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Some Robust Ridge Regression for handling Multicollinearity and Outlier

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Abstract

Ridge Regression and Robust Regression Estimators were proposed to deal with the problem of multicollinearity and outlier in a classical linear regression model respectively. This paper proposes a robust ridge regression estimator (RRR) for solving the problem of multicollinearity and outlier in a classical linear regression model simultaneously. The technique of the estimator requires using the robust estimators (M, MM, S, LTS, LAD, LMS) to estimate the ridge parameter instead of using the Ordinary Least Squares (OLS) estimator. The Robust Ridge Estimators performed better than OLS and the Ordinary Ridge Regression (ORR) estimator when data set suffers from both problems. Mean Square Error was used as a criterion for examining the performance of these estimators. Result was achieved by the application of the proposed estimator to a data set having the two problems.

Keywords: Ordinary Least Square Estimator, Ridge Regression Estimator, Robust Regression Estimator, Robust Ridge Regression Estimator.

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1. Introduction

Linear regression model routinely assesses the degree of relationship between one dependent variable and a set of explanatory variables. The Ordinary Least Squares (OLS) Estimator is most popularly used to estimate the parameters of regression model. The estimator has some very attractive statistical properties which have made it one of the most powerful and popular estimators of regression model. The performance of OLS estimator is inefficient in the presence of multicollinearity. The regression coefficients possess large standard errors and some even have the wrong sign Gujarati [8]. In literature, there are various methods existing to solve this problem. Among them is the ridge regression estimator first introduced by Hoerl and Kennard [11]. Ridge Regression Estimator has a smaller MSE than OLS estimator.

Consider the standard regression model:

$$Y = X\beta + \epsilon \tag{1}$$

X is an n$\times$ p matrix with full rank, Y is a n$\times$ 1 vector of dependent variable, β is a p$\times$ 1 vector of unknown parameters, and ϵ is the error term such that E$(\epsilon) = 0$ and $E(\epsilon\epsilon')$=$\sigma^2 I$.

The OLS estimator is defined as:

$$\hat{\beta} = (X'X)^{-1}X'Y \tag{2}$$

While the ridge estimator is defined as:

$$\hat{\beta} = (X'X + KI)^{-1}X'Y \tag{3}$$

Where K is a scalar ridge parameter.

Another common problem in a regression model is problem of outlier and non-normality of error term. Robust regression estimator is an important estimation technique for analyzing data that are contaminated with outliers or data with non normal error term. It can be used to detect outliers and to provide resistant (stable) results in the presence of outliers. These include M estimation proposed by Huber [13], LTS estimation by Rousseeuw [24], S estimation by Rousseeuw and Yohai [23], and MM estimation proposed by Yohai [29].

Inevitably, these two problems can exist together in a data set; see for instance [12]. When both problems exist then robust ridge regression (RRR) estimator proposed in this paper is suggested. This has also attracted the attention of some researchers. Holland [12] proposed robust M-estimator for ridge regression to handle the problem of multicollinearity and outliers. Askin and Montgomery [3] proposed ridge regression based on the M-estimates. It is computed using weighted least squares procedures. Walker [27] modified Askin and Montgomery’s approach to allow the use of GM estimators instead of M estimators. Simpson and Montgomery [26] proposed a biased-robust estimator that uses a multistage GM estimator with fully iterated ridge regression to control both influence and collinearity in the regression data set. Pfaffenberger & Dielman [22] combines least absolute value estimator with Ridge to proposed Ridge Least Absolute Value Estimator. Silvapulle [25] proposed a new class of ridge type M estimators obtained by using M estimators instead using OLS estimators. Arslan & Billor [4] proposed two alternative ridge type GM estimators to handle multicollinearity and outliers simultaneously. Authors in [28] introduced a robust regression estimator that performs well regardless of the quantity and configuration of outliers. Habshah and Marina [9] proposed Ridge MM estimator (RMM) by combining the MM estimator and ridge regression. Hatice and Ozlem [10] proposed robust ridge regression methods based on M, S, MM and GM estimators. Maronna [19] proposed robust MM estimator in ridge regression for high dimensional data.

In this study, ridge regression methods based on M, S, MM, LTS, LAD and LMS estimators are examined in the presence of both outliers and multicollinearity. Mean Square Error was used as a criterion for examining the performances of these estimators. The data sets used in this study was extracted from the study of Hussein and Ahmed [14].

2. Materials and Methods

2.1 Ridge Regression

The concept of ridge regression was introduced by Hoerl and Kennard [11]. Ridge regression is a method of biased linear estimation which has been shown to be more efficient than the OLS estimator when data exhibit multicollinearity. It reduces multicollinearity by adding a ridge parameter, K, to the main diagonal elements of X'X, the correlation matrix. The ridge estimator is defined in (3) such that $K \geq 0$. In literature several techniques for estimating the Ridge parameter K have been suggested by different researchers. Among them are [11,18,17,6,7,15,16,1,2,21,20].

In this study, the ridge parameter by Kibria [15] is used. It is defined as:

$$\widehat{K}_{GM} = \frac{\sigma^2}{\left(\prod_{i=1}^{p} {\alpha_i}^2\right)^{\frac{1}{p}}} \quad (4)$$

σ^2 and α_i^2 are generally unknown. Hoerl and Kennard in [11] suggested the replacement of σ^2 and α_i^2 by their corresponding unbiased estimators $\hat{\sigma}^2 and\ \hat{\alpha}_i^2$ where $\hat{\sigma}^2 = \frac{\sum_{i=1}^{n} e_i^2}{n-p}$.

2.2 Robust Estimators

2.2.1 M Estimators

The most common general method of robust regression is M-estimation, introduced by Huber [13]. It is nearly as efficient as OLS. Rather than minimize the sum of squared errors as the objective, the M-estimate minimizes a function ρ of the errors. The M-estimate objective function is

$$\min\sum_{i=1}^{n}\rho\left(\frac{e_i}{s}\right)=\min\sum_{i=1}^{n}\rho\left(\frac{y_i-X'\hat{\beta}_i}{s}\right) \tag{5}$$

Where s is an estimate of scale often formed from linear combination of the residuals. The function ρ gives the contribution of each residual to the objective function. A reasonable ρ should have the following properties: $\rho(e)\geq 0$, $\rho(0)=0$,$\rho(e)=\rho(-e)$, and $\rho(e_i)\geq\rho(e_i')$ for $|e_i|\geq|e_i'|$

the system of normal equations to solve this minimization problem is found by taking partial derivatives with respect to β and setting them equal to 0, yielding,

$$\sum_{i=1}^{n}\psi\left(\frac{y_i-X'\hat{\beta}_i}{s}\right)X_i=0 \tag{6}$$

Where ψ is a derivative of ρ. The choice of the ψ function is based on the preference of how much weight to assign outliers. Newton-Raphson and Iteratively Reweighted Least Squares (IRLS) are the two methods to solve the M-estimates nonlinear normal equations. IRLS expresses the normal equations as:

$$X'WX\hat{\beta}=X'Wy \tag{7}$$

2.2.2 S Estimator

Rousseeuw and Yohai [23] introduced S estimator which is derived from a scale statistics in an implicit way, corresponding to s (θ) where s(θ) is a certain type of robust M-estimate of the scale of the residuals $e_1(\theta),\dots,e_n(\theta)$. They are defined by minimization of the dispersion of the residuals: minimize S ($e_1(\theta),\dots,e_n(\hat{\theta})$ with final scale estimate $\hat{\sigma}$ = S($e_1(\theta),\dots,e_n(\hat{\theta})$). The dispersion $e_1(\theta),\dots,e_n(\hat{\theta})$ is defined as the solution of

$$\frac{1}{n}\sum_{i=1}^{n}\rho\left(\frac{e_i}{s}\right)=k \tag{8}$$

K is a constant and $\rho\left(\frac{e_i}{s}\right)$ is the residual function. Rousseeuw & Yohai suggested Tukey's biweight function given by:

$$\rho(x)=\begin{cases}\frac{x^2}{2}-\frac{x^4}{2c^2}+\frac{x^6}{6c^4}\ for|x|\leq c\\ \frac{c^2}{6}\ for|x|>c\end{cases} \tag{9}$$

Setting c=1.5476 and K=0.1995 gives 50% breakdown point (Rousseeuw & Leroy, 1984).

2.2.3 MM Estimator

MM-estimation is special type of M-estimation developed by Yohai [29]. MM-estimators combine the high asymptotic relative efficiency of M-estimators with the high breakdown of class of estimators called S-estimators. It was among the first robust estimators to have these two properties simultaneously. The MM refers to the fact that multiple M-estimation procedures are carried out in the computation of the estimator. Yohai described the three stages that define an MM-estimator:

Stage 1 A high breakdown estimator is used to find an initial estimate, which we denote $\tilde{\beta}$. the estimator need to be efficient. Using this estimate the residuals, $r_i(\beta) = y_i - x_i^T\tilde{\beta}$ are computed.

Stage 2 Using these residuals from the robust fit and $\frac{1}{n}\sum_{i=1}^{n}\rho\left(\frac{r_i}{s}\right) = k$ where k is a constant and the objective function ρ, an M-estimate of scale with 50% BDP is computed. This s $(r_1(\tilde{\beta}), \dots, r_n(\tilde{\beta}))$ is denoted s_n. The objective function used in this stage is labeled ρ_0.

Stage 3 The MM-estimator is now defined as an M-estimator of β using a redescending score function, $\varphi_1(u) = \frac{\partial\rho_1(u)}{\partial u}$, and the scale estimate s_n obtained from stage 2. So an MM-estimator $\hat{\beta}$ defined as a solution to

$$\sum_{i=1}^{n} x_{ij}\ \varphi_1\left(\frac{y_i - x_i^T\tilde{\beta}}{s_n}\right)=0 \qquad j=1,\dots,p. \tag{10}$$

2.2.4 LTS Estimator

Rousseeuw [24] developed the least trimmed squares (LTS) estimation method. Extending from the trimmed mean, LTS regression minimizes the sum of trimmed squared residuals. This method is given by,

$$\hat{\beta}_{LTS} = argminQ_{LTS}(\beta) \tag{11}$$

where $Q_{LTS}(\beta) = \sum_{i=1}^{h} e_i^2$ such that $e_{(1)}^2 \le e_{(2)}^2 \le e_{(3)}^2 \le \dots \le e_{(n)}^2$ are the ordered squares residuals and h is defined in the range $\frac{n}{2} + 1 \le h \le \frac{3n+p+1}{4}$, with n and p being sample size and number of parameters respectively. The largest squared residuals are excluded from the summation in this method, which allows those outlier data points to be excluded completely. Depending on the value of h and the outlier data configuration. LTS can be very efficient. In fact, if the exact numbers of outlying data points are trimmed, this method is computationally equivalent to OLS.

2.2.5 LMS Estimator

The least median of squares (LMS) estimator is defined as the p-vector

$$\hat{\beta}_{LMS} = argminQ_{LMS}(\beta) \quad (12)$$

Where $Q_{LMS}(\beta) = e_h^2$ such that $e_{(1)}^2 \leq e_{(2)}^2 \leq e_{(3)}^2 \leq \cdots \leq e_{(n)}^2$ are the ordered squares residuals and h is defined in the range $\frac{n}{2} + 1 \leq h \leq \frac{3n+p+1}{4}$. The breakdown value for the LMS estimate is also $\frac{n-h}{n}$. However the LTS estimate has several advantages over the LMS estimate.

2.2.6 LAD Estimator

Least Absolute Value (LAV) regression is also known by several other names, including Minimum Absolute Deviation regression, Least Absolute Deviation (LAD) regression, and Minimum Sum of Absolute Errors regression Dielman [5] developed the LAD estimator which minimizes the sum of the absolute values of the residuals with respect to the coefficient vector b:

$$min \sum_{i=1}^{n} |y_i - x_i b|. \quad (13)$$

A property of the LAD estimator is that there are K residuals that are exactly zero. LAD is robust to an outlier in the y-direction. However, LAD estimator does not protect against outlying x (leverages).

2.3 Robust Ridge Regression

This is a combination of ridge and robust regression to handle the problem of multicollinearity and outliers simultaneously. This will dampen the effects of both problems in a classical linear regression model. To compute Robust Ridge Estimator, the formula used is:

$$\hat{\beta}_{Robustridge} = (X'X + K_R I)^{-1} X'Y \quad (14)$$

Where K_R is called the robust ridge parameter. It is obtained from robust regression methods instead of using OLS. This will be computed as given above, only that $\hat{\sigma}^2 and\ \hat{\alpha}_i^2$ are replaced with $\hat{\sigma}_{robust}^2\ and\ \hat{\alpha}_{robust}^2$ respectively.

2.4 Data Used in this Study

Data set taken from Hussein and Ahmed [14] was used to examine the performance of the considered estimators. This contains three (3) regressors and one (1) response variable.

2.5 Criterion for Investigation

To investigate whether the ridge estimator is better than the OLS estimator, the MSE was calculated using the following equation:

$$MSE(\hat{\beta}_{robust\ \ ridge}) = \hat{\sigma}^2_{robust}\ \sum_{i=1}^{p} \frac{t_i}{(t_i+K)^2} + K^2 \sum_{i=1}^{p} \frac{\alpha_i^2}{(t_i+K)^2} \quad (15)$$

$$MSE(\hat{\beta}_{OLS}) = \hat{\sigma}^2 \sum_{i=1}^{p} \frac{1}{t_i} \quad (16)$$

Where $t_1, t_2, \dots, t_p$ are the eigenvalues of $X'X$, K is the ridge parameter obtained from OLS and robust estimates. α_i is the ith element of the vector $\alpha = Q'\beta$.

3. Results and Discussion

The model does not include the intercept term because the data was standardized. The results of robust diagnostic check of the data as shown in Table 1 revealed the presence of outliers and leverages. The following observations: 12, 14, 15, 16, 17, 18, 19, 20, 21 were identified as the outlying points in the X-space while observations 12, 13, 14,15,30,31 as the outlying points in the Y-space. Observation 12, 14, 15 are bad leverages. This necessitated the use of the robust regression and this provides a more stable regression estimates than OLS as seen in Table 2. The result in Table 2 shows that the estimates of LTS, S and MM estimator are fairly close and provides more stable regression estimates when compared with other robust estimators. It is also observed that the scale estimates ($\hat{\sigma}$) of LTS, MM, S are more efficient than others. Though the scale estimates of M estimators is not too different from the first three estimators mentioned. The VIF in Table 2 revealed the presence of multicollinearity since VIF's > 10. Also, the coefficient of $\hat{\beta}_3$ has a negative sign which is not consistent with the prior expectation, this indicated the presence of multicollinearity and hence necessitate the use of Ridge regression estimator rather than using OLS. It could therefore be inferred from Table 1 and 2 that the data set suffered the problem of multicollinearity and outliers simultaneously. Hence, since the data set suffered both problem of multicollinearity and outlier, the ridge parameter K is computed from the estimates of the following robust estimators: (M, MM, S, LTS, LAD, and LMS) and the performance is compared with the ridge parameter computed using OLS (Ordinary ridge regression). The ridge regression estimates based on the robust estimators in this study is called robust ridge regression. The results were presented in Table 3. The negative sign in $\hat{\beta}_3$ is corrected and found to be consistent with prior expectation which shows that the effect of multicollinearity has been handled. The regression estimates of robust ridge estimates based on LTS, MM, S and M are fairly closed than those obtained based on OLS, LAD and LMS. Table 4 revealed that the problem of multicollinearity has been solved using all the estimators but in terms of the MSE of the coefficeints robust ridge estimates based on LTS, S and MM performs better than other estimators. OLS has the least performance among the estimators with a large MSE.

Table 1: Robust Regression Diagnostics

Observation	Index	Mahalanobis	Robust MCD Distance	Leverage	Standardized Robust Residual	Outlier
12	1	1.3526	9.3644	*	6.3730	*
13	2	0.8171	2.6697		4.3987	*
14	3	0.9511	5.856	*	7.1817	*
15	4	4.2907	30.3189	*	14.3779	*
16	5	2.5975	17.3025	*	-0.677	
17	6	0.9385	5.0736	*	0.4202	
18	7	2.0496	10.6951	*	-1.0365	
19	8	1.2055	6.0795	*	0.4342	
20	9	3.971	23.1163	*	-0.5587	
21	10	2.9798	17.6757	*	-0.3455	
30	11	2.7834	2.6734		-14.2728	*
31	12	2.2603	2.0341		-14.7688	*

Table 2: OLS and Robust Estimates

Coefficient	OLS	LTS	S	MM	M	LAD	LMS	VIF
$\hat{\beta}_1$	0.2079	0.0180	0.0118	0.0232	0.4539	0.4313	0.1171	128.26
$\hat{\beta}_2$	0.9206	1.6952	1.7192	1.6913	0.7254	0.7373	1.7212	103.43
$\hat{\beta}_3$	-0.1340	-0.6148	-0.6259	-0.6178	-0.1116	-0.1027	-0.7165	70.87
$\hat{\sigma}$	0.1073	0.0410	0.045	0.0466	0.0501	0.1376	0.1508	
K_{GM}	0.1297	0.0238	0.0372	0.0259	0.0227	0.1853	0.0826	

Table 3: Ordinary Ridge and Robust Ridge Estimates

Coefficient	ORR	LTS	S	MM	M	LAD	LMS
$\hat{\beta}_1$	0.3189	0.3272	0.3279	0.3275	0.3269	0.3131	0.3239
$\hat{\beta}_2$	0.3521	0.4754	0.4310	0.4662	0.4807	0.3367	0.3747
$\hat{\beta}_3$	0.2823	0.1842	0.2235	0.1923	0.1795	0.2868	0.2693
MSE(β)	3.4539	**0.1609**	0.1695	0.1684	0.1735	0.1937	0.2247

Table 4: VIF OF OLS and Robust Ridge Estimates

Coefficient	ORR	LTS	S	MM	M	LAD	LMS
$\hat{\beta}_1$	0.2989	4.2693	2.0686	3.7314	4.6003	0.1977	0.5674
$\hat{\beta}_2$	0.3571	4.9462	2.4843	4.3573	5.3051	0.2281	0.6940
$\hat{\beta}_3$	0.4333	5.8335	3.0292	5.1779	6.2290	0.2680	0.5398

4. Conclusion

The OLS and the robust estimators could not perform well in the presence of multicollinearity and outlier. The estimators could not correctly estimate the regression coefficients. The performance of OLS, LAD and LMS were close because of the presence of bad leverages. M estimation is a commonly used method for outlier detection and robust regression when contamination is mainly in the response direction. Its performance cannot be compared with the high breakdown value estimators (LTS, S, and M). These high breakdown estimators are good estimators especially when data sets have bad leverages. In this study, the performance of LTS, S and MM were not statistically different in terms of their scale ($\hat{\sigma}$) and MSE(β). The mean square error revealed that ridge regression estimated based on OLS is the least compared to ridge regression based on the robust estimators. Ridge estimates based on LTS perform better than all other estimators followed by Ridge estimates based on MM and S since they have the smallest mean square error in their order. The other ridge estimates based on M, LAD and LMS also perform better than the ridge estimates based on OLS. In conclusion, it has been seen that when data set exhibit both problem of multicollinearity and outlier the robust ridge regression estimator are better than OLS and the counterparts ridge or robust estimators.

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