(r e^{i\theta[\Lambda, J_-]} \right) = \partial_+ \left(\rho e^{i\alpha' \int \frac{1}{r^2} \partial_- \Lambda d\sigma^-} \right). \quad (3.45)$$

The extra terms depending on J_- , J_-^* are required to guarantee the equality between the correlation functions before and after the duality transformation. To leading order the currents are:

$$\partial_+ \left(r e^{\pm i\alpha'} \int \frac{1}{r^2} \partial_- \Lambda d\sigma^- \right), \quad \partial_- \left(r e^{\pm i\alpha'} \int \frac{1}{r^2} \partial_+ \Lambda d\sigma^+ \right). \quad (3.46)$$

Note however that in solving (3.43) there will be corrections to all orders in α' in order to obtain the correct OPE's for the dual currents \tilde{P}_\pm . These currents can be used to write the emission vertex operators in the dual theory and they are almost always non-local. Since the OPE's of \tilde{P}_\pm are preserved, the spectrum of the original and the dual theories are equivalent. Nevertheless, we have to be careful regarding the operator mapping.

In conclusion, there is no problem with space-time supersymmetry from the point of view of CFT, but the correct operators that needed to be used to represent the emission vertices of low energy particles in the dual theory are often non-local, and do not admit a straightforward α' expansion unless we write the dual states in terms of those which follow from the correspondence as dictated by the duality transformation. When there are curvature singularities the approach based on the effective low energy theory has many limitations and to obtain reliable information we should go back to the underlying string theory. Finally to find the graviton, gravitino, etc vertex operators in the dual picture we could have solved the anomalous dimension operators in the dual background (including the dilaton and the background gauge fields), as was done for tachyons in [24]. We believe that the two approaches are equivalent.

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