A Causal Order for Spacetimes with C^0 Lorentzian Metrics: Proof of Compactness of the Space of Causal Curves

R.D. Sorkin
Dept. of Physics, Syracuse University, Syracuse, NY13244–1130, USA
and
Instituto de Ciencias Nucleares, UNAM, A. Postal 70-543 México, D.F. 04510, México
E-Mail: rdsorkin@mailbox.syr.edu

and

E. Woolgar
Dept. of Mathematics, University of Saskatchewan, 106 Wiggins Ave., Saskatoon, SK, Canada S7N 5E6
and
Winnipeg Institute for Theoretical Physics, Dept. of Physics, University of Winnipeg, 515 Portage Ave., Winnipeg, MB, Canada R3B 2E9
E-Mail: woolgar@math.usask.ca

Abstract

We recast the tools of "global causal analysis" in accord with an approach to the subject animated by two distinctive features: a thoroughgoing reliance on order-theoretic concepts, and a utilization of the Vietoris topology for the space of closed subsets of a compact set. We are led to work with a new causal relation which we call K^+, and in terms of it we formulate extended definitions of concepts like causal curve and global hyperbolicity. In particular we prove that, in a spacetime $\mathcal{M}$ which is free of causal cycles, one may define a causal curve simply as a compact connected subset of $\mathcal{M}$ which is linearly ordered by K^+. Our definitions all make sense for arbitrary C^0 metrics (and even for certain metrics which fail to be invertible in places). Using this feature, we prove for a general C^0 metric, the familiar theorem that the space of causal curves between any two compact subsets of a globally hyperbolic spacetime is compact. We feel that our approach, in addition to yielding a more general theorem, simplifies and clarifies the reasoning involved. Our results have application in a recent positive energy theorem [1], and may also prove useful in the study of topology change [2]. We have tried to make our treatment

Published *Class. Quant. Grav.* **13**: 1971-1994 (1996); e-print archive: gr-qc/9508018.

self-contained by including proofs of all the facts we use which are not widely available in reference works on topology and differential geometry.

1. Introduction

Global geometrical properties of manifolds often condition the behavior of curves within the manifold. For example, in a riemannian manifold, the metric property of completeness implies the existence of a minimum length (and therefore geodesic) curve joining any two points of the manifold (Hopf-Rinow theorem [3]). In a lorentzian manifold, the situation is somewhat different. Here properties of order and topology tend to take precedence over strictly metric properties, and one is primarily interested, not in arbitrary curves, but in causal ones. In the lorentzian case there are two analogues of a minimal geodesic: a longest timelike curve and (*cf.* [4], [1]) a "fastest causal curve" (necessarily a null geodesic). An analogue of the Hopf-Rinow result does exist, but the causal property of global hyperbolicity replaces that of Cauchy completeness as the basic assumption. Moreover the compactness of the space of causal curves between two points of the manifold now becomes an essential part of the argument. This compactness thus plays a crucial role in the global analysis of lorentzian manifolds. In particular, many of the singularity theorems of general relativity rely directly on it, as does a recent proof of positivity of the total energy in general relativity [1].

For this last application however, the standard compactness theorem does not suffice, because it presupposes the existence of convex normal neighborhoods, which in turn requires that the metric be of differentiability class C^2 (or at least C^{2-}).* On the other hand, since the compactness theorem itself envisions causal curves which are not even differentiable, one might expect that this degree of smoothness should not be necessary. Below we will confirm this expectation by proving a compactness theorem which holds even when the metric is only C^0.

In fact, most of our results will be meaningful and true even when the metric is degenerate in places, and thus is not strictly lorentzian (nor even strictly C^0 insofar as the inverse metric is concerned). This may be of importance in the study of topology

* whereas the proof in [1] takes place in conformally extended spacetime, which is not smooth at the point at spatial infinity, i^0.

change, because topology-changing manifolds (non-product cobordisms) in general require such degeneracies [2].

In making do without recourse to smoothness assumptions, we are led to a set of definitions and proofs which rely more directly on order-theoretic ideas than has been customary heretofore. The resulting framework is not only of wider applicability than previous ones, but we believe it affords a proof of our main theorem which is conceptually simpler as well. In the same way, we suspect that the bulk of the theory of "global causal analysis" could instructively be recast along such order-theoretic and general-topological lines, the main attraction in doing so being the prospect of a more elementary and more perspicuous development. This, and not greater generality *per se*, is what primarily motivated the work reported herein. (Indeed, if accommodation to C^0 metrics were the only goal, we suspect that existing approaches could be adapted to that end without undue fuss.)

Two technical modifications animating our treatment are the use of an unfamiliar topology in studying convergence of families of curves, and the introduction of a new causal relation (which we have dubbed K^+ in analogy with the notations I^+ and J^+) in terms of which our notions of causal curve and of global hyperbolicity are defined. The main advantage of K^+ in comparison with I^+ and J^+ is that it is topologically closed as well as transitive, which helps for example in the proofs of Lemmas 14 and 15 and Theorems 20 and 21 below. The topology just alluded to is the so-called Vietoris topology, which is naturally defined not just on curves, but on the space $2^{\mathcal{X}}$ of all closed non-empty subsets of a topological space $\mathcal{X}$. Its primary advantage is the availability of the compactness result, Theorem 1, but it has as well the minor virtue that the space of causal curves becomes automatically Hausdorff in a natural way, even when the endpoints are not held fixed. We note that the use of the Vietoris topology in proving compactness might be adapted to spaces of more general objects than causal curves (null hypersurfaces, closed future/past sets, *etc.*), although we will not pursue this here.

An outline of the remainder of this paper is as follows. Section 2 is a summary of the general facts we will need involving ordered sets on one hand, and the Vietoris topology on the other hand. The key properties of the latter are that a Vietoris limit of connected subsets of a bicompact space is connected, and that $2^{\mathcal{M}}$ is bicompact whenever $\mathcal{M}$ itself is (bicompact $\equiv$ compact and Hausdorff). In connection with ordered spaces, we give general criteria for a poset to be isomorphic to the real unit interval $[0, 1]$, and for a space which carries both an order and a topology to be simultaneously isomorphic to $[0, 1]$ in both respects. Complete proofs for these results will be found in Appendix A.

Section 3 is in a sense the heart of the paper. In it we define our basic causality relation $K^+ = \prec$ to be the smallest closed* and transitive relation containing I^+; and based on this definition, we introduce the elements of our framework for "C^0 causal analysis", including: a causality condition called "K-causality" which requires that K^+ be a partial order; a generalized definition of causal curve and of global hyperbolicity; the fact that a K-causal manifold is automatically "locally causally convex"; and a simple characterization of a causal curve — as a compact connected subset linearly ordered by K^+ — which is valid in an arbitrary K-causal spacetime, and which we use when proving our main theorem in the following section. The proof of the main theorem itself in Section 4 is short and straightforward, and amounts to showing that the Vietoris limit of a sequence of causal curves is itself a causal curve. This being demonstrated, the compactness properties of the Vietoris topology lead directly to the desired conclusion, namely that the space of causal curves between two compact subsets of a globally hyperbolic C^0-lorentzian manifold is itself compact.

Appendix B is devoted to a comparison of our definitions with ones which are standard in the C^2 situation. In particular we prove that, when restricted to C^2 metrics, our definitions of causal curve and of global hyperbolicity of a manifold agree with the usual ones.

Concerning notation, a *C^p-lorentzian manifold* $\mathcal{M}$ means herein a time-oriented (paracompact) manifold of differentiability class C^{p+1} ($p \geq 0$), equipped with a symmetric tensor field g_{ab} of differentiability class C^p and signature $(-1, 1, 1, 1, \ldots)$. (A time-orientation for such a manifold can be given by choosing a nowhere-vanishing timelike vector field u^a — provided one exists.) Any manifold which occurs in this paper will be assumed, without explicit mention, to be at least C^0-lorentzian. We use the abstract index convention, denoting such indices by early Latin letters. Finally, we use throughout the convention that the symbol Γ denotes the image of γ, where $\gamma : [0, 1] \to \mathcal{M}$ is a *path* or parameterized curve.

Finally, we note here that none of our considerations will actually depend on making a choice of representative metric within a given conformal equivalence class.

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* Closure here means topological closure with respect to the manifold topology. One could also entertain an order-theoretic notion of closure by letting $x_n \nearrow x$ (respectively $x_n \searrow x$) mean that x is the supremum (respectively infimum) of the x_n, and defining closure in terms of this kind of convergence. This possibility raises the prospect that order will eventually absorb both geometry *and* topology in the basic formulations of the subject.

2. Some Definitions and Theorems Concerning Ordered Sets and the Vietoris Topology

In this preliminary section we assemble some definitions and theorems of a general character which will be useful later. As none of these theorems relate specifically to lorentzian manifolds, we do not prove them here; but, for completeness, and as some of the theorems are not easily located in the literature, we do give full proofs in Appendix A.

We begin with the Vietoris topology, which is a topology on the space $2^{\mathcal{X}}$ of non-empty closed subsets of an arbitrary topological space $\mathcal{X}$. Later we will characterize it in terms of convergence; here we define it by giving the open sets directly, or rather by giving a *base* for them (a base being a collection of open sets whose arbitrary unions furnish the topology itself). Let $\{\mathcal{A}_0, \mathcal{A}_1, \ldots, \mathcal{A}_n\}$ be a finite collection of open subsets of $\mathcal{X}$. We define

$$\mathcal{B}(\mathcal{A}_0; \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = 2^{\mathcal{A}_0} \cap \left(2^{\mathcal{X}} \backslash 2^{\mathcal{X} \backslash \mathcal{A}_1}\right) \cap \left(2^{\mathcal{X}} \backslash 2^{\mathcal{X} \backslash \mathcal{A}_2}\right) \cap \ldots \cap \left(2^{\mathcal{X}} \backslash 2^{\mathcal{X} \backslash \mathcal{A}_n}\right) \quad ,$$

where $\mathcal{A}_1 \backslash \mathcal{A}_2$ is the set of all elements of $\mathcal{A}_1$ which are not elements of $\mathcal{A}_2$. Equivalently, if $\mathcal{C}$ is a closed subset of $\mathcal{X}$ then

$$\mathcal{C} \in \mathcal{B}(\mathcal{A}_0; \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \Longleftrightarrow \mathcal{C} \subseteq \mathcal{A}_0 \text{ and } \mathcal{C} \text{ meets } \mathcal{A}_i \text{ for } i = 1, \ldots, n \quad .$$

In other words, $\mathcal{C}$ must be large enough to intersect each of the $\mathcal{A}_i$ and small enough never to stray outside of $\mathcal{A}_0$. The sets $\mathcal{B}(\mathcal{A}_0; \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$ for finite n constitute a base for the Vietoris topology on $2^{\mathcal{X}}$. We can also give a somewhat simpler *sub-base* for this topology. It consists simply of sets of the form $B(\mathcal{A}; \mathcal{X})$ *together with* sets of the form $B(\mathcal{X}; \mathcal{A})$. The former are the sets whose elements are the closed subsets $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{A}$ being an open subset of $\mathcal{X}$. The latter are the sets of closed sets $\mathcal{C} \subseteq \mathcal{X}$ that meet the open set $\mathcal{A} \subseteq \mathcal{X}$. The finite intersections of sets of these two types produce the base sets $\mathcal{B}(\mathcal{A}_0; \mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$. Note that the empty subset of $\mathcal{X}$ is not an element of the space $2^{\mathcal{X}}$, ‡ and correspondingly, the limit of a sequence of non-empty closed sets is always non-empty itself. We will refer to limits with respect to the Vietoris topology as *Vietoris limits.*

The next two propositions, both intuitively plausible, will feed directly into the proof of the compactness theorem which is our principal result in Section 4.

Theorem 1: The topological space $2^{\mathcal{X}}$ is bicompact whenever $\mathcal{X}$ itself is.

‡ Because of this, the notation $2^{\mathcal{X}} - 1$ might be more evocative, albeit more clumsy. Still another notation is $\mathcal{F}_0(\mathcal{X})$, which is used in [5].

Lemma 2:

(i) In a compact space, every Vietoris limit is a compact set.

(ii) In a bicompact space, a Vietoris limit of connected sets is connected.

In addition to the use of the Vietoris topology, our approach rests on a liberal employment of order-theoretic concepts. For that reason, we gather here a number of definitions relating to ordered sets, and we state two general theorems giving criteria for an ordered space to be isomorphic to the closed unit interval $[0, 1]$. Both these theorems are proved in Appendix A.

Definition 3: A *partial order* is a relation $\prec$ defined on a set S which satisfies the axioms of

(i) asymmetry: $p \prec q$ and $q \prec p \Rightarrow p = q$, and

(ii) transitivity: $p \prec q$ and $q \prec r \Rightarrow p \prec r$.

The partial order is *reflexive* iff it observes the convention that every element of S precedes itself ($p \prec p \; \forall p \in S$), and *irreflexive* iff it observes the convention that no element precedes itself. (In the irreflexive case, condition (i) would be more naturally stated just as the condition that $p \prec q$ and $q \prec p$ never occur together.)

A partial order is often called simply an *order*, and a set endowed with an order is called an *(partially) ordered set* or *poset*.

Definitions 4: Let P be a reflexive poset with the order relation denoted by $\prec$. A *linearly ordered* subset or *chain* Q in P is a subset such that $x, y \in Q \Rightarrow$ either $x \prec y$ or $y \prec x$ (as is the case with $Q = \mathbb{R}$, for example). The corresponding order is called a *linear order* (also a *total order*). For any $x, y \in P$, we define the *order-open interval* $\langle x, y \rangle$ to be $\{r | x \prec r \prec y, \; | x \neq r \neq y\}$ and the *order-closed interval* $\langle\langle x, y \rangle\rangle$ to be simply $\{r | x \prec r \prec y\}$, with the half-open intervals $\langle\langle p, q \rangle$ and $\langle p, q \rangle\rangle$ defined analogously.* A *minimum* element of a poset (necessarily unique) is one which precedes every other element, and dually for a *maximum* element. We will always denote the former by 0 and the latter by 1. For subsets A and B of P, we write $A \prec B$ to mean that $a \prec b \; \forall a \in A, b \in B$. A subset $S \subseteq P$ is *order-dense* in P iff it meets every non-empty order-open interval $\langle x, y \rangle$. A function $f : X \to Y$, for posets X and Y, is a *tonomorphism* or *order-isomorphism*, and we say that

* The terms "open" and "closed" seem appropriate for the linear orders we will be dealing with in this paper. More generally, the words "exclusive" and "inclusive" might be more descriptive.

X and Y are *tonomorphic* or *order-isomorphic* via f, iff f is an order-preserving bijection, in which case we have, clearly, $x_1 \prec_X x_2 \Leftrightarrow f(x_1) \prec_Y f(x_2)$.

Theorem 5: In order that a poset X be tonomorphic to the closed unit interval $[0,1] \subseteq \mathrm{I\!R}$, it is necessary and sufficient that

(i) X be linearly ordered with both a minimum and a maximum element (denoted $0, 1$ respectively),

(ii) X have a countable order-dense subset, and

(iii) every partition of X into disjoint subsets $A \prec B$ be either of the form $A = \langle\langle 0, x\rangle\rangle$, $B = \langle x, 1\rangle\rangle$ or of the form $A = \langle\langle 0, x\rangle$, $B = \langle\langle x, 1\rangle\rangle$.

Condition (iii) is just the requirement that both A and B be intervals, as defined above, with one of them half-open and the other one closed. Its failure is often expressed by saying that X has either "jumps" (which occur when $A = \langle\langle 0, a\rangle\rangle$ and $B = \langle\langle b, 1\rangle\rangle$) or "gaps" (which occur when A lacks a supremum and B lacks an infimum).

The following result gives conditions for the isomorphism of Theorem 5 to be topological as well as order-theoretic.

Theorem 6: Let Γ be a set provided with both a linear order and a topology such that

- with respect to the topology it is compact and connected and contains a countable dense subset,
- with respect to the order it has both a minimum and a maximum element, and
- (with respect to both) it has the property that $\langle\langle x, y\rangle\rangle$ is topologically closed $\forall x, y \in \Gamma$.

Then Γ is isomorphic to the interval $[0,1] \subseteq \mathrm{I\!R}$ by a simultaneous order and topological isomorphism.

3. Causal Analysis in Terms of the Relation K^+

In this section we set forth the elements of a framework for doing global causal analysis with C^0 metrics. In particular, we introduce the relation K^+, we define global hyperbolicity and causal curve, and we give a simple "intrinsic" characterization of the latter in Theorem

20. We begin with some definitions, most of which are standard in the literature, except that here they pertain to general C^0 metrics.

Definitions 7: Let $\mathcal{M}$ be a C^0-lorentzian manifold with metric g_{ab}, and let u^a be any vector field defining its time orientation. A timelike or lightlike vector v^a is *future-pointing* if $g_{ab}v^a u^b < 0$ and *past-pointing* if $g_{ab}v^a u^b > 0$. Now let $I = [0,1] \subseteq \mathbb{R}$. A *future-timelike path* in $\mathcal{M}$ is a piecewise C^1, continuous function $\gamma : [0,1] \to \mathcal{M}$ whose tangent vector $\gamma^a(t) = (d\gamma(t)/dt)^a$ is future-pointing timelike whenever it is defined (in particular γ possesses a future-pointing timelike tangent vector almost everywhere); a *past-timelike* path is defined dually. The image of a future- or past-timelike path is a *timelike curve*. Let $\mathcal{O}$ be an open subset of $\mathcal{M}$. If there is a future-timelike curve in $\mathcal{O}$ from p to q, we write $q \in I^+(p, \mathcal{O})$, and we call $I^+(p, \mathcal{O})$ the *chronological future* of p relative to $\mathcal{O}$. If $q \in I^+(p, \mathcal{O})$, we write $p <_{\mathcal{O}} q$. Past-timelike paths and curves, the sets $I^-(p, \mathcal{O})$, and the symbol $>_{\mathcal{O}}$ are defined analogously. If $\mathcal{O} = \mathcal{M}$, we may omit it, so $I^+(p, \mathcal{M}) = I^+(p)$, and so forth.

Notice that a curve is defined herein to be automatically compact: it will be convenient not to allow the domain of γ to be an open (or half-open) interval in $\mathbb{R}$.

When the manifold obeys the "chronology condition" that it contain no closed timelike curves, the relation $<$ is a *partial order*. Actually, it is a consequence of the definition of $I^+(p)$ that it does not contain p when the chronology condition holds; hence $<$ is a partial order observing the irreflexive convention.

That our basic causality relation be an order will be crucial for results like Theorem 20, but that attribute alone will not be enough. It will also be important that the order be *closed*, by which we mean that it be *topologically* closed when regarded as a subset of $\mathcal{M} \times \mathcal{M}$. Unfortunately I^+ does not enjoy this property, and so we need to complete it in some manner to a relation which we will call $K^+ = \prec$.

To see that $<$ is not closed on $\mathcal{M}$, observe that if it were closed, then we would have $x < y$ for the limits of any convergent sequences $x_n \to x$ and $y_n \to y$ such that $x_n < y_n$ for each n; but for example, in 2-dimensional Minkowski space $\mathbf{M}^2$, there are (in terms of Galilean coordinates (t, z)) sequences $x_n = (0,0) \to x = (0,0)$ and $y_n = (1, 1 - \frac{1}{n}) \to y = (1,1)$, for which $x_n < y_n$ but $x \not< y$ (see Figure 1).

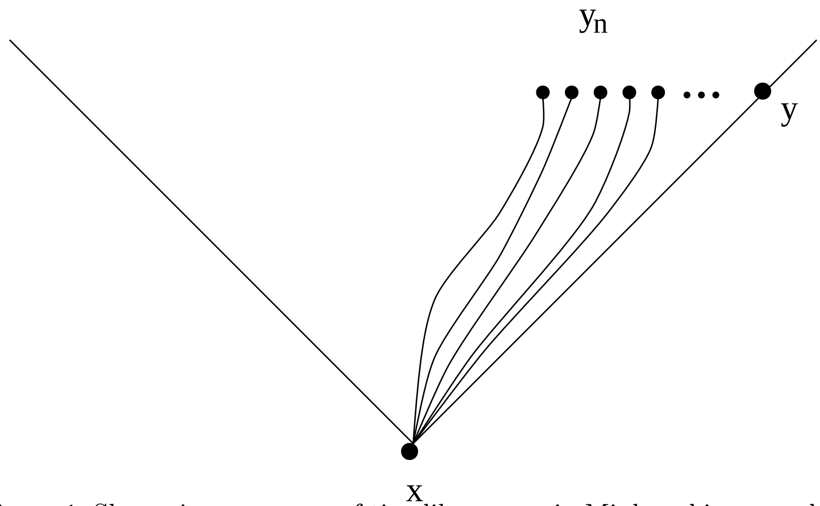

Figure 1: Shown is a sequence of timelike curves in Minkowski space, sharing a common past endpoint x. Their future endpoints y_n approach the lightcone of x, and so $y = \lim y_n$ is not in $I^+(x)$, illustrating the fact that the relation I^+ is not topologically closed. Here, y is simply the future endpoint of a *null* curve emanating from x, but in more general situations (as in the manifolds of Figures 2a–c, for example) there may be no causal curve from x to y at all.

One possibility for constructing K^+ would just be to adjoin the lightlike relations to $<$ (after all, x and y are lightlike related in the above example), but that would not ensure closure in more general situations. Another possibility would be to simply replace $<$ by its *closure* in $\mathcal{M} \times \mathcal{M}$, but that would not always be transitive for every manifold on which one might want to define causal curves. It seems most straightforward, therefore, just to define $K^+ = \prec$ in terms of the properties we need (but *cf.* the second footnote in the Introduction):

Definition 8: K^+ is the smallest relation containing I^+ that is transitive and (topologically) closed.

That is, we define the relation K^+, regarded as a subset of $\mathcal{M} \times \mathcal{M}$, to be the intersection of all closed subsets $R \supseteq I^+$ with the property that $(p,q) \in R$ and $(q,r) \in R$ implies $(p,r) \in R$. (Such sets R exist because $\mathcal{M} \times \mathcal{M}$ is one of them.) One can also describe K^+ as the closed-transitive relation *generated by* I^+. When $(p,q) \in K^+$, we write $q \in K^+(p)$ or $p \prec q$ (read "p precedes q"), and we write $q \in K^+(p,\mathcal{O})$ or $p \prec_{\mathcal{O}} q$, for the analogous relation defined within $\mathcal{O}$. We also write $K(p,q)$ for the order-closed interval $K^+(p) \cap K^-(q) \equiv \langle\langle p,q\rangle\rangle$. Note that $K^+(p,\mathcal{O})$ is always closed in $\mathcal{O}$, as an immediate consequence of the closure of $\prec_{\mathcal{O}}$ itself.

Remark 8a: In a Minkowski spacetime, K^+ agrees with the ordinary "algebraic" relation of causal precedence, for which p precedes q iff the vector $q-p$ is future-timelike or future-lightlike. This follows immediately from the fact that the ordinary relation is closed and transitive. More generally, $\prec_{\mathcal{O}}$ agrees with Minkowskian causal precedence for any convex open subset $\mathcal{O}$ of a Minkowski spacetime.

Figures 2a–c illustrate properties of K^+ in some spacetimes that are locally Minkowski.

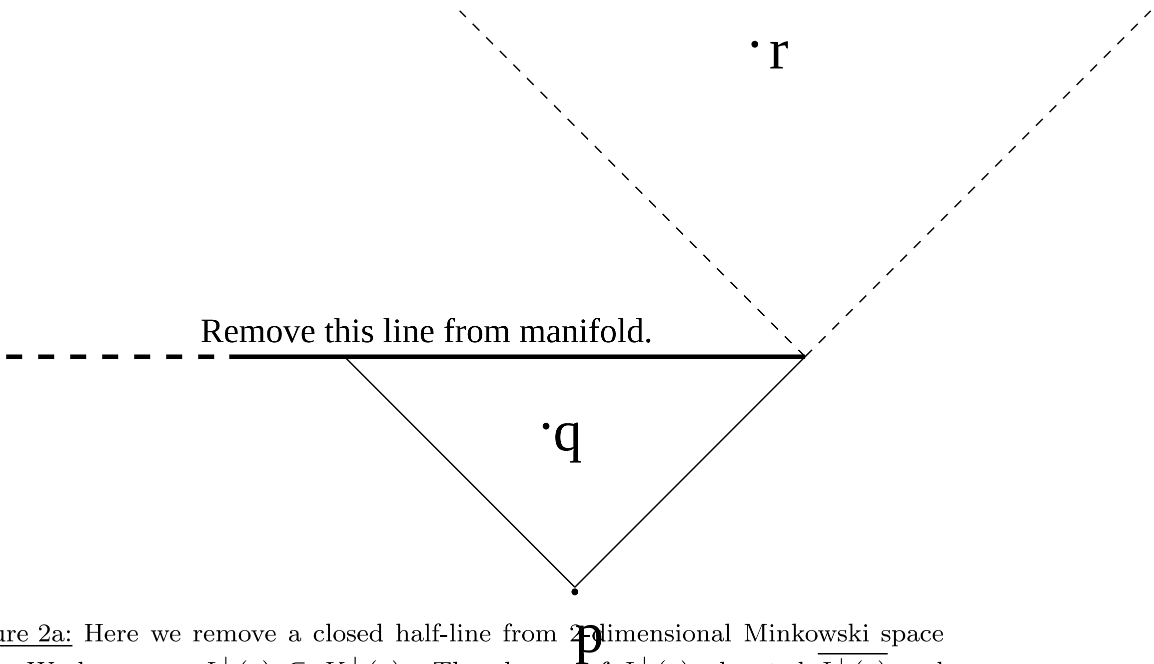

Figure 2a: Here we remove a closed half-line from 2-dimensional Minkowski space $\mathbf{M}^2$. We have $q \in I^+(p) \subseteq K^+(p)$. The closure of $I^+(p)$, denoted $\overline{I^+(p)}$ and contained in $K^+(p)$, lies entirely below the removed half-line, and consists of every point between or on the thin solid lines. While r is in $K^+(p)$, it is not in $\overline{I^+(p)}$; the same is true for any point between or on the dashed lines. Indeed, r is in the interior of $K^+(p)$, although p is on the boundary of $K^-(r)$ and the pair (p,r) itself is on the boundary of the relation $\prec$.

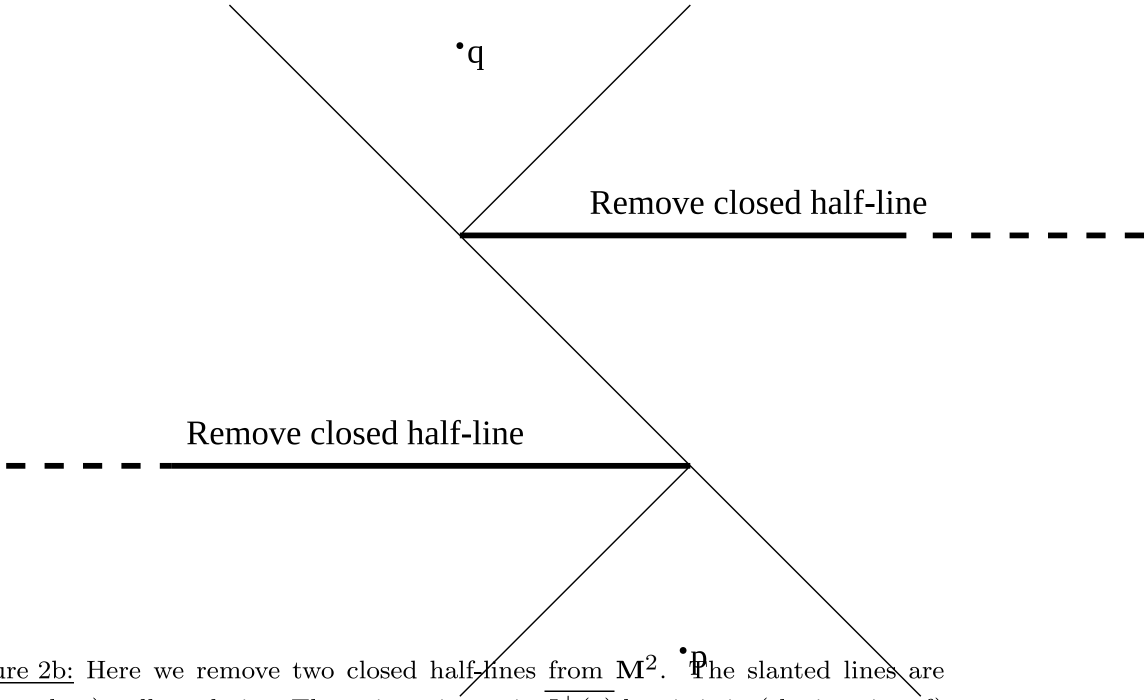

Figure 2b: Here we remove two closed half-lines from $\mathbf{M}^2$. The slanted lines are (incomplete) null geodesics. The point q is not in $\overline{I^+(p)}$ but it is in (the interior of) $K^+(p)$. Indeed, the pair (p, q) is in the interior of the relation $\prec$.

Although we have defined $\prec$ for a general manifold, its usefulness is confined essentially to spacetimes (or their subsets) fulfilling the asymmetry axiom (i) of Definition 3 above. This motivates the following definition, which we record here together with the definitions of two other causal regularity conditions we will invoke repeatedly.

Definition 9: An open set $\mathcal{O}$ is K-*causal* iff the relation $\prec$ induces a (reflexive) partial ordering on $\mathcal{O}$; *i.e.* iff the asymmetry axiom, that $p \prec q$ and $q \prec p$ together imply $p = q$, holds for $p, q \in \mathcal{O}$.

Definition 10: A subset of $\mathcal{M}$ is K-*convex* (also called *causally convex with respect to* K^+) iff it includes the interval $K(p, q)$ between every pair of its members, that is, iff it contains along with p and q any $r \in \mathcal{M}$ for which $p \prec r \prec q$.

Definition 11: A K-causal open set $\mathcal{O} \subseteq \mathcal{M}$ is *globally hyperbolic* iff, for every pair of points $p, q \in \mathcal{O}$, the interval $K(p, q)$ is compact and contained in $\mathcal{O}$.

Notice that Definition 11 requires a globally hyperbolic set to be K-convex. Notice also that the above definitions have obvious analogues for any other causality relation R, such as $R = I^+$. In particular, a set S will be called R-convex iff p, $q \in S \Rightarrow R(p, q) \subseteq S$.

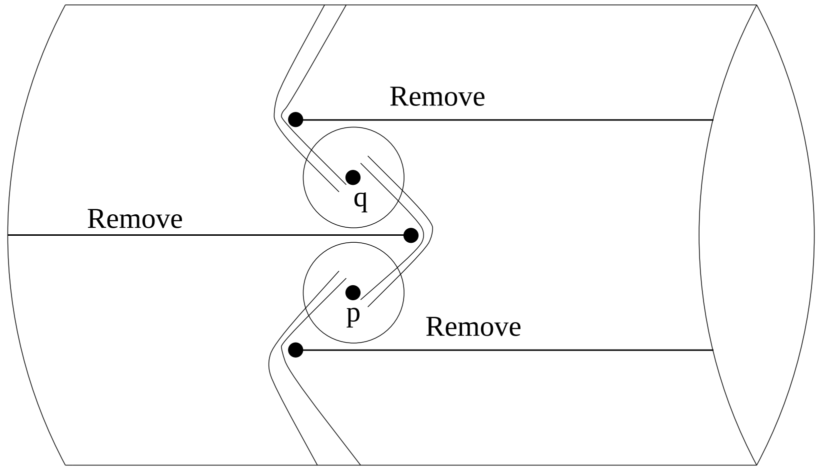

Figure 2c: Shown is a timelike cylinder derived from $\mathbf{M}^2$ with Cartesian coordinates (t, z) by periodically identifying t (say with period 4, so $(-2, z)$ and $(2, z)$ denote the same point). From this spacetime, remove three closed half-lines, each parallel to the z-axis: the half-lines starting at $(\pm 1, -1)$ and moving off to the right, and the half-line starting from $(0, 0)$ and moving off to the left. The resulting manifold has families of timelike curves as depicted, which implies that $p \prec q \prec p$, even though no causal curve can actually join p to q. This example also illustrates that causal convexity with respect to I^+ need not imply causal convexity with respect to K^+.

The next two lemmas relate $\prec_{\mathcal{U}}$ to $\prec_{\mathcal{O}}$ in a situation where $\mathcal{U} \subseteq \mathcal{O}$. The second of these lemmas will be needed only in circumstances where it will be a trivial corollary of Lemma 15 below, but we give it in full strength here, because its proof — though a bit lengthy — is actually more elementary than the proof we have for Lemma 15.

Let $\mathcal{U}$ and $\mathcal{O}$ be open subsets of $\mathcal{M}$ with $\mathcal{U} \subseteq \mathcal{O}$, and let $p, q \in \mathcal{U}$. Then:

Lemma 12: $p \prec_{\mathcal{U}} q$ implies $p \prec_{\mathcal{O}} q$;

Lemma 13: $p \prec_{\mathcal{O}} q$ implies $p \prec_{\mathcal{U}} q$ if $\mathcal{U}$ is causally convex with respect to $\prec_{\mathcal{O}}$.

Proofs: Without loss of generality, we can take $\mathcal{O} = \mathcal{M}$, and we do so to ease the notational burden on the proofs. It is immediate from the definitions that the *restriction* $\prec|\mathcal{U}$ of $\prec$ to $\mathcal{U}$ is closed and transitive, and it also plainly contains the relation $<_{\mathcal{U}}$. Therefore, as $\prec_{\mathcal{U}}$ is the *smallest* relation with these properties, we have $\prec_{\mathcal{U}} \subseteq \prec|\mathcal{U}$, which is the content of Lemma 12.

Conversely, the content of Lemma 13 is the statement that $\prec|\mathcal{U} \subseteq \prec_{\mathcal{U}}$, which, in view of Lemma 12, actually means that they are equal. To prove this, we will construct a relation $\preceq$ on $\mathcal{M}$ which will obviously restrict to $\prec_{\mathcal{U}}$ on $\mathcal{U}$, and we will prove that $\preceq\, = \,\prec$. We define the relation $\preceq$ as follows:

(i) If $p, q \in \mathcal{U}$, then $p \preceq q \Leftrightarrow p \prec_{\mathcal{U}} q$.

(ii) If either or both of p, q are not in $\mathcal{U}$, then $p \preceq q \Leftrightarrow p \prec q$.

Obviously $\preceq |\mathcal{U} = \prec_{\mathcal{U}}$ by (i). Also, (i), (ii), and Lemma 12 imply the inclusion $\preceq\, \subseteq\, \prec$. Thus we need only prove the reverse inclusion, which we will do by showing that $\preceq$ is closed and transitive and contains I^+.

Closure of $\preceq$ is straightforward. Let $p_n \to p$ and $q_n \to q$, with $p_n \preceq q_n$. If both p and $q \in \mathcal{U}$, then the openness of $\mathcal{U}$ implies that eventually $p_n \in \mathcal{U}$, $q_n \in \mathcal{U}$, whence $p_n \prec_{\mathcal{U}} q_n$ by (i), whence $p \prec_{\mathcal{U}} q$ since $\prec_{\mathcal{U}}$ is closed, whence $p \preceq q$, again by (i). If, on the contrary, one of p, q is not in $\mathcal{U}$, then by (ii) we need only show $p \prec q$, but this follows immediately from the closure of $\prec$ since, as observed just above, $p_n \preceq q_n$ always implies $p_n \prec q_n$, whether or not p_n, $q_n \in \mathcal{U}$.

Next we check transitivity. We assume that $p \preceq r \preceq q$, and we check that $p \preceq q$. If $p, r, q \in \mathcal{U}$, then transitivity follows from transitivity of $\prec_{\mathcal{U}}$. If p and r are in $\mathcal{U}$ and q is not, then $p \preceq r \Rightarrow p \prec r$ as we know, and $r \prec q$ by part (ii) of the definition of $\preceq$, whence $p \prec q$ by transitivity of $\prec$, whence $p \preceq q$ by again applying part (ii) of the definition of $\preceq$; likewise if $r, q \in \mathcal{U}$ and $p \notin \mathcal{U}$. The potentially dangerous case of $p, q \in \mathcal{U}$, $r \notin \mathcal{U}$ does not arise, since $p \prec r \prec q \Rightarrow r \in \mathcal{U}$ by the causal convexity of $\mathcal{U}$. If no more than one of p, r, q are in $\mathcal{U}$, then $p \preceq r \preceq q$ implies $p \prec r \prec q$, which implies $p \prec q$, which in turn implies $p \preceq q$, once again by part (ii) of the definition of $\preceq$.

Lastly, we must show that $\preceq\, \supseteq\, <$. So suppose there exists a timelike curve γ from p to q. (Recall that '$p < q$' means '$q \in I^+(p)$'.) Because $p < q \Rightarrow p \prec q \Rightarrow p \preceq q$ when either p or q is not in $\mathcal{U}$, the only non-trivial case occurs when p, $q \in \mathcal{U}$. But then, since $\mathcal{U}$ is causally convex, γ must remain within it; otherwise there would exist $r \notin \mathcal{U}$ such that $p < r < q$, whence we would have $p \prec r \prec q$, contrary to causal convexity. Therefore $q \in I^+(p, \mathcal{U})$, *i.e.* $p \prec_{\mathcal{U}} q$, whence $p \preceq q$ by (i).

Thus, $\preceq$ is closed and transitive, it contains $<$, and it is contained within $\prec$. Since $\prec$ is defined to be the smallest closed and transitive relation containing $<$, we have $\preceq\, = \,\prec$. □

Precisely because the relation of K^+ to curves is rather indirect, certain lemmas which would be straightforward for J^+ are more difficult to establish for K^+. The next two results are of this nature, and we resort to transfinite induction to prove them. (Notice, incidentally, that no assumption of K-causality is involved.)

Lemma 14: Let B be a subset of $\mathcal{M}$ with compact boundary, and let $x \prec y$ with $x \in B$, $y \notin B$ (or vice versa). Then $\exists w \in \partial B$ such that $x \prec w \prec y$.

Proof: We can regard the relation K^+ as having been built up via transfinite induction (see *e.g.* [6]) from I^+ by adding pairs $x \prec y$ as required. Specifically, we look at each stage of the induction for a pair which is not yet present, but which is implied by either transitivity or closure, and if we find one, we add it. Since there are at most $2^{\aleph_0}$ possible pairs, the process must terminate.

Let the steps in this process be labeled by the ordinal number α, with the corresponding relation as built up at stage α denoted by $\prec^\alpha$. We now make the inductive hypothesis H_α that, if at stage α there are points $x \prec^\alpha y$ with $x \in B$, $y \notin B$, then there exists $w \in \partial B$ such that $x \prec w \prec y$ (meaning that w will lie between x and y in the *ultimate* relation $\prec$).

Clearly H_α is valid for $\alpha = 0$, so we need only check that it holds for α assuming it holds for all $\beta < \alpha$.

Here there are two cases, depending on whether α is a limit ordinal or not. If α is a limit ordinal, then $x \prec^\alpha y$ really just means that $x \prec^\beta y$ for some $\beta < \alpha$, so H_α follows trivially.

If α is not a limit ordinal, let $\alpha = \beta + 1$, and observe that we can assume that the relation $x \prec^\alpha y$ is precisely the one which was added in passing from β to α. There are then two sub-cases depending on why this relation was added:

(i) If $x \prec^\alpha y$ was added by transitivity, then there must have been a z for which $x \prec^\beta z \prec^\beta y$. If $z \in B$, then the inductive hypothesis H_β implies the existence of $w \in \partial B$ such that $z \prec w \prec y$, whence $x \prec z \prec w \prec y \Rightarrow x \prec w \prec y$, as desired. On the other hand, if z lies outside of B, then $H_\beta \Rightarrow \exists w \in \partial B$ such that $x \prec w \prec z \prec y \Rightarrow x \prec w \prec y$, again as desired.

(ii) If $x \prec^\alpha y$ was added because of the existence of sequences $x_n \to x$, $y_n \to y$ with $x_n \prec^\beta y_n$ then $H_\beta \Rightarrow \forall n\ \exists w_n \in \partial B$ such that $x_n \prec w_n \prec y_n$. (Strictly speaking, we can be sure that (eventually) $x_n \in B$, $y_n \notin B$ only if x is in the interior of B and y is in the interior of its complement; but in fact this is the only situation we need consider, because the lemma itself holds trivially if either x or y is in ∂B.) Then, as ∂B is compact, we can pass to a convergent subsequence of the w_n and assume therefore that $w_n \to w$ for some $w \in \partial B$. From this and the closure of K^+ it follows directly that $x \prec w \prec y$, thus completing the proof. $\square$

Lemma 15: Let B be any open set in $\mathcal{M}$ with compact closure. If there are $x, y \in B$ with $x \prec y$ but not $x \prec_B y$, then $\exists z \in \partial B$ such that $x \prec z \prec y$.

Proof: We will employ a closely similar transfinite induction as before, with the inductive hypothesis H_α being that if $x \prec^\alpha y$ but $x \not\prec_B y$ then $\exists z \in \partial B$ such that $x \prec z \prec y$. For this proof we adopt the shorthand that $x \prec y$ is *unnatural* (with respect to B) iff $x \not\prec_B y$.

Clearly H_α holds for $\alpha = 0$ since $K_0^+ = I^+$. Now suppose H_β holds for all $\beta < \alpha$, and let $x \prec^\alpha y$ be unnatural. Clearly if $x \prec^\beta y$ for some $\beta < \alpha$ then $H_\beta \Rightarrow \exists z \in \partial B$ such that $x \prec z \prec y$, and we are done. So, as before, the only non-trivial case is where $x \prec^\alpha y$ was added at stage α, which means that we have $\alpha = \beta + 1$ and $x \not\prec^\beta y$. Then, as before, there are two sub-cases.

In the sub-case where $x \prec^\alpha y$ was added for transitivity, there must exist q such that $x \prec^\beta q \prec^\beta y$. If $q \notin B$ then, by the previous lemma, $x \prec q \Rightarrow \exists z \in \partial B$ such that $x \prec z \prec q \Rightarrow x \prec z \prec q \prec y \Rightarrow x \prec z \prec y$, as required. If, on the other hand $q \in B$, then (by the transitivity of $\prec_B$) at least one of $x \prec q$ or $q \prec y$ must be unnatural. If $x \prec q$ is unnatural then $H_\beta \Rightarrow \exists z \in \partial B$ such that $x \prec z \prec q \Rightarrow x \prec z \prec q \prec y \Rightarrow x \prec z \prec y$; if $q \prec y$ is unnatural then we see similarly that $H_\beta \Rightarrow \exists z \in \partial B$ such that $q \prec z \prec y \Rightarrow x \prec q \prec z \prec y \Rightarrow x \prec z \prec y$.

In the sub-case where $x \prec^\alpha y$ was added because there exist sequences $x_n \to x$, $y_n \to y$ with $x_n \prec^\beta y_n$, we can conclude that $x_n \prec y_n$ is unnatural for sufficiently great n (otherwise we would have $x_n \prec_B y_n$ for arbitrarily great n, whence $x \prec_B y$ by the topological closure of $\prec_B$). Thus we can assume without loss of generality that $x_n \prec y_n$ is unnatural for all n. The inductive hypothesis H_β then provides us for all n with $z_n \in \partial B$ such that $x_n \prec z_n \prec y_n$, and by the compactness of ∂B, we can pass to a subsequence such that the z_n converge to some z in ∂B. But then the closure of $\prec$ implies that $x \prec z \prec y$ as required. $\square$

The last two results allow us to prove that every K-causal manifold is what might be called *locally K-convex* or "strongly causal with respect to K^+:"

Lemma 16: If $\mathcal{M}$ is K-causal then every element of $\mathcal{M}$ possesses arbitrarily small K-convex open neighborhoods (*i.e.* $\mathcal{M}$ is *locally K-convex*).

Proof: Given any neighborhood N of p in $\mathcal{M}$, we can find an open sub-neighborhood $B \equiv U_0$ of compact closure, and within B a nested family of open sub-neighborhoods $\{U_n\}$ which shrink down to p and which are all causally convex with respect to $\prec_B$. (For B sufficiently small, such a family can be obtained by taking neighborhoods which are causally convex with respect to some flat metric within B whose light cones are everywhere wider than those of the true metric g_{ab}. That this construction works follows immediately from Remark 8a, the continuity of g_{ab}, and the definitions of $\prec_B$ and of causal convexity.)

Now suppose the theorem fails at p. Then for some neighborhood N of p, none of the U_n just described can be causally convex with respect to $\prec$ itself. This means that for each n there are $x_n, z_n \in U_n$ such that $x_n \prec y_n \prec z_n$ with $y_n \notin U_n$. Now either y_n belongs to B or it doesn't. If it does, then (in the terminology of the proof of Lemma 15) either $x_n \prec y_n$ or $y_n \prec z_n$ must be unnatural with respect to B (because U_n is causally convex within B). Suppose that $x_n \prec y_n$ is unnatural (the other possibility being analogous). Then Lemma 15 provides us with $w_n \in \partial B$ such that $x_n \prec w_n \prec y_n \prec z_n$, whence $x_n \prec w_n \prec z_n$. On the other hand, if y_n lies outside of B then a precisely analogous application of Lemma 14 furnishes us, once again, with a $w_n \in \partial B$ such that $x_n \prec w_n \prec z_n$.

In either case, we get a sequence w_n of elements of the compact set ∂B, which lie causally between x_n and z_n and, passing to a subsequence if necessary, an element $w \in \partial B$ to which the w_n converge. But this would imply $p = \lim x_n \prec w \prec \lim z_n = p$, contradicting the K-causality of $\mathcal{M}$. □

In the remainder of this section, we define the notion of causal curve using $\prec$, and we derive a simple intrinsic characterization of this notion which is valid in an arbitrary K-causal manifold. Among the lemmas leading up to this result, Lemma 19 seems of some interest in itself as an independent simple characterization of causal curve.

Definition 17: A *causal curve* Γ from p to q is the image of a C^0 map $\gamma : [0,1] \to \mathcal{M}$ with $p = \gamma(0)$, $q = \gamma(1)$, and such that for each $t \in (0,1)$ and each open $\mathcal{O} \ni \gamma(t)$, there is a positive number ϵ (depending on $\mathcal{O}$) such that

$$t' \in (t, t+\epsilon) \Rightarrow \gamma(t) \prec_{\mathcal{O}} \gamma(t') \quad , \text{ and} \qquad \text{(i)}$$

$$t' \in (t-\epsilon, t) \Rightarrow \gamma(t') \prec_{\mathcal{O}} \gamma(t) \quad . \qquad \text{(ii)}$$

Moreover, condition (i) above is required to hold for $t = 0$, and condition (ii) for $t = 1$. The point $p = \gamma(0)$ is called the *initial endpoint* of γ, and $q = \gamma(1)$ is called its *final endpoint*.

A causal curve is thus defined to be the image of what might be called a locally increasing mapping, and rewording the definition accordingly results in a slightly more compact formulation as follows.

A path γ is *causal* iff for every $t \in [0,1]$ and every open neighborhood $\mathcal{O}$ of $\gamma(t)$ there exists a neighborhood of t in $[0,1]$ within which $t' < t'' \Rightarrow \gamma(t') \prec_{\mathcal{O}} \gamma(t'')$; a causal curve is the image of a causal path.

We may remark also that the requirement in Definition 17 that γ be C^0 is actually redundant, being implicit in conditions (i) and (ii).

Lemma 18: If $\Gamma = \text{image}(\gamma)$ is a causal curve, then for every neighborhood $\mathcal{O} \supseteq \Gamma$ and every pair $t_i < t_f \in [0,1]$, we have $\gamma(t_i) \prec_{\mathcal{O}} \gamma(t_f)$.

Proof: Without loss of generality we can take $t_i = 0$ and $t_f = 1$; let us also set $p = \gamma(0)$. Let $t_0 = \sup\{t \in [0,1] \mid p \prec_{\mathcal{O}} \gamma(t)\}$. Because $\prec_{\mathcal{O}}$ is closed, $r := \gamma(t_0)$ also follows p (*i.e.* $p \prec_{\mathcal{O}} r$). But then we must have $t_0 = 1$ since otherwise, by Definition 17, there would exist $t_1 > t_0$ with $p \prec_{\mathcal{O}} r \prec_{\mathcal{O}} \gamma(t_1)$, whence $p \prec_{\mathcal{O}} \gamma(t_1)$ by transitivity, contradicting the definition of t_0. $\square$

Notice that the converse of Lemma 18 is in general false: $p \prec q$ need not imply the existence of a causal curve from p to q, even for q in the interior of $K^+(p)$. Notice also that, in the absence of K-causality, Lemma 18 does not in general endow Γ with a linear order, because the asymmetry axiom can fail for $\prec$ restricted to Γ, even if we replace $\prec$ by the intersection of the $\prec_{\mathcal{O}}$ for all $\mathcal{O} \supseteq \Gamma$.

In a K-causal spacetime, Definition 17 of causal curve can be stated more simply: a causal curve is the image of a continuous order-preserving map from $[0,1]$ to $\mathcal{M}$. We obtain this equivalence in the following lemma.

Lemma 19: A subset Γ of a K-causal spacetime is a causal curve if and only if it is the image of a continuous increasing function γ from $[0,1]$ to $\mathcal{M}$.

(By increasing, we mean that $t < t' \Rightarrow \gamma(t) \prec \gamma(t'),\ \forall t, t' \in [0,1]$.)

Proof: The forward implication ("only if" clause) follows immediately from Lemma 18. To establish the reverse implication, we only need show that, for every $t \in [0,1]$ and every open neighborhood $\mathcal{O}$ of $p = \gamma(t)$, there exists a neighborhood of t in $[0,1]$ within which $t' < t'' \Rightarrow \gamma(t') \prec_{\mathcal{O}} \gamma(t'')$. But given any such $\mathcal{O}$, Lemma 16 provides an open subset $\mathcal{U}$ of $\mathcal{O}$ which is causally convex and contains p. Then, since γ is continuous, there exists a neighborhood $[r, s]$ of t in $[0, 1]$ whose image by γ lies within $\mathcal{U}$; while for $r \le t' \le t'' \le s$ we have by assumption that $\gamma(t') \prec \gamma(t'')$. Hence $\gamma(t') \prec_{\mathcal{U}} \gamma(t'')$ by Lemma 13, which in turn entails $\gamma(t') \prec_{\mathcal{O}} \gamma(t'')$, by Lemma 12. □

By the definition of K-causality, the function γ of Lemma 19 can be taken, without loss of generality, to be injective. Thus, the distinction between a causal curve and a causal path becomes essentially irrelevant in a K-causal spacetime; that is, a causal curve Γ becomes identifiable with the corresponding causal path $\gamma : [0, 1] \to \mathcal{M}$, up to reparameterization induced by a diffeomorphism of $[0, 1]$.

Theorem 20: A subset Γ of a K-causal manifold is a causal curve iff it is compact, connected and linearly ordered by $\prec = K^+$.

Proof: To prove the theorem, we will establish that if $\Gamma \subseteq \mathcal{M}$ is compact, connected and linearly ordered by $\prec$ then it is homeomorphic (by an order-preserving correspondence) to the unit interval in $\mathbb{R}$. The theorem will then be immediate from Lemma 19.

Let us apply Theorem 6. To verify the hypotheses of that theorem, we need to show that Γ contains a countable dense subset, that it has minimum and maximum elements, and that $\{z \in \Gamma | x \prec z \prec y\}$ is closed for all x and y in Γ.

The existence of a countable dense subset follows directly from the fact that Γ is a compact subset of a finite dimensional manifold. (As such, Γ can be covered by a finite number of subsets of $\mathcal{M}$ homeomorphic to open balls in $\mathbb{R}^n$, therefore it is second-countable, therefore it has a countable dense subset.)

The fact that Γ contains minimum and maximum elements follows from the circumstance that Γ is compact and $\prec$ is closed. This implies that the set $\Omega := \cap_{x \in \Gamma}(\Gamma \cap K^-(x))$ is non-empty (being the intersection of a nested family of non-empty closed subsets of the compact set Γ). But, since any element of Ω clearly precedes every element of Γ, we must have $\Omega = \{0\}$, where 0 is the desired minimum element. Likewise, we obtain $1 \in \Gamma$ such that $p \prec 1,\ \forall p \in \Gamma$.

That, finally, every interval in Γ is topologically closed follows immediately from the fact that the interval in Γ bounded by $x, y \in \Gamma$ can be expressed as the intersection of three closed sets, $K^+(x) \cap K^-(y) \cap \Gamma$. (Recall that $K^\pm(x)$ is closed since $\prec$ itself is.) □

Remark: The proof shows that Γ would be a curve, even without the condition of K-causality; but this condition is not redundant because, in its absence, one can find examples where Γ turns out to be a spacelike curve rather than a causal one.

4. A Bicompactness Theorem for Spaces of Causal Curves

Theorem 23 of this section will be the main fruit of our work in this paper. Given the general characterization of a causal curve (in a K-causal spacetime) as a compact connected subset linearly ordered by $\prec$, the derivation of Theorem 23 is relatively simple and straightforward. All that is needed, basically, is to establish that the limit of a sequence of causal curves is a causal curve, which is the content of the next proposition.

Theorem 21: In a K-causal C^0-lorentzian manifold $\mathcal{M}$, a compact Vietoris limit of a sequence (or net) of causal curves is also a causal curve.

Proof: Let us verify the conditions of Theorem 20 for the limit Γ of a sequence of causal curves Γ_n.

In order to establish that Γ is linearly ordered by $\prec$, let p and q be points on Γ, and let $\mathcal{O}_1 \ni p$ and $\mathcal{O}_2 \ni q$ be open neighborhoods. Then $\Gamma \in \mathcal{B}(\mathcal{M}; \mathcal{O}_1, \mathcal{O}_2)$, together with $\Gamma_n \to \Gamma$, implies $\Gamma_n \in \mathcal{B}(\mathcal{M}; \mathcal{O}_1, \mathcal{O}_2)$ for n sufficiently great. By letting $\mathcal{O}_1$ shrink down around p and $\mathcal{O}_2$ shrink down around q, we obtain sequences of points $p_n, q_n \in \Gamma_n$ such that $p_n \to p$ and $q_n \to q$. Since the Γ_n are causal curves, Lemma 19 tells us that, for each n, $p_n \prec q_n$ or vice versa. Hence one of these holds for an infinite number of n, and the fact that $\prec$ is closed then implies that $p \prec q$ or vice versa. Thus every two points of Γ are related by $\prec$, as required.

That Γ is compact is immediate from the first part of Lemma 2. That it is connected follows from the second part, in virtue of the fact that Γ has a compact neighborhood within which the Γ_n must ultimately lie. (Such a neighborhood exists because Γ, being compact, can be covered by a finite number of open sets of compact closure. The union $\mathcal{O}$ of these sets is then an open neighborhood of Γ which ultimately contains the Γ_n since ultimately $\Gamma_n \in \mathcal{B}(\mathcal{O}; \mathcal{M})$. Its closure $\overline{\mathcal{O}}$ is the desired compact neighborhood of Γ.) □

We will also use the following, unsurprising lemma.

Lemma 22: In a K-causal manifold, let the causal curve Γ be the Vietoris limit of a sequence (or net) of causal curves Γ_n with initial endpoints p_n and final endpoints q_n. Then the p_n converge to the initial endpoint of Γ and the q_n to its final endpoint.

Proof: Since Γ is compact, it has a compact neighborhood $\mathcal{N}$ within which the Γ_n eventually lie; hence we can assume without loss of generality that there exists $p \in \mathcal{N}$ to which the p_n converge. In fact, p must belong to Γ, as follows from the more general fact that Γ *contains any point* x *at which the* Γ_n *accumulate* in the sense that they eventually meet any neighborhood of x. (Proof: If some point x is not in Γ, then, as Γ is compact, x possesses a closed neighborhood $\mathcal{K}$ disjoint from Γ. The complement of $\mathcal{K}$ is then a neighborhood of Γ, and therefore must eventually include all the Γ_n. Hence the latter do not accumulate at x, being disjoint from its neighborhood $\mathcal{K}$.) Finally, it is easy to see that p must be the minimum element of Γ. Indeed, any $r \in \Gamma$ is the limit of some sequence of elements $r_n \in \Gamma_n$ (as follows from the fact that every neighborhood $\mathcal{O} \ni r$ must eventually meet all of the Γ_n), and this, together with the fact that $p_n \prec r_n, \forall n$ (by Theorem 20), implies that $p \prec r$. Thus p must be the initial endpoint of Γ according to Theorem 20. The statement for q follows dually. $\square$

With this technicality out of the way, Theorem 21 implies:

Theorem 23: Let $\mathcal{O}$ be a globally hyperbolic open subset of a C^0-lorentzian manifold $\mathcal{M}$, and let $\mathcal{P}$ and $\mathcal{Q}$ be compact subsets of $\mathcal{O}$. Then the space of causal curves from $\mathcal{P}$ to $\mathcal{Q}$ is bicompact.

Proof: Let Γ_n be a net of causal curves from $\mathcal{P}$ to $\mathcal{Q}$. Since $\mathcal{P}$ and $\mathcal{Q}$ are compact, the initial endpoints p_n and final endpoints q_n accumulate at some points $p \in \mathcal{P}$ and $q \in \mathcal{Q}$. Since $\mathcal{O}$ is open, we can find $p' \in I^-(p) \cap \mathcal{O}$ and $q' \in I^+(q) \cap \mathcal{O}$. Set $\mathcal{K} := K(p', q')$; it is compact by Definition 11 of global hyperbolicity. We can assume without loss of generality that all the p_n belong to $I^+(p') \subseteq K^+(p')$ and the q_n to $I^-(q') \subseteq K^-(q')$, hence that all the Γ_n lie in $\mathcal{K}$. Since $\mathcal{K}$ is compact, Theorem 1 provides a subnet Γ_m converging to some compact set $\Gamma \subseteq \mathcal{K}$, which Theorem 21 assures us is actually a causal curve. Finally, Lemma 22 assures us (in view of the circumstance that $\mathcal{O}$ is K-causal by the definition of global hyperbolicity) that p is in fact the initial endpoint of Γ and q is its final endpoint;

i.e. Γ is a causal curve from p to q. Thus, the space of causal curves from $\mathcal{P}$ to $\mathcal{Q}$ is compact.

Additionally, since $\mathcal{K}$ is compact Hausdorff, the Vietoris topology on $2^{\mathcal{K}}$ is also Hausdorff, again by Theorem 1. Thus, the space of causal curves from $\mathcal{P}$ to $\mathcal{Q}$ is Hausdorff as well. $\square$

One can also prove, conversely to Theorem 23, that compactness of the space of causal curves between arbitrary pairs of points p, q implies global hyperbolicity. Similarly one can readily prove generalized forms of many other results which are familiar in the C^2 setting, for example it is not hard to see that in a globally hyperbolic manifold, $K^+ = J^+ = \overline{I^+}$, where $J^+(p)$ is defined as the union of all causal curves emanating from p (*cf.* Lemma 25 in Appendix B).

Acknowledgments

RDS would like to acknowledge partial support from the National Science Foundation, grant NSF PHY 9307570, and from Syracuse University Research Funds. EW would like to thank E. Tymchatin for a discussion concerning Theorem 6.

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Appendix A: Proofs of the Theorems in Section 2

Herein, for completeness, we prove the general mathematical theorems which were collected in Section 2. We deal with the Vietoris topology first, and then the theorems concerning ordered sets.

In preparation for the proof of Theorem 1, we need the following lemma ("Alexander's Theorem").

Lemma 24: The topological space $\mathcal{X}$ is compact iff every cover of $\mathcal{X}$ by open sets belonging to some fixed *sub-base* for its topology has a finite subcover.

Proof: The "only if" clause being trivial, let us assume that every cover by sub-basic sets has a finite subcover and prove that then any net x_n in $\mathcal{X}$ has an accumulation point x. Suppose there were no accumulation point. Then for each $x \in \mathcal{X}$ there would exist a sub-basic open set $\mathcal{O} \ni x$ from which x_n would eventually be absent. It follows immediately that x_n would then have to be eventually absent from any finite union of these $\mathcal{O}$s, whence no such union could include all of $\mathcal{X}$ (since each x_n has to be *somewhere*). Thus it would require an infinite number of the sub-basic sets $\mathcal{O}$ to cover $\mathcal{X}$, contrary to hypothesis. □

Using this lemma we can prove the compactness of $2^{\mathcal{X}}$ very simply, following [5].

Theorem 1: If the topological space $\mathcal{X}$ is bicompact, then so also is the space $2^{\mathcal{X}}$ of non-empty closed subsets of $\mathcal{X}$.

Proof: By definition, the sets of the form $\mathcal{B}(\mathcal{O};\mathcal{X})$ and $\mathcal{B}(\mathcal{X};\mathcal{O})$ provide a subbase for the Vietoris topology. Then let

$$2^{\mathcal{X}} = \bigcup_{i\in I} \mathcal{B}(\mathcal{X};\mathcal{O}_i) \ \cup \ \bigcup_{k\in K} \mathcal{B}(\mathcal{O}_k;\mathcal{X}) \tag{1}$$

be a cover of $2^{\mathcal{X}}$ by such sets, where i and k range over distinct index sets I and K (possibly empty). We claim that the $\mathcal{O}_i$ together with at most one of the $\mathcal{O}_k$ provide an open cover of $\mathcal{X}$ itself. In fact if

$$\mathcal{Y} := \mathcal{X} \backslash \bigcup_{i\in I} \mathcal{O}_i$$

is not empty then (being closed) it is an element of $2^{\mathcal{X}}$, hence contained in one of the $\mathcal{B}(\mathcal{O}_k;\mathcal{X})$, say $\mathcal{Y} \in \mathcal{B}(\mathcal{O}_0;\mathcal{X})$. (This follows from (1) since $\mathcal{Y}$ is not in any of the $\mathcal{B}(\mathcal{X};\mathcal{O}_i)$, being disjoint from all of the $\mathcal{O}_i$ by construction.) But this means precisely that $\mathcal{Y} \subseteq \mathcal{O}_0$, whence $\mathcal{O}_0$ combines with the $\mathcal{O}_i$ to cover $\mathcal{X}$, as alleged.

Now since $\mathcal{X}$ is compact by assumption, we can find a finite subset of the $\mathcal{O}_i$ which, together with $\mathcal{O}_0$, cover $\mathcal{X}$. Letting $I_0 \subseteq I$ be the corresponding finite set of indices, we have then

$$\mathcal{X} = \bigcup_{i\in I_0} \mathcal{O}_i \ \cup \ \mathcal{O}_0 \tag{2}$$

(where again, either $\mathcal{O}_0$ or the $\mathcal{O}_i$ might be absent because one of the indexing sets I or K might be empty).

We claim that the corresponding open sets $\mathcal{B}(\mathcal{O}_0;\mathcal{X})$, and $\mathcal{B}(\mathcal{X};\mathcal{O}_i)$ for $i \in I_0$, provide the desired finite subcover of (1). In fact, let $\mathcal{S} \in 2^{\mathcal{X}}$ be any closed subset of $\mathcal{X}$. Either there is an $i \in I_0$ such that $\mathcal{S}$ meets $\mathcal{O}_i$, or there is not. In the former case, $\mathcal{S}$ is by definition an element of $\mathcal{B}(\mathcal{X};\mathcal{O}_i)$, while in the latter case, we see from (2) that $\mathcal{S}$ must be a subset of $\mathcal{O}_0$, whence by definition an element of $\mathcal{B}(\mathcal{O}_0;\mathcal{X})$. Thus

$$2^{\mathcal{X}} = \bigcup_{i\in I_0} \mathcal{B}(\mathcal{X};\mathcal{O}_i) \ \cup \ \mathcal{B}(\mathcal{O}_0;\mathcal{X}),$$

and we have shown that $2^{\mathcal{X}}$ is compact.

Lastly, we note that, since $\mathcal{X}$ is bicompact, it is regular, whence $2^{\mathcal{X}}$ is Hausdorff by an easily proved property of the Vietoris topology (Prop. 2.2.3. of [5]). $\square$

Lemma 2:
(i) In a compact space every Vietoris limit is a compact set.
(ii) In a bicompact space, a Vietoris limit of connected sets is connected.

Proof: The first statement is trivial since any closed subset of a compact set is compact. For the second, observe that, if the limit Γ of some net Γ_n were not connected, then it would be a disjoint union† $\Gamma = \Gamma' \sqcup \Gamma''$, where both Γ' and Γ'' would be closed in Γ, and therefore compact, since Γ itself is compact by (i). Because $\mathcal{X}$ is Hausdorff, Γ' and Γ'' would then possess disjoint open neighborhoods, $\mathcal{O}_1$ and $\mathcal{O}_2$, in $\mathcal{X}$. But that would mean that $\Gamma \in \mathcal{B}(\mathcal{O}_1 \cup \mathcal{O}_2; \mathcal{O}_1, \mathcal{O}_2)$, whence, by the definition of the Vietoris topology, Γ_n would also belong to $\mathcal{B}(\mathcal{O}_1 \cup \mathcal{O}_2; \mathcal{O}_1, \mathcal{O}_2)$ for sufficiently large n. But then for such n, Γ_n itself would not be connected, contrary to hypothesis. $\square$

Next we turn to ordered spaces. For the following theorem, see *e.g.* reference [9].

Theorem 5: In order that a poset X be tonomorphic (*i.e.* order-isomorphic) to the closed unit interval $I = [0, 1] \subseteq \mathbb{R}$, it is necessary and sufficient that
(i) X be linearly ordered with both a minimum and a maximum element (denoted $0, 1$ respectively),
(ii) X have a countable order-dense subset, and
(iii) every partition of X into disjoint subsets $A \prec B$ be either of the form $A = \langle\langle 0, x\rangle\rangle$, $B = \langle x, 1\rangle\rangle$ or of the form $A = \langle\langle 0, x\rangle$, $B = \langle\langle x, 1\rangle\rangle$.

Proof: Condition (ii) provides us with a countable order-dense subset $C_0 \subseteq X$ which we clearly may assume to contain the 0 and 1 elements of X. Similarly, the rational numbers between 0 and 1 (inclusive) are a countable order-dense subset $Q_0 \subseteq I = [0, 1]$. We will first set up a bijection $f : C_0 \to Q_0$ and then complete it to obtain an isomorphism between X and I.

We will build up f inductively by alternately choosing elements of f and f^{-1}. To begin with, we may assume that both C_0 and Q_0 (being countable)

† We use $\sqcup$ to denote disjoint union.

are presented explicitly as sequences or *lists* $C = (c_0, c_1, c_2, \ldots) = (0, 1, \ldots)$ and $Q = (q_0, q_1, q_2, \ldots) = (0, 1, \ldots)$, where in each list, the order of the elements is immaterial except for the initial two. (No confusion should result from the fact that 0 and 1 are being used to denote both real numbers and elements of X.) We will obtain f by inductively rearranging C and Q into lists $\widetilde{C} = (\widetilde{c}_0, \widetilde{c}_1, \widetilde{c}_2, \ldots) = (0, 1, \ldots)$ and $\widetilde{Q} = (\widetilde{q}_0, \widetilde{q}_1, \widetilde{q}_2, \ldots) = (0, 1, \ldots)$, such that the resulting induced correspondence $f(\widetilde{c}_i) = \widetilde{q}_i$ $(i = 0, 1, 2, \ldots)$ is tonomorphic from $C_0 \subseteq X$ to $Q_0 \subseteq [0, 1] \subseteq \mathbb{R}$.

We may express this rearrangement algorithmically as a process in which we begin with empty $\widetilde{C}$ and $\widetilde{Q}$ and gradually build them up iteratively by *transferring* elements from C and Q until the latter are exhausted, taking care that at each stage, the resulting partially defined f is maintained as a tonomorphism. To do this we simply repeat the following two-step "loop" a countable number of times.

Step 1: Remove the initial remaining element c from C and place it at the end of $\widetilde{C}$. Then transfer from Q to the end of $\widetilde{Q}$ any element q (say the earliest one in the list) which will make f remain a tonomorphism. Such a q certainly exists because Q is order-dense in I and the new element c is situated (with respect to the order of X) between a well-defined pair of the elements occurring earlier in the list $\widetilde{C}$.

Step 2: This is the mirror image of Step 1, with the roles of C and Q interchanged. Remove the initial remaining element q from Q and place it at the end of $\widetilde{Q}$. Then transfer from C to the end of $\widetilde{C}$ any element c (say the earliest one in the list) which will make f remain a tonomorphism. Such a c certainly exists because C is order-dense in X and the new element q is situated (with respect to the order of $\mathbb{R}$) between a well-defined pair of the elements occurring earlier in the list $\widetilde{Q}$.

After looping through these two steps a countable number of times, we will have emptied both original lists and replaced them with re-ordered lists $\widetilde{C} = (0, 1, \widetilde{c}_2, \widetilde{c}_3, \ldots)$ and $\widetilde{Q} = (0, 1, \widetilde{q}_2, \widetilde{q}_3, \ldots)$, thereby obtaining a densely defined, order-preserving mapping from X to I, or more precisely, an order-preserving bijection $f : C_0 \to Q_0$. To complete the proof we need to extend f to all of X.

Therefore, consider any $x \in X \backslash C_0$. Such an x partitions C_0 into two sets A and B such that $A < x < B$ (taking $<$ to be the order relation on X); this decomposition is called a *cut*. The corresponding cut in Q determines a unique real number r, and we set $f(x) = r$. We claim that the resulting extended f is an isomorphism.

To confirm this, let us first verify injectivity. Consider $x' < x''$. Since C_0 is dense, there is an $x \in C_0$ such that $x' < x < x''$ (unless $\langle x', x'' \rangle$ were empty, but that would mean that X admitted a decomposition $X = \langle\langle 0, x' \rangle\rangle \sqcup \langle\langle x'', 1 \rangle\rangle$ contrary to hypothesis). Therefore, x' and x'' induce unequal cuts in X, since the cut due to x' will have x in the superior set of the cut, while the x'' cut places x in its inferior set. But distinct cuts in X correspond to distinct cuts in $\mathbb{R}$; the cut defined by $f(x')$ will have $f(x)$ in its superior set while that defined by $f(x'')$ will have $f(x)$ in its inferior set. Therefore, f is injective. Notice that we have also proved $f(x') < f(x) < f(x'')$, whence $f(x') < f(x'')$, and our extended f is also order-preserving.

To prove surjectivity, we need only ask if all the irrationals in $I = [0, 1]$ are in the image of f, since it is clear that the rationals in $[0, 1]$ are. Now any irrational $r \in [0, 1]$ makes a cut in Q, say into (R, S) with $R < S$. The inverse images by f R and S give a cut (A, B) in $C_0 \subseteq X$. If there were no $x \in X$ inducing this cut, then, setting $\overline{A} = \{x \in X | (\exists a \in A)(x < a)\}$ and $\overline{B} = \{x \in X | (\exists b \in B)(b < x)\}$, we would obtain a partition $(\overline{A}, \overline{B})$ of X which would violate condition (iii) of the theorem. Thus, there is an $x \in X$ for every irrational $r \in [0, 1]$, so f is surjective.

Lastly, we must verify that $x < x' \Leftrightarrow f(x) < f(x')$. But '$\Rightarrow$' has already been established above, while, in the present setting, '$\Leftarrow$' follows from '$\Rightarrow$' since, by hypothesis, any two elements of X are comparable. Thus, $f : X \to [0, 1] \subseteq \mathbb{R}$ is a tonomorphism. $\square$

Finally, we use the result just derived in the proof of the following theorem (*cf.* [10]), which provides a sufficient condition for an ordered topological space to be isomorphic to $[0, 1]$.

Theorem 6: Let Γ be a set provided with both a linear order and a topology such that

- with respect to the topology it is compact and connected and contains a countable dense subset,
- with respect to the order it has both a minimum and a maximum element, and
- (with respect to both) it has the property that $\langle\langle x, y \rangle\rangle$ is topologically closed $\forall x, y \in \Gamma$.

Then Γ is isomorphic to the interval $[0, 1] \subseteq \mathbb{R}$ by a simultaneous order- and topological isomorphism.

Proof: Let S be the countable (topologically) dense subset. To establish that Γ is order-isomorphic to $[0, 1] \subseteq \mathbb{R}$, we verify the hypotheses of Theorem 5 by first checking that S is order-dense and then verifying condition (iii) of that theorem.

First, to prove that S is order-dense, observe that any non-empty order-open interval $I = \langle x, y \rangle$ is also topologically open, because its complement, $\langle\langle 0, x \rangle\rangle \cup \langle\langle y, 1 \rangle\rangle$, is the union of two order-closed intervals, and order-closed intervals are also topologically closed by hypothesis. Thus I contains points of S as required.

To verify condition (iii) of Theorem 5, let us first show that any "past set" (or "order ideal") in Γ has a supremum (a past set being a subset A such that $x \prec y \in A \Rightarrow x \in A$). To that end, observe first that, if A is a past set, then A, as ordered by $\prec$, is a directed set and therefore defines a *net* in the topological space Γ. As Γ is compact this net has an accumulation point a, which we will show is the desired supremum. In fact, if $x \in A$ then by definition all of the points of the net A eventually $\succ x$, *i.e.* they belong to $\langle\langle x, 1 \rangle\rangle$; hence their accumulation point a belongs to $\langle\langle x, 1 \rangle\rangle$, because the latter set is topologically closed by hypothesis, and a closed set contains all of its accumulation points. Thus $\forall x \in A$, $x \prec a$, and we conclude that a is an upper bound of A. On the other hand if c is an arbitrary upper bound of A, then $A \subseteq \langle\langle 0, c \rangle\rangle$ which, being closed, must contain the accumulation point a as before, whence $a \prec c$. Hence, a is the least upper bound of A, as promised.

Now let $\Gamma = A \sqcup B$ with $A \prec B$. Since A is manifestly a past set in this situation, it has a supremum a, as we have just seen; and dual reasoning tells us that B also has an infimum b. Then either $a = b$ or not.

In the former case, either $a = b \in A$ and the first alternative of condition (iii) holds, or $a = b \in B$ and the second alternative of condition (iii) holds.

On the other hand if a and b are distinct, then a jump occurs: $A = \langle\langle 0, a \rangle\rangle$ and $B = \langle\langle b, 1 \rangle\rangle$, which means that Γ would be the disjoint union of a pair of order-closed intervals. But, as order-closed intervals are topologically closed by hypothesis, this would mean that Γ was topologically disconnected, contrary to assumption.

So far we have shown that Γ is order-isomorphic to $[0, 1] \subseteq \mathbb{R}$. Now we must prove that it is not only tonomorphic to $[0, 1]$, but homeomorphic as well.

To that end, fix a tonomorphism $f : \Gamma \to [0,1]$ and let $[r,s] \subseteq [0,1] \subseteq \mathbb{R}$ be a closed real interval. We have $f^{-1}([r,s]) = \langle\langle f^{-1}(r), f^{-1}(s)\rangle\rangle$, which is closed by hypothesis. Now the family of all closed intervals in $[0,1]$ generates the topology of $[0,1]$ and, since the inverse images of these intervals under f are all closed in Γ, f is continuous. But a continuous bijection between compact spaces is a homeomorphism, so f is a joint homeo- and tonomorphism of Γ to $[0,1] \subseteq \mathbb{R}$, as required. □

Appendix B: Comparison with Definitions Used in the C^2 Context

In a C^2-lorentzian manifold every point possesses a so-called convex normal neighborhood, and the framework presented in references such as [3], [7] and [8] becomes applicable. Since C^2 is a special case of C^0, the framework of this paper also applies, of course, and we may compare it with the C^2 one. In fact, although some of the definitions do differ, most of the important ones coincide, including the definitions of causal curve and of globally hyperbolic manifold. Here we summarize (in a few cases without proof) the most important agreements and disagreements we know of. Unless otherwise stated, any terms which we use without definition will have the meaning assigned to them in ref. [7]; and *the manifold in question will always be taken to be C^2-lorentzian as a standing assumption.*

The most important disagreements in definition stem from our use of K^+ where the corresponding C^2 definition would use J^+, the latter being in general a proper subset of the former. However in many situations (including in all globally hyperbolic spacetimes) these two causal relations coincide. In the C^2 context, $J^+(p)$ denotes the set of points reached from p by smooth curves with future-directed null or timelike tangent; and $\mathcal{M}$ is said to be *causally simple* iff $J^+(\mathcal{K})$ and $J^-(\mathcal{K})$ are closed for every compact set $\mathcal{K} \subseteq \mathcal{M}$. (Equally, $\mathcal{M}$ is causally simple iff $J^+(p)$ is the closure of $I^+(p)$ and $J^-(p)$ is the closure of $I^-(p)$ for each p.) When $\mathcal{M}$ is causally simple, there is no difference between J^+ and K^+:

Lemma 25: In a causally simple C^2-lorentzian manifold, $K^+(p) = J^+(p)$ and $K^-(p) = J^-(p)$.

Proof: Write $p \leq q \Leftrightarrow q \in J^+(p)$. Trivially, $\leq$ is transitive and contains the chronology relation I^+ so we check closure. Let $x_n \leq y_n$ for each n, where $x_n \to x$ and $y_n \to y$. Let us prove by contradiction that $x \leq y$. If not, then

$x \notin J^-(y)$, whence, since $J^-(y)$ is closed by causal simplicity, x would possess a compact neighborhood $\mathcal{K}$ disjoint from $J^-(y)$, whence $y \notin J^+(\mathcal{K})$. But since $J^+(\mathcal{K})$ is closed, y would have to have an open neighborhood disjoint from it, contradicting the existence of timelike curves from x_n to y_n for arbitrarily large n. Hence $x \leq y$ and $\leq$ is closed. Lastly, since by causal simplicity $J^+(p)$ is the closure of $I^+(p)$, we cannot find a smaller closed relation than $\leq$ which includes I^+. Therefore, $\leq$ is $\prec$, and $K^+(p) = J^+(p)\ \forall p$. Likewise, $J^-(p) = K^-(p)$. $\square$

Next consider the definition of causal curve. Not everyone phrases this in precisely the same manner, but it is easy to verify that, for example, the definition given in [7] coincides with our Definition 17, as one sees by noting that $\gamma(t) \prec_{\mathcal{O}} \gamma(t')$ in Definition 17 is equivalent to $\gamma(t') \in J^+(\gamma(t), \mathcal{O})$ by the causal simplicity of convex normal neighborhoods.

Now consider the definition of global hyperbolicity, a concept made by joining together a causality condition with a compactness requirement. Within our framework, global hyperbolicity is defined in terms of K^+ and signifies that cycles are absent and intervals are compact (Definition 11). Within the C^2 framework, the accepted definition (in one of several equivalent forms which are in use) refers to J^+ rather than K^+. With that difference, it again requires that intervals be compact, but the acyclicity condition (intrinsically weaker in the case of J^+) is strengthened to local causal convexity. (Usually one says "strong causality", but that condition is nothing but local causal convexity with respect to J^+ (or equivalently I^+).) In other words, the C^2 framework uses the

Alternative Definition 26: The C^2-lorentzian manifold $\mathcal{M}$ is globally hyperbolic iff it is locally J-convex and $J(p,q)$ is compact $\forall\, p, q \in \mathcal{M}$.

In defining global hyperbolicity of an open *subset* $\mathcal{O} \subseteq \mathcal{M}$, one requires as well that $\mathcal{O}$ be J-convex.

To compare this with Definition 11, we must in particular compare K-causality with local J-convexity. But since $I^+ \subseteq K^+$ we have as a simple corollary of Lemma 16:

Lemma 27: If $\mathcal{M}$ is K-causal, then it is locally J-convex.

Proof: Apply Lemma 16 and observe that causal convexity relative to K^+ immediately implies causal convexity relative to I^+. $\square$

The converse of Lemma 27 is not true. For example the manifold depicted in Figure 2c is locally J-convex, but it is not K-causal because $p \prec q$ and $q \prec p$. Thus K-causality is strictly stronger than "strong causality."*

Using Lemma 27, we can prove that global hyperbolicity of $\mathcal{M}$ in our sense coincides with the usual meaning (which we will refer to as being globally hyperbolic "in the C^2 sense").

Lemma 28: A C^2-lorentzian manifold $\mathcal{M}$ is globally hyperbolic in the sense of Definition 26 iff it is globally hyperbolic in the sense of Definition 11.

Proof: First assume that $\mathcal{M}$ is globally hyperbolic in the C^2 sense, in which case it is also causally simple. Then, by Lemma 25, $J^{\pm} = K^{\pm}$. Hence $K(p,q) = J(p,q) := J^+(p) \cap J^-(q)$, which is compact by Definition 26. Furthermore, $\mathcal{M}$ is K-causal since if there were unequal points which K-preceded each other, then they would also J-precede each other (since $J = K$), contradicting the local J-convexity of $\mathcal{M}$. Hence $\mathcal{M}$ is globally hyperbolic in our sense.

Conversely assume that $\mathcal{M}$ fulfills the conditions of Definition 11. Then it is certainly locally J-convex by Lemma 27, so we only have to prove that $J(p,q)$ is compact for any points $p, q \in \mathcal{M}$. Since $J(p,q)$ is included in the compact set $K(p,q)$, it suffices to show that it is closed. Then let $x_j \to x$ be any convergent sequence with $x_j \in J(p,q)$ for all j. For each j there is by definition a causal path γ_j such that $\gamma_j(0) = p$ and $\gamma_j(1) \to x$. Theorem 23 and Lemma 22 then guarantee us a causal curve Γ from p to x, proving that $x \in J^+(p)$, as required. (Recall that our definition of causal curve coincides with the C^2 definition.) Similarly, $x \in J^-(q)$, and we are done. □

Because of this equivalence, we have not, in the main text, introduced any new word to differentiate our concept of global hyperbolicity from the C^2 one. We should point out, however, that the equivalence demonstrated in Lemma 28 applies to global hyperbolicity of an entire manifold $\mathcal{M}$, and not necessarily to arbitrary open subsets of $\mathcal{M}$, the reason

* R. Low [11] has pointed out to us that the causal relation known as "Seifert's J_S^+" is closed and transitive (and contains I^+). Therefore it contains $\prec$. Since the condition that J_S^+ be a partial order is known as "stable causality," it follows that stable causality implies K-causality. Whether these two conditions are actually equivalent is an interesting question. Both K-causality and stable causality fail on the manifold of Figure 2c.

being that for a subset of $\mathcal{O} \subseteq \mathcal{M}$, J-convexity does not always imply K-convexity. (Once again Figure 2c provides a counterexample.)

Finally, consider the definition of convergence for sequences of causal curves. In the C^2-lorentzian context, at least two distinct notions of convergence are in use, as summarized in [3], and both can be phrased in topological terms. The first is convergence with respect to what reference [5] calls the "upper topology," and the second is convergence with respect to the "lower topology." In the upper topology (sometimes called the C^0 topology), a sequence (or net) of curves Γ_n converges to the curve Γ iff the Γ_n are eventually *included* in every open set $\mathcal{O} \subseteq \mathcal{M}$ which includes Γ itself. In the lower topology, the same sequence (or net) converges to Γ iff the Γ_n eventually *meet* (have non-empty intersection with) every open set $\mathcal{O} \subseteq \mathcal{M}$ which meets Γ.

The upper and lower topologies are essentially equivalent in locally J-convex manifolds (Prop. 2.21 in [3]), but taken separately they have the drawback of not being Hausdorff unless supplemented by conditions controlling the endpoints of the curves (see for example [8]). By requiring both types of convergence at once, we obtain a topology which is automatically Hausdorff, and is in fact the Vietoris topology, or more precisely the topology on the space of causal curves induced by the Vietoris topology on $2^{\mathcal{M}}$. In this paper we have interpreted convergence in the sense of Vietoris because, by doing so, we were able to prove the existence of accumulation curves by first constructing accumulation *sets*, and then showing that these sets are in fact causal curves.

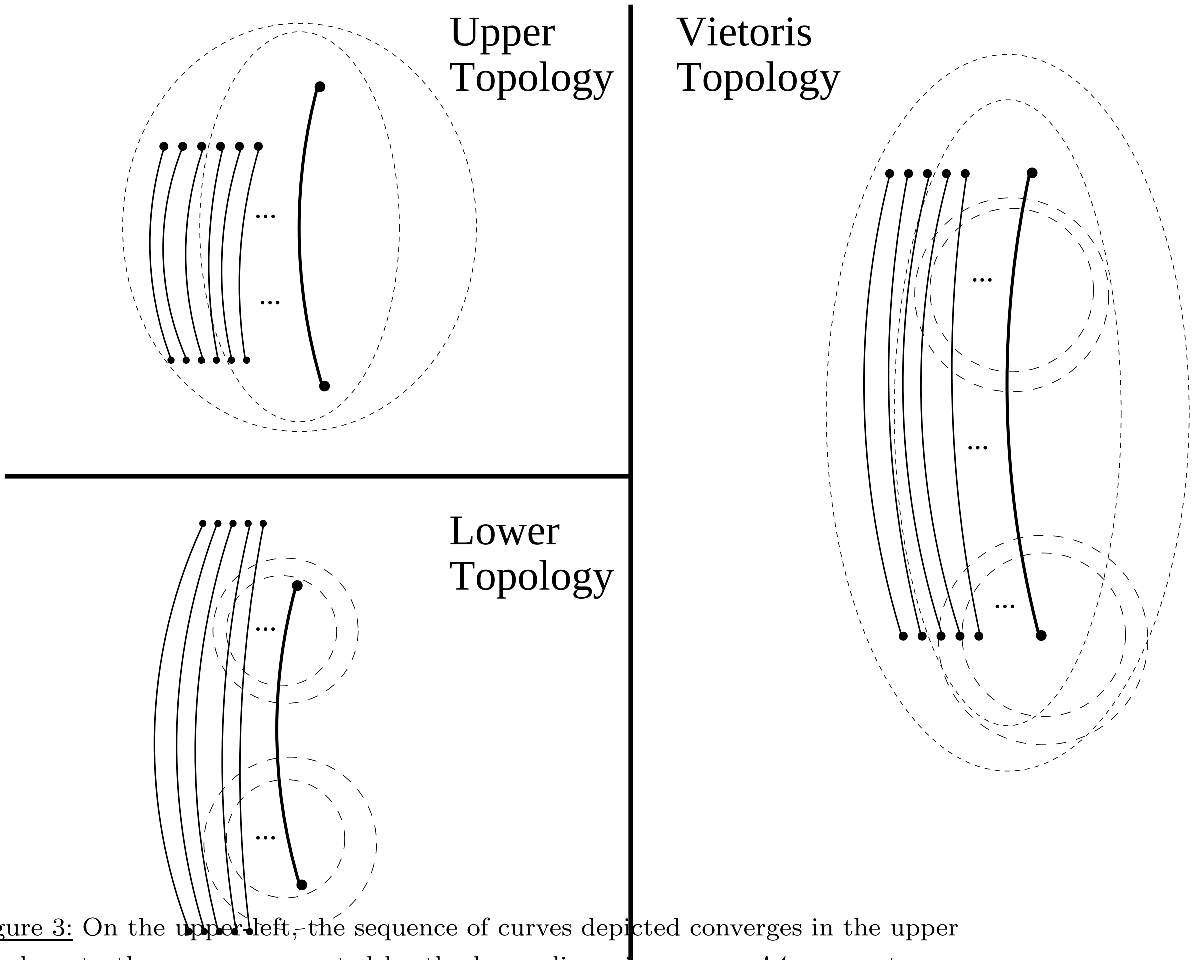

Figure 3: On the upper-left, the sequence of curves depicted converges in the upper topology to the curve represented by the heavy line, since every $\mathcal{M}$-open set containing the latter also contains all but finitely many curves of the sequence. The endpoints of the sequence curves need not converge to the endpoints of the limit curve.

The sequence of curves depicted on the lower-left converges in the lower topology to the curve represented by the heavy line, since every $\mathcal{M}$-open set that meets the latter also meets all but finitely many curves of the sequence. Again, without endpoint conditions, the endpoints of the sequence curves need not converge to the endpoints of the limit.

The right-hand sequence of curves converges in the Vietoris topology to the curve depicted by the heavy line, since the sequence meets both the criterion for convergence in the upper topology and the criterion for convergence in the lower topology. Notice that convergence of the endpoints is enforced automatically (Lemma 22).

Now in general the upper, lower, and Vietoris topologies all differ, but in a locally J-convex region, they induce precisely the same topology on the space $C(p,q)$ of causal curves from a given point p to a given point q; and this is all that is relevant in comparing our Theorem 23 with the corresponding theorem from the C^2 context. In fact, all we need in order to show that the C^2 theorem follows from ours is the trivial observation that the Vietoris topology is in every situation either equal to or finer than the upper and lower topologies.

Having taken note of this, we can easily derive the standard C^2 theorem as a corollary of Theorem 23. In other words, we can demonstrate that within any open subset $\mathcal{O} \subseteq \mathcal{M}$ which is globally hyperbolic in the standard C^2 sense (as described after Definition 26), the space of curves $C(p,q)$ is compact with respect to the upper topology. To see this, notice first that $\mathcal{O}$, regarded as a manifold in its own right, will be globally hyperbolic in the sense of Definition 26, and therefore also globally hyperbolic in our sense, by Lemma 28. But then Theorem 23 implies that $C(p,q)$ is compact in the Vietoris topology, hence compact in any weaker topology, including in particular the upper topology.