Crystalizing the Spinon Basis

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Abstract
The quasi/n2dparticle structure of the higher spin XXZ model is studied. We obtained a new description of crystals associated with the level $k$ integrable highest weight $U_q(\mathfrak{sl}_2)$ modules in terms of the creation operators at $q = 0$ (the crystalline spinon basis). The fermionic character formulas and the Yangian structure of those integrable modules naturally follow from this description. We have also derived the conjectural formulas for the multi quasi-particle states at $q = 0$.

Introduction
In this paper, we consider the integrable XXZ spin chain with spin $k/2$ of $\mathfrak{sl}_2$. The space of states is the infinit tensor product (the space of local fields)

$$\mathcal{W} = \cdots \otimes C^{k+1} \otimes C^{k+1} \otimes C^{k+1} \otimes C^{k+1} \otimes \cdots.$$ 

In remarkable papers [7, 25], using the Bethe Ansatz, Fadeev and Takhtajan discovered that the one-particle excitation in the anti-ferromagnetic regime of XXX chain is always (a kink of) spin $1/2$. According to this picture, one can expect another description of space of states[7, 23] such as (the space of asymptotic particles)

$$\mathcal{F} = \sum_{n=0}^{\infty} \left[ \sum_{\pi \in \text{path}} C((z_1, \cdots, z_n)) \otimes [\otimes^n C^2]_p \right]^{\text{Symm}},$$

where Symm is the symmetrization with respect to the S-matrix. On the other hand, stimulated by the deep results by Smirnov[24], the third description of the space of states in terms of the representations of $U_q(\hat{\mathfrak{sl}_2})$ (the space of non-local symmetries) was proposed[5, 10]

$$\mathcal{H} = \sum_{i,j=0}^{k} V(\lambda_i) \hat{\otimes} V(\lambda_j)^*.$$ 

Here $V(\lambda_i)$ is the integrable highest weight representation of $U_q(\hat{\mathfrak{sl}_2})$ and $\hat{\otimes}$ means some extended tensor product. The particle picture is recovered using the $q$-vertex operators acting among them.
In [3, 4], a new description of the Hilbert space of the (chiral) WZW conformal field theory was obtained. The spin 1/2 vertex operators were identified with the particles (spinons) and the basis of the integrable $\widehat{sl}_2$ modules are determined in terms of the spinons (spinon basis). Their picture is clear, but technically it is quite hard because of very complicated algebra of spinons.

The aim of this paper is to establish the remarkable equivalence

$$\mathcal{H} = \mathcal{F},$$

rigorously, focusing on its combinatorial aspects, that is at the level of crystals. In a sense this can be considered as the crystallization of the $q$-deformation of [3, 4].

One of (and the most remarkable) advantages to consider $q$-analogue is that one can go into $q \to 0$ limit where everything becomes clear and transparent (the "crystal" theory). It is expected that in the limit $q \to 0$ the resulting algebraic structures for "crystaline spinon" will be much simpler than that of $q = 1$.

Here we must recall that there are two kinds of $q$-deformation of the vertex operators, called type I and type II in the terminology of [5]. They almost have the same properties as far as we consider them separately. However, since it is crucial, in the study of spin chains, to consider them simultaneously, they inevitably reveal different faces.

While the type I $q$-vertex operators have a well defined $q = 0$ limit and play the central role for the equivalence $\mathcal{H} = \mathcal{W}$, the existence of $q \to 0$ limits of the creation operators (those are represented as type II $q$-VO) is not clear. Indeed it is known that type II $q$-VO produces poles at $q = 0$, hence the $q \to 0$ limit does not exist in a naı̈ve sense.

Nevertheless, it is conjectured [5] that those poles summed up to a meromorphic function in spectral parameters and the $q \to 0$ limits of the creation operators are well-defined, if they act on the true vacuum, the ground state of the model. In fact what we need here is this type II $q$-vertex operators.

One way to avoid this subtle problem is to construct the creation operators and their action on the path basis directly at $q = 0$, rather than taking the limit $q \to 0$. For the level $k = 1$ case, this kind of description of $q = 0$ creation operators was considered previously in [5], and we will generalize it to higher levels $k \geq 2$. Though there still remains a deep gap between $q = 0$ and $q \neq 0$ descriptions, thus obtained statements of combinatorial nature (such as character formulas) are independent of $q$ and available to generic $q$ including 1. This is reminiscent of "Combinatorial Bethe Ansatz" due to [19].

The paper is organized as follows. Section 1 is a quick review of the path realization that shows the equivalence $\mathcal{H} = \mathcal{W}$. Then in section 2, we define and study the algebra of the creation operators at $q = 0$, and give a precise combinatorial description of the space $\mathcal{F}$ including subtleties on the statistics. In section 3, we formulate and prove our main theorem (Theorem 3) that gives
bijections (isomorphism of crystals) between space of fields $\mathcal{W}$ and space of particles $\mathcal{F}$. In section 4 and 5, we discuss about the Yangian structure and spinon (or fermionic) character formulas. These kinds of structures, that are conjectured by recent TBA analysis (see for example [17, 20, 18]), will be obtained as corollaries of the results in section 3. In section 6 the conjectural formulas for the $q = 0$ limit of the quasi-particles of the spin $k/2$ XXZ model are given in terms of path basis. This is also an easy consequence of the commutation relations of creation operators at $q = 0$. Section 7 is devoted to comments and discussions.

1 Review of the path realization

Here we review a path realization of crystals. Unless otherwise stated we follow the notations of § 2 of [10]. Other notations and fundamental properties of crystals which we use in this paper we refer [16, 22]. We denote by $B^{(k)} = \{b^{(k)}_l \mid 0 \leq l \leq k\}$ the associated crystal to the crystal base of the $k+1$ dimensional $U_q(\widehat{sl}_2)$ module $V^{(k)}$. The actions of $\tilde{\epsilon}_i, \tilde{f}_i$ $(i = 0, 1)$ on $B^{(k)}$ are specified by

$$\tilde{f}_i b^{(k)}_l = \tilde{\epsilon}_i b^{(k)}_l = b^{(k)}_{l+1}, \quad 0 \leq l \leq k-1,$$

$$\tilde{\epsilon}_i b^{(k)}_l = \tilde{f}_i b^{(k)}_l = b^{(k)}_{l-1}, \quad 1 \leq l \leq k,$$

and $\tilde{\epsilon}_i b^{(k)}_0 = 0$ (otherwise) for $x = e, f, i = 0, 1$. We call a crystal defined from $U_q(\widehat{sl}_2) \widehat{sl}_2$ crystal. A $\widehat{sl}_2$ crystal is called $(\widehat{sl}_2)_x$ crystal $(i = 0, 1)$ if we forget the actions of $\tilde{\epsilon}_i, \tilde{f}_i, \{h_i\}$ being the simple coroots. Let $P$ and $P^*$ denote the weight and the dual weight lattice of $\widehat{sl}_2$, $P = Z \Lambda_0 \oplus Z \Lambda_1 \oplus Z \delta$, $P^* = Z h_0 \oplus Z h_1 \oplus Z d$, $\{h_i, \Lambda_j\} = \delta_{ij}, \{d, \delta\} = 1, \{h_i, \delta\} = \{d, \Lambda_j\} = 0$.

Let us set $\mathcal{P}^{(m, m')} = B(\lambda_m) \otimes B(\lambda_{m'}^*)$, which is the crystal associated to $V(\lambda_m) \otimes V(\lambda_{m'}^*)$. The ground state path $\bar{f}_m$ is defined by

$$\bar{f}_m(l) = m + (k - 2m)e(l), \quad l \in Z,$$

where $e(l) = 0(l : \text{even}) = 1(l : \text{odd})$. Define the $(m, m')$ ground state path by

$$\bar{f}_{m,m'}(l) = \begin{cases} 
\bar{f}_m(l) & l \geq 1, \\
\bar{f}_{m'}(l) & l < 0.
\end{cases}$$

Then

**Proposition 1** There is a bijection

$$\mathcal{P}^{(m, m')} = \{p = (p(l))_{l \in Z} \mid p(l) \in \{0, \cdots, k\}, p(l) = \bar{f}_{m,m'}(l) (|l| >> 0)\}.$$
We call an element in the right hand side of this \((m, m')\) path. From now on we identify \(\mathcal{P}(m, m')\) with the set of \((m, m')\) paths.

A \((m, m')\) path can be described in terms of domain walls. For \(p \in \mathcal{P}(m, m')\) we can associate the uniquely determined data

\[ n \in \mathbb{Z}_{\geq 0}, \quad (l_0, \cdots, l_1) \in \mathbb{Z}^n, \]

and \((m_n, \cdots, m_0) \in \{0, \cdots, k\}^{n+1}\)

such that

\[
\begin{align*}
    l_0 & \geq \cdots \geq l_1, \\
    m_n &= m, \quad m_0 = m', \\
    |m_j - m_{j-1}| & = 1 \quad \text{for any } j, \\
    \text{if } l_{s+1} > l_{s+1}' - 1 = \cdots = l_s > l_{s-1}, \text{ then } |m_{s+1} - m_s| = s', \\
    p(l) &= \bar{p}_m(l) \quad l_{r+1} \geq l \geq l_r + 1.
\end{align*}
\]  

(1)

If \(l_r = l_{r+1}\), we understand that there are no corresponding condition of the form (1). Conversely these data uniquely determines the \((m, m')\) path. For a fixed sequence \((m_n, \cdots, m_0)\) we denote this path by \([l_0, \cdots, l_1]\).

Let us describe the actions of \(\hat{\epsilon}_i\) and \(\hat{f}_i\) on \([l_0, \cdots, l_1]\). Set

\[ s_j = \text{sign}(p(l_j) + p(l_j + 1) - k) \quad 1 \leq j \leq n \]

and \(r_j = \frac{1}{2}(1 + s_j)\), where \(\text{sign}(j) = 1(j > 0) = -1(j < 0)\). Note that \(p(l_j) + p(l_j + 1) - k \neq 0\). Let us associate the element \(b\) of \(B(1)^{\otimes n}\) with the path \(p\) by

\[ b = b^{(1)}_{l_1} \otimes \cdots \otimes b^{(1)}_{l_1}. \]

(3)

Suppose that

\[ \hat{\epsilon}_i b = b^{(1)}_{l_1} \otimes \cdots \otimes \hat{\epsilon}_i b^{(1)}_{r_j} \otimes \cdots \otimes b^{(1)}_{r_j}. \]

(4)

Then

\[ \hat{\epsilon}_i [[l_0, \cdots, l_1]] = \begin{cases} 0, & \text{if } \hat{\epsilon}_i b = 0, \\
[l'_0, \cdots, l'_1], & \text{if } \hat{\epsilon}_i b \neq 0, \end{cases} \]

(5)

where \(l'_s = l_s\) \((s \neq j)\) and

\[ l'_j = \begin{cases} l_j + 1, & \text{for } x = f, \\
 l_j - 1, & \text{for } x = e. \end{cases} \]

(6)

The weight of a \((m, m')\) path \(p\) is given by

\[ w(t(p)) = (m - m' + 2s(p))(\Lambda_1 - \Lambda_0) - h(p)\delta, \]

\[ s(p) = \sum_{i \in \mathbb{Z}} (\bar{p}_{m, m'}(i) - p(i)), \]

\[ h(p) = \sum_{i \in \mathbb{Z}} i \left( \hat{H}(p(l + 1), p(i)) - \hat{H}(\bar{p}_{m, m'}(l + 1) - \bar{p}_{m, m'}(l)) \right), \]

4
where the function $\tilde{H}(j, j')$ is defined by

$$
\tilde{H}(j, j') = \begin{cases} 
- j' & \text{if } j + j' \leq k, \\
- j - k & \text{if } j + j' > k.
\end{cases}
$$

## 2 Creation algebra at $q = 0$

**Definition 1** The algebra $\mathcal{A}$ is generated by $\{ \varphi_{j_i}^p \mid j \in \mathbb{Z}, p \in \{0, 1\} \}$ over $\mathbb{Z}$ subject to the following defining relations:

$$
\begin{align*}
\varphi_{j_1}^{p_1} \varphi_{j_2}^{p_2} + \varphi_{j_2}^{p_1} \varphi_{j_1}^{p_2} &= 0, & j_1 &= j_2 \mod 2 \text{ and } (p_1, p_2) = (1, 0), \\
\varphi_{j_1}^{p_1} \varphi_{j_2}^{p_2} + \varphi_{j_2}^{p_1} \varphi_{j_1}^{p_2} &= 0, & j_1 &\neq j_2 \mod 2 \text{ and } (p_1, p_2) = (1, 0), \\
\varphi_{j_1}^{p_1} \varphi_{j_2}^{p_2} + \varphi_{j_2}^{p_1} \varphi_{j_1}^{p_2} &= 0, & j_1 &= j_2 \mod 2 \text{ and } (p_1, p_2) \neq (1, 0), \\
\varphi_{j_1}^{p_1} \varphi_{j_2}^{p_2} + \varphi_{j_2}^{p_1} \varphi_{j_1}^{p_2} &= 0, & j_1 &\neq j_2 \mod 2 \text{ and } (p_1, p_2) \neq (1, 0), \\
\varphi_{j_1}^{p_1} &= 1 \varphi_{j_1}^p = \varphi_j^p.
\end{align*}
$$

As a special case of the above defining equations we have

$$
\begin{align*}
\varphi_{j_1}^{p_1} \varphi_{j_2}^{p_2} &= \varphi_{j_2}^{p_1} \varphi_{j_1}^{p_2} = 0, & \text{if } (p_1, p_2) = (1, 0), \\
\varphi_{j_1}^{p_1} \varphi_{j_2}^{p_2} &= \varphi_{j_2}^{p_1} \varphi_{j_1}^{p_2} = 0, & \text{if } (p_1, p_2) \neq (1, 0).
\end{align*}
$$

We call $\mathcal{A}$ the crystaline creation algebra. The derivation of the above commutation relations is explained in appendix A. The algebra $\mathcal{A}$ is naturally graded by

$$
\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{A}_n,
$$

$$
\mathcal{A}_n = \sum_{\{p_n, \cdots, p_1, j_n, \cdots, j_1\}} Z \varphi_{j_n}^{p_n} \cdots \varphi_{j_1}^{p_1}, \quad \mathcal{A}_0 = \mathbb{Z}. \quad (7)
$$

Let us introduce the functions $H, K, \epsilon$ by

$$
H(0, 0) = H(0, 1) = H(1, 1) = 0, \quad H(1, 0) = 1
$$

$$
K(p_1, p_2) = 1 - H(p_1, p_2), \quad \epsilon(j) = \frac{1 - (-1)^j}{2}.
$$

Then the commutation relations can be written compactly as

$$
\varphi_{j_2}^{p_2} \varphi_{j_1}^{p_1} + \varphi_{j_1}^{p_1} \varphi_{j_2}^{p_2} + 2K(p_2, p_1)\epsilon(j_1 + j_2) \varphi_{j_1}^{p_1} \varphi_{j_2}^{p_2} + 2K(p_2, p_1)\epsilon(j_1 + j_2) \varphi_{j_2}^{p_2} \varphi_{j_1}^{p_1} = 0.
$$

Let us set

$$
B = \bigcup_{n=0}^{\infty} \{ \varphi_{j_n}^{p_n} \cdots \varphi_{j_1}^{p_1} \mid j_i \in \mathbb{Z}, p_i \in \{0, 1\}, \}
$$

$$
B(p_n, \cdots, p_1) = \{ \varphi_{j_n}^{p_n} \cdots \varphi_{j_1}^{p_1} \mid (j_n, \cdots, j_1) \text{ satisfies the condition (8)} \},
$$

$$
B_{\geq 0}(p_n, \cdots, p_1) = \{ \varphi_{j_n}^{p_n} \cdots \varphi_{j_1}^{p_1} \in B(p_n, \cdots, p_1) \mid j_i \geq 0 \}.
$$
If $n = 0$, we define $(p_0, \cdots, p_1) = \phi$ and $B(\phi) = \{1\}$. The condition is
\[
j_0 - 2j_0(p_0, \cdots, p_1) \geq \cdots \geq j_2 - 2j_2(p_2, p_1) \geq j_1, \tag{8}
\]
\[
l(p_1, \cdots, p_1) = \sum_{i=1}^{l-1} H(p_{i+1}, p_i).
\]

**Theorem 1** $B(p_0, \cdots, p_1)$ is a $Z$ linear base of $A_n$.

We prove the theorem in a more general setting. Let $J$ be a set. Let us consider the associative algebra over $Z$ with unit generated by $\{\psi_A(n)[A \in J, n \in Z]\}$ whose defining relations are
\[
\psi_A(n)\psi_B(m) + \psi_A(m + s_{AB})\psi_B(n - s_{AB}) = 0,
\]
where $s_{AB}$ is some fixed integer. In what follows, we fix a sequence $(A_1, \cdots, A_n)$ and put $s_{i,i+1} = s_{A_i, A_{i+1}}$. In general let us define
\[
s_{i,j} = \begin{cases} 
  s_{i,i+1} + s_{i+1,i+2} + \cdots + s_{j-1,j}, & (i < j), \\
  0, & (i = j), \\
  -s_{j,i}, & (i > j).
\end{cases}
\]

**Lemma 1** For any permutation $\sigma$ of $(1, 2, \cdots, n)$,
\[
\psi_{A_1}(m_1) \cdots \psi_{A_n}(m_n) = \text{sgn}(\sigma)\psi_{A_{\sigma(1)}}(m_{\sigma(1)}) \cdots \psi_{A_{\sigma(n)}}(m_{\sigma(n)}),
\]
where
\[
m_{i,j} = m_i + s_{i,j}.
\]

**Proof**
By the induction on the length of $\sigma$, the lemma is easily proved.

**Definition 2** In a product
\[
\Psi = \psi_{A_1}(m_1) \cdots \psi_{A_n}(m_n),
\]
the index pair $(m_i, m_j)$ ($i < j$) is said to be normal (zero, abnormal) iff
\[
m_i > m_j + s_{i,j} \quad (m_i = m_j + s_{i,j}, \quad m_i < m_j + s_{i,j}).
\]
Furthermore, $\Psi$ is normal iff all pairs are normal.

**Lemma 2** The notion of normal (zero, abnormal) of two indices $(m_i, m_j)$ is independent of the position of them as far as their order is preserved. If the order is exchanged, then the normal pair is transformed into abnormal one and zero pair to zero pair.
Proof
Under any permutation \( \sigma \) such that \( \sigma^{-1}(i) = i' \) and \( \sigma^{-1}(j) = j' \), one has
\[
m_{i',i} = m_i + s_{i',i}, \quad m_{j',j} = m_j + s_{j',j}.
\]
Then, for the new pair \((m_{i',i}, m_{j',j})\) at position \((i', j')\), one has
\[
m_{i',i} - m_{j',j} = m_i - m_j + s_{i',i} - s_{j',j} = m_i - m_j - s_{i,j},
\]
and
\[
m_{j',j} - m_{i',i} + s_{j',j} = -(m_i - m_j - s_{i,j}).
\]
Hence the normal (zero, abnormal) pair is transformed into normal (zero, abnormal) for \( i' < j' \), or abnormal (zero, normal) for \( i' > j' \).

**Corollary 1** A product of the form
\[
\Psi = \psi_{A_1}(m_1) \cdots \psi_{A_n}(m_n),
\]
is 0 iff there exist (at least) a pair \((m_i, m_j)\) such that
\[
m_i = m_j + s_{i,j}.
\]
Otherwise, it can be transformed into normal form.

**Corollary 2** The set of normal forms \(\{\psi_{A_1}(m_1) \cdots \psi_{A_n}(m_n)\}\) forms a Z linear base of the algebra.

Proof
The corollary is proved by the standard argument of constructing a representation of the algebra using Corollary 1.

Proof of Theorem 1
Take \( J = \{0, 1\}^2 \) and
\[
s_{(p_1, i_1)(p_2, i_2)} = -K(p_1, p_2) + H(i_2, i_1).
\]
Then the above algebra is isomorphic to \( A \) by \( \varphi_{B_0+1} = \psi_{[p,i]}(n) \), where \( i = 0, 1 \). In this case the normality condition is exactly the condition (8). Hence Theorem 1 follows from Corollary 2.

**Definition 3** Let us define the weight of a nonzero element of \( B \) by
\[
wt(\varphi_{p_1}^{p_2} \cdots \varphi_{j_1}^{p_2}) = \sum_{l=1}^{n} wt(\varphi_{j_1}^{p_1})
\]
\[
wt(\varphi_j^{p}) = -[\frac{j}{2}] \delta + (1 - 2 \alpha(j))(A_1 - A_0),
\]
where \([j]\) is the Gauss symbol. We set \( wt1 = 0 \).
Let us introduce the actions of \( \hat{e}_i, \hat{f}_i \) \((i = 0, 1)\) on the set \( B(p) = \{ \varphi_j^p \mid j \in \mathbb{Z} \} \) \((p = 0, 1)\) by

\[
\hat{f}_1 \varphi_j^p = \varphi_j^{p+1}, \quad \hat{f}_0 \varphi_j^p = \varphi_j^p, \\
\hat{e}_1 \varphi_j^p = \varphi_j^{p-1}, \quad \hat{e}_0 \varphi_j^p = \varphi_j^p, \\
\tilde{x}_i \varphi_j^p = 0 \quad \text{otherwise},
\]

where \( x = \epsilon, f \). By these actions and the weight of \( \varphi_j^p \), \( B(p) \) is an affine crystal [16, 22] isomorphic to \( \text{Aff}(B(1)) \). In general

**Theorem 2** Let \( n \geq 1 \) and \((p_n, \cdots, p_1) \in \{0, 1\}^n \). There is a unique \( \widehat{sl}_2 \) crystal structure in \( B(p_n, \cdots, p_1) \) such that the natural map

\[
\text{Aff}(B(1))^\otimes n \to \left( B(p_n, \cdots, p_1) \cup B(p_n, \cdots, p_1) \cup \{0\} \right)/\pm \\
\varphi_j^{p_n} \otimes \cdots \otimes \varphi_j^{p_1} \to \varphi_j^{p_n} \cdots \varphi_j^{p_1}
\]

commutes with the actions of \( \hat{e}_i, \hat{f}_i \) \((i = 0, 1)\).

We consider \( B(0) = \{1\} \) to be the trivial crystal, \( \tilde{x}_i 1 = 0 \) for \( x = \epsilon, f \) and \( i = 0, 1 \).

**Proof of Theorem 2**

Take any \((p_2, p_1) \in \{0, 1\}^2 \) and fix it. Let us set, as a subset of the tensor algebra generated by \( B(p) \) \((p = 0, 1)\),

\[
\mathcal{I} = \{ \varphi_j^{p_2} \otimes \varphi_j^{p_1} + \varphi_j^{p_2} \otimes \varphi_j^{p_1} \mid j, j_i \in \mathbb{Z} \}
\]

Then it is sufficient to prove that

\[
\tilde{x}_i \mathcal{I} \subset \mathcal{I} \cup \{0\}
\]

for \( x = \epsilon, f \), \( i = 0, 1 \). Here in general for the elements \( b_1, b_2 \) of a crystal we understand \( \tilde{x}_i (b_1 + b_2) = \tilde{x}_i b_1 + \tilde{x}_i b_2 \). By direct calculations this property is easily proved.

### 3 Crystaline spinon base

**Definition 4** Let us call \((p_n, \cdots, p_1) \in \{0, 1\}^n \) a level \( k \) restricted path from \( 0 \) to \( l \) of length \( n \) if

\[
(-1)^{p_1} + \cdots + (-1)^{p_s} \in \{0, \cdots, k\} \quad \text{for} \quad 1 \leq s \leq n, \\
(-1)^{p_1} + \cdots + (-1)^{p_n} = l.
\]

We denote by \( \mathcal{P}_{res, n}^k(0, l) \) the set of restricted paths from \( 0 \) to \( l \) of length \( n \) and set

\[
\mathcal{P}_{res, n}^k = \bigcup_{l=0}^k \mathcal{P}_{res, n}^k(0, l).
\]
We understand that \( P^h_{\mathfrak{r}, s, \emptyset} = \{ \phi \} \).

The following theorem provides a new parametrization of the crystal base of the integrable highest or lowest weight \( U_q(\widehat{sl}_2) \) modules.

**Theorem 3** There is an isomorphism of affine crystals
\[
\bigcup_{n=0}^{\infty} \bigcup_{\{p_n, \cdots, p_1\}\in P_{\mathfrak{r}, s, n}(0, 1)} B(p_n, \cdots, p_1) \cong B(\lambda_1) \otimes B(\lambda_0)^* 
\]
given by
\[
1 \quad \mapsto \quad [ ] = b_{\lambda_0} \otimes b_{-\lambda_0},
\]
\[
\varphi_j^{p_n} \cdots \varphi_j^{p_1} \quad \mapsto \quad [[j_0 - p_n, \cdots, j_1 - p_1]].
\]

**Corollary 3** The map in Theorem 3 induces the isomorphism of \((\widehat{sl}_2)_1\) crystals with affine weights:
\[
\bigcup_{n=0}^{\infty} \bigcup_{\{p_n, \cdots, p_1\}\in P_{\mathfrak{r}, s, n}(0, 1)} B_{\geq 0}(p_n, \cdots, p_1) \cong B(\lambda_1),
\]
where we make the identification:
\[
B(\lambda_1) \cong B(\lambda_1) \otimes b_{-\lambda_0},
\]
\[
b \quad \mapsto \quad b \otimes b_{-\lambda_0}.
\]
Here \( b_{-\lambda_0} \) is the lowest weight element in \( B(\lambda_0)^* \).

Note that there are no naturally defined \( \widehat{sl}_2 \) crystal structure on \( B_{\geq 0}(p_n, \cdots, p_1) \).

The map in Theorem 3 determines a representation of \( \mathcal{A} \) as in the proof of Corollary 2. The action of \( \varphi_j^{p} \) on \( \oplus Zb \) is described in the following manner, where \( b \) runs over all the elements in \( \bigcup_{n=0}^{\infty} B(\lambda_1) \otimes B(\lambda_0)^* \). Let \( b = [[j_0 - p_n - 1, \cdots, j_1 - p_1]] \) be as in Theorem 3 and \( \varphi = \varphi_j^{p_n} \varphi_j^{p_{n-1}} \cdots \varphi_j^{p_1} \). If \( \varphi \neq 0 \) then, by Lemma 1 and Lemma 2, there exist unique \( \mu \in \{ \pm 1 \} \) and normal sequence \( (j_n, \cdots, j_1) \) such that
\[
\varphi = \mu \varphi_j^{p_n} \varphi_j^{p_{n-1}} \cdots \varphi_j^{p_1}.
\]
We define \( \varphi_j^{p} b = 0 \) if \( (p_n, p_{n-1}, \cdots, p_1) \) is not a restricted path. Suppose that \( (p_n, p_{n-1}, \cdots, p_1) \in P_{\mathfrak{r}, s, n} \). Then
\[
\varphi_j^{p} b = \begin{cases} 0, & \text{if } \varphi = 0, \\
\mu[[j_0 - p_n, j_{n-1} - p_{n-1}, \cdots, j_1 - p_1]], & \text{if } \varphi \neq 0.
\end{cases}
\]
In particular for the element in \( B(p_n, \cdots, p_1) \) we have
\[
\varphi_j^{p_n} \cdots \varphi_j^{p_1} [[ ]] = [[j_n - p_n, \cdots, j_1 - p_1]].
\]
This representation is considered as a Fock type representation of the crystalline creation algebra, the vacuum being $[\lambda_0]$. We call the basis given in Theorem 3 and Corollary 3 the crystalline spinon basis.

Proof of Theorem 3
The bijectivity of the map is obvious by the description of the paths in terms of domain walls and the normality condition (8). So what we should prove is the following two statements.

(i) The map commutes with the actions of $\hat{e}_i$, $\hat{f}_i$ ($i = 0, 1$).

(ii) The map preserves weights.

Let us prove (i) first.

Lemma 3 Let $p = (p(n))_{n \in \mathbb{Z}} = [[j_0 - p_n, \ldots, j_1]]$ be an element of $\bigcup_{\lambda_0} B(\lambda) \odot B(\lambda_0)^*$. If

$$\frac{j_r + t}{p_r} > \frac{j_r + t_{-1}}{p_r} = \cdots = \frac{j_r}{p_r} > \frac{j_{r-1}}{p_r} - \frac{j_r - p_r}{1}$$

then

$$p(j_r - p_r) + p(j_r - p_r + 1) = k - l(-1)^{j_r}.$$

Note that the assumption of the lemma implies $p_r + t_{-1} = \cdots = p_r$ and $j_r + t_{-1} = \cdots = j_r$. By the rule (2) - (4), the above domain wall corresponds to $b_{\lambda_0}^{(1)} \in B(\lambda_0)^*$. 

Proof
The statement is easily proved by direct calculations.

Let us consider the classical crystal morphism

$$c^{p_i} : Aff(B^{(1)} \odot n) \longrightarrow B^{(1)} \odot n,$$

$$\varphi_{j_n}^{p_n} \otimes \cdots \otimes \varphi_{j_1}^{p_1} \mapsto b_{\lambda_0}^{(1)} \otimes \cdots \otimes b_{\lambda_0}^{(1)}.$$

Since $c^{p_i}$ commutes with the actions of $\hat{e}_i$ and $\hat{f}_i$

$$\hat{x}_i(\varphi_{j_n}^{p_n} \otimes \cdots \otimes \varphi_{j_1}^{p_1}) = \varphi_{j_n}^{p_n} \otimes \cdots \otimes \hat{x}_i \varphi_{j_1}^{p_1} \otimes \cdots \otimes \varphi_{j_1}^{p_1}$$

is equivalent to

$$\hat{x}_i(b_{\lambda_0}^{(1)} \otimes \cdots \otimes b_{\lambda_0}^{(1)}) = b_{\lambda_0}^{(1)} \otimes \cdots \otimes \hat{x}_i b_{\lambda_0}^{(1)} \otimes \cdots \otimes b_{\lambda_0}^{(1)}.$$

Hence (i) follows from the definition of the actions of $\hat{e}_i$, $\hat{f}_i$ on $B(p)$ and (2)-(6).

Next let us prove (ii).

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Lemma 4 Let \( p \in \cup_{m,m_1} \mathcal{P}^{(m,m_1)} \) be a path with \( n \) domain walls counting multiple ones. Then there exists \( P \) which is a composition of \( \hat{e}_i \) and \( \hat{f}_i \) \((i = 0, 1)\) such that
\[
Pp = [[l_n, \cdots, l_1]], \quad l_n > \cdots > l_1 \geq 0.
\]

Proof
Since \( B^{(1)} \) is a perfect crystal of level 1, \( B^{(1)\otimes n} \) is connected. Hence there exists a composition \( P_1 \) of \( \hat{e}_i \), \( \hat{f}_i \) \((i = 0, 1)\) such that
\[
c^{(n)}(P_1 p) = b_0^{(1)\otimes n}.
\]
If we denote \( P_1 p = [[l_n, \cdots, l_1]] \) then
\[
\hat{f}_1 \hat{f}_1 p \hat{f}_1 P_1 p = [[l_n + 2, \cdots, l_1 + 2]].
\]
Hence \( p \) is connected to \([r_n, \cdots, r_1]\) such that
\[
r_1 \geq 0, \quad c^{(n)}[r_n, \cdots, r_1] = b_0^{(1)\otimes n}.
\]
So let us assume that \( p \) is of this form from the beginning. Then
\[
(\hat{f}_0 \hat{f}_1^2)(\hat{f}_0^3 \hat{f}_1^4) \cdots (\hat{f}_0^{n-2} \hat{f}_1^{n-1}) p = [[r_n + n - 1, \cdots, r_2 + 1, r_1]] \quad n: \text{odd},
\]
\[
\hat{f}_1(\hat{f}_0^3 \hat{f}_1^3) \cdots (\hat{f}_0^{n-2} \hat{f}_1^{n-1}) p = [[r_n + n - 1, \cdots, r_2 + 1, r_1]] \quad n: \text{even}
\]
which proves the lemma.

We prove the weight preservation by induction on the number of domain walls. By Lemma 4 and the statement (i) above it is sufficient to prove the weight preservation for such a path \([l_n, \cdots, l_1]\) that
\[
l_n > \cdots > l_1 \geq 0.
\]

0) For the vacuum (=zero domain wall) state, the statement is obvious since \( wt([[]]) = wt(1) = 0 \). Now, we will prove that
\[
wt(p_2) - wt(p_1) = wt(O),
\]
for any \((b, c)\)- and \((a, c)\)-path, \( p_1 \) and \( p_2 \) such as
\[
p_1(l) = \begin{cases} \bar{p}_b(l), & n < l, \\ q(l), & l \leq n, \end{cases}
\text{and} \quad p_2(l) = \begin{cases} \bar{p}_a(l), & m < l, \\ q(l), & l \leq n, \end{cases}
\]
and the corresponding creation operator \( O \) (see 3 below). Here we can assume \( m > n \geq 0 \).
1) First, calculate the $h$-function. For $h(p_1)$, one decomposes the sum into four parts $h(p_1) = h_1^{(4)} + h_1^{(3)} + h_1^{(2)} + h_1^{(1)}$ as $(n < l)$, $(n = l)$, $(n > l > 0)$ and $(0 > l)$, then

$$
h_1^{(4)} = \sum_{n < l} [\hat{H}(p_1(l + 1), p_1(l)) - \hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l))] = 0,
$$

$$
h_1^{(3)} = n[\hat{H}(\bar{p}_a(n + 1), q(n)) - \hat{H}(\bar{p}_a(n + 1), \bar{p}_a(n))],
$$

$$
h_1^{(2)} = \sum_{n > l > 0} [\hat{H}(q(l + 1), q(l)) - \hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l))],
$$

$$
h_1^{(1)} = \sum_{0 > l} [\hat{H}(q(l + 1), q(l)) - \hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l))].
$$

Similarly, by decomposing to six parts, one has $h(p_2) = h_2^{(6)} + \cdots + h_2^{(1)}$, where

$$
h_2^{(6)} = \sum_{m < l} [\hat{H}(p_2(l + 1), p_2(l)) - \hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l))] = 0,
$$

$$
h_2^{(5)} = m[\hat{H}(\bar{p}_a(m + 1), \bar{p}_a(m)) - \hat{H}(\bar{p}_a(m + 1), \bar{p}_a(m))],
$$

$$
h_2^{(4)} = \sum_{m > l > 0} [\hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l)) - \hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l))],
$$

$$
h_2^{(3)} = n[\hat{H}(\bar{p}_a(n + 1), q(n)) - \hat{H}(\bar{p}_a(n + 1), \bar{p}_a(n))],
$$

$$
h_2^{(2)} = \sum_{n > l > 0} [\hat{H}(q(l + 1), q(l)) - \hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l))],
$$

$$
h_2^{(1)} = \sum_{0 > l} [\hat{H}(q(l + 1), q(l)) - \hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l))].
$$

Taking the difference of these two, one obtain

$$
h(p_2) - h(p_1) = m[\hat{H}(\bar{p}_a(m + 1), \bar{p}_a(m)) - \hat{H}(\bar{p}_a(m + 1), \bar{p}_a(m))] + \sum_{m > l > 0} [\hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l)) - \hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l))].
$$

The first line of the R.H.S. is evaluated as

$$
m[\hat{H}(\bar{p}_a(m + 1), \bar{p}_a(m)) - \hat{H}(\bar{p}_a(m + 1), \bar{p}_a(m))] = \begin{cases} m[\bar{p}_a(m) - \bar{p}_a(m)], & \bar{p}_a(m) \geq \bar{p}_a(m), \\ 0, & \bar{p}_a(m) < \bar{p}_a(m). \end{cases}
$$

The second line is

$$
\sum_{n > l > 0} [\hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l)) - \hat{H}(\bar{p}_a(l + 1), \bar{p}_a(l))] = \begin{cases} -(a - b)m/2, & \text{even } m, \\ (a - b)(m - 1)/2, & \text{odd } m. \end{cases}
$$

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Hence

\[ h(p_2) - h(p_1) = \begin{cases} 
(a - b)m/2, & \text{even } m, \ a > b, \\
-(a - b)m/2, & \text{even } m, \ a < b, \\
(a - b)(m - 1)/2, & \text{odd } m, \ a > b, \\
-(a - b)(m + 1)/2, & \text{odd } m, \ a < b.
\end{cases} \]

2) By similar and easier calculation, one can show

\[ s(p_2) - s(p_1) = \sum_{m \geq 1} \left[ \tilde{p}_m(l) - \tilde{p}_m(l) \right] = \begin{cases} 
0, & \text{even } m, \\
b - a, & \text{odd } m.
\end{cases} \]

Together with the results of 1), one obtain

\[ \text{wt}(p_2) - \text{wt}(p_1) = \begin{cases} 
(a - b)(\Lambda_1 - \Lambda_0) - (a - b)m/2 \delta, & \text{even } m, \ a > b, \\
-(a - b)(\Lambda_1 - \Lambda_0) + (a - b)m/2 \delta, & \text{even } m, \ a < b, \\
-(a - b)(\Lambda_1 - \Lambda_0) - (a - b)(m - 1)/2 \delta, & \text{odd } m, \ a > b, \\
-(a - b)(\Lambda_1 - \Lambda_0) + (a - b)(m + 1)/2 \delta, & \text{odd } m, \ a < b.
\end{cases} \]

3) On the other hand, in spinon basis, the path \( p_2 \) is obtained from \( p_1 \) by applying the following (products of) operators

\[ \mathcal{O} = \begin{cases} 
(\varphi^{a-\delta}_m)^{a-b}, & \text{odd } m, \\
(\varphi^{1+a}_{m+1})^{b-a}, & \text{even } m.
\end{cases} \]

It is easy to see that the difference of \( \text{wt} \), \( \text{wt}(p_2) - \text{wt}(p_1) \) is given exactly by the weight of these operators, hence the theorem is proved.

4 Yangian like structure

In this section we give a decomposition of the crystal of integrable irreducible highest weight \( U_q(s\hat{l}_2) \) modules as \((s\hat{l}_2)_1\) crystals which has a natural description in terms of crystalline spinon basis. Those \((s\hat{l}_2)_1\) crystals can be considered as describing the Yangian module structure in the \( q \to 1 \) limit. In fact one half of the character formula (11) corresponds to this structure which is known to describe the Yangian contribution [4].

**Definition 5** For \( (p_n, \cdots, p_1) \in \{0, 1\}^n \) and \((j_n, \cdots, j_1)\) which satisfy the condition

\[ j_{i+1} - j_i \geq H(p_{i+1}, p_i) \quad 1 \leq i \leq n - 1 \quad (9) \]

define

\[ \mathcal{B}^{p_n; \cdots; p_1}_{j_n; \cdots; j_1} = \{ \varphi^{p_n}_{j_n + i_n} \cdots \varphi^{p_1}_{j_1 + i_1} \in B(p_n, \cdots, p_1) | i_1, \cdots, i_n = 0, 1 \}. \]

Here we set \((j_n, \cdots, j_1) = \phi \) if \( n = 0 \), and \( \mathcal{B}^{\phi}_{\phi} = \{ 1 \} \).
Note that the set \( \mathcal{B}^{\pi_1,\ldots,\pi_1}_l \cup \{0\} \) is preserved by the actions of \( \tilde{e}_1 \) and \( \tilde{f}_1 \) and all nonzero elements of it has the same weight with respect to \( d \). As a corollary of Theorem \( 3 \) we have

**Theorem 4** There is an isomorphism of \((\widehat{\mathfrak{sl}_2})_1\) crystal

\[
B(\lambda_i) \cong \bigcup_{n=0}^{\infty} \bigcup_{(p_n,\ldots,p_1) \in \mathcal{P}_{n+1}} \bigcup_{(j_n,\ldots,j_1)} \mathcal{B}^{\pi_1,\ldots,\pi_1}_{j_n,\ldots,j_1},
\]

where the disjoint union in \((j_n,\ldots,j_1)\) is over all set which satisfies the condition (9) and \( j_i \geq 0 \).

Let us determine the structure of \( \mathcal{B}^{\pi_1,\ldots,\pi_1}_l \) as a \((\widehat{\mathfrak{sl}_2})_1\) crystal. The crucial point for it is that an element of the form \( \varphi_\pi^{j_2 \pi + i} \cdots \varphi_\pi^{j_1 + i} \) can be zero. We first study the most degenerate cases.

**Proposition 2** If \( j_l - j_{l-1} = H(p_l, p_{l-1}) \) for any \( l \geq 2 \), then

\[
\mathcal{B}^{\pi_1,\ldots,\pi_1}_{j_n,\ldots,j_1} \cong B^{(n)}
\]
as a \((\widehat{\mathfrak{sl}_2})_1\) crystal.

**Proof**

Using

\[
\varphi_\pi^{j_l} \varphi_\pi^{j_{l-1}} = 0
\]

and the definition of \( \tilde{f}_1, \tilde{e}_1 \), the statement is easily verified.

In order state the general case we need to introduce some terminology.

**Definition 6** For \((p_n,\ldots,p_1) \in \{0,1\}^n\) the \((j_n,\ldots,j_1) \in Z^n\) is said to be a string if

\[
\text{\(j_l - j_{l-1} = H(p_l, p_{l-1})\) \(2 \leq l \leq n\).} \quad (10)
\]

We call \( n \) the length of the string.

For a fixed \((p_n,\ldots,p_1)\) we can uniquely decompose \((j_n,\ldots,j_1)\) into strings in such a way that

(i) there exist \( M \in Z_{\geq 1} \) and \( 0 = r_0 < \cdots < r_M = n \) such that \((j_r,\ldots,j_{r-1})\) is a string for any \( s = 1,\ldots,M \),

(ii) \( j_{r_s+1} - j_{r_s} > H(p_{r_s+1}, p_{r_s}) \) for any \( s = 1,\ldots,M-1 \).

Setting \( m_s = r_s - r_{s-1} \), we call this decomposition the string decomposition of type \((m_M,\ldots,m_1)\).
Theorem 5 Let us fix \((p_0, \ldots, p_1)\). If the string decomposition of \((j_0, \ldots, j_1)\) is of type \((m_0, \ldots, m_1)\), then there is an isomorphism

\[
\mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} \simeq \mathcal{B}^{m_0, \ldots, m_1}
\]

of \(\widehat{\mathfrak{s}l_2}\) crystals.

Proof
Let us use the notations of the above definition for the string decomosition of \((j_0, \ldots, j_1)\) such as \(r_n\). Set

\[
\mathcal{B} = \{ \varphi_{j_0, \ldots, j_1}^{n_0} \circ \cdots \circ \varphi_{j_0, \ldots, j_1}^{n_1} \mid \varphi_{j_0, \ldots, j_1}^{n_0}, \ldots, \varphi_{j_0, \ldots, j_1}^{n_1} = 0, 1 \}.
\]

By definition the product map \(\mathcal{B} \longrightarrow \mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} \cup \{0\} \) is factorized as

\[
\begin{array}{ccc}
\mathcal{B} \cup \{0\} & \longrightarrow & \mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} \cup \{0\} \\
\downarrow & & \downarrow \\
\mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} \cup \{0\} \circ \cdots \circ \mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} \cup \{0\} & \overset{F}{\longrightarrow} & \mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} \cup \{0\}.
\end{array}
\]

By the definition of the crystal structure on \(\mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1}\), each map of the above diagram commutes with \(\tilde{e}_i\) and \(\tilde{f}_i\). It is easily verified that \(F\) is a bijection. Then the theorem follows from Proposition 2.

To each \(\mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} \simeq \mathcal{B}^{m_0, \ldots, m_1}\), let us define the \(U_q(\widehat{\mathfrak{s}l_2})\) module \(V^{p_0, \ldots, p_1}_{j_0, \ldots, j_1}\) by

\[
V^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} = \mathcal{B}^{m_0, \ldots, m_1}.
\]

where \(U_q(\widehat{\mathfrak{s}l_2})\) is the subalgebra generated by \(e_1, f_1, h_1\). Then, by the uniqueness of the crystal base (Theorem 3 of [13]), as a corollary of Theorem 4 and Theorem 5 we have

Corollary 4 Suppose that \(q \in \{ q \in C \mid q^n \neq 1, n = 1, 2, \ldots \} \cup \{1\}\). Then there is an isomorphism of \(U_q(\widehat{\mathfrak{s}l_2})\) modules

\[
V(\lambda) \simeq \bigotimes_{n=0}^{\infty} \left( \bigoplus_{p_0, \ldots, p_1} \mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} \right) \otimes \left( \bigoplus_{j_0, \ldots, j_1} V^{p_0, \ldots, p_1}_{j_0, \ldots, j_1} \right).
\]

So far we do not have a natural full Yangian type operations on each set \(\mathcal{B}^{p_0, \ldots, p_1}_{j_0, \ldots, j_1}\) or on the \(U_q(\widehat{\mathfrak{s}l_2})\) module \(V^{p_0, \ldots, p_1}_{j_0, \ldots, j_1}\). One reason of this will be explained in the following way. We consider not a \(\mathfrak{s}l_2\) module but a \(U_q(\widehat{\mathfrak{s}l_2})\) module. The full Yangian action may not survive in the deformation but it is possible for the \(\mathfrak{s}l_2\) structure to outlive. If we see appropriately we can extract the full Yangian structure from it.
5 Character formulas

Here, we derive the character formulas for the level $k$ integrable $\widehat{sl}_2$-modules $V(\lambda_j) \ (\lambda_j = (k - j)\Lambda_0 + j\Lambda_1, \ 0 \leq j \leq k)$ using the description of the crystalline spinon basis in section 3 and 4.

Let us denote $(p_n, \cdots, p_1) \in \{0, 1\}^n$ the level $k$ restricted path $(m_n, \cdots, m_0)$ beginning $m_0 = 0$ ending $m_n = j$ by making identification as $m_i = m_{i-1} + 1 - 2p_i$.

In section 3 (corollary 3), we obtained a spinon parametrization of the basis of $B(\lambda_j)$,
\[ \varphi^{q_n}_{j_n} \cdots \varphi^{q_1}_{j_1}, \quad n \geq 0, \]
where the conditions for the indices are
\[ j_i \geq j_{i-1} + 2 \quad \text{if} \ (p_i, p_{i-1}) = (1, 0), \]
\[ j_i \geq j_{i-1} \quad \text{otherwise}, \]
\[ j_1 \geq 0. \]

The weight $wt$ of which is given by\(^1\)
\[ -d \equiv \langle -d, wt \rangle = \sum_{i=1}^{n} n_i, \]
\[ h_1 \equiv \langle h_1, wt \rangle = \sum_{i=1}^{n} (1 - 2r_i) = n - 2m(r), \]
where $j_i = 2n_i + r_i \ (r_i \in \{0, 1\})$, and $m(r) = \sharp \{ i \mid r_i = 1 \}$.

For fixed $p = (p_n, \cdots, p_1)$ and $r = (r_n, \cdots, r_1)$, the above conditions on $(j_n, \cdots, j_1)$ can be rewritten in terms of mode sequences $(n_0, \cdots, n_1)$ as
\[ n_1 + r_1 \geq n_{i-1} + r_{i-1} + H(r_i, r_{i-1}) + H(p_i, p_{i-1}), \quad (1 \leq i \leq n) \]
where $r_0 = p_0 = 0$ and the $H$-function is
\[ H(0, 0) = H(0, 1) = H(1, 1) = 0, \quad H(1, 0) = 1. \]

Especially, the minimal allowed mode sequence is
\[ n_{min, l} = \sum_{i=1}^{l} (H(r_i, r_{i-1}) + H(p_i, p_{i-1})) - r_i. \]

The general mode sequence can be parametrized as
\[ n_i = n_{min, l} + i_1 + \cdots + i_l, \quad (i_i \geq 0), \]
\(^1\)In conformal field theory language, $L_0 = -\Delta + \Delta(j), \ (\Delta(j) = j(j+2)/4(k+2))$ is the conformal weight and $J_0^0 = h_1/2$ is the $U(1)$ charge.
and weight $-d$ of which is given by

$$\sum_{i=1}^{n} n_i = \sum_{i=1}^{n} i_i + \sum_{i=1}^{n} [(n - l + 1)H(r_i, r_{i-1}) + (n - l + 1)H(p_i, p_{i-1}) - r_i].$$

In summary, the crystalline spinon basis can be parametrized by three sequences $i = (i_0, \ldots, i_n) \in (Z_{\geq 0})^n$, $p = (p_0, \ldots, p_n) \in \{0, 1\}^n$ (restricted) and $r = (r_0, \ldots, r_n) \in \{0, 1\}^n$, with weights

$$-d = h(i) + h(p) + h(r) - m(r),$$
$$h_1 = n - 2m(r),$$
$$h(i) = \sum_{i=1}^{n} i_i,$$
$$h(p) = \sum_{i=1}^{n} [(n - l + 1)H(p_i, p_{i-1}),$$
$$h(r) = \sum_{i=1}^{n} [(n - l + 1)H(r_i, r_{i-1}),$$

Using this description of basis, one obtain the following character formula.

**Theorem 6** The character $c_{h_j}(q, z) \equiv tr_{\mathcal{V}(\lambda_j)}(q^{-d} z^{h_j})$ is given by

$$c_{h_j}(q, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{(q)_n (q)_{n-m}(q)_m} \sum_{p} q^{h(p)} z^n 2^{-2m},$$

where $p = (p_0, \ldots, p_n)$ is level $k$ restricted fusion path from $m_0 = 0$ to $m_n = j$.

**Proof.**

Using the parametrization of the crystalline basis as above, one obtain

$$c_{h_j}(q, z) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{(q)_n (q)_{n-m}(q)_m} \sum_{p} q^{h(p) + h(r)} q^{-m(r)} z^n 2^{-2m(r)} ,$$

where we have used the formula

$$\sum_{i_1 \geq 0} q^{i_1 + 2i_2 + \cdots + n i_n} = \frac{1}{(q)_n},$$

and

$$\sum_{r = (r_0, \ldots, r_n)} q^{h(r)} q^{-m(r)} z^{-2m(r)} = \sum_{m=0}^{n} \frac{(q)_n}{(q)_{n-m}(q)_m} z^{-2m}.$$
Thus the formula is proved.

This result is exactly the spinon character formula proposed in [4].

In the paper [4], the remaining sum over the restricted fusion path $p$ has also been done explicitly. We quote their final result

$$c h_j(q,z) = q^{-j/4} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{1}{(q)_{n-m}(q)_m} z^{n-2m} q^{-n^2/4} \Phi_{A_k}^p(u, q),$$

and

$$\Phi_{K}^{m_k}(u, q) = \sum_{m_2, \ldots, m_k} q^{\frac{1}{2} m_k K m} \prod_{i \geq 2} \left[ \frac{1}{2} ((2 - K) \cdot m + u)_i \right].$$

Here $K$ is the Cartan matrix of $A_k$-type, $(u)_i = \delta_{i,j+1}$ and

$$\left[ \begin{array}{c} n \\ m \end{array} \right] = \frac{(q)_n}{(q)_{n-m}(q)_m}.$$

The $m$ sum is taken over all non-negative odd [resp. even] integers for $(m_j, m_j - 2, \ldots)$ [resp. otherwise].

6 Quasi-particles at $q = 0$

In this section, as another application of the crystalline creation algebra, we derive the conjectural formula for the multi quasi-particle states of the higher spin XXZ models at $q = 0$. In the case of spin 1/2 XXZ chain, the formulas are consistent with the results of the Bethe Ansatz calculations[5].

**Proposition 3** For $(p_n, \cdots, p_1) \in \mathcal{P}_k^{r,s,n}$ and $(\epsilon_n, \cdots, \epsilon_1) \in \{0, 1\}^n$ we have

$$\varphi^{p_\epsilon}_e(z_n) \cdots \varphi^{p_\epsilon}_e(z_1)[[ ]] = \sum_{(m_1, \cdots, m_n) \in M} C(m_n \cdots m_1)[[2m_n + \epsilon_n - p_n, \cdots, 2m_1 + \epsilon_1]],$$

where

$$C(m_n \cdots m_1) = \prod_{i=1}^{n} z_i^{p_i} \det(z_i^{m_j - m_i})_{1 \leq i, j \leq n},$$

and

$$\nu_j = \sum_{i=j}^{n-1} (-K(p_{i+1}, p_i) + H(\epsilon_i, \epsilon_{i+1}))$$

The sum in $(m_1, \cdots, m_n)$ is over the modes $M$ such as

$$M = \{(m_1, \cdots, m_n) \mid 2m_n + \epsilon_n - 2l_n \geq \cdots \geq 2m_1 + \epsilon_1 \}.$$

Here $l_n = I_n(p_n, \cdots, p_1)$ etc.
Proof
Let us use the notations in the proof of Theorem 1. Namely, for \( \mu = 0, 1 \),
\[
\varphi_{2\mu}^p = \psi_{|p, p|}(n).
\]
Let us set \( A_j = (p_n - j + 1, e_n - j + 1) \). Then, using Lemma 1, we have
\[
\begin{align*}
\varphi_{2\mu}^p (z_n) \cdots \varphi_{2\mu}^1 (z_1) & = \sum_{m_1, \ldots, m_n \in \mathbb{Z}} \psi_{A_1} (m_1) \cdots \psi_{A_n} (m_n) [z_{m_1}^n \cdots z_{m_n}^n] \\
& = \sum_{(m_1, \ldots, m_n) \in M} \sum_{\sigma \in S_n} \text{sgn}(\sigma) z_{m_1, \sigma(1)}^n \cdots z_{m_n, \sigma(n)}^n \psi_{A_1} (m_1) \cdots \psi_{A_n} (m_n) [z_{m_1}^n \cdots z_{m_n}^n] \\
& = \sum_{(m_1, \ldots, m_n) \in M} \text{det}(z_{m_1, j + m_n - j+1}^n) [2m_0 + e_n - p_n, \ldots, 2m_1 + e_1].
\end{align*}
\]
Noting that
\[
\text{det}(z_{m_1, j + m_n - j+1}^n) = \text{det}(z_{i_1}^{m_1} + s_n - i_1, n - j + 1) = \prod_{l=1}^{n} z_{l}^{s_n - i_1}
\]
and \( u_j = s_{n-j+1,1} = -s_{j, n-j+1} \) we obtain the desired formula.

Let \( \{ G (b) | b \in B(\lambda_0) \otimes B(\lambda_0)^* \} \) be the global crystal base of \( V(\lambda_0) \otimes V(\lambda_0)^* \) \([15]\)
and \( \{ \langle G (b) | G (b') \rangle = \delta_{bb'} \) \). Let \( \varphi_{\lambda_0}^p (z) \) be the creation operator defined in \([10]\) acting on the actual representation. We denote by \( \text{vac}_0 \) the ground state in \( V(\lambda_0) \otimes V(\lambda_0)^* \) \([10]\). Then

**Conjecture 1** Assume the same data as in Proposition 3. Let \( b = [[2m_0 + e_n - p_n, \ldots, 2m_1 + e_1]] \) be an element of \( \bigcup_{i=1}^{k} B(\lambda_i) \otimes B(\lambda_0) \). Then the \( n \)-quasi-particles have the well defined \( q \rightarrow 0 \) limit and
\[
\langle G (b) | \varphi_{\lambda_0}^{p_{i_1}} (z_n) \cdots \varphi_{\lambda_0}^{p_{i_1}} (z_1) | \text{vac}_0 \rangle |_{q \rightarrow 0} = \prod_{j=1}^{n} z_{j}^{p_{i_1}} C (m_0, \ldots, m_1),
\]
where \( i_0 = 0, i_l - i_{l-1} = 1 - 2p_l \) and \( r_j = \Delta(i_j) - \Delta(i_{j-1}) \).

## 7 Discussion
In this paper we have given a spinon type description of the crystal of level \( k \)
integrable highest or lowest weight \( U_q (\widehat{sl}_2) \) modules. The algebra of creation
operators for higher spin XXZ model at \( q = 0 \) has been introduced and it plays a
central role. As consequences of this description we derived the fermionic
type character formulas and found the Yangian like structure in the integrable
highest weight modules or in the level zero modules. Moreover this description
clarifies the structure of the crystal associated with the tensor products of the integrable highest and lowest weight modules of the same level in a different manner from that in [10]. In the following part we shall discuss the possible generalizations and remaining problems.

To generalize our formalism to the case of arbitrary affine quantum algebras will be possible. The problems there are the identification of creation operators with type II VO’s and to calculate the commutation relations of them. As to the first problem Reshetikhin[23] gave a conjecture that the elementary excitations of the model defined by the affine Lie algebra $G$ will be parametrized by the fundamental weights of $G$. We remark that the second process will be skipped with the aid of the general theory of crystals[16, 6]. The treatment of the RSOS models will also be possible[12].

So far we could not prove the existence of the $q \to 0$ limit of the creation operators. However it is reasonable and certain that this limit actually exists (see for example [21]). If the limit exists, the corresponding components in the products of $q$-vertex operators to our crystalline spinon basis form a base of the $U_q(\widehat{sl}_2)$ module. To understand the "Yangian" representation $V_p^\otimes \cdots \otimes V_1$ in the integrable highest weight modules the study of these problems will be important.

**Note**

While preparing this manuscript we came to know that Nakanishi et al.[27] also proved the character formula which is equivalent to one in section 5 rigorously based on the path description of the character. They also discuss the Yangian structure from the combinatorial point of view which is closely related to our results in section 4. We would like to thank T.Nakanishi for sending us his private note.

**A Appendix**

Here we consider the naive limit of the $q$-vertex operators at $q = 0$. The consideration here motivates the definition of section 2, though it tells nothing about the existence.

In the vertex operator formalism, the creation operators are represented as the type II vertex operator $\varphi^\otimes_{\mu}(z)$. The commutation relation of them are given by

$$\varphi^\otimes_{\mu}(z_1)\varphi^\otimes_{\lambda}(z_2) = R_{\lambda\cdot V^*} \varphi^\otimes_{\lambda}(z_1)\varphi^\otimes_{\mu}(z_2)\varphi^\otimes_{\mu}(z_1)W(\lambda, \mu, \nu, z_1, z_2).$$

For the definitions and notations see [5, 10, 12]. $R_{\lambda\cdot V^*}$ is the trigonometric $R$-matrix of $U_q(\widehat{sl_2})$ for 2-dimensional representation $V^*$. $W$ is essentially the
elliptic Boltzmann weight due to $[1]$. Infinitely many poles in $W$ corresponds to
the multivaluedness of usual ($q = 1$) CFT chiral vertex operators $[26]$. \(^2\)

Being regarded as a commutation relation, it looks too terrible to treat. For instance
when $q = 1$, the algebra brings some generalized commutation relation
in the sense of the vertex operator algebra which takes very complicated form in
general. However, if one consider the opposite limit $q \to 0$, remarkable simple
structure comes out. In fact one can show that

$$R_{V \cdot V^*}(z) \to -z^{-1/2} \begin{bmatrix} 1 & z \\ 1 & 1 \end{bmatrix},$$

$$W(\lambda, \lambda_\pm | z) \to z^{\pm 1/2},$$

$$W(\lambda, \lambda_\pm | z) \to z^{-1/2},$$

and other $W$’s vanish. Note that the $W$ are independent of the initial weight $\lambda$
in this limit, hence, one can regard it as a vertex type weight. Accordingly, we
change the notation as

$$\varphi^{\pm}_{\lambda, c}(z) \to \varphi^{\pm}_c(z),$$

then the commutation relation takes the form

$$\varphi^{p_1}_{r_1}(z_1) \varphi^{p_2}_{r_2}(z_2) = -\left(\frac{z_1}{z_2}\right)^e \varphi^{p_1}_{r_1}(z_2) \varphi^{p_2}_{r_2}(z_1),$$

where the exponent $\epsilon$ is given by

$$\epsilon = H(p_1, p_2) + H(\epsilon_2, \epsilon_1) - 1.$$  

This algebra can be regarded as a twisted version of anti-commutation relation.
The algebra in the main text (Definition 1) can be derived by mode expansion
as follows

$$\varphi^p_0(z) = \sum_{n \in \mathbb{Z}} \varphi^{p*}_{2n} z^n,$$

$$\varphi^p_1(z) = \sum_{n \in \mathbb{Z}} \varphi^{p*}_{2n+1} z^n.$$

### B Appendix

\(^2\)One can take another $q \to 1$ limit in which $R$ and $W$ reduce to rational and trigonometic
$R$ respectively by rescaling the spectral parameter $z = q^{-2\pi i/\tau}$. This limit is exactly what we
expected for the "particles", that is (nilpotent half of) the Faddeev-Zamolodchikov algebra.
Table 1: crystalline spinon basis of $V(A_1)$

<table>
<thead>
<tr>
<th>$(-d, h_1)$</th>
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<th>[[domain wall]]</th>
<th>(fusion path)</th>
<th>[j-path]</th>
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Table 2: crystalline spinon basis of $V(A_1)$

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Table 3: crystalline spinon basis of $V(2A_1)$

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<th>[j-path]</th>
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Table 4: crystalline spinon basis of $V(A_0 + A_1)$

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Table 5: crystalline spinon basis of $V(2A_1)$

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References


[6] Date, E., Jimbo, M., Okado, M., Crystal base and q vertex operators, CMP 155, 47-69(1993)


