Sum Rules for $B(M1, 0_1^+ \to 1^+)$ Strength in IBM-3 and IBM-4

P. Halse[1], P. Van Isacker[1], and B. R. Barrett[2]

[1] *GANIL, BP 5027, F-14021 Caen Cedex, France*

[2] *Physics Department, University of Arizona, Tucson, AZ 85721, U.S.A.*

ABSTRACT

Sum rules for $B(M1, 0_1^+ \to 1_i^+)$ strength are derived for even-even nuclei in the isospin-invariant forms of the IBM, IBM-3 and IBM-4, in the cases where the respective natural internal symmetries, isospin $U(3)$ and $U(6) \supset SU(4)$, are conserved. Subsequently, the total strength is resolved into its component partial sums to the allowed isospins (and $SU(4)$ representations in IBM-4). In cases where the usual IBM dynamical symmetries are also valid, a complete description of all $B(M1, 0_1^+ \to 1_i^+)$ is given. In contrast to IBM-2, there is fragmentation of the strength even in the dynamical symmetry cases, for $T \neq 0$, over two states in IBM-3, and over three states in IBM-4. The presence of pn bosons in the ground state of the extended versions reduces the expected strength from that for IBM-2, allowing in principle the possibility of using $B(M1)$ data for a given nucleus to infer which version is the most appropriate.

PACS codes: 12.60.Ev, 21.60.Fw

Keywords: interacting boson model, IBM-3 and IBM-4, M1 sum rules, scissor mode

One of the most important areas of research arising from use of the proton-neutron Interacting Boson Model (IBM-2) is the investigation of properties of nuclear levels corresponding to boson states of mixed symmetry in the pn and orbital degrees of freedom [1]. (Such structures are also present in other collective models.) Prominent amongst these are the so-called "scissor" modes in even-even nuclei, corresponding in the geometrical models to an oscillation in the angle between the symmetry axes of the deformed proton and neutron distributions, whose $J^\pi{=}1^+$ level is strongly excited by a largely orbital $M1$ process in (e,e') from the ground state [2]. Recently, an expression for the summed $B(M1)$ strength in IBM-2 has been derived [3]. It is found to depend upon the mean number of d-bosons in the ground state, and so can be used to estimate that number from the $B(M1)$ data.

For nuclei where the dominant shell model states involve valence protons and neutrons in the same orbits, the manifest isospin invariance suggests the inclusion of this feature also in the IBM. Two versions have received most attention: IBM-3 [4], the minimal isospin invariant model completing an isospin triplet of sd bosons by the addition of a $T{=}1$, $M_T{=}0$ complement (sometimes referred to as δ) to the π ($M_T{=}1$) and ν ($M_T{=}$-1) bosons of IBM-2, and so allowing classification by an isospin $U(3)$ group containing a boson realisation of the usual isospin $SU(2)$; IBM-4 [5], further augmented by the addition of a $T{=}0$, $S{=}1$ boson (sometimes referred to as σ), allowing classification by an isospin-spin $U(6)$ group that can be reduced via a Wigner supermultiplet $SU(4)$ to separate $SU(2)$ groups for the isospin and spin.

In this letter, we present IBM-3 and IBM-4 $B(M1)$ sum rules for even-even nuclei in cases where the above internal symmetries are conserved, and subsequently resolve the total strength into its components to each internal symmetry representation (isospin, $SU(4)$ label). Finally, for the usual IBM dynamical symmetries, individual $B(M1)$'s are completely specified. Thus we obtain various expressions appropriate to the symmetry-limit cases, which may be used to give "benchmark" values as have often proved useful in IBM work. In addition, these will be seen to allow the possibility of using the $B(M1)$ data to infer which version of the IBM is the most appropriate; the uncertainties involved are discussed.

M1 Sum Rules

Although the IBM-4 magnetic dipole operator could in principle contain many terms involving combinations of the various orbital, spin, and isospin operators, the one-boson analogues of those in the nucleon operator are expected to be the most important,

$$\mathbf{T}^{(1)} = \sqrt{\frac{3}{4\pi}}\,(g_{l0}\mathbf{L} + \frac{1}{2}g_{l1}\sum_k T_0(k)\mathbf{L}(k) + g_{s0}\mathbf{S} + g_{s1}\mathbf{Y}_0), \tag{1}$$

$$g_{l1} = g_\pi - g_\nu. \tag{2}$$

This restricted form has indeed proved satisfactory in previous applications [6, 7].

The summed $B(M1)$ strength can be equated to a ground state expectation value

$$\sum_i B(M1, 0_1^+ \to 1_i^+) = \langle 0_1^+ | \mathbf{T}^{(1)} \cdot \mathbf{T}^{(1)} | 0_1^+ \rangle, \tag{3}$$

where the dot denotes the angular momentum scalar product. Natural $U(6) \supset SU(4)$ and TS labels for the IBM-4 N-boson isospin-T ground state are $[N]$ $(0T0)$ T 0 [5], in which case analysis of selection rules reveals that only the (tl).(tl) term in the product contributes:

$$\sum_i B(M1, 0_1^+ \to 1_i^+) = \frac{3}{16\pi} g_{l1}^2 \langle 0_1^+ | \left(\sum_k T_0(k) \mathbf{L}(k) \right) \cdot \left(\sum_k T_0(k) \mathbf{L}(k) \right) | 0_1^+ \rangle. \tag{4}$$

The situation for IBM-3, where the natural magnetic dipole operator is obtained by omitting the spin terms from that for IBM-4, thus differs only in the labels for the ground state, which are $[N]$ T with respect to $U(3) \supset SU(2)$ [4]. Indeed, the expression (4) also accommodates the case of IBM-2, where $\frac{1}{2} T_0$ would be written as F_0, and the ground state carries the (F-spin) $U(2)$ label $[N]$. In all cases, we have

$$M_T = N_\pi - N_\nu, \quad T = |M_T|. \tag{5}$$

The assumed total symmetry of the ground state allows the replacement [6]

$$\begin{aligned} \left(\sum_k T_0(k) \mathbf{L}(k) \right) \cdot \left(\sum_k T_0(k) \mathbf{L}(k) \right) &= \sum_k T_0(k)^2 \mathbf{L}(k) \cdot \mathbf{L}(k) + \sum_{k \neq k'} T_0(k) T_0(k') \mathbf{L}(k) \cdot \mathbf{L}(k') \\ &\to \frac{1}{N} \sum_k T_0(k)^2 \sum_k \mathbf{L}(k) \cdot \mathbf{L}(k) \\ &+ \frac{1}{N(N-1)} \sum_{k \neq k'} T_0(k) T_0(k') \sum_{k \neq k'} \mathbf{L}(k) \cdot \mathbf{L}(k'). \end{aligned} \tag{6}$$

Continuing, we have for expectation values in the $L = 0, |M_T| = T$ ground state

$$\sum_{k \neq k'} \mathbf{L}(k) \cdot \mathbf{L}(k') = \mathbf{L} \cdot \mathbf{L} - \sum_k \mathbf{L}(k) \cdot \mathbf{L}(k) \to -\sum_l l(l+1) n_l, \tag{7}$$

$$\sum_{k \neq k'} T_0(k) T_0(k') = T_0^2 - \sum_k T_0(k) T_0(k) \to T^2 - (N - \langle N_{pn} \rangle), \tag{8}$$

where N_{pn} is the number operator for pn (M_T=0) bosons, i.e. δ and σ in IBM-4, δ in IBM-3, the expectation value being trivially zero in IBM-2. Thus we have

$$\sum_i B(M1, 0_1^+ \to 1_i^+) = \frac{3}{16\pi} g_{l1}^2 \, \Sigma_l l(l+1) \langle n_l \rangle \frac{(N-T)(N+T) - N\langle N_{pn} \rangle}{N(N-1)}. \tag{9}$$

$$\sum_i B(M1, 0_1^+ \to 1_i^+) = \frac{9}{8\pi}\, g_{l1}^2\, \langle n_d \rangle\, \frac{(N-T)(N+T) - N\langle N_{pn}\rangle}{N(N-1)}. \tag{10}$$

For IBM-2, with N_{pn}=0, this reduces to the expression derived previously by Ginocchio [3],

$$\sum_i B(M1, 0_1^+ \to 1_i^+) = \frac{9}{4\pi}\, g_{l1}^2\, \langle n_d \rangle \frac{2N_\pi N_\nu}{N(N-1)}. \tag{11}$$

It is apparent that the inclusion of the pn bosons reduces the expected $B(M1)$ strength, which opens up the possibility of using data on $B(M1)$'s to infer which version of the model is the most appropriate; we return to this point below.

Now consider the derivation of $\langle N_{pn} \rangle$ in IBM-4 and IBM-3; in fact, it is of interest to calculate the separate values $\langle N_\delta \rangle$ and $\langle N_\sigma \rangle$ in IBM-4. It is convenient to introduce

$$N_\sigma = \frac{1}{2}\,(N - \Delta), \quad \Delta = N_{(10)} - N_\sigma, \tag{12}$$

$$N_\delta = \frac{1}{3}\,\left(N_{(10)} + \sum_k Q_0(k) \right). \tag{13}$$

where Q_0 is the isospin quadrupole operator, normalised to have matrix elements -1, 2, -1 for π, δ, ν respectively.

The homomorphism $SU(4) \sim SO(6)$ suggests that the structure $U(6) \supset SU(4)$, as well as $U(3) \supset SO(3)$, should be associated with a complementary boson quasispin group $SU(1,1)$ [8], allowing the use of reduction formulae in the evaluation of matrix elements. Indeed, explicit realisations are given by the canonical forms, where Ω equals half the number of internal states, $\Omega = 3$ and $3/2$ for IBM-4 and IBM-3 respectively, and B^+ creates the $SO(2\Omega)$ scalar (seniority zero) pair:

$$S_+ = \sqrt{\Omega}\, B^+ = \begin{cases} \sqrt{\frac{3}{2}}\, B^{+(0)} & = \frac{1}{2}\, \mathbf{b}^{+(10)} \cdot \mathbf{b}^{+(10)} \qquad (\mathrm{IBM}-3) \\ \sqrt{3}\, B^{+(000)(00)} & = \frac{1}{2}\, (\mathbf{b}^{+(10)} \cdot \mathbf{b}^{+(10)} - \mathbf{b}^{+(01)} \cdot \mathbf{b}^{+(01)}) \quad (\mathrm{IBM}-4) \end{cases},$$

$$S_- = (S_+)^+\,, \quad S_0 = \frac{1}{2}(N + \Omega). \tag{14}$$

Under commutation with the relevant generators, the operators Δ (in IBM-4) and Q_0 (in IBM-3 and IBM-4) both transform according to the finite dimensional representations labeled by $(S = 1,\ M = 0)$, while the states $|[N](0T0)\rangle$ and $|[N]T\rangle$ transform according to the infinite dimensional unitary representations labeled by $(S = (T+\Omega)/2, M = (N+\Omega)/2)$. Thus, using analytic continuations of the usual $SU(2)$ 3-j symbols [8],

$$\begin{aligned} \langle SM|(10)|SM\rangle &= (-)^{S-M} \begin{pmatrix} -S & -S & 1 \\ M & -M & 0 \end{pmatrix} \Big/ \begin{pmatrix} -S & -S & 1 \\ S & -S & 0 \end{pmatrix} \times \langle SS|(10)|SS\rangle \\ &= \frac{M}{S}\, \langle SS|(10)|SS\rangle. \end{aligned} \tag{15}$$

$$\langle [N](0T0)(T0)|\Delta|[N](0T0)(T0)\rangle = \frac{T(N+3)}{T+3}, \tag{16}$$

$$\langle [N](0T0)(T0)|Q_0|[N](0T0)(T0)\rangle = -\frac{T(N+3)}{T+3}, \tag{17}$$

$$\langle [N]T|Q_0|[N]T\rangle = -\frac{T(N+3/2)}{T+3/2} = -\frac{T(2N+3)}{2T+3}, \tag{18}$$

so that

$$\langle [N](0T0)(T0)|N_\sigma|[N](0T0)(T0)\rangle = \frac{3(N-T)}{2(T+3)}, \tag{19}$$

$$\langle [N](0T0)(T0)|N_\delta|[N](0T0)(T0)\rangle = \frac{N-T}{2(T+3)}, \tag{20}$$

$$\langle [N](0T0)(T0)|N_{pn}|[N](0T0)(T0)\rangle = \frac{2(N-T)}{T+3}, \tag{21}$$

$$\langle [N]T|N_\delta|[N]T\rangle = \frac{N-T}{2T+3}, \tag{22}$$

where Eqn.(22) may also be obtained using the standard matrix elements of Q_0 in the $SU(3) \supset SO(3)$ representations $(N0)\ T$ [8].

Thus final expressions for the $B(M1)$ sum rules in IBM-3 and IBM-4 are

$$\begin{aligned}
\mathrm{IBM-3}: \ \sum_i B(M1, 0_1^+ \to 1_i^+) &= \frac{9}{8\pi}\, g_{l1}^2\, \langle n_d\rangle\, \frac{N-T}{N(N-1)}\left(N+T\ -\ \frac{N}{2T+3}\right) \\
&= \frac{9}{8\pi}\, g_{l1}^2\, \langle n_d\rangle\, \frac{(N-T)(2N(T+1)+T(2T+3))}{(N-1)N(2T+3)}, \qquad (23) \\
\mathrm{IBM-4}: \ \sum_i B(M1, 0_1^+ \to 1_i^+) &= \frac{9}{8\pi}\, g_{l1}^2\, \langle n_d\rangle\, \frac{N-T}{N(N-1)}\left(N+T\ -\ \frac{2N}{T+3}\right) \\
&= \frac{9}{8\pi}\, g_{l1}^2\, \langle n_d\rangle\, \frac{(N-T)(N(T+1)+T(T+3))}{(N-1)N(T+3)}. \qquad (24)
\end{aligned}$$

Resolution over Isospins and Wigner Supermultiplets

Comparison of the final state internal symmetry representations contained in the $U(2\Omega)$ representation $[N-1,1]$ with those arising in the Kronecker products for the ground state (isospin T) and $\sum_k T_0(k)\mathbf{L}(k)$, yields

$$\mathrm{IBM-3}: \ [N-1,1] \quad T\ (T \neq 0 \text{ or } N), \quad T+1\ (T \neq N), \tag{25}$$

$$\begin{aligned}
\mathrm{IBM-4}: \ [N-1,1] \quad & (0T0)(T0)\ (T \neq 0 \text{ or } N) \\
& (1T1)(T0)\ (T \neq 0 \text{ or } N), \quad (T+1,0)(T \neq N). \qquad (26)
\end{aligned}$$

The ratios of $B(M1)$ strength to subspaces defined by the representations of the various symmetry labels, including the experimentally accessible isospin, involve simply ratios of the group coupling coefficients.

$$\frac{\sum_i B(M1, 0_1^+ \to (1^+, T)_i)}{\sum_i B(M1, 0_1^+ \to (1^+, T+1)_i)} = \frac{\langle (N0)T\ (10)1 | (N-1,1)T \rangle^2}{\langle (N0)T\ (10)1 | (N-1,1)T+1 \rangle^2} \times \frac{\langle TT\ 10 | TT \rangle^2}{\langle TT\ 10 | T+1\ T \rangle^2} = \frac{T(2T+3)(N+T+1)}{(T+2)N}, \tag{27}$$

so that

$$\sum_i B(M1, 0_1^+ \to (1^+, T)_i) = \frac{9}{8\pi}\, g_{l1}^2\, \langle n_d \rangle\, \frac{T(N-T)(N+T+1)}{(T+1)N(N-1)}, \tag{28}$$

$$\sum_i B(M1, 0_1^+ \to (1^+, T+1)_i) = \frac{9}{8\pi}\, g_{l1}^2\, \langle n_d \rangle\, \frac{(T+2)(N-T)}{(T+1)(2T+3)(N-1)}. \tag{29}$$

For IBM-4, $SU(4) \supset SU(2) \times SU(2)$ and Clebsch-Gordan coefficients yield

$$\frac{\sum_i B(M1, 0_1^+ \to (1^+, (1T1)T)_i)}{\sum_i B(M1, 0_1^+ \to (1^+, (1T1)T+1)_i)} = \frac{\langle (0T0)T0\ (101)10 | (1T1)T0 \rangle^2}{\langle (0T0)T0\ (101)10 | (N-1,1)T+1, 0 \rangle^2} \times \frac{\langle TT\ 10 | TT \rangle^2}{\langle TT\ 10 | T+1\ T \rangle^2} = \frac{3T}{T+4}. \tag{30}$$

Ratios involving the $B(M1)$ sum to $(0T0)$ could also be derived, using in addition coefficients for $U(6) \supset SU(4)$. However, we note that the complete resolution of the strength can be simply obtained in this case by evaluating the $B(M1)$ sum to $(1T1)\ T+1$ via the ground state expectation value of $T^{(1)}\, T_- T_+\, T^{(1)}/2(T+1)$ (for $M_T = +T$); this follows the derivation presented above. One finds

$$\sum_i B(M1, 0_1^+ \to (1^+, (0T0)T)_i) = \frac{9}{16\pi}\, g_{l1}^2\, \langle n_d \rangle\, \frac{2T(N-T)(N+T+4)}{(T+4)N(N-1)}, \tag{31}$$

$$\sum_i B(M1, 0_1^+ \to (1^+, (1T1)T)_i) = \frac{9}{16\pi}\, g_{l1}^2\, \langle n_d \rangle\, \frac{3T(T+2)(N-T)}{(T+1)(T+3)(T+4)(N-1)}, \tag{32}$$

$$\sum_i B(M1, 0_1^+ \to (1^+, (1T1)T+1)_i) = \frac{9}{16\pi}\, g_{l1}^2\, \langle n_d \rangle\, \frac{(T+2)(N-T)}{(T+1)(T+3)(N-1)}. \tag{33}$$

$B(M1)$'s in Dynamical Symmetries

In cases where the sd space dynamical symmetries $(U(5), O(6), SU(3))$ are valid, transitions proceed to at most one orbital representation [1]. However, from the presentation above it is seen that there is still generally fragmentation of the M1 strength, in contrast to IBM-2, over two states in IBM-3, and three in IBM-4, unless $T = 0$ when only one transition is allowed in both versions. Furthermore, analytic values are available for $\langle n_d \rangle$ [1],

$$\mathrm{U(5)}: \langle \mathrm{n_d} \rangle = 0, \tag{34}$$

$$\mathrm{SO(6)}: \langle \mathrm{n_d} \rangle = \frac{N(N-1)}{2(N+1)}, \tag{35}$$

$$\mathrm{SU(3)}: \langle \mathrm{n_d} \rangle = \frac{4N(N-1)}{3(2N-1)}, \tag{36}$$

relevant expression (Eqns.(28,29) and (31-33)).

Discussion

Since the above expressions for $M1$ strength differ between IBM-2, -3, and -4, they furnish a possible means of inferring from $M1$ data which version is the most appropriate for a given nucleus. However, it should be noted that: 1) The extended versions are indicated by successively reduced strength, and so might be wrongly implicated by some $M1$'s being undetected. 2) The effects of departures from the assumed internal symmetries are not known; in particular, $SU(4)$ breaking in IBM-4 may lead to involvement of the strong isovector spin term. 3) Another extension of the IBM(-2), to include g-bosons, leads to increased $M1$ strength [10], which could offset or even reverse any decrease due to the presence of pn bosons.

Acknowledgements

One of us (BRB) acknowledges the hospitality of GANIL, where this work was carried out, and PVI thanks Joe Ginocchio for a discussion concerning this work. This research is supported in part by the National Science Foundation, Grant No. INT-9415451 and by the Centre National de Recherche Scientifique.

1. F. Iachello and A. Arima, *The Interacting Boson Model*, Cambridge University Press, Cambridge, 1987, and references therein.
2. A. Richter, Nucl. Phys. A522 (1991) 139c, and references therein.
3. J. N. Ginocchio, Phys. Lett. B 265 (1991) 6.
4. J. P. Elliott and A. P. White, Phys. Lett 97B (1980) 169.
5. J. P. Elliott and J. A. Evans, Phys. Lett 101B (1981) 216.
6. P. Halse, J. P. Elliott, and J. A. Evans, Nucl. Phys. A417 (1984) 301.
7. P. Halse, Nucl. Phys. A445 (1985) 93.
8. B. G. Wybourne, *Classical Groups for Physicists*, Wiley, New York, 1974.
9. J. D. Vergados, Nucl. Phys. A111 (1968) 681.
10. B. R. Barrett and P. Halse, Phys. Lett 155B (1985) 133.