Abstract

Clique-width is a graph parameter with many algorithmic applications. For a positive integer $k$, the $k$-th power of a graph $G$ is the graph with the same vertex set as $G$, in which two distinct vertices are adjacent if and only if they are at distance at most $k$ in $G$. Many graph algorithmic problems can be expressed in terms of graph powers. We initiate the study of graph classes of power-bounded clique-width. A graph class is said to be of power-bounded clique-width if there exists an integer $k$ such that the $k$-th powers of graphs in the class form a class of bounded clique-width. We identify several graph classes of power-unbounded clique-width and give a sufficient condition for clique-width to be power-bounded. Based on this condition, we characterize graph classes of power-bounded clique-width among classes defined by two connected forbidden induced subgraphs. We also show that for every integer $k$, there exists a graph class of power-bounded clique-width such that the $k$-th powers of graphs in the class form a class of unbounded clique-width.

Keywords: Clique-width; power of a graph; hereditary graph class

1 Introduction

Clique-width is a graph parameter with many algorithmic applications: numerous problems that are generally $\text{NP}$-hard admit polynomial-time solutions when restricted to graphs of bounded clique-width (see, e.g., [10, 14, 18, 21, 22, 35, 45, 52]). Unfortunately, it is $\text{NP}$-complete to determine, for a given graph $G$ and an integer $k$, if the clique-width of $G$ is at...
most $k$. For specific values of $k$, polynomial-time algorithms have so far been found only for $k \leq 3$, while for higher values the complexity remains unknown. Examples of graph classes of bounded clique-width include trees, cographs, and distance-hereditary graphs. Many other graph classes have been shown to be of bounded clique-width; for many others, it has been shown that the clique-width is unbounded (see, e.g., and references quoted therein). Clique-width is also equivalent to several other graph width parameters, in the sense that these parameters are bounded on the same sets of graphs. This is the case for:

- NLC-width (introduced in 1994 by Wanke),
- symmetric clique-width (introduced in 2004 by Courcelle),
- rank-width (introduced in 2006 by Oum and Seymour), and
- boolean-width (introduced in 2011 by Bui-Xuan, Telle and Vatshelle).

For a positive integer $k$, the $k$-th power of a graph $G$ is the graph denoted by $G^k$ and obtained from $G$ by adding to it all edges between pairs of vertices at distance at most $k$. Graph powers are basic graph transformations with a number of results about their properties in the literature (see, e.g., [5, 47]). They are also a useful modeling tool, as several graph algorithmic problems can be expressed in terms of graph powers, for instance:

1. Determining whether a given graph has an efficient dominating set can be reduced to the maximum weight independent set problem in the square of the graph [6, 42].
2. Determining whether a given graph has an efficient edge dominating set (equivalently: a dominating induced matching) can be reduced to the maximum weight independent set problem in the square of the line graph of the graph [6].
3. Distance-$k$ colorings, distance-$k$ dominating sets, perfect $k$-codes, $k$-identifying codes in graphs directly correspond to colorings [4], dominating sets [28], efficient dominating sets [1], and identifying codes [34] in the $k$-th power of the graph, respectively.

An algorithm for coloring powers of graphs of bounded clique-width was given by Todinca.

In view of the usefulness of graph powers for modeling graph algorithmic problems and of algorithmic tractability of many problems on graphs of bounded clique-width, it is of interest to identify tuples $(G, k, \ell)$ where $G$ is a graph and $k$ and $\ell$ are positive integers such that $\text{cw}(G^k) \leq \ell$. Notice that a sufficiently large power of every graph is a disjoint union of complete graphs. Since the clique-width of any disjoint union of complete graphs is at most 2, this implies that a sufficiently large power of every graph is of bounded clique-width. This observation motivates the following question:

**Question 1.** Which graph classes $X$ have the property that there exists an integer $k$ such that the $k$-th power of $X$, that is, the set of graphs

$$X^k := \{G^k \mid G \in X\},$$
is of bounded clique-width?

Notice that $k$ must be a constant independent of $G \in X$. Graph classes with the above property will be the central topic of this paper, and are introduced in the following.

**Definition 1.** A graph class $X$ is said to be of power-bounded clique-width if there exists a positive integer $k$ such that $X^k$ is of bounded clique-width. If no such $k$ exists, we say that $X$ is of power-unbounded clique-width.

For a graph class $X$ of power-bounded clique-width, we denote by $\pi(X)$ the smallest positive integer $k$ such that $X^k$ is of bounded clique-width. Clearly, $\pi(X) = 1$ if and only if $X$ is of bounded clique-width. Hence, in some sense, the parameter $\pi(X)$ measures how far $X$ is from having bounded clique-width.

A natural further restriction in the definition of graph classes of power-bounded clique-width can be obtained by requiring that, once the clique-width of a certain power of the given class is bounded, it is also bounded for all higher powers.

**Definition 2.** A graph class $X$ of power-bounded clique-width is said to be of strongly power-bounded clique-width if for every positive integer $k \geq \pi(X)$, the class $X^k$ is of bounded clique-width.

We do not know whether there exists a graph class of power-bounded clique-width that is not of strongly power-bounded clique-width.

Various results from the literature imply that each of the following graph classes of increasing generality is of strongly power-bounded clique-width:

- **Paths.** (Heggernes et al. showed in [29] that for every $k \geq 1$, $\text{cw}(G) = k + 2$ if $G$ is the $k$-th power of a path with at least $(k + 1)^2$ vertices.)
- **Trees.** (Gurski and Wanke showed in [26] that for every tree $T$ and every $k \geq 1$, we have $\text{cw}(T^k) \leq k + 2 + \max\{\lfloor k/2 \rfloor - 2, 0\}$.)
- **Graphs of treewidth at most $\ell$.** (Gurski and Wanke also showed in [26] that for a graph $G$ with treewidth at most $\ell$ and a positive integer $k$, we have $\text{cw}(G^k) \leq 2(k + 1)^{\ell+1} - 2$.)
- **Graphs of clique-width at most $\ell$.**

The fact that every graph class of bounded clique-width is of strongly power-bounded clique-width is a consequence of the following.

**Proposition 1.1.** For every graph $G$ and positive integers $k, \ell$, if $\text{cw}(G) \leq \ell$, then $\text{cw}(G^k) \leq 4(k + 1)^\ell$.

**Proof.** This follows from an analogous result on NLC-width due to Suchan and Todinca [51] and a linear relation between clique-width and NLC-width due to Johansson [32]. (For the definition of NLC-width, see Wanke [53].) More precisely, denoting by $\text{nlcw}(G)$ the NLC-width of a graph $G$, every graph $G$ with $\text{nlcw}(G) \leq \ell$ satisfies $\text{nlcw}(G^k) \leq 4(k + 1)^\ell$. 


2(k + 1)\ell \leq 4(k + 1)\ell. 

Another example of a family of graph classes of power-bounded clique-width is given by graph classes of bounded diameter. Since we will use this simple observation in some of our proofs, we state it formally and give the short proof below. For our purpose, it will be convenient to define the diameter of a graph class \( X \) as

\[ \text{diam}(X) = \sup\{\text{diam}(C) \mid C \text{ is a connected component of some graph } G \in X\} \]

and to say that a graph class \( X \) is of bounded diameter if \( \text{diam}(X) < \infty \).

**Observation 1.** For every set of graphs \( X \) of bounded diameter, we have

\[ \pi(X) \leq \text{diam}(X). \]

In particular, every graphs class of bounded diameter is of power-bounded clique-width.

**Proof.** Let \( k = \text{diam}(X) \). Then, for every \( G \in X \), the graph \( G^k \) is a disjoint union of complete graphs, and hence \( \text{cw}(G^k) \leq 2 \). Consequently, \( \text{cw}(X^{\text{diam}(X)}) \leq 2 \) and the conclusion follows. \qed

In this paper, we initiate the study of graph classes of power-bounded clique-width. We show that for graph classes of bounded vertex degree and for minor-closed graph classes, power-bounded clique-width is equivalent to bounded treewidth. We exhibit several well-known graph classes of power-unbounded clique-width, including bipartite permutation graphs, unit interval graphs, and hypercube graphs, and prove a sufficient condition for power-bounded clique-width. Using this condition, we develop our main result: a complete characterization of graph classes of power-bounded clique-width, defined by two connected forbidden induced subgraphs. Finally, we show that there exist hereditary graph classes \( X \) with arbitrary large values of \( \pi(X) \).

The paper is structured as follows. In Section 2, we review the necessary preliminaries. In Section 3, we establish the connection between power-boundedness of the clique-width and boundedness of the treewidth for graphs of bounded degree and minor-closed graph classes. In Section 4, we identify several graph classes of power-unbounded clique-width. In Section 5, we give a sufficient condition for power-boundedness of the clique-width. In Section 6, we reduce the problem of characterizing hereditary graph classes of power-bounded of the clique-width to prime induced subgraphs of graphs in the class. In Section 7, we combine the results from earlier sections to derive a characterization of power-boundedness of the clique-width within graph classes defined either by one forbidden induced subgraph (Theorem 7.4) or by two connected forbidden induced subgraphs (Theorem 7.6). Finally, in Section 8, we construct graph classes \( X \) with arbitrarily large values of \( \pi(X) \). We conclude with some open questions in Section 9.
2 Preliminaries

All graphs in this paper are finite, simple and undirected. In some of the arguments in
the paper, we will use the obvious fact that if \( X \subseteq Y \), then \( \pi(X) \leq \pi(Y) \). In particular,
if \( X \subseteq Y \) and \( Y \) is of power-bounded clique-width, then so is \( X \).

Clique-width. A labeled graph is a graph in which every vertex has a label from \( \mathbb{N} \). A labeled graph is a \( k \)-labeled graph if every label is from \( [k] := \{1, 2, \ldots, k\} \). The clique-width of a graph \( G \) is the minimum number of labels needed to construct \( G \) using the following four operations: (i) Creation of a new vertex \( v \) with label \( i \) (denoted by \( i(v) \)); (ii) Disjoint union of two labeled graphs \( G \) and \( H \) (denoted by \( G \oplus H \)); (iii) Joining by
an edge each vertex with label \( i \) to each vertex with label \( j \) (\( i \neq j \), denoted by \( \eta_{i,j} \) or \( \eta_{j,i} \)); (iv) relabeling each vertex with label \( i \) with label \( j \) (denoted by \( \rho_{i\rightarrow j} \)). It can be easily seen that every graph can be defined by an algebraic expression using these four
operations. For instance, a chordless path on five consecutive vertices \( u, v, x, y, z \) can be
defined as follows:

\[
\eta_{3,2}(3(3) \oplus \rho_{2\rightarrow 3}(\eta_{3,2}(3(y)) \oplus \rho_{3\rightarrow 2}(\rho_{2\rightarrow 1}(\eta_{3,2}(3(x)) \oplus \eta_{2,1}(2(v) \oplus 1(u)))))\).
\]

Such an expression is called a \( k \)-expression if it uses at most \( k \) different labels. The clique-
width of \( G \), denoted \( \text{cw}(G) \), is the minimum \( k \) for which there exists a \( k \)-expression defining \( G \). If a graph \( G \) is of clique-width at most \( k \), then a \((2^k + 1)\)-expression for it can be
computed in time \( O(|V(G)|^3) \) using the rank-width \([31,46]\).

An older and more well-studied parameter than clique-width is the treewidth of graphs,
denoted by \( \text{tw}(G) \). We do not define it here, but refer instead to [2] for several equivalent
characterizations.

Graph classes. A graph class is a set of graphs closed under isomorphism. Given a
graph class \( X \), the clique-width of \( X \) is \( \text{cw}(X) = \sup\{\text{cw}(G) \mid G \in X\} \). We say that \( X \) is of bounded clique-width if \( \text{cw}(X) < \infty \). The maximum degree \( \Delta(X) \) of \( X \) is defined similarly, \( \Delta(X) = \sup\{\Delta(G) \mid G \in X\} \), and we say that \( X \) is of bounded degree if \( \Delta(X) < \infty \). Given
two graphs \( G \) and \( H \), graph \( H \) is said to be an induced subgraph of \( G \) if \( H \) can be obtained
from \( G \) by a sequence of vertex deletions, a subgraph of \( G \) if \( H \) can be obtained from \( G \) by a sequence of vertex and edge deletions, and a minor of \( G \) if \( H \) can be obtained from \( G \) by a sequence of vertex deletions, edge deletions, and edge contractions. For a set \( M \) of graphs, we say that a graph \( G \) is \( M \)-free if no member of \( M \) is an induced subgraph of \( G \). Similarly, for a graph \( H \), we say that \( G \) is \( H \)-free if it is \( \{H\} \)-free. The set of all \( M \)-free graphs (where \( M \) is either a graph or a set a graphs) will be denoted by \( \text{Free}(M) \).

A graph class is hereditary if it is closed under taking induced subgraphs. It is well-known
(and not difficult to see) that a graph class \( X \) is hereditary if and only if \( X = \text{Free}(M) \)
for some set \( M \) of graphs. Two important families of hereditary classes are minor-closed
classes (i.e., graph classes closed under taking minors) and monotone classes. A graph
class is monotone if it is closed under taking subgraphs. For graph classes not defined in
this paper, we refer to [5].

Given a graph \( G \), an independent set in \( G \) is a set of pairwise non-adjacent vertices,
and a clique is a set of pairwise adjacent vertices. For a subset of vertices \( X \subseteq V(G) \), we

will denote by $G - X$ the graph obtained from $G$ by deleting from it vertices in $X$ and all edges incident with them, and by $G[X]$ the subgraph of $G$ induced by $X$, that is, $G[X] = G - (V(G) \setminus X)$. For two vertices $x, y$ in a connected graph $G$, we denote by $\text{dist}_G(x, y)$ the \textit{distance between $x$ and $y$}, that is, the length (number of edges) of a shortest $x, y$-path in $G$. The \textit{diameter} of a connected graph $G$ is defined as $\text{diam}(G) = \max_{x,y \in V(G)} \text{dist}_G(x, y)$, and we define the diameter of a disconnected graph $G$ to be the maximum diameter of a connected component of $G$. By $P_n$, $C_n$, and $K_n$, we denote the path, the cycle, and the complete graph on $n$ vertices, respectively. For two vertex-disjoint graphs $G_1$ and $G_2$, the \textit{disjoint union} of $G_1$ and $G_2$ is the graph $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$, and the \textit{join} of $G_1$ and $G_2$ is the graph obtained by adding to the disjoint union of $G_1$ and $G_2$ all edges of the form $\{uv \mid u \in V(G_1), v \in V(G_2)\}$. The disjoint union of $k$ graphs isomorphic to a graph $H$ will be denoted by $kH$. The \textit{complement} of a graph $G = (V, E)$ is the graph $\overline{G}$ with the same vertex set as $G$, in which two distinct vertices are adjacent if and only if they are non-adjacent in $G$. A graph is said to be \textit{co-connected} if its complement is connected. A \textit{linear forest} is an acyclic graph of maximum degree at most 2, that is, a disjoint union of paths.

\textbf{Modular decomposition.} A subset $M$ of vertices in a graph $G$ is said to be a \textit{module} if every vertex $v \in V(G) \setminus M$ is either adjacent to all vertices in $M$, or non-adjacent to all vertices of $M$. A module is said to be \textit{trivial} if $M = V$ or $|M| \leq 1$, and a graph $G$ is \textit{prime} if it does not contain any nontrivial module. Examples of prime graphs include $K_1, K_2, 2K_1$, and $P_3$. The idea of decomposing a graph with respect to its modules has been first described in the 1960s by Gallai \cite{gallai1967disjoint}, and also appeared in the literature under various other names such as \textit{prime tree decomposition} \cite{golumbic1993algorithmic}, \textit{X-join decomposition} \cite{golumbic1994linear}, or \textit{substitution decomposition} \cite{golumbic2002combinatorial}; see also \cite{golumbic2005combinatorial}. An important property of modules is that if $G$ is connected and co-connected, then its vertex set admits a unique partition into pairwise disjoint maximal modules. The \textit{characteristic graph} of such a graph $G$ is the subgraph $C$ of $G$ induced by an arbitrary set of vertices obtained by choosing one vertex from each maximal module. It is easy to see that $C$ is unique up to isomorphism, which justifies the slight abuse of the terminology (“the” characteristic graph of $G”). Moreover, the characteristic graph is always prime.

We now recall some known results about treewidth and clique-width that we will use in some of our proofs.

\textbf{Proposition 2.1} (Johansson \cite{johansson1993clique}, Courcelle-Olaru \cite{courcelle2000linear}). \textit{If $H$ is an induced subgraph of $G$ then $\text{cw}(H) \leq \text{cw}(G)$.}

\textbf{Proposition 2.2} (Courcelle-Olaru \cite{courcelle2000linear}). \textit{For every graph $G$ we have $\text{cw}(G) = \max\{\text{cw}(H) \mid H$ is a prime induced subgraph of $G\}$.}

\textbf{Proposition 2.3} (Boliac-Lozin \cite{boliac2008clique}). \textit{There exists a function $f$ such that for every graph $G$ and every subset $U \subseteq V(G)$, we have $\text{cw}(G) \leq f(\text{cw}(G - U), |U|)$.}

\textit{A subgraph complementation} on a graph $G$ is the operation of replacing an induced subgraph of $G$ with its complement.
Proposition 2.4 (Kamiński et al. [33]). If $X$ is a graph class of unbounded clique-width, then the class of graphs obtained from graphs in $X$ by applying a constant number of subgraph complementations is also of unbounded clique-width.

Theorem 2.5 (Courcelle-Olariu [15]). There exists a function $f$ such that for every graph $G$, we have $\text{tw}(G) \leq f(\Delta(G), \text{cw}(G))$.

Theorem 2.6 (Corneil-Rotics [12]). For every graph $G$, we have $\text{cw}(G) \leq 3 \cdot 2^{\text{tw}(G) - 1}$.

Theorem 2.7 (Gurski-Wanke [25]). For every graph $G$ and its line graph $L(G)$, we have $(\text{tw}(G) + 1)/4 \leq \text{cw}(L(G)) \leq 2\text{tw}(G) + 2$.

Theorem 2.8 (Gurski-Wanke [24]). If $G$ is a graph that does not contain $K_{t,t}$ as a subgraph, then $\text{tw}(G) \leq 3(t - 1)\text{cw}(G) - 1$.

Theorem 2.9 (Robertson-Seymour [49]). For every planar graph $H$ there is a number $N$ such that every graph with no minor isomorphic to $H$ has treewidth at most $N$.

The following two propositions are well known (see, e.g., Lemma 16 and Corollary 89 in Bodlaender [2], respectively).

Proposition 2.10. If $H$ is a minor of $G$, then $\text{tw}(H) \leq \text{tw}(G)$.

Proposition 2.11. The $n \times n$ grid is of treewidth exactly $n$.

3 Graphs of bounded degree and minor-closed classes

We show in this section that for graphs of bounded degree and minor-closed graph classes, power-bounded clique-width is equivalent to bounded treewidth.

Proposition 3.1. For every integer $\Delta \geq 1$ and every graph class $X$ of maximum degree at most $\Delta$, the following conditions are equivalent:

1. $X$ is of power-bounded clique-width.
2. All powers of $X$ are of bounded clique-width.
3. $X$ is of bounded clique-width.
4. $X$ is of bounded treewidth.

Proof. The implication 4 $\Rightarrow$ 3 holds by Theorem 2.6. The implication 3 $\Rightarrow$ 2 holds by Proposition 1.1. The implication 2 $\Rightarrow$ 1 holds by definition.

To prove the implication 1 $\Rightarrow$ 4, suppose for a contradiction that $X$ is a class of graphs of maximum degree at most $\Delta$ that is of power-bounded clique-width but of unbounded treewidth. Since edge additions cannot decrease the treewidth, each power $X^k$ is also of unbounded treewidth. However, since $X$ is of maximum degree at most $\Delta$, the power class $X^k$ is of maximum degree at most $\Delta \cdot \left(\sum_{i=0}^{k-1} (\Delta - 1)^i\right)$. By Theorem 2.5, the class $X^k$ is of unbounded clique-width as well. Since $k$ was arbitrary, we conclude that $X$ is of power-unbounded clique-width, a contradiction.

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For an integer \( n \geq 1 \), the \( n \times n \) grid is the graph with vertex set \( \{1, \ldots, n\}^2 \), in which two vertices \((i, j)\) and \((k, \ell)\) are adjacent if and only if \(|i - k| + |j - \ell| = 1\). A square grid is a graph isomorphic to some \( n \times n \) grid.

**Corollary 3.2.** For every \( k \geq 1 \), the set of graphs obtained from square grids by replacing each edge with a path with \( k \) edges, is of power-unbounded clique-width.

**Proof.** Let \( G_{n,k} \) be the graph obtained from the \( n \times n \) grid by replacing each edge with a path with \( k \) edges. Since the \( n \times n \) grid \( G_{n,1} \) is a minor of \( G_{n,k} \), and grids are of unbounded treewidth (see Proposition 2.11), the set of graphs \( \{G_{n,k} \mid n \geq 1\} \) is also of unbounded treewidth, by Proposition 2.10. The conclusion now follows from Proposition 3.1.

Recall that the girth of a graph \( G \) is defined as the shortest length of a cycle in \( G \) (or infinity if \( G \) is acyclic). The following consequence of Corollary 3.2 is immediate.

**Corollary 3.3.** For every \( k \geq 3 \), the class of graphs of girth at least \( k \) is of power-unbounded clique-width.

Now we show an analogue of Proposition 3.1 for proper minor-closed graph classes.

**Proposition 3.4.** For every proper minor-closed graph class \( X \), the following conditions are equivalent:

1. \( X \) is of power-bounded clique-width.
2. All powers of \( X \) are of bounded clique-width.
3. \( X \) is of bounded clique-width.
4. \( X \) is of bounded treewidth.

**Proof.** For the implications \( 4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1 \), we proceed as in the proof of Proposition 3.1. For the implication \( 1 \Rightarrow 4 \), suppose that \( X \) is a proper minor-closed graph class of power-bounded clique-width. By Corollary 3.2, \( X \) excludes some grid \( G \). Since \( X \) is minor closed, no graph in \( X \) has a minor isomorphic to \( G \). The fact that \( X \) is of bounded treewidth now follows from Theorem 2.9.

### 4 Examples of graph classes of power-unbounded clique-width

In this section, we identify further examples of graph classes of power-unbounded clique-width besides grids and graphs of large girth. We start with bipartite permutation graphs and unit interval graphs, shown by Lozin [37] to be minimal graph classes of unbounded clique-width (in the sense that every proper hereditary subclass of either unit interval or bipartite permutation graphs is of bounded clique-width).

Recall that a graph \( G \) is a bipartite permutation graph if it is both bipartite and permutation, where a graph \( G = (V, E) \) is bipartite if its vertex set can be partitioned into two independent sets, and permutation if there exists a permutation \( \pi = (\pi_1, \ldots, \pi_n) \) of the set \( \{1, \ldots, n\} \) where \( V = \{v_1, \ldots, v_n\} \) such that \( v_iv_j \in E \) if and only if \( (\pi_i - \pi_j)(i - j) < 0 \).
Proposition 4.1. The class of bipartite permutation graphs is of power-unbounded clique-width.

Proof. Let $H_n$ be the graph whose set of vertices $V(H_n)$ is defined by $V(H_n) = \{v_{i,j} \mid 0 \leq i \leq n-1, 0 \leq j \leq n-1\}$ and whose set of edges is defined by $E(H_n) = \{v_{i,j}v_{i,j+1} \mid 0 \leq j \leq n-2, 1 \leq i_1 \leq n-1, 0 \leq i_2 \leq i_1 - 1\}$. It follows from Theorem 1 of [7] that $H_n$ is a bipartite permutation graph.

We denote by $I_n$ the graph whose vertex set is defined by $V(I_n) = V(H_n)$ and whose set of edges is $E(I_n) = E(H_n) \cup \{v_{i,j}v_{i,j+1} \mid 0 \leq j \leq n-1, 0 \leq i_1 \leq n-1, 0 \leq i_2 \leq n-1, i_1 \neq i_2\}$. Fix an integer $k \geq 2$. The subgraph of $H_{n(n-1)+1}^k$ induced by $\{v_{i,k,j} \mid 0 \leq i \leq n-1, 0 \leq j \leq n-1\}$ is isomorphic to $I_n$; in fact, $v_{i,k,j} \mapsto v_{i,j}$ is an isomorphism between the two graphs. Since it was proved in [23] that $\text{cw}(I_n) \geq n$, Proposition 2.1 implies that $\text{cw}(H_{n(n-1)+1}^k) \geq n$ for every $n \geq 1$ and $k \geq 2$. This proves that the class of bipartite permutation graphs is of power-unbounded clique-width. \qed

Proposition 4.2. The class of path powers is of power-unbounded clique-width.

Proof. We need to argue that for each integer $k \geq 1$, the class of $k$-th powers of path powers is of unbounded clique-width. It was proved in [29] that for each positive integer $s$ and each $n \geq (s+1)^2$, we have $\text{cw}(P_n^s) = s + 2$. Therefore, for each pair of positive integers $k$ and $N$, there exist two positive integers $j$ and $n$ such that the $k$th-power of the path $P_n^j$ has clique-width more than $N$. Indeed, we can take $j$ so that $jk + 2 > N$ and $n = (jk + 1)^2$, obtaining $\text{cw}((P_n^j)^k) = \text{cw}(P_n^{jk}) = jk + 2 > N$. \qed

Recall that a graph $G$ is a unit interval graph if it is the intersection graph of a collection of unit intervals on the real line. It is well known (see, e.g., Theorem 9(ii) of [36]) that path powers are unit interval graphs. Hence, we have the following.

Corollary 4.3. The class of unit interval graphs is of power-unbounded clique-width.

Our next example is the class of hypercube graphs. For an integer $d \geq 1$, the $d$-dimensional hypercube graph is the graph $Q_d$ with vertex set given by all $2^d$ binary sequences of length $d$, in which two vertices are adjacent if and only they differ, as sequences, in exactly one coordinate.

Proposition 4.4. The class of hypercube graphs is of power-unbounded clique-width.

Proof. Hypercube graphs are of unbounded treewidth [9] and they do not contain $K_{3,3}$ as a subgraph (in fact, not even $K_{2,3}$). Therefore, by Theorem 2.8, they are of unbounded clique-width.

Now, we show that for every two positive integers $k$ and $d$, there exists an integer $d'$ such that the $k$-th power of the $d'$-dimensional hypercube $Q_d^{k}$ contains the $d$-dimensional hypercube $Q_d$ as an induced subgraph. Since the clique-width of any graph is bounded from below by the clique-width of any of its induced subgraphs (Proposition 2.1), this will imply that for every $k$, the class of $k$-th powers of hypercubes is of unbounded clique-width, thus proving the proposition.
The integer \( d' \) can be defined as \( d' = dk \). To find an induced copy of \( Q_d \) in \( Q^k_{dk} \), consider the subgraph of \( Q^k_{dk} \) induced by the vertices in \( W \) where \( x = (x_1, x_2, \ldots, x_{kd}) \in W \) if and only if for every \( i \in \{1, \ldots, d\} \), we have

\[
x_{(i-1)k+1} = x_{(i-1)k+2} = \cdots = x_{(i-1)k+k}.
\]

Intuitively, this means that we partition the set of all \( kd \) coordinates into \( d \) blocks of \( k \) consecutive coordinates each, and put in \( W \) exactly those vertices of \( Q^k_{kd} \) that agree on each of the blocks. Every vertex of \( W \) is completely determined with its values on each of the \( d \) blocks. Hence \( |W| = 2^d \), and there is a natural bijection \( f : V(Q_d) \to W \), one that maps every vertex \( (x_1, \ldots, x_d) \in \{0,1\}^d \) to the vertex in \( W \) with values \( x_i \) on all coordinates in the \( i \)-th block, for all \( i \). We claim that \( f \) is an isomorphism between \( Q_d \) and \( Q^k_{kd}[W] \). Suppose first that \( xy \in E(Q_d) \). This means that \( x \) and \( y \) differ in exactly one coordinate, say \( i \)-th. Hence, \( f(x) \) and \( f(y) \) differ in exactly \( k \) coordinates (namely in those in the \( i \)-th block), and therefore \( f(x) \) and \( f(y) \) are adjacent in \( Q^k_{kd}[W] \). Conversely, suppose that \( x \) and \( y \) are vertices of \( Q_d \) such that \( f(x) \) and \( f(y) \) are adjacent in \( Q^k_{kd}[W] \). Since \( f(x) \) and \( f(y) \) are adjacent vertices in the \( k \)-th power of \( Q_{dk} \), they differ in at most \( k \) coordinates. On the other hand, since \( f(x) \) and \( f(y) \) are distinct vertices of \( W \), they differ in at least \( k \) coordinates. Hence, they differ in exactly \( k \) coordinates, that is, in exactly one block. This implies that \( x \) and \( y \) differ in exactly one coordinate and therefore \( xy \in E(Q_d) \). This shows that the subgraph of \( Q^k_{dk} \) induced by \( W \) is isomorphic to \( Q_d \) and completes the proof.

\( \square \)

5 A sufficient condition for power-bounded clique-width

By Observation 1, every graph class of bounded diameter is of power-bounded clique-width. We now extend this observation by giving a sufficient condition for power-bounded clique-width that is also applicable for graph classes of unbounded diameter. A 2-path in a graph \( G \) is an induced path in \( G \) all the vertices of which are of degree 2 in \( G \). Two vertices \( u \) and \( v \) of a graph \( G \) are twins if they have exactly the same set of neighbors, other than \( u \) and \( v \). The twin relation of \( G \) is the relation \( \sim_G \) on \( V(G) \) in which two vertices are related if and only if they are twins:

\[
u \sim_G v \quad \text{if and only if} \quad N(u) \setminus \{v\} = N(v) \setminus \{u\}.
\]

Note that twins may be either adjacent or non-adjacent. It is easy to see that the twin relation is an equivalence relation, every equivalence class is either a clique or an independent set, and for every two equivalence classes, there are either all edges or no edges between them. Thus, the quotient graph of the twin relation, denoted by \( G/\sim \), is well defined: its vertex set is the set of all equivalence classes of \( \sim \), and two distinct classes \( U \) and \( W \) are adjacent if and only if there is an edge in \( G \) joining a vertex of \( U \) to a vertex of \( W \) (equivalently: every vertex of \( U \) is adjacent in \( G \) to every vertex of \( W \)).

For positive integers \( k \) and \( d \), let \( X^{k,d} \) be the set of all graphs \( G \) that contain a set \( \mathcal{P} \) of at most \( k \) 2-paths such that the diameter of each connected component of \( G - \bigcup_{P \in \mathcal{P}} V(P) \) is at most \( d \).
Theorem 5.1. For every pair of positive integers \( k \) and \( d \), the graph class \( X^{k,d} \) is of power-bounded clique-width.

Proof. Let \( G \in X^{k,d} \) and let \( \{P_1, \ldots, P_r\} \) with \( r \leq k \) be a set of 2-paths in \( G \) such that the diameter of each connected component of \( H := G - \cup_{i=1}^r V(P_i) \) is at most \( d \). It is sufficient to show that \( \text{cw}(G^d) \leq p \) for some integer \( p \) depending only on \( k \) and \( d \), but not on \( G \). Since the clique-width of a disconnected graph is the maximum clique-width of its connected components, we may assume that \( G \) is connected. Moreover, since paths (and thus also linear forests) are of clique-width at most 3 (see Theorem 2.6), we may assume that \( H \) is non-empty. Let \( C_1, \ldots, C_m \) denote the vertex sets of the components of \( H \). Let \( A = V(H) \) and \( B = \cup_{i=1}^r V(P_i) \). Since \( G \) is connected and \( B \) has at most \( k \) components, each of which has neighbors in at most two components of \( H \), we have \( m \leq k + 1 \).

Let us analyze the structure of \( G^d \). First, the assumption on the diameter implies that the subgraph of \( G^d \) induced by each \( C_i \) is complete. Moreover, every vertex \( u \in B \) such that \( d_G(u, A) > d \) has no neighbors in \( G^d \) in \( A \), and also no neighbors in any component of \( G[B] \) other than the component of \( G[B] \) containing \( u \). For a vertex \( x \in A \) and an endpoint \( w \) of a component (path) in \( G[B] \), let us define

\[
f(x, w) = \begin{cases} 
    d_G(x, w), & \text{if } d_G(x, w) \leq d; \\
    d + 1, & \text{otherwise.}
\end{cases}
\]

Let us denote by \( F(x) \) the array of values \( f(x, w) \) for all endpoints \( w \) of components (paths) of \( G[B] \) (in some fixed order). Clearly, this assignment of arrays to vertices in \( A \) results in at most \( (d+1)^{2k} \) different arrays. Let us now define on each set \( C_i \) an equivalence relation \( \sim_i \) by the rule \( x \sim_i y \) if and only if \( F(x) = F(y) \). Every such relation will have at most \( (d+1)^{2k} \) equivalence classes. Moreover, for every \( i \), every two vertices \( x, y \in C_i \) such that \( x \sim_i y \) are twins in \( G^d \). Indeed:

- \( N_{G^d}[x] \cap C_i = N_{G^d}[y] \cap C_i = C_i \), since \( G^d[C_i] \) is complete.
- For every \( j \in \{1, \ldots, r\} \setminus \{i\} \) and every vertex \( z \in C_j \), we have \( xz \in E(G^d) \) if and only if there exists a component \( P \) of \( G[B] \) with endpoints \( a \) and \( b \) such that \( f(x, a) + d_G(a, b) + F(z)_b \leq d \). If this is the case, then, since \( F(x) = F(y) \), we also have \( f(y, a) + d_G(a, b) + F(z)_b \leq d \), which implies that \( z \) is adjacent to \( y \) in \( G^d \). Hence, \( N_{G^d}[x] \cap C_j \subseteq N_{G^d}[y] \cap C_j \), and by symmetry, \( N_{G^d}[x] \cap C_j = N_{G^d}[y] \cap C_j \).
- Now, let \( z \in B \). Such a vertex belongs to a path \( P \) in \( G \) and it is adjacent to \( x \) in \( G^d \) if and only if there exists an endpoint \( a \) of \( P \) such that \( f(x, a) + d_G(a, z) \leq d \). Again, this condition implies that \( f(y, a) + d_G(a, z) \leq d \), hence \( z \) is also adjacent to \( y \) in \( G^d \). We conclude that \( N_{G^d}[x] \cap B = N_{G^d}[y] \cap B \).

The above three conditions imply that \( N_{G^d}[x] = N_{G^d}[y] \).

Proposition 2.2 implies that \( \text{cw}(G^d) = \max \{ \text{cw}(H) \mid H \text{ is a prime induced subgraph of } G^d \} \). In particular, since no prime induced subgraph of \( G^d \) with at least three vertices contains two twin vertices (as they would form a non-trivial module), this implies that \( \text{cw}(G^d) = \text{cw}(G^d/\sim) \), where \( G^d/\sim \) is the quotient graph of the twin relation of \( G^d \). Hence,
it is sufficient to show that the clique-width of $G^d/\sim$ is bounded. The above discussion implies that the graph $G^d/\sim$ contains at most $(d+1)^{2k}$ vertices from each $C_i$. Let $Y$ denote the set of equivalence classes of relation $\sim$ corresponding to the vertices of $G$ at distance at most $d$ from $A$. Then,

$$|Y| \leq (k+1) \cdot (d+1)^{2k} + k \cdot (2d),$$

where the first and second summand give upper bounds on the numbers of equivalence classes corresponding to vertices in $A$ and in $B$, respectively. Moreover, it follows from the definition of $Y$ that the graph $G' = G^d/\sim - Y$ is isomorphic to the $d$-th power of a linear forest $F$ (where $F$ is the subgraph of $G$ induced by vertices at distance at least $d+1$ from $A$). Since $\text{cw}(Q) = d+2$ if $Q$ is the $d$-th power of a path with at least $(d+1)^2$ vertices \cite{29}, this implies that the clique-width of $G'$ is bounded from above by a function of $d$. Since $G^d/\sim$ differs from $G'$ by only constantly many vertices, the clique-width of $G^d/\sim$ is also bounded by a function of $d$ and $k$, as follows from Proposition \ref{2.3}. This completes the proof.

For positive integers $k$ and $d$, let $Y^{k,d}$ be the set of all graphs $G$ such that there exists a graph $H$ with the following properties:

(i) Every connected component of $H$ is of diameter at most $d$, and

(ii) There exists a set $F \subseteq E(H)$ with $|F| \leq k$ such that $G$ is the graph obtained from $H$ by replacing each edge $e \in F$ with a path of length at least 1.

**Corollary 5.2.** For every pair of positive integers $k$ and $d$, the graph class $Y^{k,d}$ is of power-bounded clique-width.

**Proof.** Let $G \in Y^{k,d}$ and let $H$ and $F$ be as in the corollary. Let $|F'|$ be the set of edges in $F$ that are replaced with paths of length at least 2, and let $k' = |F'|$. The graph $H' = H - F$ is of diameter at most $(k'+1)d$ \cite{50}. Thus, $G \in X^{k,(k'+1)d}$ and the conclusion follows from Theorem \ref{5.1} (note that $k' \leq k$ is a constant).

**Remark 1.** The result of Theorem \ref{5.1} is sharp, in the sense that neither of the two conditions can be dropped:

(1) If the diameter of $G - \cup_{P \in \mathcal{P}} V(P)$ can be unbounded, then the class can be of power-unbounded clique-width, as exemplified by the class of grids (cf. Corollary \ref{3.2}).

(2) If the number of 2-paths in $\mathcal{P}$ is unbounded, then the class can be of power-unbounded clique-width, even if the graph $G - \cup_{P \in \mathcal{P}} V(P)$ is complete. Indeed, for every $k$, let $G_{n,k}$ be the graph obtained from the complete graph $K_n$ by attaching to it $\binom{n}{2}$ chordless paths of length $2k$, each connecting a different pair of vertices of $K_n$. Then, the $k$-th power of $G_{n,k}$ contains the graph $K_n^*$ as an induced subgraph. Here, $K_n^*$ denotes the graph obtained from a complete graph on $n$ vertices by gluing a triangle on every edge. Formally, $V(K_n^*) = [n] \cup \binom{[n]}{2}$ where $[n] = \{1, \ldots, n\}$ and $\binom{[n]}{2} = \{Y \subseteq [n] \mid |Y| = 2\}$, and $E(K_n^*) = \binom{[n]}{2} \cup \{\{i, Y\} \mid Y \in \binom{[n]}{2} \land i \in Y\}$. As shown in \cite{41}, the clique-width of graphs $K_n^*$ is unbounded. Hence the family of graphs \{$G_{n,k} \mid n \geq 2$\} is of power-unbounded clique-width.
6 Reduction to prime graphs

We will apply Theorem 5.1 in Section 7 to characterize graph classes of power-bounded clique-width among hereditary graph classes defined by two connected forbidden induced subgraphs. To do this, we need another technical result stating that for hereditary graph classes, whether the class is of power-bounded clique-width or not depends only on the prime graphs in the class. This result will be developed in this section. (Recall that “prime” here means with respect to modular decomposition.)

**Theorem 6.1.** Let $X$ be a hereditary graph class and let $X'$ be the set of all prime graphs in $X$. Then, for every positive integer $k$, the graph class $X^k$ is of bounded clique-width if and only if $(X')^k$ is of bounded clique-width.

**Corollary 6.2.** Let $X$ be a hereditary graph class and let $X'$ be the set of all prime graphs in $X$. Then, $X$ is of power-bounded clique-width if and only if $X'$ is of power-bounded clique-width.

We will prove Theorem 6.1 in this section, following a sequence of three preliminary lemmas.

**Lemma 6.3.** Let $G$ be a connected and co-connected graph, let $x$ and $y$ be two vertices from different maximal modules of $G$, and let $P$ be a shortest $x, y$-path. Then, $P$ intersects every maximal module of $G$ at most once.

*Proof.* Let $P = (x = v_0, v_1, \ldots, v_r = y)$ be a shortest path in $G$ from $x$ to $y$ such that the maximal modules $M_x$ and $M_y$ containing $x$ and $y$ respectively are distinct. Suppose for a contradiction that there exists a maximal module $M$ with $|M \cap V(P)| \geq 2$. Let $v_i$ and $v_j$ be the first and the last vertex on $P$ that belong to $M$. Then $i < j$, and either $v_i = x$ or $v_j = y$. Without loss of generality, assume that $v_i \neq x$. Then $v_{i-1}$ is well defined and not in $M$. Since $v_{i-1}v_i \in E(G)$ and $v_j \in M$, we have $v_{i-1}v_j \in E(G)$, contrary to the minimality of $P$. \hfill $\square$

Given two graphs $G$ and $H$, we say that $H$ is an isometric subgraph of $G$ if $H$ is a connected subgraph of $G$ and for every two vertices $x, y \in V(H)$, we have $\text{dist}_H(x, y) = \text{dist}_G(x, y)$.

**Lemma 6.4.** For every connected and co-connected graph $G$, its characteristic graph is an isometric subgraph of $G$.

*Proof.* Let $C$ be the characteristic graph of $G$ and let $x, y \in V(C)$. We need to show that $\text{dist}_C(x, y) = \text{dist}_G(x, y)$. Since $C$ is a subgraph of $G$, we have $\text{dist}_G(x, y) \leq \text{dist}_C(x, y)$. Let $P = (x = x_0, x_1, \ldots, x_d = y)$ be a shortest path connecting $x$ and $y$ in $G$. By Lemma 6.3, path $P$ intersects every module of $G$ at most once. Hence, we can define, for each $i \in \{0, 1, \ldots, d\}$, vertex $x'_i \in V(C)$ as the unique vertex in $V(C)$ that belongs to the same module as $x_i$. Since for each $i \in \{0, 1, \ldots, d - 1\}$, we have $x_ix_{i+1} \in E(G)$, we infer that $x'_ix'_{i+1} \in E(C)$. Thus, $(x = x'_0, x'_1, \ldots, x'_d = y)$ is an $x$-$y$ path in $C$, which implies that $\text{dist}_C(x, y) \leq d = \text{dist}_G(x, y)$. \hfill $\square$
Lemma 6.5. For every $k \geq 1$, every module in $G$ is a module in $G^k$.

Proof. Let $M$ be a module in $G$, and suppose for a contradiction that $M$ is not a module in $G^k$. This means that there exists a vertex $w \in V(G) \setminus M$ and two vertices $x, y \in M$ such that $wx \in E(G^k)$ but $wy \notin E(G^k)$. Thus, we infer that $d_G(w, x) \leq k$, while $d_G(w, y) > k$. Let $P = (w = v_0, v_1, \ldots, v_r = x)$ be a shortest path in $G$ from $w$ to $x$. Then $r \geq 1$, and, by Lemma 6.3, $v_{r-1} \notin M$. Since $v_{r-1}x = v_{r-1}v_r \in E(G)$ and $\{x, y\} \subseteq M$, we also have $v_{r-1}y \in E(G)$. Hence, replacing $x$ with $y$ in $P$ yields an $x$-$y$ path in $G$ of length $k$, contrary to the assumption $d_G(w, y) > k$. \hfill \Box

Now we have everything ready to prove Theorem 6.1.

Proof of Theorem 6.1. Clearly, $X' \subseteq X$, which implies $(X')^k \subseteq X^k$. Therefore, if $X^k$ is of bounded clique-width, then so is $(X')^k$.

Suppose now that $(X')^k$ is of bounded clique-width. We may assume that $X'$ contains a graph on at least 4 vertices, since otherwise $X' \subseteq \{K_1, K_2, \overline{K_2}\}$, and $X$ is a subclass of $P_4$-free graphs and therefore of bounded clique-width, so we may apply Proposition 1.1.

Let $\ell$ be an integer such that for every $G \in X'$, we have $\text{cw}(G^k) \leq \ell$. We will show that for every $G \in X$, it holds $\text{cw}(G^k) \leq \max\{2, \ell\}$. Let $G \in X$. If $k = 1$, then Proposition 2.2 applies and we have $\text{cw}(G) = \max\{\text{cw}(H) \mid H$ is a prime induced subgraph of $G\} \leq \ell$.

Suppose now that $k \geq 2$. We will prove the desired inequality $\text{cw}(G^k) \leq \max\{2, \ell\}$ by induction on $|V(G)|$. If $G$ is disconnected, then so is $G^k$, and we can assume inductively that $\text{cw}(H) \leq \max\{2, \ell\}$ holds for every connected component $H$ of $G^k$, which implies the desired inequality for $G^k$. If the complement of $G$ is disconnected, then $\text{diam}(G) \leq 2$, and hence $G^k$ is complete, and the result follows. So let us assume that $G$ is connected and co-connected. We may also assume that $\text{diam}(G) \geq 3$, since otherwise $G^k$ is complete. Let $M_1, \ldots, M_r$ be the maximal modules of $G$, and let $C$ be the characteristic graph of $G$. Since $\text{diam}(G) \geq 3$, graph $G$ contains an induced $P_4$, and consequently $G^k$ contains an induced $P_4$. It follows that, since $C$ is prime and $C \notin \{K_1, K_2, \overline{K_2}\}$, $C$ contains no isolated vertices. This implies that every module $M_i$ becomes a clique in $G^k$. By Lemma 6.5, $M_i$ is a module in $G^k$.

By Proposition 2.2, the clique-width of $G^k$ equals the maximum clique-width of its prime induced subgraphs. Thus, in order to show that $\text{cw}(G^k) \leq \max\{2, \ell\}$, it suffices to show that for every prime induced subgraph $H$ of $G^k$, we have $\text{cw}(H) \leq \max\{2, \ell\}$. Let $H$ be a prime induced subgraph of $G^k$. Since every $M_i$ is a module in $G^k$, graph $H$ contains at most one vertex from each $M_i$. Thus, we may assume that $V(H) \subseteq V(C)$, where $C$ is the characteristic graph of $G$. By definition, two vertices $x, y \in V(H)$ are adjacent in $H$ if and only if $\text{dist}_G(x, y) \leq k$. By Lemma 6.4, this is equivalent to the condition $\text{dist}_C(x, y) \leq k$. In particular, since $V(H) \subseteq V(C)$ and $xy \in E(H)$ if and only if $xy \in C^k$, this means that $H$ is an induced subgraph of $C^k$. Since $C$ is a prime induced subgraph of a graph in $X$ (namely, of $G$), we have $C \in X'$ and therefore $\text{cw}(C^k) \leq \ell$. Since $H$ is an induced subgraph of $C^k$, Proposition 2.1 implies that $\text{cw}(H) \leq \text{cw}(C^k) \leq \ell \leq \max\{2, \ell\}$. \hfill \Box
7 Hereditary graph classes of power-bounded clique-width

Let us denote $S_k := \text{Free}\left(\{K_{1,4}, C_3, \ldots, C_k, H_1, \ldots, H_k\}\right)$, where $H_i$ are the graphs depicted in Fig. 1.

Figure 1: Graphs $H_i$

Denote the class of line graphs of graphs in $S_k$ by $T_k$.

Proposition 7.1. For every $k \geq 3$, the classes $S_k$ and $T_k$ are of power-unbounded clique-width.

Proof. Since graphs in $S_k$ are $\{K_3, K_{1,4}\}$-free, every graph in $S_k$ is of maximum degree at most 3. Hence, Proposition 3.1 applies, and the fact that $S_k$ is of power-unbounded clique-width follows from a result from [39] showing that $S_k$ is of unbounded clique-width.

To show the second part of the proposition, observe that the fact that the class in $S_k$ is of unbounded clique-width implies that it is also of unbounded treewidth, and consequently Theorem 2.7 implies that $T_k$ is of unbounded clique-width. Since every graph in $S_k$ is of maximum degree at most 3, every graph in $T_k$ is of maximum degree at most 5. Hence, Proposition 3.1 implies that the class $T_k$ is of power-unbounded clique-width. $\square$

To extend Proposition 7.1, let us recall the following two parameters, introduced in [38]:

- $\kappa(G)$ is the maximum $k$ such that $G \in S_k$. If $G$ belongs to no class $S_k$, we define $\kappa(G)$ to be 0, and if $G$ belongs to all classes $S_k$, then $\kappa(G)$ is defined to be $\infty$. Also, for a set of graphs $M$, we define $\kappa(M) = \sup\{\kappa(G) \mid G \in M\}$.

- $\lambda(G)$ is the maximum $\ell$ such that $G \in T_\ell$. If $G$ belongs to no class $T_\ell$, then $\lambda(G) := 0$, and if $G$ belongs to every $T_\ell$, then $\lambda(G) := \infty$. For a set of graphs $M$, we define $\lambda(M) = \sup\{\lambda(G) \mid G \in M\}$.

According to the definition, in order for $\kappa(G)$ to be infinite, $G$ must belong to every class $S_k$. This is the case if and only if every connected component of $G$ is of the form $S_{i,j,k}$ represented on the left in Figure 2 (where the values of $i,j,k \geq 0$ may depend on component). Let us denote the class of all such graphs by $S$. More formally, $S := \bigcap_{k \geq 3} S_k$.

Moreover, $\lambda(G) = \infty$ if and only if $G$ is the line graph of a graph in $S$. Let us denote the class of all such graphs by $T$. In other words, $T$ is the class of graphs every connected component of which has the form $T_{i,j,k}$ represented on the right-hand side in Figure 2.

The following result follows from the proofs of Theorems 2 and 6 in [38].
Figure 2: Graphs $S_{i,j,k}$ (left) and $T_{i,j,k}$ (right)

Lemma 7.2. Let $M$ be a set of graphs. If $\kappa(M) < \infty$, then there is an integer $k$ such that $S_k \subseteq \text{Free}(M)$. If $\lambda(M) < \infty$, then there is an integer $k$ such that $T_k \subseteq \text{Free}(M)$.

Consequently:

Theorem 7.3. Let $M$ be a set of graphs. If $\kappa(M) < \infty$ or $\lambda(M) < \infty$, then the class of $M$-free graphs is of power-unbounded clique-width.

7.1 Monogenic graph classes

A graph class is said to be monogenic if it is defined by a single forbidden induced subgraph. We now characterize monogenic graph classes of power-bounded clique-width.

Theorem 7.4. For every graph $H$, the class of $H$-free graphs is of power-bounded clique-width if and only if $H$ is a linear forest.

Proof. If $H$ is a linear forest then there exists a positive integer $k$ such that $H$ is an induced subgraph of path $P_k$. Hence, the class of $H$-free graphs is a subclass of the class of $P_k$-free graphs, which in particular implies that the diameter of $H$-free graphs is bounded. Observation 1 now implies that the class of $H$-free graphs is of power-bounded clique-width.

Conversely, suppose that $H$ is not a linear forest. Suppose first that $H$ is a forest. Then, $H$ contains a vertex of degree at least 3. In particular, $H$ contains a claw as an induced subgraph. Hence the class of claw-free graphs, and in particular, the class of unit interval graphs, is a subclass of $H$-free graphs. The power-unboundedness of the clique-width now follows from Corollary 4.3. Suppose now that $H$ is not a forest. Denoting by $k$ the girth of $H$, observe that every graph with girth at least $k + 1$ is $H$-free. By Corollary 3.3, the class of graphs of girth at least $k + 1$ is of power-unbounded clique-width, hence the same holds also for the larger class of $H$-free graphs.

7.2 Bigenic graph classes

A graph class is said to be bigenic if it is defined by two forbidden induced subgraphs. In this section, we prove our main result: a complete characterization of graph classes of
power-bounded clique-width among bigenic classes defined by two connected forbidden induced subgraphs (Theorem 7.6). This result will be derived from the following proposition, in which we identify several bigenic graph classes of power-bounded and power-unbounded clique-width.

**Proposition 7.5.** Let $A$ and $B$ be two graphs, and let $X$ be the class of $\{A, B\}$-free graphs. Then, the following holds:

(i) If $\{A, B\} \cap S = \emptyset$ or $\{A, B\} \cap T = \emptyset$, then $X$ is of power-unbounded clique-width.

(ii) If $A \in S \cap T$ (that is, $A$ is a linear forest), then $X$ is of power-bounded clique-width.

(iii) If $A \in S \setminus T$, $B \in T \setminus S$ and $A$ contains an induced $S_{2,2,2}$, then $X$ is of power-unbounded clique-width.

(iv) If $A \in S \setminus T$, $B \in T \setminus S$ and $B$ contains an induced $T_{2,2,2}$, then $X$ is of power-unbounded clique-width.

(v) If $A \in S \setminus T$ contains an induced $2S_{1,1,1}$ and $B \in T \setminus S$, then $X$ is of power-unbounded clique-width.

(vi) If $A \in S \setminus T$ is $\{S_{2,2,2}, 2S_{1,1,1}\}$-free and $B \in T \setminus S$ is $\{T_{2,2,2}, 2T_{1,1,1}\}$-free, then $X$ is of power-bounded clique-width.

**Proof.** Suppose that $\{A, B\} \cap S = \emptyset$ or $\{A, B\} \cap T = \emptyset$. Then $\kappa(\{A, B\}) < \infty$ or $\lambda(\{A, B\}) < \infty$, and by Theorem 7.3, $X$ is of power-unbounded clique-width.

If $A \in S \cap T$, then $A$ is a linear forest, and hence, by Theorem 7.4, the class of $A$-free graphs is of power-bounded clique-width. Since $X$ is a subclass of the class of $A$-free graphs, it is also of power-bounded clique-width.

Let $A \in S \setminus T$ and $B \in T \setminus S$ be such that $A$ contains an induced $S_{2,2,2}$. Graph $B$ is not a linear forest; in particular, $B$ contains an induced triangle. We claim that every bipartite permutation graph is $\{A, B\}$-free. Indeed: (i) every bipartite permutation graph is $S_{2,2,2}$-free (see, e.g., [19, 20, 30]), and hence also $A$-free; (ii) since every bipartite permutation graph is bipartite, it is triangle-free, and thus also $B$-free. By Proposition 4.1, the clique-width is power-unbounded in $X$.

Let $A \in S \setminus T$ and $B \in T \setminus S$ be such that $B$ contains an induced copy of $T_{2,2,2}$. Graph $A$ is not a linear forest; in particular, $A$ contains an induced claw, that is, $S_{1,1,1}$. We claim that every unit interval graph is $\{A, B\}$-free. Indeed, every unit interval graph is $(S_{1,1,1}, T_{2,2,2})$-free [48], and hence also $\{A, B\}$-free. By Corollary 4.3, the clique-width is power-unbounded in $X$.

Let $A \in S$ be such that $A$ contains an induced $2S_{1,1,1}$, and $B \in T \setminus S$. As shown by Lozin and Volz [40], there exists a family $\{G_n\}_n$ of connected bipartite $2P_3$-free graphs of diameter 3 such that $cw(G_n) \geq n$. The fact that these graphs have diameter 3 implies that every two vertices in the same part of a bipartition of $G_n$ have a common neighbor in the other part.

Given a graph $G_n$ from this family, define another bipartite graph $H_n$ as follows: Fix a bipartition $(U, W)$ of the vertex set of $G_n$. For each vertex $w \in W$, attach to $w$ a new
path $P^w$ of length (number of edges) $n$ in such a way that $w$ is an endpoint of $P^w$. Let $H_n$ denote the obtained graph. We claim that $H_n$ is $2S_{1,1,1}$-free (and consequently $A$-free).

Suppose that a subset $K \subseteq V(H_n)$ induces a copy of $2S_{1,1,1}$. Let $a$ and $b$ be the two vertices of degree 3 in $G[K]$. Clearly, $a, b \in V(G_n)$. Since every vertex in $V(G_n)$ has at most one neighbor (in $H_n$) outside $V(G_n)$, it follows that at least two edges incident to $a$ in $K$ belong to $G_n$, and similarly, at least two edges incident to $b$ in $K$ belong to $G_n$. But then, $G_n$ contains an induced $2P_3$, a contradiction. Thus, $H_n$ is $2S_{1,1,1}$-free. Since $G_n$ is bipartite, so is $H_n$. In particular, $H_n$ is triangle-free and hence also $B$-free. This shows that $H_n \in X$.

Let $Y = \{H_n \mid n \geq 2\}$. We will show that the clique-width of graphs in $Y^k$ is unbounded for every positive integer $k$. For $k = 1$, this follows from the fact that $\text{cw}(H_n) \geq \text{cw}(G_n) \geq n$, using Proposition 2.1. So let $k \geq 2$, and let $n \geq k$. Let $D_k$ denote the set of vertices of $H_n$ that are at distance exactly $k$ from $U$. Since $k \geq 2$, we have $D_k \cap (U \cup W) = \emptyset$.

It can be seen that the subgraph of $H_n^k$ induced by $U \cup D_k$ is a split graph isomorphic to the graph obtained from $G_n$ by adding all the edges between vertices in $U$. In particular, $H_n^k[U \cup D_k]$ is isomorphic to the result of a subgraph complementation on $G_n$. Therefore, by Proposition 2.4, for every fixed $k$, the clique-width of graphs in the set $\{H_n^k[U \cup D_k] \mid n \geq k\}$ is unbounded. It follows that $Y^k$ is of unbounded clique-width, hence $Y$ is a class of power-unbounded clique-width. Consequently, since $Y \subseteq X$, so is $X$.

Suppose now that $A \in S \setminus T$ is $\{S_{2,2,2}, 2S_{1,1,1}\}$-free and $B \in T \setminus S$ is $\{T_{2,2,2}, 2T_{1,1,1}\}$-free. Then, exactly one component of $A$, say $C$, is not a path, and exactly one component of $B$, say $D$, is not a path. Since $C$ is $S_{2,2,2}$-free and $D$ is $T_{2,2,2}$-free, there exists a positive integer $k$ such that $A$ is an induced subgraph of $S_{1,k,k}$ and $B$ is an induced subgraph of $T_{1,k,k}$. Therefore, it is sufficient to show the statement for the case when $A = S_{1,k,k}$ and $B = T_{1,k,k}$, for all $k \geq 3$.

Let $k \geq 3$, and let $G$ be a $\{S_{1,k,k}, T_{1,k,k}\}$-free graph. By Corollary 6.2, we may assume that $G$ is prime. In particular, $G$ is connected.

**Claim 1.** Let $P$ be a shortest path between two vertices in $G$. Then, for every vertex $x$ on $P$ which is at distance more than $k$ from each endpoint of $P$, the degree of $x$ in $G$ is 2.

**Proof of claim.** Suppose for a contradiction that such a vertex $x$ has a neighbor $x'$ outside $P$. It follows from the minimality of $P$ that every two neighbors of $x'$ on $P$ are at distance at most two on $P$. If $x$ is the only neighbor of $x'$ on $P$, then an induced $S_{1,k,k}$ arises in $G$, a contradiction. So $x'$ has at least two neighbors on $P$. If $x'$ has exactly two neighbors on $P$, then they cannot be adjacent, since otherwise $G$ would contain an induced $T_{1,k,k}$. This leaves only two possibilities: either $x'$ has exactly two neighbors on $P$ that are at distance two on $P$, or $x'$ has exactly three neighbors on $P$ that appear consecutively on $P$. In either case, $x'$ is adjacent to two vertices on $P$ at distance two on $P$. If necessary, let us redefine $x$ so that it denotes one of these two vertices, and let the other one be $y$. Let $z$ be the common neighbor of $x$ and $y$ on $P$. Let $A$ denote the set of common neighbors of $x$ and $y$ in $G$. Then, $|A| \geq 2$. Since $G$ is prime, there exists a vertex, say $w$, in $V \setminus A$, that has both a neighbor and a non-neighbor in $A$. By the minimality of $P$, vertex $w$ is not on $P$. Without loss of generality, we may assume that $wz \in E(G)$ and $wx' \notin E(G)$. Applying
similar reasoning as above, with the role of \(x'\) interchanged by \(w\), we may assume that \(w\) is adjacent to a vertex, say \(t\), on \(P\), that is at distance two on \(P\) from \(z\).

Let \(u\) and \(v\) be the endpoints of \(P\) labeled so that \(d_G(u, x) < d_G(u, y)\). The common neighbor of \(t\) and \(z\) on \(P\) can be either \(x\) or \(y\). Without loss of generality, assume that it is \(x\). Let \(P'\) and \(P''\) be the unique pair of vertex disjoint subpaths of \(P\) of respective lengths \(k\) and \(k - 3\) such that \(t\) is an endpoint of \(P'\) and \(y\) is an endpoint of \(P''\). Let \(H\) be the subgraph of \(G\) induced by \(\{t, x, x', y, w\} \cup V(P') \cup V(P'')\). Then, \(H\) is either isomorphic to \(S_{1,k,k}\) (if \(xw \notin E(G)\)) or contains an induced copy of \(T_{1,k,k}\) (otherwise). In either case, we reach a contradiction with the \(\{A, B\}\)-freeness of \(G\).

We split the rest of the proof into two cases.

**Case 1:** There exist two vertices, say \(u\) and \(v\), such that \(d_G(u) \geq 3\), \(d_G(v) \geq 3\) and \(d_G(u, v) > 7k + 4\).

Let \(P\) be a shortest \(u-v\) path.

**Claim 2.** For every \(x \in V(G) \setminus V(P)\), we have \(d_G(x, \{u, v\}) \leq 2k\).

**Proof of claim.** Suppose for a contradiction that there exists a vertex \(x \in V(G) \setminus V(P)\) such that \(d_G(x, \{u, v\}) = 2k + 1\).

Without loss of generality, we may assume that \(d(x, u) \leq d(x, v)\). Let \(u'\) be the vertex of \(P\) at distance \(2k + 1\) from \(u\). Let \(P'\) be a shortest \(x-u'\) path in \(G\) and let \(u''\) be the first vertex of \(P'\) on \(P\).

We claim that \(u''\) is at distance at most \(k\) from \(u\). Suppose for a contradiction that \(u''\) is at distance greater than \(k\) from \(u\). Since \(u''\) has degree at least 3 in \(G\), Claim 1 ensures that \(u''\) is at distance at most \(k\) from \(v\). The length \(\ell(P')\) of \(P'\) can be bounded from above as follows:

\[
\ell(P') = d_G(x, u') \leq d_G(x, u) + d_G(u, u') = 2k + 1 + 2k + 1 = 4k + 2.
\]

Consequently, the length \(\ell(P)\) of \(P\) can be bounded from above as follows:

\[
\ell(P) \leq d_G(u, u') + d_G(u', u'') + d_G(u'', v) \leq 2k + 1 + \ell(P') + k \leq 7k + 3,
\]

a contradiction.

We claim that \(u''\) is at distance more than \(k\) from each endpoint of \(P'\). Indeed,

\[
d_G(u'', x) \geq d_G(u, x) - d_G(u, u'') \geq 2k + 1 - k = k + 1
\]

and

\[
d_G(u'', u') \geq d_G(u, u') - d_G(u, u'') \geq 2k + 1 - k = k + 1.
\]

Now, since \(P'\) is a shortest \(x, u'\)-path, Claim 1 implies that the degree of \(u''\) in \(G\) is equal to 2, which contradicts the definition of \(u''\). Note that here we have used the assumption of Case 1 which guarantees that \(u\)—and hence also \(u''\)—is of degree at least 3. This completes the proof of Claim 2.
Let $A$ denote the set of vertices at distance at most $2k$ from $\{u, v\}$. Claim 2 implies that $G - A$ is a path. Moreover, since $G - A$ is a subpath of $P$, every internal vertex of $G - A$ is of degree 2 in $G$. Thus, $G$ can be obtained from a graph of bounded diameter by subdividing one of its edges. Corollary 5.2 implies that $G$ is of power-bounded clique-width.

**Case 2:** Every two vertices in $G$ of degree at least 3 are at distance at most $7k + 4$ from each other.

If every vertex of $G$ has degree at most 2, $G$ is a path or a cycle and hence $G$ is of clique-width at most 4. So we may assume that $G$ has a vertex, say $u$, of degree at least 3.

Let $B$ be the set of vertices in $G$ at distance at most $8k + 4$ from $u$. Then, $B$ will contain all vertices of $G$ of degree at least 3, together with all vertices that are at distance at most $k$ from some vertex of degree at least 3. In particular, the subgraph $F$ of $G$ induced by $V(G) \setminus A$ consists only of vertices of degree at most 2 in $G$; in particular, $F$ is a linear forest.

We claim that $F$ has at most one connected component. Suppose for a contradiction that $F$ has at least two connected components. Let $s$ and $t$ be two vertices in different components of $F$. Then, any shortest path $P$ between $s$ and $t$ must pass through $B$, and, since $B$ induces a connected graph, $P$ will contain a vertex, say $x$, of degree at least 3. However, this is a contradiction with Claim 1.

Thus, $G$ can be obtained from a graph of bounded diameter by subdividing one of its edges. Corollary 5.2 implies that $G$ is of power-bounded clique-width.

This completes the proof of Proposition 7.5.

Proposition 7.5 implies the following characterization of graph classes of power-bounded clique-width defined by two connected forbidden induced subgraphs.

**Theorem 7.6.** Let $A$ and $B$ be two connected graphs, and let $X$ be the class of $\{A, B\}$-free graphs. Then $X$ is of power-bounded clique-width if and only if either one of $A$ and $B$ is a path, or one of $A$ and $B$ is isomorphic to some $S_{1,j,k}$, and the other one to some $T_{1,j,k}$.

**Proof.** Suppose that $X$ is of power-bounded clique-width. By Proposition 7.5(i), we may assume that $A \in \mathcal{S}$ and $B \in \mathcal{T}$. We may assume that neither of $A$ and $B$ is a path (or we are done). Since $A$ and $B$ are connected, $A$ is of the form $S_{i,j,k}$ (for some $i, j, k$), and $B$ is of the form $T_{i,j,k}$ (for some $i, j, k$). By Proposition 7.5(iii), we have that $A$ is of the form $S_{1,j,k}$ (for some $j, k$). Similarly, Proposition 7.5(iv) implies that $B$ is of the form $T_{1,j,k}$ (for some $j, k$).

Suppose now that either one of $A$ and $B$ is a path, or one of $A$ and $B$ is isomorphic to some $S_{1,j,k}$, and the other one to some $T_{1,j,k}$. If one of $A$ and $B$ is a path, then Proposition 7.5(iii) implies that $X$ is of power-bounded clique-width. Otherwise, $A$ is $\{S_{2,2,2}, 2S_{1,1,1}\}$-free and $B$ is $\{T_{2,2,2}, 2T_{1,1,1}\}$-free, hence $X$ is of power-bounded clique-width by Proposition 7.5(vi).

An example of a graph class of unbounded clique-width described in Theorem 7.6 is the class of $\{\text{claw, bull}\}$-free graphs. (The claw is the graph $S_{1,1,1}$, while the bull is the graph $T_{1,2,2}$.) The fact that the class of claw-free bull-free graphs is of unbounded clique-width
follows from the fact that it contains all complements of triangle-free graphs (in particular, all complements of square grids), hence Proposition 2.4 applies.

8 Graph classes with arbitrarily large value of $\pi(X)$

Recall that for a class $X$ of power-bounded clique-width, we denote by $\pi(X)$ the smallest positive integer $k$ such that $X^k$ is of bounded clique-width. In this section, we show that $\pi(X) = 3$ for the class of split graphs, and construct hereditary graph classes $X$ with arbitrary large values of $\pi(X)$. A graph is a split graph if its vertex set can be partitioned into a clique and an independent set. Makowsky and Rotics showed in [41] that the clique-width of split graphs is unbounded. On the other hand, since every split graph is $P_5$-free, it is of diameter at most 3, and Observation 1 implies that $\pi(\{\text{Split graphs}\}) \leq 3$. This bound is sharp.

**Proposition 8.1.** $\pi(\{\text{Split graphs}\}) = 3$.

**Proof.** Let $X$ be the set of split graphs. We only need to show that the class $X^2$ contains graphs of arbitrarily large clique-width. This follows from the observation that every graph is an induced subgraph of the square of some split graph. Indeed, given a graph $G = (V, E)$, let $H$ be the split graph such that $V(H) = V \cup E$, $V$ is an independent set, $E$ is a clique, and $v \in V$ and $e \in E$ are adjacent if and only if $v$ is incident with $e$ in $G$. We claim that for every two distinct vertices $u, v \in V$, we have $uv \in E(G)$ if and only if $d_H(u, v) \leq 2$. Indeed, if $uv \in E(G)$, then the edge $e = uv$ is a common neighbor of $u$ and $v$ in $H$, hence $d_H(u, v) \leq 2$. The converse direction can be proved similarly. This implies that the subgraph of $H^2$ induced by $V$ equals $G$. □

For positive integers $k$ and $\ell$, let $C_{k, \ell}$ denote the largest hereditary graph class such that for all $G \in C_{k, \ell}$, we have $cw(G^k) \leq \ell$. Equivalently,

$$C_{k, \ell} = \{G \mid cw(H^k) \leq \ell \text{ for all induced subgraphs } H \text{ of } G\}.$$

For example, for every $\ell \geq 1$, the class $C_{1, \ell}$ is the class of graphs of clique-width at most $\ell$, while for every $k \geq 1$, the class $C_{k, 1}$ is the class of edgeless graphs. Also, notice that since $cw(G^k) \leq \ell$ for all $G \in C_{k, \ell}$, we have $\pi(C_{k, \ell}) \leq k$.

**Remark 2.** The inequality $\pi(X) \leq \text{diam}(X)$ from Observation 1 can be strict, as the following example shows. Let $X = C_{2,2}$. Then, $\text{diam}(X) \geq 3$ since $P_3 \in X$. On the other hand, $\pi(X) \leq 2$.

It follows directly from the definition that $\pi(C_{k, \ell}) \leq k$. We now construct a family of graph classes in which the above inequality is attained with equality.

**Proposition 8.2.** For all even $k \geq 4$, we have $\pi(C_{k,2}) = k$.

**Proof.** We only need to show that $\pi(C_{k,2}) \geq k$. Let $G$ be a set of connected split graphs of unbounded clique-width (for example, the graphs $K_n^*$ defined in Remark 1). Let $G \in G$ be a connected split graph with split partition $(K, I)$, where $K = \{w_1, \ldots, w_r\}$ is a clique and
that the set of graphs $G$ complete. Note that since $G$ was connected, the diameter of $G_1$ is at most 2. Let $G_2$ be the graph obtained from $G_1$ by adding, for each vertex $v_i \in I$, a path $P_i$ of length $\frac{k}{2}$ having vertices $v_i = v_i^0, v_i^1, \ldots, v_i^{\frac{k}{2}}$, and for each vertex $w_j \in K$, a path $Q_j$ of length $\frac{k}{2} - 2$ having vertices $w_j = w_j^0, w_j^1, \ldots, w_j^{\frac{k}{2} - 2}$. Let $I' = \{v_i^{\frac{k}{2}}, v_i^{\frac{k}{2}+1}, \ldots, v_i^h\}$, $K' = V(G_2) \setminus I'$. The vertices of $I'$ are mutually at distance $k + 1$ in $G_2$, so they form an independent set in $G_2^k$. The vertices of $K'$ are mutually at distance at most $k$ in $G_2$, so they form a clique in $G_2^k$. The vertices of $I'$ are at distance at most $k$ from the vertices of $K'$ in $G_2$, so the graph $G_2^k$ is a complete split graph $(I', K')$, and thus $\text{cw}(G_2^k) = 2$.

Let $H$ be a connected induced subgraph of $G_2$. If $H$ has no vertex from $K$, then $H^k$ is complete. If $H$ has no vertex from $I$, then $H^k$ is either complete or a complete split graph. So assume that $V(H) \cap I \neq \emptyset$ and $V(H) \cap K \neq \emptyset$. Note that the distance in $H$ between a vertex of $I$ and a vertex of $K$ is at most 3. So, defining $I'' = \{v_i^{\frac{k}{2}}, v_i^{\frac{k}{2}+1}, \ldots, v_i^h\} \cap V(H)$, $K' = V(H) \setminus I''$, we have that the vertices of $K'$ form a clique in $H^k$, and the vertices of $I''$ form an independent set in $H^k$. Let $I''' \subseteq I'$ be the set of vertices $v_i^{\frac{k}{2}}$ such that $v_i^0$ has no neighbors in $K \cap V(H)$, and let $K'' \subseteq K'$ be the set of vertices $w_j^{\frac{k}{2} - 2}$ such that $w_j^0$ has no neighbors in $I \cap V(H)$. In $H^k$, every vertex in $I'' \setminus I'''$ is adjacent to every vertex in $K'$, while every vertex in $I'''$ is adjacent to every vertex in $K' \setminus K''$ and non-adjacent to every vertex in $K''$. Therefore, the graph $H^k$ is a cograph, and thus $\text{cw}(H^k) \leq 2$.

The above implies that $\{G_2 \mid G \in \mathcal{G}\} \subseteq C_{k,2}$.

Let us now analyze $\text{cw}(G_2^h)$ for $1 \leq h \leq k - 1$. Let us first consider $h$ even. Let $I' = \{v_1^{\frac{k}{2}}, \ldots, v_s^{\frac{k}{2}}\}$, $K' = \{w_1^{\frac{k}{2} - 1}, \ldots, w_r^{\frac{k}{2} - 1}\}$. Then $I'$ is an independent set in $G_2^h$, $K'$ is a clique in $G_2^h$, and $v_1^{\frac{k}{2}}$ is adjacent to $w_1^{\frac{k}{2} - 1}$ in $G_2^h$ if and only if $v_1^0$ is adjacent to $w_1^0$ in $G$. Thus, $G_2^h[I' \cup K'] \cong G$, and $\text{cw}(G_2^h) \geq \text{cw}(G)$ by Proposition 2.1. This implies that the set of graphs $\{G_2^h \mid G \in \mathcal{G}\}$ is of unbounded clique-width. Consider now $h$ odd, $h > 1$. Let $I' = \{v_1^{\frac{k+1}{2}}, \ldots, v_s^{\frac{k+1}{2}}\}$, $K' = \{w_1^{\frac{k+1}{2} - 1}, \ldots, w_r^{\frac{k+1}{2} - 1}\}$. Then $I'$ is an independent set in $G_2^h$, $K'$ is a clique in $G_2^h$, and $v_1^{\frac{k+1}{2}}$ is adjacent to $w_1^{\frac{k+1}{2} - 1}$ in $G_2^h$ if and only if $v_1^0$ is adjacent to $w_1^0$ in $G$. Thus, $G_2^h[I' \cup K'] \cong G$, and $\text{cw}(G_2^h) \geq \text{cw}(G)$, implying that the set of graphs $\{G_2^h \mid G \in \mathcal{G}\}$ is of unbounded clique-width. For $h = 1$, observe that the graph $G_2^1[K \cup I] = G_1$ is isomorphic to the graph obtained from $G$ by a single subgraph complementation. Since $\text{cw}(G_2) \geq \text{cw}(G_1)$, Proposition 2.1 implies that the set of graphs $\{G_2 \mid G \in \mathcal{G}\}$ is of unbounded clique-width.

Therefore, $\pi(C_{h,2}) \geq \pi(\{G_2 \mid G \in \mathcal{G}\}) = k$. $
$

9 Conclusion

In conclusion, we would like to mention some open questions related to the topic of this paper. Note that $\text{cw}(X^{k+1})$ can be bigger than $\text{cw}(X^k)$. For example, if $X$ is the set of paths, then $\text{cw}(X) = 3$ while $\text{cw}(X^2) = 4$ [29]. However, we do not know if $\text{cw}(X^{k+1})$ is
always bounded from above by some function of \( \text{cw}(X^k) \).

**Problem 1.** Is it true that every graph class of power-bounded clique-width is also of strongly power-bounded clique-width? Equivalently, is there a function \( f \) such that for every graph \( G \) and every positive integer \( k \), we have: if \( \text{cw}(G^k) \leq \ell \), then \( \text{cw}(G^{k+1}) \leq f(k, \ell) \)?

A positive answer to the above question would follow from a positive answer to the following one.

**Problem 2.** Is there a function \( f \) such that for every graph \( G \) and every positive integer \( k \), we have \( \text{cw}(G^{k+1}) \leq f(\text{cw}(G^k)) \)?

On the other hand, a positive resolution to Problem 1 would imply a positive answer to the following two problems.

**Problem 3.** Is it true that every graph class \( X \) of power-bounded clique-width has only finitely many powers of unbounded clique-width?

**Problem 4.** Is it true that every graph class \( X \) of power-bounded clique-width has infinitely many powers of bounded clique-width?

In relation with Problem 3, it can be seen that for every graph class \( X \) for which we proved power-boundedness of the clique-width, our proofs in fact show that \( X \) has only finitely many powers of unbounded clique-width.

Finally, we mention a question related to a property of the sequence of the clique-widths of powers of a fixed graph. Given a graph \( G \), consider the sequence \( (\text{cw}(G^k))_{k=1}^{\text{diam}(G)} \). Since \( \text{cw}(P_4) = 3 > 2 = \text{cw}(P^2_4) \) and \( \text{cw}(P_9) = 3 < 4 = \text{cw}(P^2_9) \), in general such a sequence will be neither monotonically decreasing nor monotonically increasing. However, we do not know of a graph for which the sequence of the clique-widths of its powers would not be unimodal. Recall that a finite sequence of real numbers \( (a_1, \ldots, a_n) \) is said to be unimodal if there exists an integer \( k \in \{1, \ldots, n\} \) such that \( a_j \leq a_{j+1} \) for all \( j \in \{1, \ldots, k-1\} \) and \( a_j \geq a_{j+1} \) for all \( j \in \{k, \ldots, n-1\} \).

**Problem 5.** Is it true that for every graph \( G \), the sequence

\[
(\text{cw}(G^k))_{k=1}^{\text{diam}(G)}
\]

is unimodal?

References


