

PERIODIC QUOTIENTS OF HYPERBOLIC AND LARGE GROUPS

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ABSTRACT. Let G be either a non-elementary (word) hyperbolic group or a large group (both in the sense of Gromov). In this paper we describe several approaches for constructing continuous families of periodic quotients of G with various properties.

The first three methods work for any non-elementary hyperbolic group, producing three different continua of periodic quotients of G . They are based on the results and techniques, that were developed by Ivanov and Olshanskii in order to show that there exists an integer n such that G/G^n is an infinite group of exponent n .

The fourth approach starts with a large group G and produces a continuum of pairwise non-isomorphic periodic residually finite quotients.

Speaking of a particular application, we use each of these methods to give a positive answer to a question of Wiegold from Kurovka Notebook.

1. INTRODUCTION

Let G be an infinite finitely generated group. For $n \in \mathbb{N}$, by G^n we denote the (normal) subgroup generated by the n -th powers of elements of G . Recall that G is called *periodic* if each element of G has a finite order. The group G is said to be of *bounded exponent* if there is $n \in \mathbb{N}$ such that $x^n = 1$ for all $x \in G$, or, in other words, $G^n = \{1\}$; the least integer n for which this equality holds is called the *exponent* of G . If for a periodic group G such n does not exist, it is said to be *periodic of unbounded exponent*. A group is called *elementary* if it has a cyclic subgroup of finite index.

The question whether the quotient group G/G^n (which is of bounded exponent n) can be infinite has a long history in Group Theory. In the case when $G = F_m$ is a non-abelian free group, it was asked by W. Burnside in 1902 [8] and answered in the affirmative much later by P. Novikov and S. Adian [30] (see also [1]) for any sufficiently large odd exponent n . Yet later, A. Olshanskii [31] suggested a different, geometric approach to this problem, and, afterwards, in [34], applied similar methods to show that for every non-elementary torsion-free hyperbolic (in the sense of M. Gromov [18]) group G there exists an integer $n \in \mathbb{N}$ such that G/G^n is infinite. In [19] S. Ivanov proved that F_m/F_m^n is infinite for any sufficiently large even exponent n divisible by 2^9 (infiniteness of groups F_m/F_m^n for large even values

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of n was also proved by I. Lysënok [27]). Finally, in [20] Ivanov and Olshanskii combined the techniques from [34] and [19] and obtained the following theorem:

Theorem A. *For every non-elementary hyperbolic group G there exists a positive even integer $n = n(G)$ such that the following are true:*

- (a) *The quotient group G/G^n is infinite.*
- (b) *Suppose $n = n_1 n_2$, where n_1 is odd and n_2 is a power of 2. Then every finite subgroup of G/G^n is isomorphic to an extension of a finite subgroup K of G by a subgroup of the direct product of two groups one of which is a dihedral group of order $2n_1$ and the other is a direct product of several copies of a dihedral group of order $2n_2$.*

Remark 1. As it was shown in [20, Sec. 3], the exponent n from the previous statement can be chosen to be any sufficiently large integer which is divisible by $2^{k_0+5}n_0$, where

- a) $n_0/2$ is the least common multiple of the exponents of the holomorphs $Hol(K)$ over all finite subgroups K of G ;
- b) k_0 is the minimal integer with $2^{k_0-3} > \max |K|$ over all finite subgroups K of G .

The question which now naturally arises concerns the number (up to isomorphism) of periodic quotients for a fixed G .

In [32] the second author of the present article proved that there exists a continuum of 2-generated non-isomorphic groups with the property that every proper subgroup is cyclic of prime order. In [10] G. Deryabina showed that for every odd prime p there is continuum of non-isomorphic simple groups, generated by two elements, such that all their proper subgroups are cyclic p -groups.

First examples of infinite finitely generated periodic residually finite groups were found by E. Golod in [13]. The techniques that he used allow, in fact, to obtain uncountably many of such groups (we thank L. Bartholdi for pointing this out). Explicitly, constructions of continuous families of finitely generated periodic residually finite groups first appear in the papers [14],[15] of R. Grigorchuk (see also [29, 2.10.5]).

Speaking of periodic groups of bounded exponent, one should mention a result of V. Atabekyan [3], who demonstrated that the free group F_2 , of rank 2, possesses a continuum of pairwise non-isomorphic simple quotient-groups of given period n for each sufficiently large odd n . The third author of the present article established an analogous result in the situation when n is a sufficiently large even integer divisible by 2^9 (see [38]).

The first objective of this paper is to present several ways in which one can construct periodic quotients of a given non-elementary hyperbolic group G . We suggest three different methods each of which gives a continuum of such non-isomorphic quotients.

The second and the third approaches rely heavily on the techniques and statements used by Ivanov and Olshanskii in their proof of Theorem A in [20]. The first approach also uses their result but in a less thorough manner. These approaches are used to produce the following three theorems. (Recall that a group H is said to be a *central extension* of the group K if there exists a subgroup $N \triangleleft H$, contained in the center $Z(H)$ of H , such that $H/N \cong K$.)

Theorem 1. *Let G be a non-elementary hyperbolic group and let an integer n be chosen according to Remark 1. Then G possesses a continuum of pairwise non-isomorphic periodic quotients Q_{1i} , $i \in I_1$, of exponent $2n$, each of which is a central extension of the group G/G^n .*

Theorem 2. *Let G be a non-elementary hyperbolic group and let an integer n be chosen according to Remark 1. Then G possesses a continuum of pairwise non-isomorphic periodic quotients Q_{2i} , $i \in I_2$, of unbounded exponent, such that for every $i \in I_2$ the order of an arbitrary element $g \in Q_{2i}$ divides $2^l n$ for some $l = l(g) \in \mathbb{N}$.*

Every non-elementary hyperbolic group G contains a unique maximal finite normal subgroup $E(G)$ ([33, Prop. 1]). The quotient $\hat{G} = G/E(G)$, in addition to being non-elementary and hyperbolic, also satisfies $E(\hat{G}) = \{1\}$.

Theorem 3. *Assume that G is a non-elementary hyperbolic group and denote $\hat{G} = G/E(G)$. Choose a positive integer \hat{n} arising after an application of Remark 1 to \hat{G} . Then G has a continuum of pairwise non-isomorphic centerless quotients of exponent $2\hat{n}$.*

Observe that if $Z(K) = \{1\}$ and H_1, H_2 are two central extensions of K then $H_1/Z(H_1) \cong K \cong H_2/Z(H_2)$.

We will show that if the original group G is centerless, then so is the group G/G^n . Thus the collections of quotients produced by Theorem 1 and Theorem 3 are significantly different because for any $i, j \in I_2$ one has $Q_{1i}/Z(Q_{1i}) \cong Q_{1j}/Z(Q_{1j})$.

Our second objective is to obtain an analogue of the above results for large groups. Recall that a group G is said to be *large* if it has a normal subgroup $N \triangleleft G$ of finite index, for which there exists an epimorphism $\varphi : N \rightarrow F$, where F is a non-abelian free group. A recent result of M. Lackenby [25] asserts that if G is a finitely generated large group and $g_1, \dots, g_r \in G$, then the quotient $G/\langle\langle g_1^n, \dots, g_r^n \rangle\rangle$ is large for infinitely many $n \in \mathbb{N}$. Lackenby's proof is topological. A different, combinatorial, proof of this result (not requiring the large group to be finitely generated) was found by A. Olshanskii and D. Osin in [37]; we will employ their methods to prove Theorem 4 below.

Theorem 4. *Suppose that G is a finitely generated group having a normal subgroup N , of finite index, that maps onto a non-abelian free group. Let p be an arbitrary prime number. Then G possesses 2^{\aleph_0} of periodic residually finite pairwise non-isomorphic quotients H_j , $j \in J$, such that for every $j \in J$, the natural image of N in H_j is a p -group.*

The output of the previous theorem is very different from the continuous families given by Theorems 1,2,3 as it produces periodic quotients with the additional property of residual finiteness. We recall that an infinite periodic finitely generated group of bounded exponent cannot be residually finite due to E. Zelmanov's positive solution of the restricted Burnside problem [39]-[40]. In the authors' opinion, the periodic quotients from the claim of Theorem 2 are not very likely to be residually finite. For instance, if G is a non-elementary hyperbolic group possessing no proper finite index subgroups (it was shown I. Kapovich and D. Wise [22] and, independently, by the second author [35], that the existence of such a group is equivalent to the existence of a non-residually finite hyperbolic group, which is a well-known open problem), then no non-trivial quotient of G can be residually finite. Thus an

application of Theorem 2 to G will produce a continuum of non-(residually finite) periodic groups.

Let G be a finitely generated group and $p \in \mathbb{N}$ be a prime number. Define a series of finite index characteristic subgroups $\delta_i^p(G)$ of G by $\delta_0^p(G) = G$ and

$$\delta_i^p(G) = [\delta_{i-1}^p(G), \delta_{i-1}^p(G)] (\delta_{i-1}^p(G))^p \leq \delta_{i-1}^p(G), \quad i \in \mathbb{N}.$$

If G is finitely generated, then for every prime p and $t \in \mathbb{N}$, the number $|G/\delta_t^p(G)|$ is a natural invariant of G . Another point in which the fourth approach differs from the others, is that it uses these invariants in order to distinguish between the quotients of G . Thus we show that for every prime p , there exist continually many finitely generated residually finite p -groups that can be distinguished via natural inner algebraic invariants.

Let $\Omega = \{0, 1\}^{\mathbb{N}}$ be the set of all infinite sequences of 0's and 1's. In Section 8 we prove

Corollary 1. *Assume that G is a finitely generated group, $N \triangleleft G$ is a normal subgroup mapping onto F_2 and G/N is a finite p -group for some prime $p \in \mathbb{N}$. Then for every $\omega \in \Omega$, G has quotient-group H_ω such that*

- (a) H_ω is a periodic p -group;
- (b) $\bigcap_{i=0}^{\infty} \delta_i^p(H_\omega) = \{1\}$, thus H_ω is residually finite;
- (c) if $\omega \neq \omega' \in \Omega$ then there is $t \in \mathbb{N}$ such that $|H_\omega/\delta_t^p(H_\omega)| \neq |H_{\omega'}/\delta_t^p(H_{\omega'})|$; consequently $H_\omega \not\cong H_{\omega'}$.

Finally, after combining Corollary 1 with a theorem of B. Baumslag and S. Pride [5], we deduce

Corollary 2. *Assume that a group G admits a presentation with $k \geq 2$ generators and at most $k - 2$ relators. Then for any prime p , G has a collection of quotients $\{H_\omega \mid \omega \in \Omega\}$ such that*

- H_ω is a periodic p -group for every $\omega \in \Omega$;
- H_ω is residually finite for every $\omega \in \Omega$;
- if $\omega \neq \omega' \in \Omega$ then there is $t \in \mathbb{N}$ such that $|H_\omega/\delta_t^p(H_\omega)| \neq |H_{\omega'}/\delta_t^p(H_{\omega'})|$; consequently $H_\omega \not\cong H_{\omega'}$.

2. A PARTICULAR APPLICATION

Before proving Theorems 1-4 we are going to consider one application. We will show that each of these theorems can be used to answer the following question of J. Wiegold :

Question. ([24, 16.101]) Does the group $G = \langle x, y \mid x^2 = y^4 = (xy)^8 = 1 \rangle$ have uncountably many quotients each of which is a 2-group?

As mentioned by Wiegold, the group G can be homomorphically mapped onto a 2-group Γ that is a subgroup of index 2 in the first Grigorchuk's group \mathcal{G} (introduced in [17]). It is known that every proper quotient of \mathcal{G} is finite [16], in other terms, \mathcal{G} is *just infinite*. In a private communication with the first author, R. Grigorchuk asserted that a similar property holds for Γ . Thus, the group Γ has only countably many distinct quotients. (More generally, L. Bartholdi has recently proved that

every finitely generated subgroup of Grigorchuk's group \mathcal{G} has at most countably many distinct quotients [4].)

In order to apply our approaches to the group G , we need to observe that G is a subgroup of index 2 in the group

$$T = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, (ab)^2 = (bc)^4 = (ac)^8 = 1 \rangle.$$

And since $1/2 + 1/4 + 1/8 < 1$, T is a reflection group of isometries of the hyperbolic plane \mathbb{H}^2 , whose fundamental domain is a triangle with angles $\pi/2, \pi/4$ and $\pi/8$. In fact, G consists of all isometries from T preserving the orientation on \mathbb{H}^2 (see [11, 7.3.1]).

Hence the group G acts properly discontinuously and cocompactly on \mathbb{H}^2 , therefore it is non-elementary (word) hyperbolic. It is well-known that every finite subgroup of T lies inside of a finite parabolic subgroup [7, Ex. 2d, p. 130]. Hence, it must be a subgroup of a dihedral group $D(2k)$, of order $2k$, where $k = 2, 4$ or 8 . Since G contains no reflections, every finite subgroup of G will be cyclic of order dividing 8. Now, if K is a cyclic group of order 2^m , $m \in \mathbb{N}$, then $|Aut(K)| = 2^{m-1}$, hence $Aut(K) = K \rtimes Aut(K)$ has order 2^{2m-1} . By Remark 1, the exponent n from the assumptions of Theorems 1 and 2 can be chosen as a sufficiently large power of 2. Therefore both of these theorems give an affirmative answer to Wiegold's question.

By a classical theorem of E. Cartan [9, p. 267] (see also [6, Ch. II.2, Cor. 2.8]) every finite subgroup K of isometries of \mathbb{H}^2 fixes a non-empty subset that is convex in a strong sense: for any two distinct points fixed by K , K will fix the entire bi-infinite geodesic in \mathbb{H}^2 passing through these points. The fixed-point set of a normal subgroup is invariant under the action of the entire group, hence the fixed-point set $Fix(E(G))$ of $E(G)$ cannot consist of a single point (because G is infinite and acts properly discontinuously). By the same reason, it cannot be a bi-infinite geodesic. Therefore, $E(G)$ fixes at least three points in general position, hence $Fix(E(G)) = \mathbb{H}^2$. And since the action of G is faithful, one can conclude that $E(G) = \{1\}$. Thus Theorem 3 will produce a continuum of pairwise non-isomorphic centerless quotients of G of exponent $2n$ (for the same n as above).

It remains to show that Theorem 4 can also be applied to G . It is well-known that hyperbolic triangle groups, such as G , are large ([11, 7.3.1]). For our purposes, however, we will need to exhibit a stronger property, namely, the existence of a normal subgroup $N \triangleleft G$, mapping homomorphically onto a non-abelian free group, such that $|G : N| = 2^l$ for some $l \in \mathbb{N}$.

Denote by $A = \langle x \rangle_2 * \langle y \rangle_4$ the free product of cyclic groups of orders 2 and 4. Let B be the cartesian subgroup of A , i.e., the kernel of the natural homomorphism from A onto the direct product $\langle x \rangle_2 \times \langle y \rangle_4$. The cartesian subgroup of a free product is a free group (this follows from Kurosh subgroup theorem and the fact that the cartesian subgroup trivially intersects the free factors). Note that B is freely generated by its elements $[x, y], [x, y^2]$ and $[x, y^3]$ (where $[u, v] = uvu^{-1}v^{-1}$). Set $C = B^2$ to be the normal subgroup of A generated by the squares of elements of B . Then $|B : C| = 8$, $(xy)^4 = [x, y][x, y^2]^{-1}[x, y^3] \in B \setminus C$ and $(xy)^8 \in C$. By Schreier's formula, C is a free group of rank $(3-1)8+1 = 17$. Since $|A : \langle xy \rangle C| = 8$, we can use [37, Lemma 2.3] to show that $D = \langle\langle (xy)^8 \rangle\rangle^A \triangleleft C$ is a normal closure in C of at most 8 elements. Hence the subgroup $E = C/D \triangleleft A/D = G$ has a presentation with 17 generators and 8 relators. Since $17 - 8 \geq 2$, by a result of Baumslag and Pride [5], for every sufficiently large $k \in \mathbb{N}$, E has a normal subgroup N' , of index

k , such that N' can be mapped onto a non-abelian free group. Of course, one can take $k = 2^{l'}$ for some $l' \in \mathbb{N}$. Observe that $N = \bigcap_{g \in G} gN'g^{-1} \triangleleft G$ has finite index in N' , and, therefore, it can also be mapped onto non-abelian free group. It remains to note that E/N is a finite 2-group because $gN'g^{-1} \triangleleft E$ and $|E : gN'g^{-1}| = 2^{l'}$, for every $g \in G$. Consequently $|G : N| = |G : E| \cdot |E : N| = 2^6 \cdot |E : N| = 2^l$ for some $l \in \mathbb{N}$.

Therefore an application of Theorem 4 to G and N provides a continuum of pairwise non-isomorphic residually finite quotients of G each of which is a 2-group.

One can now see that Wiegold's problem mentioned above can be solved in a number different ways. In the course of writing this paper, R. Grigorchuk informed the authors of yet another possible method to solve this problem, using the techniques developed in his paper [14].

3. PRELIMINARIES

Fix a group G with a finite symmetrized generating set \mathcal{A} . If $g \in G$, $|g|_{\mathcal{A}}$ will denote the length of a shortest word W over \mathcal{A} representing g in G . This gives rise to the standard left-invariant distance function $d(\cdot, \cdot)$ on G defined by $d(x, y) = |x^{-1}y|_{\mathcal{A}}$ for any $x, y \in G$. Afterwards this can be extended to the metric $d(\cdot, \cdot)$ on the Cayley graph $\Gamma(G, \mathcal{A})$ in the usual way. For subset a Q of $\Gamma(G, \mathcal{A})$, the closed ε -neighborhood is defined by

$$\mathcal{N}_{\varepsilon}(Q) = \{x \in \Gamma(G, \mathcal{A}) \mid \exists y \in Q \text{ such that } d(x, y) \leq \varepsilon\}.$$

For any two points $x, y \in \Gamma(G, \mathcal{A})$, $[x, y]$ will denote a geodesic segment between them.

Fix a number $\delta \geq 0$. A geodesic n -gon in $\Gamma(G, \mathcal{A})$ is called δ -slim if each of its sides is contained in the closed δ -neighborhood of the others. Using the definition of E. Rips, we will say that G is δ -hyperbolic if every geodesic triangle in its Cayley graph is δ -slim. As a consequence of this definition, it is easy to obtain that each geodesic quadrilateral in $\Gamma(G, \mathcal{A})$ is 2δ -slim. Further on we shall assume that the group G is non-elementary and δ -hyperbolic for some given $\delta \geq 0$. There is a number of other (equivalent up to changing δ) definitions of δ -hyperbolicity – see [2], for example. Since some of the Lemmas quoted below utilize them, we will suppose that our δ is sufficiently large so that G also satisfies all of these other definitions.

The length of a word W over the alphabet \mathcal{A} will be denoted by $\|W\|$. Suppose W represents an element $g \in G$. Then we define $|W|_{\mathcal{A}} = |g|_{\mathcal{A}}$. If p is a path in the Cayley graph, then p_- , p_+ and $\|p\|$ will denote its starting point, ending point and the length respectively. In the case when p is a simplicial path, $lab(p)$ will stand for the word written on p and p^{-1} – for the inverse path to p , that is, $p_-^{-1} = p_+$, $p_+^{-1} = p_-$ and $lab(p^{-1}) \equiv lab(p)^{-1}$. Given some numbers $\bar{\lambda}, \bar{c}$ satisfying $0 < \bar{\lambda} \leq 1$, $\bar{c} \geq 0$, we will say that p is $(\bar{\lambda}, \bar{c})$ -quasigeodesic if for any subpath q of p one has $\bar{\lambda}\|q\| - \bar{c} \leq d(q_-, q_+)$. A word W will be called $(\bar{\lambda}, \bar{c})$ -quasigeodesic provided some (equivalently, any) path p , with $lab(p) \equiv W$, is $(\bar{\lambda}, \bar{c})$ -quasigeodesic in $\Gamma(G, \mathcal{A})$.

The next two lemmas are standard properties of a hyperbolic metric space:

Lemma 1. ([12, 5.6, 5.11], [2, 3.3]) *There is a constant $\nu = \nu(\delta, \bar{\lambda}, \bar{c})$ such that for any $(\bar{\lambda}, \bar{c})$ -quasigeodesic path p in $\Gamma(G, \mathcal{A})$ and a geodesic q with $p_- = q_-$, $p_+ = q_+$, one has $p \subset \mathcal{N}_{\nu}(q)$ and $q \subset \mathcal{N}_{\nu}(p)$.*

Lemma 2. ([28, Lemma 4.1]) Consider a geodesic quadrilateral $x_1x_2x_3x_4$ in the Cayley graph $\Gamma(G, \mathcal{A})$ with $d(x_2, x_3) > d(x_1, x_2) + d(x_3, x_4)$. Then there are points $u, v \in [x_2, x_3]$ such that $d(x_2, u) \leq d(x_1, x_2)$, $d(v, x_3) \leq d(x_3, x_4)$ and the geodesic subsegment $[u, v]$ of $[x_2, x_3]$ lies 2δ -close to the side $[x_1, x_4]$.

It is well known (see, for example, [33]) that in a hyperbolic group G every element g of infinite order is contained in a unique maximal elementary subgroup $E_G(g)$, and

$$(1) \quad E_G(g) = \{x \in G \mid xg^m x^{-1} = g^n \text{ for some } m, n \in \mathbb{Z} \setminus \{0\}\} = \\ \{x \in G \mid xg^n x^{-1} = g^{\pm n} \text{ for some } n \in \mathbb{N}\}.$$

The subgroup $E_G^+(g) = \{x \in G \mid xg^n x^{-1} = g^n \text{ for some } n \in \mathbb{N}\}$ has index at most 2 in $E_G(g)$.

4. CENTRAL EXTENSIONS

Fix a presentation $\langle \mathcal{A} \parallel R \in \mathcal{R}_0 \rangle$ of the initial non-elementary hyperbolic group $G = G(0)$, where the set of generators \mathcal{A} is finite and \mathcal{R}_0 is the set of all relators in G , that is, \mathcal{R}_0 consists of all words from over the alphabet $\mathcal{A}^{\pm 1}$ that represent the identity element in G .

Choose the exponent n by Theorem A and consider the group $G(\infty) = G/G^n$.

Let us recall a few things from [20]. It is shown that

$$(2) \quad G(\infty) = \langle \mathcal{A} \parallel R, R \in \mathcal{R}_0, A_1^n, A_2^n, \dots \rangle,$$

where the word A_j , $j \in \mathbb{N}$, is called the *period of rank j*. Moreover, each group of rank i

$$G(i) \stackrel{\text{def}}{=} \langle \mathcal{A} \parallel R, R \in \mathcal{R}_0, A_1^n, A_2^n, \dots, A_i^n \rangle$$

is also non-elementary hyperbolic. For every $i \in \mathbb{N}$, the word A_i has infinite order in $G(i-1)$ and there exists a unique maximal finite subgroup $\mathcal{F}(A_i)$ which is normalized by A_i in $G(i-1)$ (more precisely, $\mathcal{F}(A_i)$ is the torsion subgroup of $E_{G(i-1)}^+(A_i)$). One of the properties, proved in [20], states that $A_i^{n/2} T A_i^{-n/2} T^{-1} = 1$ in $G(i-1)$ for any $i \in \mathbb{N}$ and any word T representing an element of $\mathcal{F}(A_i)$ (see the adaptation of Lemma 18.5 from [19] in [20, Section 14]).

A word J is called an $\mathcal{F}(A_i)$ -involution if J normalizes the subgroup $\mathcal{F}(A_i)$ of $G(i-1)$, $J^2 \in \mathcal{F}(A_i)$ and $J^{-1} A_i J = A_i^{-1} T$ for some $T \in \mathcal{F}(A_i)$. By Lemma 19.2 [20],[19], $A_i^{n/2} J A_i^{n/2} J^{-1} = 1$ in the group $G(i-1)$. If for a given $i \in \mathbb{N}$ there exists an $\mathcal{F}(A_i)$ -involution, the period A_i is said to be *even*; otherwise, A_i is said to be *odd*. Equivalently, A_i is odd if and only if $E_{G(i-1)}(A_i) = E_{G(i-1)}^+(A_i)$.

Consider a diagram Δ over the presentation (2) on some orientable surface S (say, a plane or a sphere). Then one can fix the same (say, clockwise) direction of the contours of each cell in Δ . If Π is a cell corresponding to the relation $A_j^n = 1$, $j \in \mathbb{N}$, one defines its rank $r(\Pi)$ to be j ; by definition, $r(\Pi) = 0$ if Π corresponds to a relator $R \in \mathcal{R}_0$. A vertex $o \in \partial\Pi$ is called a *phase vertex* if, starting at o and going along the contour of Π in the clockwise direction, we read the word $A_j^{\pm n}$. The number $r(\Delta) = \max\{r(\Pi) \mid \Pi \in \Delta\}$ is called the *strict rank* of the diagram Δ . If the strict rank of Δ is i , the *type* $t(\Delta)$ of Δ is a sequence $(\tau_i, \tau_{i-1}, \dots, \tau_0)$ where τ_j is the number of cells of rank j in Δ . For the convenience of inductive reasoning we will impose the *short-lex* order on the set of all types.

Let Π_k , $k = 1, 2$, be two distinct cells of Δ of rank j . The pair of cells Π_1 and Π_2 is said to be a *reducible j-pair* if there is a simple path t in the diagram Δ , connecting two phase vertices o_1 and o_2 on the their boundary contours, such that one of the following holds:

- 1) The period A_j is even and the words written on $\partial\Pi_1$ and $\partial\Pi_2$ starting with the vertices o_1 and o_2 are the same (i.e., $A_j^{\pm n}$), and $lab(t)$ is an $\mathcal{F}(A_j)$ -involution.
- 2) The words written on $\partial\Pi_1$ and $\partial\Pi_2$ starting with o_1 and o_2 are mutually inverse (that is, they are either A_j^n and A_j^{-n} or A_j^{-n} and A_j^n respectively), and $lab(t)$ represents an element of $\mathcal{F}(A_j)$ in $G(j - 1)$.

In the case when the original group G is free, the following statement can be compared with [36, Lemma 5.2].

Lemma 3. *Let $(\tau_i, \tau_{i-1}, \dots, \tau_0)$ be the type of an arbitrary spherical diagram Δ over the presentation (2). Then for and any $k \in \{1, 2, \dots, i\}$ the integer τ_k is even.*

Proof. Fix $k \in \{1, \dots, i\}$ and use induction on $t(\Delta)$. If Δ has no cells of positive rank the claim is trivial. Otherwise, the adaptation of [19, Lemma 6.2] described in Section 7 of [20] claims that Δ has a reducible j -pair of cells Π_1 and Π_2 for some $1 \leq j \leq i$. Let t denote the corresponding simple path between some phase vertices o_1 and o_2 of the contours $\partial\Pi_1$ and $\partial\Pi_2$, and set $T \equiv lab(t)$. Denote by Γ the subdiagram of Δ consisting of Π_1 , Π_2 and t . Then the word written on the boundary of Γ starting o_1 is $A_j^{\epsilon n} T A_j^{\hat{\epsilon} n} T^{-1}$ where $\epsilon, \hat{\epsilon} \in \{1, -1\}$. By the definition of a reducible j -pair and the properties, mentioned above, in $G(j - 1)$ we have

$$A_j^{\epsilon n/2} T A_j^{\hat{\epsilon} n/2} T^{-1} = 1.$$

By van Kampen's Lemma there exists a disk diagram Γ' over the presentation (2) with $r(\Gamma') \leq j - 1$ whose boundary label is $A_j^{\epsilon n/2} T A_j^{\hat{\epsilon} n/2} T^{-1}$. Gluing Γ' with a copy of itself along the two different occurrences of T on their boundaries one obtains a new disk diagram Γ'' (with $r(\Gamma'') = r(\Gamma')$), whose boundary label is letter-by-letter equal to $A_j^{\epsilon n} T A_j^{\hat{\epsilon} n} T^{-1}$ which is the boundary label of Γ . Obviously, the number χ_k of cells of rank k in Γ'' is twice that number for Γ' , and, hence, it is even.

Now one can perform a standard diagram surgery, by cutting Γ out of Δ and replacing it with Γ'' , to obtain a new spherical diagram Δ' . Since $t(\Delta') < t(\Delta)$ we can use the induction hypothesis to show that the number ψ_k of cells of rank k in Δ' is even. Due to the construction, one has $\tau_k = \psi_k - \chi_k + 2$ (if $k = j$) or $\tau_k = \psi_k - \chi_k$ (if $k \neq j$). In either case τ_k will be even. \square

Let F be the free group on \mathcal{A} and let N_0 and N denote the kernels of the natural homomorphisms $F \rightarrow G$ and $F \rightarrow G(\infty)$ respectively. The *mutual commutator* $[F, N]$ is the normal subgroup of F generated by all commutators of the form $[x, y]$ where $x \in F$ and $y \in N$.

Lemma 4. *Suppose the word $W \equiv A_{i_1}^{n\tau_1} A_{i_2}^{n\tau_2} \dots A_{i_s}^{n\tau_s}$, $1 \leq i_1 < i_2 < \dots < i_s$, represents an element of the subgroup $[F, N]N^2N_0$ in F . Then the integers τ_1, \dots, τ_s are all even.*

Proof. Clearly, by definitions of the subgroups $[F, N]$ and N^2 , one can write W as a product $\prod_t U_t A_{j_t}^{\pm n} U_t^{-1} h$ where $h \in N_0$, and for every l the number of occurrences of $A_l^{\pm n}$ in this product is even. Therefore, as in the proof of van Kampen's Lemma

(see, for example, [26]), one can construct a disk diagram Γ over the presentation (2), whose boundary contour is labelled by W and which has an even number ψ_l of cells of every positive rank l . On the other hand, the equality from the assumptions of the lemma gives rise to a diagram Δ_2 over the presentation (2) with the same boundary contour and having exactly τ_k cells of rank i_k for each $k = 1, 2, \dots, s$. Gluing together Δ_1 with a mirror copy of Δ_2 along their boundary contours, one will obtain a spherical diagram Δ_3 . Applying Lemma 3 to Δ_3 one gets that the number $(\tau_k + \psi_{i_k})$ of the cells of rank i_k in it is even. Therefore τ_k must also be even. \square

Set $L = G^n \triangleleft G$ and $M = [G, L]L^2 \triangleleft G$; the full preimages of these subgroups in F are N and $[F, N]N^2N_0$ respectively. Then the group G/M is a central extension of the group $G(\infty) = G/L$ by the central subgroup L/M . The analogue of the next theorem in the case when G is a free group of finite rank was proved in [36, Thm. 5].

Corollary 3. *The group L/M is a direct product of the groups of order 2 generated by the natural images of A_j^n , $j \in \mathbb{N}$.*

Proof. Evidently, L/M is an abelian group generated by the images \hat{a}_j of A_j^n , $j \in \mathbb{N}$, and these images have order at most 2 in it. Now, Lemma 4 asserts that if an element of the form $\hat{a}_{i_1}^{\tau_1} \cdots \hat{a}_{i_s}^{\tau_s}$ is trivial in L/M , then τ_k is even for each $k = 1, \dots, s$. Hence L/M is a direct product of the subgroups $\langle \hat{a}_j \rangle$. \square

The following observation is a consequence of the fact that there can be at most countably many different homomorphisms from a given finitely generated group G to a fixed countable group.

Remark 2. Let G be a finitely generated group, I be a set of cardinality continuum and $\{N_i\}_{i \in I}$ be a family of normal subgroups of G such that $N_i \neq N_j$ whenever $i \neq j$. Then the set of quotients $\{G/N_i\}_{i \in I}$ contains continually many pairwise non-isomorphic groups.

Proof of Theorem 1. Clearly, the order of any element $g \in G/M$ divides $2n$. It is known that an infinite hyperbolic group always contains an element of infinite order ([12, 8.3.36]), hence $G(\infty) \neq G(i)$ for every $i \in \mathbb{N}$. Consequently the set of periods $\{A_i\}$ is infinite. Thus, according to Theorem 3, the abelian group L/M has continually many distinct subgroups. As L/M is contained in the center of the group $H = G/M$, H also has a continuum of different central normal subgroups Z_i , $i \in I$. Note that since G is finitely generated, then so is H . By Remark 2, there is a continuum of non-isomorphic groups among the quotients $\{H/Z_i \mid i \in I\}$, all of which are central extensions of $G(\infty)$. Therefore, the statement of the theorem is true. \square

5. QUOTIENTS OF UNBOUNDED EXPONENT

In the present section we will prove Theorem 2. Suppose that the non-elementary hyperbolic group G has a presentation $\langle \mathcal{A} \parallel R \in \mathcal{R}_0 \rangle$, where \mathcal{A} is finite and \mathcal{R}_0 is the set of all relators in G .

By [20, Lemma 18], there is a constant $k_0 \in \mathbb{N}$ such that $2^{k_0-3} > \max |K|$ for any finite subgroup K of G . By n_0 denote a constant such that $n_0/2$ is the least common multiple of the exponents of holomorphs $Hol(K)$ over all finite subgroups K of G .

Let us recall the construction of the quotient G/G^n from [20], where n is a large integer divisible by $2^{k_0+5}n_0$.

Introduce a total order \prec on the set of words in the alphabet \mathcal{A} so that $\|X\| < \|Y\|$ implies $X \prec Y$. Set $G(0) = G$ and define groups $G(i)$ by induction on i .

As a matter of notational convenience, in what follows we shall often identify a word W over the alphabet \mathcal{A} with an element that it represents in $G(0)$. For a word W , representing an element of infinite order $w \in G(0)$, denote by $F(W)$ the maximal finite subgroup of $E^+(W)$. Such a W is called *simple* in $G(0)$ (see [20, Sec. 4]) if its coset in $E_G(W)/F(W)$ generates the cyclic subgroup $E_G^+(W)/F(W)$ and none of the conjugates of elements $W^{\pm 1}F$ (where the word F represents an element from $F(W)$) has length less than $\|W\|$.

Assuming that the group $G(i-1)$, $i \geq 1$ is already defined, consider the least (with respect to the order \prec) word A of infinite order in $G(i-1)$, with additional requirement that A is simple in $G(0)$ if $\|A\| < C$ (see [20, Lemma 13] for the meaning of parameter C). Declare such a word A to be the period A_i of rank i , and define the group $G(i)$ by imposing the relation $A_i^n = 1$ on the group $G(i-1)$:

$$G(i) = \langle \mathcal{A} \parallel R \in \mathcal{R}_0 \cup \{A_1^n, A_2^n, \dots, A_i^n\} \rangle.$$

The period A_i exists for every $i \geq 1$, and the group G/G^n is the direct limit of groups $G(i)$:

$$G/G^n = G(\infty) = \langle \mathcal{A} \parallel R \in \mathcal{R}_0 \cup \{A_1^n, A_2^n, \dots\} \rangle.$$

The construction of Ivanov and Olshanskii [20] may be modified in the following way. Instead of imposing on each step a relation $A^n = 1$ for a fixed exponent n , we will introduce periodic relations with large exponent that may vary from step to step. In the case when $G(0)$ a non-abelian free group such modification has already appeared in [36].

Given a sequence $\omega = (\omega_j)_{j=1}^\infty$ of 0's and 1's, we modify the choice of defining relations for the groups $G(i)$ from [20] as follows.

Set $n(0) = n$ and, for $j \geq 1$, define $n(j) = (1 + \omega_j)n(j-1)$. Inductively define groups $G_\omega(i)$: set $G_\omega(0) = G$ and assuming that the period A_i of rank i is chosen, define $G_\omega(i)$ to be the quotient of $G_\omega(i-1)$ by the normal closure of $A_i^{n(i)}$:

$$G_\omega(i) = \langle \mathcal{A} \parallel R \in \mathcal{R}_0 \cup \{A_1^{n(1)}, A_2^{n(2)}, \dots, A_i^{n(i)}\} \rangle.$$

The direct limit of groups $G_\omega(i)$ with respect to canonical epimorphisms is denoted by $G_\omega(\infty)$.

Note that for any sequence ω of 0's and 1's the sequence $(n(j))$ is non-decreasing and $n(j) \geq n$ for every $j \geq 0$. We now refer to [36], and modify the arguments from [20] in a way similar to the one given in [36]. Namely, the order of the period A_i in $G_\omega(\infty)$ is now $n(i)$ instead of n ; in all the estimates, the terms $\|A_i^n\|$ and $n\|A_i\|$ are substituted by $\|A_i^{n(i)}\|$ and $n(i)\|A_i\|$ respectively; finite subgroups of $G_\omega(i)$ are isomorphically embedded into a direct product of a direct power of the dihedral group $D(2n(i))$ (instead of $D(2n)$) and an elementary associated with G group (see [20]); n is replaced by $n(i)$ in the group identities of the analogue of Lemma 15.10 [20],[19]; in the inductive step from $G_\omega(i)$ to $G_\omega(i+1)$ (in arguments from [19, §§ 18,19] and their analogues from [20]) n is replaced by $n(i+1)$.

After all of these modifications, we obtain that for any sequence ω the period A_i exists for every $i \geq 1$ and the group $G_\omega(\infty)$ is infinite and periodic.

Let ω, ω' be two different sequences of 0's and 1's. Then the kernels of the canonical homomorphisms $G \rightarrow G_\omega(\infty)$ and $G \rightarrow G_{\omega'}(\infty)$ are different. Indeed, let j be the smallest index where ω and ω' differ, say $\omega_j = 0, \omega'_j = 1$. We remark here that for all $i, 0 \leq i < j$, the sets of defining words of the groups $G_\omega(i)$ and $G_{\omega'}(i)$ coincide in view of minimality of j and the fact that the order \prec was fixed a-priori. This means, in turn, that the same word A_j is the period of rank j in both $G_\omega(\infty)$ and $G_{\omega'}(\infty)$. By the analogue of Lemma 10.4 [20] for $G_\omega(\infty)$ (for $G_{\omega'}(\infty)$), the order of the period A_j in $G_\omega(\infty)$ (in $G_{\omega'}(\infty)$) is equal to $n(j)$ ($2n(j)$ respectively).

Thus, the groups $G_\omega(\infty)$ and $G_{\omega'}(\infty)$ are quotients of the finitely generated group G by different normal subgroups provided $\omega \neq \omega'$. Remark 2 permits us to conclude that the set of pairwise non-isomorphic groups among $\{G_\omega(\infty)\}_\omega$ is of cardinality continuum.

6. APERIODIC ELEMENTS IN HYPERBOLIC GROUPS

The purpose of this section is to establish a few auxiliary results, that will be used in Section 7 in order to develop the third approach and prove Theorem 3.

Throughout this section we will assume that G is a non-elementary hyperbolic group with a fixed finite symmetrized generating set \mathcal{A} .

Let W_1, W_2, \dots, W_l be words in \mathcal{A} representing elements w_1, w_2, \dots, w_l of infinite order, where $E_G(w_i) \neq E_G(w_j)$ for $i \neq j$. For any given $M \geq 0$ consider the set $S(W_1, \dots, W_l; M)$ consisting of words

$$W \equiv W_{i_1}^{m_1} W_{i_2}^{m_2} \dots W_{i_s}^{m_s}$$

where $s \in \mathbb{N}$, $i_k \neq i_{k+1}$ for $k = 1, 2, \dots, s-1$ (each i_k belongs to $\{1, \dots, l\}$), and $|m_k| > M$ for $k = 2, 3, \dots, s-1$. The following lemma will be useful:

Lemma 5. ([33, Lemma 2.3]) *There exist constants $\lambda_1 = \lambda_1(W_1, W_2, \dots, W_l) > 0$, $c_1 = c_1(W_1, W_2, \dots, W_l) \geq 0$ and $M_1 = M_1(W_1, W_2, \dots, W_l) > 0$ such that any path p in the Cayley graph $\Gamma(G, \mathcal{A})$ with $\text{lab}(p) \in S(W_1, \dots, W_l; M_1)$ is (λ_1, c_1) -quasigeodesic.*

Consider a closed path $p_1 q_1 p_2 q_2$ in the Cayley graph $\Gamma(G, \mathcal{A})$ such that $\text{lab}(q_1), \text{lab}(q_2^{-1}) \in S(W_1, \dots, W_l; M_2)$. Thus $q_1 = o_1 \dots o_s$ where $\text{lab}(o_k) \equiv W_{i_k}^{m_k}$, $i_k \in \{1, \dots, l\}$, $k = 1, \dots, s$, and $i_k \neq i_{k+1}$, $k = 1, \dots, s-1$. Similarly, $q_2^{-1} = \bar{o}_1 \dots \bar{o}_{\bar{s}}$ where $\text{lab}(\bar{o}_k) \equiv W_{j_k}^{\bar{m}_k}$, $j_k \in \{1, \dots, l\}$, $k = 1, \dots, \bar{s}$, and $j_k \neq j_{k+1}$, $k = 1, \dots, \bar{s}-1$. We will say that v is a phase vertex of a path o_j if the subpath of o_j from $(o_j)_-$ to v is labelled by some power of the word W_{i_j} (and similarly for \bar{o}_j).

Paths o_k and $\bar{o}_{\bar{k}}$ will be called *compatible* if there is a path u_k in $\Gamma(G, \mathcal{A})$ joining some phase vertices of o_k and $\bar{o}_{\bar{k}}$ such that $\text{lab}(u_k) W_{j_{\bar{k}}} \text{lab}(u_k)^{-1}$ represents an element of $E_G(w_{i_k})$ in G . Such a path u_k is said to be *matching*.

The following is a simplification of [33, Lemma 2.5] (we note that instead of requiring s and \bar{s} to be bounded in the proof of [33] it is enough to demand that one uses only finitely many distinct words W_1, \dots, W_l):

Lemma 6. *Suppose that q_1, q_2, p_1, p_2 are as above, and $\|p_1\|, \|p_2\| \leq C_1$ for some C_1 . Then there exist integers M_2 and $\epsilon \in \{-1, 0, 1\}$, such that the paths o_k and $\bar{o}_{k+\epsilon}$ are compatible for all $k = 2, \dots, s-1$, whenever $|m_1|, \dots, |m_s| \geq M_2$, $|\bar{m}_1|, \dots, |\bar{m}_{\bar{s}}| \geq M_2$.*

Recall (see [33]) that two elements g, h having infinite order in G are said to be *commensurable* if there exist $a \in G$ and $k, l \in \mathbb{Z} \setminus \{0\}$ such that $g^k = ah^l a^{-1}$.

Suppose, now, that the maximal finite normal subgroup $E(G)$ of a non-elementary hyperbolic group G is trivial. An element $g \in G$ is said to be *suitable* if it has infinite order and $E_G(g) = \langle g \rangle$.

Lemma 7. ([33, Lemma 3.8]) *Every non-elementary hyperbolic group G , such that $E(G) = \{1\}$, contains infinitely many pairwise non-commensurable suitable elements.*

A subgroup $H \leq G$ is called *malnormal* if $gHg^{-1} \cap H = \{1\}$ for all $g \in G \setminus H$. In the special case when G is torsion-free, the following lemma was proved by I. Kapovich in [21, Thm. C].

Lemma 8. *Let G be a non-elementary hyperbolic group with $E(G) = \{1\}$. Then there are words W_1 and W_2 , representing elements of infinite order w_1 and w_2 in G , and constants $0 < \lambda_1 \leq 1$, $c_1 \geq 0$ such that*

- any path in $\Gamma(G, \mathcal{A})$ labelled by a word from $S(W_1, W_2; 0)$ is (λ_1, c_1) -quasigeodesic;
- the subgroup $H = \langle w_1, w_2 \rangle \leq G$ is free of rank 2;
- H is malnormal in G .

Proof. By Lemma 7 one can find four pairwise non-commensurable suitable elements $w, x, y, z \in G$ represented by some words W, X, Y, Z over the alphabet \mathcal{A} respectively. Apply Lemma 5 to the set of words $S(W, X, Y, Z; M_1)$ to find the constants λ_1, c_1 and M_1 . Find $\nu_1 = \nu_1(\delta, \lambda_1, c_1)$ according to Lemma 1 and denote $C_1 = 2\delta + 2\nu_1$. Now, let M_2 be the constant from the claim of Lemma 6 applied to any closed path $p_1 q_1 p_2 q_2$ where $lab(q_i) \in S(W, X, Y, Z; M_2)$ and $\|p_i\| \leq C_1$, $i = 1, 2$. Fix an arbitrary integer $m > \max\{M_1, M_2\}$ satisfying $\lambda_1 m - c_1 > 0$, and define $w_1 = w^m x^m$, $w_2 = y^m z^m$. Note that since the elements w_1 and w_2 are represented by the words $W_1 \equiv W^m X^m$ and $W_2 \equiv Y^m Z^m$ and $m > M_1$, every word from $S(W_1, W_2; 0)$ belongs to $S(W, X, Y, Z; M_1)$. Therefore the first claim of the lemma holds.

To show that w_1 and w_2 freely generate the subgroup $H = \langle w_1, w_2 \rangle \leq G$ it is enough to check that any non-empty word $A \in S(W_1, W_2; 0)$ is non-trivial in G . But this follows immediately from the (λ_1, c_1) -quasigeodesity:

$$|A|_G \geq \lambda_1 \|A\| - c_1 \geq \lambda_1 \min\{\|W_1\|, \|W_2\|\} - c_1 \geq \lambda_1 m - c_1 > 0.$$

Arguing by contradiction, assume that H is not malnormal, i.e., $ba_1b^{-1} = a_2$ for some $b \in G \setminus H$, $a_1, a_2 \in H \setminus \{1\}$; then $ba_1^l b^{-1} = a_2^l$ for all $l \in \mathbb{N}$. Let $A_1, A_2 \in S(W_1, W_2; 0) \subset S(W, X, Y, Z; M_2)$ and B be some words representing a_1, a_2 and b . Observe that one can assume that the words A_1 and A_2 are cyclically reduced (with respect to W_1 and W_2) after replacing B with some word B_1 representing an element b_1 of the double coset HbH ; one will still have $b_1 \notin H$ because $HbH \cap H = \emptyset$. Then the words A_1^l and A_2^l will also belong to $S(W, X, Y, Z; M_2)$. Consider a closed path $p'_1 q'_1 p'_2 q'_2$ in $\Gamma(G, \mathcal{A})$ such that $lab(p'_1) \equiv B_1$, $lab(q'_1) \equiv A_1^l$, $lab(p'_2) \equiv B_1^{-1}$, $lab(q'_2) \equiv A_2^{-l}$.

As the path q'_1 is (λ_1, c_1) -quasigeodesic, one can take l to be so large that it will have a subpath q_1 whose endpoints will be at distances greater than $(\|B_1\| + \nu_1)$ from the endpoints of q'_1 and which will be labelled by one of the following words: $X^m Y^m$ (coming from an occurrence of the subword $W_1 W_2$ in the middle of A_1^l), $X^m Z^{-m}$ (coming from an occurrence of the subword $W_1 W_2^{-1}$ in the middle of A_1^l), $W^{-m} Y^m$ (from $W_1^{-1} W_2$), $W^{-m} Z^{-m}$ (from $W_1^{-1} W_2^{-1}$), $X^m W^m$ (from $W_1 W_1$), $W^{-m} X^{-m}$

(from $W_1^{-1}W_1^{-1}$), Y^mZ^m (from W_2W_2) and $Z^{-m}Y^{-m}$ (from $W_2^{-1}W_2^{-1}$). We shall consider only the first case, when $\text{lab}(q_1) \equiv X^mY^m$, because the other cases are completely similar. Thus, $q_1 = o_1o_2$ where $\text{lab}(o_1) \equiv X^m$, $\text{lab}(o_2) \equiv Y^m$.

By Lemma 1 there are points $\alpha_1, \alpha_2 \in [(q'_1)_-, (q'_1)_+]$ such that $d(\alpha_1, (q_1)_-) \leq \nu_1$ and $d(\alpha_2, (q_1)_+) \leq \nu_1$. Then by Lemma 2, applied to the geodesic quadrilateral with vertices $(p'_1)_-$, $(q'_1)_-$, $(p'_2)_-$ and $(q'_2)_-$, one can find points $\beta_1, \beta_2 \in [(p'_1)_-, (q'_2)_-]$ satisfying $d(\alpha_i, \beta_i) \leq 2\delta$, $i = 1, 2$. Applying Lemma 1 once again one will obtain points $\gamma_1, \gamma_2 \in q'_2$ with $d(\gamma_i, \beta_i) \leq \nu_1$. Let q_2 denote the subpath of q'_2 (or q'^{-1}_2) starting with γ_2 and ending with γ_1 , and $p_1 = [(q_2)_+, (q_1)_-]$, $p_2 = [(q_1)_+, (q_2)_-]$. According to the triangle inequality one has

$$\|p_1\| = d((q_2)_+, (q_1)_-) \leq d((q_1)_-, \alpha_1) + d(\alpha_1, \beta_1) + d(\beta_1, \gamma_1) \leq 2\delta + 2\nu_1 = C_1,$$

and, similarly, $\|p_2\| \leq C_1$. Note that the labels of q_1 and q'^{-1}_2 belong to the set $S(W, X, Y, Z; M_2)$, hence by Lemma 6, q'^{-1}_2 contains subpaths \bar{o}_1 and \bar{o}_2 with $(\bar{o}_1)_+ = (\bar{o}_2)_-$, $\text{lab}(\bar{o}_i) \in \{W^m, X^m, Y^m, Z^m\}^{\pm 1}$, $i = 1, 2$, such that there exist matching paths u_i which connect some phase vertices of o_i and \bar{o}_i , $i = 1, 2$. Since the elements w, x, y, z are pairwise non-commensurable, by the definition of compatible paths one immediately gets $\text{lab}(\bar{o}_1) \equiv X^{\epsilon_1 m}$ and $\text{lab}(o_2) \equiv Y^{\epsilon_2 m}$ for some $\epsilon_i \in \{-1, 1\}$, $i = 1, 2$. Moreover $\epsilon_i = 1$, $i = 1, 2$, because $E_G(x) = E_G^+(x) = \langle x \rangle$ and $E_G(y) = E_G^+(y) = \langle y \rangle$. Thus the subpath $\bar{o}_1\bar{o}_2$ of q'^{-1}_2 (and, hence, of $q'^{\pm 1}_2$) originates from an occurrence of the same subword W_1W_2 in $\text{lab}(q'^{\pm 1}_2)$.

Now, observe that since the word $\text{lab}(u_1)X\text{lab}(u_1)^{-1}$ represents an element of $\langle x \rangle$ then, applying (1), one achieves $\text{lab}(u_1) = X^{t_1}$ in G for some $t_1 \in \mathbb{Z}$. Similarly, $\text{lab}(u_2) = Y^{t_2}$ in G for some $t_2 \in \mathbb{Z}$. Since the endpoints of u_1 and u_2 are phase vertices of o_1, \bar{o}_1 and o_2, \bar{o}_2 , there are paths r_i , $i = 1, 2$, having the same starting point $(r_i)_- = (o_1)_+ = (o_2)_-$ and the same ending point $(r_i)_+ = (\bar{o}_1)_+ = (\bar{o}_2)_-$, $i = 1, 2$, such that $\text{lab}(r_1)$ represents an element of $\langle x \rangle$ and $\text{lab}(r_2)$ represents an element of $\langle y \rangle$. But $\langle x \rangle \cap \langle y \rangle = \{1\}$ and $\text{lab}(r_1) = \text{lab}(r_2)$ in G , therefore $\text{lab}(r_1)$ represents the identity element in G .

Let q''_1, q''_2 denote the subpaths of q'^{-1}_1 and q'^{-1}_2 respectively, satisfying $(q''_1)_- = (r_1)_-$, $(q''_1)_+ = (p'_1)_+$, $(q''_2)_- = (p'_1)_-$ and $(q''_2)_+ = (r_1)_+$. By construction of the paths $o_1, o_2, \bar{o}_1, \bar{o}_2$, the words $\text{lab}(q''_i)$, $i = 1, 2$, belong to $S(W_1, W_2; 0)$, and, hence, the elements which they represent in G belong to H . Finally, the equality $b_1 = \text{lab}(p'_1) = \text{lab}(q''_2)\text{lab}(r_1^{-1})\text{lab}(q''_1)$ combined with $\text{lab}(r_1^{-1}) = 1$ imply that $b_1 \in H$, contradicting the initial assumption. Thus the lemma is proved. \square

A word B over \mathcal{A} (and the element $b \in G$ represented by it) is said to be *cyclically reduced* if for any word U , conjugate to B in G , one has $\|B\| \leq \|U\|$. For $m \in \mathbb{Z}$, any subword of B^m is called *B -periodic*.

The following fact was established in [34, Lemma 27] for the case of a torsion-free hyperbolic group G by Olshanskii; in [20, Lemma 12] Ivanov and Olshanskii noted that the claim continues to hold even if G possesses elements of finite order.

Lemma 9. *Assume that G is a non-elementary hyperbolic group with a finite generating set \mathcal{A} . There exist numbers $0 < \lambda_0 \leq 1$ and $c_0 \geq 0$ such that for every cyclically reduced word B (over \mathcal{A}), representing an element of infinite order in G , any B -periodic word V is (λ_0, c_0) -quasigeodesic.*

Lemma 10. *Let G be a δ -hyperbolic group and $\bar{\lambda}, \bar{c}$ be some numbers satisfying $0 < \bar{\lambda} \leq 1$ and $\bar{c} \geq 0$. Then there exists a constant $\Lambda_1 = \Lambda_1(\delta, \bar{\lambda}, \bar{c}) \geq 0$ such that the following holds.*

Assume X_1, X_2 are some words over \mathcal{A} having infinite order in G and for each $i = 1, 2$, any path in $\Gamma(G, \mathcal{A})$ labelled by a power of X_i is $(\bar{\lambda}, \bar{c})$ -quasigeodesic. If $X_1^l = AX_2^kA^{-1}$ in G for some word A and non-zero integers k, l , then one can find words U, W and a X_1 -periodic word V such that $X_2 = UVW$ in G and $\|U\|, \|W\| \leq \Lambda_1$.

Proof. Let $\bar{\nu} = \bar{\nu}(\delta, \bar{\lambda}, \bar{c})$ be obtained from the claim of Lemma 1. Set $\Lambda_1 = 2\delta + 2\bar{\nu}$ and denote by $a \in G$ the element represented by the word A .

By the assumptions, $X_1^{ls} = AX_2^{ks}A^{-1}$ in G for every $s \in \mathbb{Z}$ (pick s so that $ks > 0$). Consider the quadrilateral with sides p_1, q_1, p_2 and q_2 in $\Gamma(G, \mathcal{A})$ where $(p_1)_- = (q_2)_+ = 1$, $\text{lab}(p_1) \equiv A$, $(q_1)_- = (p_1)_+$, $\text{lab}(q_1) \equiv X_2^{ks}$, $(p_2)_- = (q_1)_+$, $\text{lab}(p_2) \equiv A^{-1}$, $(q_2)_- = (p_2)_+$, $\text{lab}(q_2) \equiv X_1^{-ls}$. Since the paths q_1 and q_2 are $(\bar{\lambda}, \bar{c})$ -quasigeodesic, if one takes $|s|$ to be sufficiently large (compared to $|a|_{\mathcal{A}}$), it will be possible to find a subpath r of q_1 labelled by X_2 whose endpoints are at distances at least $(|a|_{\mathcal{A}} + \bar{\nu})$ from the endpoints of q_1 . According to Lemma 1 there are points u, v lying on the geodesic segment $[(q_1)_-, (q_1)_+]$ at distances at most $\bar{\nu}$ from r_- and r_+ respectively. Because of the choice of the subpath r one can now apply Lemma 2 to find points $u', v' \in [(q_2)_+, (q_2)_-]$ such that $d(u, u') \leq 2\delta$ and $d(v, v') \leq 2\delta$. Again by Lemma 1, there are points $u'', v'' \in q_2$ situated at distances at most $\bar{\nu}$ from u' and v' respectively. Hence $d(r_-, u''), d(r_+, v'') \leq 2\bar{\nu} + 2\delta = \Lambda_1$.

Now the claim of the lemma will hold if one denotes by U and W the words written on the geodesic paths $[r_-, u'']$ and $[v'', r_+]$ respectively and by V the word written on the subpath of q_2 (or q_2^{-1}) starting with u'' and ending with v'' . \square

Suppose $\Lambda > 0$ and $t \in \mathbb{N}$. Following [20] we shall say that an element $g \in G$ is (Λ, t) -periodic if there are words U, V and W over \mathcal{A} such that $g = UVW$ in G , $\max\{|U|_{\mathcal{A}}, |V|_{\mathcal{A}}\} \leq \Lambda$, and V is a Z -periodic word, for some cyclically reduced word Z having infinite order in G , with $\|V\| \geq t\|Z\|$. An element $h \in G$ is called (Λ, t) -aperiodic provided for every factorization $h = h_1h_2h_3$, where $h_1, h_2, h_3 \in G$, such that $|h|_{\mathcal{A}} = |h_1|_{\mathcal{A}} + |h_2|_{\mathcal{A}} + |h_3|_{\mathcal{A}}$, the element h_2 is not (Λ, t) -periodic. It is not difficult to see that, in geometric terms, g is (Λ, t) -aperiodic if and only if for any geodesic path p in $\Gamma(G, \mathcal{A})$, whose label represents g in G , and for any path q such that $q_-, q_+ \in \mathcal{N}_{\Lambda}(p)$ and $\text{lab}(q) \equiv V$ is a Z -periodic word (for some cyclically reduced word Z having infinite order in G), one has $\|q\| = \|V\| < t\|Z\|$.

Suppose that W_1, W_2 are some words over the alphabet \mathcal{A} representing elements $w_1, w_2 \in G$. Let $F^+(W_1, W_2)$ denote the free monoid generated by the words W_1 and W_2 (i.e., the set of all positive words in W_1 and W_2). Assume that for some $\lambda_1 > 0$ and $c_1 \geq 0$, any word from $F^+(W_1, W_2)$ is (λ_1, c_1) -quasigeodesic in G and that the canonical map $\psi : F^+(W_1, W_2) \rightarrow G$ is injective. Let $F^+ \subset G$ denote the set of elements represented by the words from $F^+(W_1, W_2)$, i.e., $F^+ = \psi(F^+(W_1, W_2))$. Note that each non-trivial $f \in F^+$ has infinite order in G .

Lemma 11. *There is a constant $\Lambda_1 \geq 0$ such that for any $\Lambda' \geq \Lambda_1$ there exist an integer $t' > 0$ and infinitely many pairwise non-commensurable elements in F^+ which are (Λ', t') -aperiodic in G .*

Proof. Let λ_0 and c_0 be the constants from Lemma 9. Define $\bar{\lambda} = \min\{\lambda_0, \lambda_1\}$ and $\bar{c} = \max\{c_0, c_1\}$, find the corresponding constant $\Lambda_1 = \Lambda_1(\delta, \bar{\lambda}, \bar{c}) \geq 0$ according to Lemma 10, and take an arbitrary $\Lambda' \geq \Lambda_1$.

In [34, Lemmas 30,31] it was shown that there exists $t' \in \mathbb{N}$ and an infinite subset $\mathcal{B} = \{b_1, b_2, \dots\} \subset F^+$ consisting of (Λ', t') -aperiodic elements in G (the proof works in the general case of an arbitrary non-elementary hyperbolic group as observed in [20, Lemma 15]).

For each $j \in \mathbb{N}$ find the (unique) word $B_j \in F^+(W_1, W_2)$ with $\psi(B_j) = b_j$. As W_1 and W_2 are words over \mathcal{A} , so is B_j . Since \mathcal{B} is infinite, after passing to its infinite subset, we can assume that $|b_{j+1}|_{\mathcal{A}} > t' \|B_j\| + 2\Lambda_1$ for every $j \in \mathbb{N}$.

Suppose that there are indices $i < j$ such that b_i is commensurable with b_j . Choose a shortest representative b of the conjugacy class of b_i in G ; then $|b|_{\mathcal{A}} \leq |b_i|_{\mathcal{A}}$. By the assumption, there exists $a \in G$, $k \in \mathbb{N}$ and $l \in \mathbb{Z} \setminus \{0\}$ such that $b^l = ab_j^k a^{-1}$.

Let A and B be shortest words over the alphabet \mathcal{A} representing a and b respectively (note that the word B is cyclically reduced in G by construction). Then $B^l = AB_j^k A^{-1}$ in G and one can use Lemma 10 to find words U, V and W such that $B_j \stackrel{G}{=} UVW$ where $\|U\|, \|W\| \leq \Lambda_1 \leq \Lambda'$ and V is a B -periodic word. The latter equality implies that

$$\|V\| \geq |V|_{\mathcal{A}} \geq |B_j|_{\mathcal{A}} - \|U\| - \|W\| \geq |b_j|_{\mathcal{A}} - 2\Lambda_1 > t' \|B_j\| \geq t' \|B\|,$$

which contradicts the (Λ', t') -aperiodicity of b_j . Therefore no two distinct elements of \mathcal{B} can be commensurable and the statement is proved. \square

Lemma 12. *Suppose G is a δ -hyperbolic group, $0 < \bar{\lambda} \leq 1$, $\bar{c} \geq 0$ and $\varkappa > 0$. For any $\Lambda > 0$ there exists $\Lambda' > 0$ such that for any $t' \in \mathbb{N}$ there is $t \in \mathbb{N}$ satisfying the following.*

Assume X_1, X_2 are some words over \mathcal{A} representing elements of infinite order $x_1, x_2 \in G$ and for each $i = 1, 2$, any path in $\Gamma(G, \mathcal{A})$ labelled by a power of X_i is $(\bar{\lambda}, \bar{c})$ -quasigeodesic. If x_1 is (Λ', t') -aperiodic, $\|X_2\| \leq \varkappa \|X_1\|$ and $x_1^l = ax_2^k a^{-1}$ for some $a \in G$, $k, l \in \mathbb{Z} \setminus \{0\}$, then x_2 is (Λ, t) -aperiodic.

Proof. Let λ_0, c_0 be from the claim of Lemma 9, $\nu_0 = \nu_0(\delta, \lambda_0, c_0)$, $\bar{\nu} = \bar{\nu}(\delta, \bar{\lambda}, \bar{c})$ be as in Lemma 1 and $\Lambda_1 = 2\delta + 2\bar{\nu}$ be from Lemma 10. Set $\Lambda' = 4\delta + 3\bar{\nu} + \Lambda_1 + \nu_0 + \Lambda$, choose an arbitrary $t' \in \mathbb{N}$ and a positive integer m so that

$$m > \frac{1}{\bar{\lambda}}(\varkappa + 2\Lambda_1 + \bar{c}) + 2,$$

and denote $t = mt'$. Assume, now, that x_1 is (Λ', t') -aperiodic.

Arguing by contradiction, suppose that x_2 is not (Λ, t) -aperiodic. Then there is a geodesic path p between 1 and x_2 , points $a, b \in p$ and a (λ_0, c_0) -quasigeodesic path r in $\Gamma(G, \mathcal{A})$ such that the label of r is a Z -periodic word (for some cyclically reduced word Z over \mathcal{A} having infinite order in G), $\|r\| \geq t \|Z\|$ and $d(a, r_-), d(b, r_+) \leq \Lambda$.

Using the assumptions together with Lemma 10, one can find a $(\bar{\lambda}, \bar{c})$ -quasigeodesic path q in $\Gamma(G, \mathcal{A})$ where $lab(q)$ is X_1 -periodic and $d(p_-, q_-), d(p_+, q_+) \leq \Lambda_1$.

Since the geodesic quadrilaterals in $\Gamma(G, \mathcal{A})$ are 2δ -slim one obtains

$$a, b \in p \subset \mathcal{N}_{2\delta}([p_-, q_-] \cup [q_-, q_+] \cup [q_+, p_+]) \subset \mathcal{N}_{2\delta+\Lambda_1}([q_-, q_+]) \subset \mathcal{N}_{2\delta+\Lambda_1+\bar{\nu}}(q).$$

Thus, there are points $a', b' \in q$ with $d(a, a'), d(b, b') \leq 2\delta + \Lambda_1 + \bar{\nu}$; consequently $d(r_-, a'), d(r_+, b') \leq 2\delta + \Lambda_1 + \bar{\nu} + \Lambda$. Using 2δ -slimness of the geodesic quadrilateral with the vertices r_-, r_+, a', b' together with the property of quasigeodesics given by Lemma 1, one gets

$$(3) \quad r \subset \mathcal{N}_{\nu_0}([r_-, r_+]) \subset \mathcal{N}_{\nu_0+2\delta}([r_-, a'] \cup [a', b'] \cup [b', r_+]) \subset \\ \mathcal{N}_{\nu_0+2\delta+2\delta+\Lambda_1+\bar{\nu}+\Lambda}([a', b']) \subset \mathcal{N}_{\nu_0+4\delta+\Lambda_1+\bar{\nu}+\Lambda+\bar{\nu}}(q) = \mathcal{N}_{\Lambda'-\bar{\nu}}(q).$$

Now let us estimate the length of q using its quasigeodesity and the triangle inequality:

$$\|q\| \leq \frac{1}{\lambda}(d(q_-, q_+) + \bar{c}) \leq \frac{1}{\lambda}(d(1, x_2) + d(1, q_-) + d(q_+, x_2) + \bar{c}) \leq \\ \frac{1}{\lambda}(\|X_2\| + 2\Lambda_1 + \bar{c}) \leq \frac{1}{\lambda}(\varkappa\|X_1\| + 2\Lambda_1 + \bar{c}).$$

Obviously, there is a path \hat{q} in $\Gamma(G, \mathcal{A})$ such that q is a subpath of \hat{q} , $\|\hat{q}\| \leq \|q\| + 2\|X_1\|$ and $lab(\hat{q}) \equiv X_1^s$ for some integer s . One has

$$|s\|X_1\| = \|\hat{q}\| \leq \|q\| + 2\|X_1\|, \text{ hence } |s| \leq \frac{1}{\lambda}(\varkappa + 2\Lambda_1 + \bar{c}) + 2 < m$$

due to the choice of m . Since $t = mt'$, one can split r in a concatenation of m subpaths $r = r_1 \dots r_m$ such that $lab(r_j)$ is a Z -periodic word and $\|r_j\| \geq t'\|Z\|$ for all $j = 1, \dots, m$. The path \hat{q} , on the other hand, is a concatenation of $|s|$ subpaths $q = \hat{q}_1 \dots \hat{q}_{|s|}$, each of which is labelled by $X_1^{\pm 1}$. Now, since $m > |s|$, one can use the Pigeon-hole Principle to find $j, j' \in \{1, \dots, m\}$, $j \leq j'$, and $k \in \{1, \dots, |s|\}$, such that $(r_j)_-, (r_{j'})_+ \in \mathcal{N}_{\Lambda'-\bar{\nu}}(\hat{q}_k) \subset \mathcal{N}_{\Lambda'}([(q_k)_-, (q_k)_+])$ (here we utilize the $(\Lambda' - \bar{\nu})$ -proximity of r to \hat{q} given by (3)). Let r' be the subpath of r starting at $(r_j)_-$ and ending at $(r_{j'})_+$. Then $lab(r')$ is a Z -periodic word and $\|r'\| \geq t'\|Z\|$. But this contradicts the (Λ', t') -aperiodicity of x_1 . The proof is finished. \square

Lemma 13. *Let G be a non-elementary hyperbolic group with $E(G) = \{1\}$. For every $\Lambda > 0$ there is $t \in \mathbb{N}$ such that G contains an infinite set $\{d_i \mid i \in \mathbb{N}\}$ of pairwise non-commensurable (Λ, t) -aperiodic elements of infinite order which are cyclically reduced and such that $E_G(d_i) = \langle d_i \rangle$ for every $i \in \mathbb{N}$.*

Proof. Let W_1, W_2, H, λ_1 and c_1 be as in Lemma 8, and let λ_0, c_0 be from Lemma 9. Set $\bar{\lambda} = \min\{\lambda_0, \lambda_1\}$, $\bar{c} = \max\{c_0, c_1\}$, $\varkappa = 1$ and find $\Lambda' > 0$ from the claim of Lemma 12. Denote $\Lambda'' = \max\{\Lambda' + \nu_1, \Lambda_1\}$, where Λ_1 and $\nu_1 = \nu_1(\delta, \lambda_1, c_1)$ are given by Lemmas 11 and 1.

According to Lemma 11, there are $t' \in \mathbb{N}$ and an infinite set $\mathcal{B} = \{b_1, b_2, \dots\} \subset F^+ = \psi(F^+(W_1, W_2)) \subset H$ of pairwise non-commensurable (Λ'', t') -aperiodic elements of infinite order in G . Observe that by (1), for any $h \in H \setminus \{1\}$ the malnormality of H implies that $E_G(h) \subset H$, hence $E_G(h)$ is cyclic as any torsion-free elementary group. In particular, for each $j \in \mathbb{N}$ there is $b'_j \in F^+$ such that $E_G(b_j) = \langle b'_j \rangle$. For each $j \in \mathbb{N}$ one can choose a word $B'_j \in F^+(W_1, W_2)$ representing b'_j such that any path in $\Gamma(G, \mathcal{A})$ labelled by a power of B'_j is (λ_1, c_1) -quasigeodesic (therefore it will also be $(\bar{\lambda}, \bar{c})$ -quasigeodesic), because such a power will still belong to $F^+(W_1, W_2) \subset S(W_1, W_2; 0)$. Now, since $b_j = (b'_j)^s$ for some $s \in \mathbb{N}$, Lemma 1 immediately implies that b'_j is $(\Lambda'' - \nu_1, t')$ -aperiodic because b_j

is (Λ'', t') -aperiodic. Thus, the infinite set $\mathcal{B}' = \{b'_1, b'_2, \dots\}$ consists of (Λ', t') -aperiodic pairwise non-commensurable elements of infinite order in G , with the additional property that $E_G(b'_j) = \langle b'_j \rangle$.

For every $b'_j \in \mathcal{B}'$ choose a cyclically reduced element $d_j \in G$ and an element $a_j \in G$ such that $b'_j = a_j d_j a_j^{-1}$. Note that the set $\mathcal{D} = \{d_1, d_2, \dots\}$ is an infinite set of pairwise non-commensurable elements of infinite order in G , and $E_G(d_j) = a_j^{-1} E_G(b'_j) a_j = \langle d_j \rangle$ is cyclic for every $j \in \mathbb{N}$.

Let t be given by Lemma 12. Choose a shortest word D_j representing d_j in G . Then $\|D_j\| \leq \|B_j\| = \varkappa \|B_j\|$ and, hence, according to Lemma 9 and the construction, one can apply Lemma 12 to show that d_j is (Λ, t) -aperiodic for every $j \in \mathbb{N}$. \square

7. QUOTIENTS OF BOUNDED EXPONENT

In this section we are going to prove Theorem 3. We will utilize the construction from [20] and explain how to obtain series of infinite quotients of a non-elementary hyperbolic group G of bounded exponent by adding periodic relations $A^{n(i)}$ with different exponents $n(i)$.

Let G be a non-elementary hyperbolic group. Fix a presentation $\langle \mathcal{A} \mid R \in \mathcal{R}_0 \rangle$ of G as in the beginning of Section 5. Since $E(G)$ is the maximal finite normal subgroup of G , the quotient $\hat{G} = G/E(G)$ is again a non-elementary hyperbolic group with the additional property $E(\hat{G}) = \{1\}$. After replacing G with \hat{G} , we will further assume that $E(G) = \{1\}$.

Let $\nu_0 = \nu_0(\lambda_0, c_0)$ be the constant provided by Lemma 1, where the pair (λ_0, c_0) is chosen according to Lemma 9. Choose the auxiliary parameters (such as $\Lambda, \alpha, \beta, \gamma, \varepsilon, \rho, \theta, n$, etc.) as in § 3 of [20]. We refer to [20], [19] for the values and estimates of the parameters; in the present section we will explicitly use the following inequalities: $\theta > \alpha + 3\beta$, $2\gamma + 1/2 < \rho$. The choice of the parameter n , which is a very large integer, is made after all the other parameters are chosen. Increasing n (that is, multiplying it by a power of 2), if necessary, and using Lemma 13, we find an infinite set $\mathcal{B} = \{B_1, B_2, \dots\}$ of cyclically reduced words satisfying the following properties:

1. Elements of the group G represented by words B_1, B_2, \dots are pairwise non-commensurable elements of infinite order that generate their respective (cyclic) elementary subgroups $E_G(B_k)$, $k = 1, 2, \dots$.
2. For every $B \in \mathcal{B}$, the word $B^{\pm 3}$ represents a $(\Lambda + \nu_0, \beta n)$ -aperiodic element.

Passing to a subsequence, we can assume that $\|B_1\| > n^2$ and $\|B_{k+1}\| > n^2 \|B_k\|$ for $k \geq 1$. Given a sequence $\omega = (\omega_k)_{k=1}^\infty$ of 0's and 1's, we are going to construct a periodic quotient $G_\omega(\infty)$ of G of exponent $2n$.

We set $G_\omega(0) = G$, introduce a total order \prec and define simple words as in Section 5. Assuming that the presentation of $G_\omega(i-1)$ has already been constructed, choose the period A_i (of rank i) as in Section 5.

We distinguish the following two cases:

- (1) A non-trivial power of some word $B_k \in \mathcal{B}$ is conjugate in $G_\omega(i-1)$ to an element $A_i^\ell F$, for some $k \geq 1, \ell \in \mathbb{Z}, \ell \neq 0, F \in \mathcal{F}(A_i)$. (It follows from Lemma 11, Lemma 16 below and Lemma 18.5(c) [19],[20] in rank $i-1$ that such k is unique.) In this case we shall say that the period A_i is *special*.

- (2) None of non-trivial powers of words from \mathcal{B} is conjugate in $G_\omega(i-1)$ to an element of the form $A_i^\ell F$, for some $\ell \in \mathbb{Z}, \ell \neq 0, F \in \mathcal{F}(A_i)$.

Set $n(i) = n$ if case (1) holds and $\omega_k = 0$. Otherwise set $n(i) = 2n$.

The group $G_\omega(i)$ is obtained by imposing on $G_\omega(i-1)$ the relation $A_i^{n(i)} = 1$:

$$G_\omega(i) = \langle \mathcal{A} \parallel R \in \mathcal{R}_0 \cup \{A_1^{n(1)}, A_2^{n(2)}, \dots, A_i^{n(i)}\} \rangle.$$

Analysis of groups $G_\omega(i)$ is done using geometric interpretation of deducing consequences from defining relations according to the scheme of [20] (and [19]). We refer the reader to [20], [19] for the definitions of a bond, a contiguity subdiagram and its standard contour, a degree of contiguity, a reduced diagram, (strict) rank and type of a diagram, a simple word in rank i (some of these concepts were defined in Section 4).

References to lemmas from [20], as well as lemmas from [19] and their analogs in [20], will be made preserving the numeration of [20] and of [19].

We note that our modification of the construction implies obvious changes in formulations and proofs from [20],[19]: a cell of rank i now corresponds to the relation $A_i^{n(i)} = 1$ and $n(i)$ appears instead of n in all the estimates of the length of its boundary; the order of the period A_i in $G_\omega(\infty)$ is now $n(i)$ instead of n ; finite subgroups of $G_\omega(i)$ are isomorphically embedded into a direct product of a direct power of the dihedral group $D(4n)$ (instead of $D(2n)$) and an elementary associated with G group (see [20]); in the inductive step from $G_\omega(i)$ to $G_\omega(i+1)$ (in arguments from [19, §§ 18,19] and their analogues from [20]) n is replaced by $n(i+1)$. The crucial part that needs explanation is validity of equations in Lemma 15.10 [19],[20].

The following lemma about structure of diagrams precedes Lemma 3.1 [19],[20].

Lemma 14. *Let $B \in \mathcal{B}$ and let Γ be a contiguity subdiagram of a cell Π to a B -periodic section p of the contour of a reduced diagram Δ of rank i , such that $\text{lab}(\Gamma \wedge p)$ is a subword of $B^{\pm 3}$. Then $r(\Gamma) = 0$ and the contiguity degree of Π to p via Γ is less than β .*

Proof. We prove this lemma by contradiction. Assume that the triple (p, Γ, Π) is a counterexample, where the contiguity subdiagram Γ has a minimal type. Let the standard contour of Γ be $\partial\Gamma = d_1 p_1 d_2 q$, where $p_1 = \Gamma \wedge p$, $q = \Gamma \wedge \Pi$. The bonds defining Γ are 0-bonds (otherwise (p, Γ, Π) is not minimal). This means that $\max\{\|d_1\|, \|d_2\|\} < \Lambda$. Assuming that $r(\Gamma) > 0$, by Lemma 5.7 [19],[20] ($\tau(\Gamma) < \tau(\Delta)$), there is a θ -cell. However, by minimality of (p, Γ, Π) , the degree of contiguity of any cell from Γ to p_1 is less than β and the contiguity degree of any cell from Γ to q is less than α by Lemma 3.4 [19],[20] (again, $\tau(\Gamma) < \tau(\Delta)$). Contiguity degrees of cells of positive ranks to sections d_1 and d_2 of $\partial\Gamma$ are bounded from above by β in view of inequality $\max\{\|d_1\|, \|d_2\|\} < \Lambda$ and Lemma 6.1 [19],[20]. The fact that $\theta > \alpha + 3\beta$ implies that Γ does not have cells of positive ranks, i.e. Γ is a diagram over the presentation of G . Note that the endpoints of the path p_1 are within ν_0 from the geodesic (in the Cayley graph of G) segment that connects the endpoints of p . Assuming that the contiguity degree of Π to p via Γ is greater than or equal to β , we arrive at a contradiction with $(\Lambda + \nu_0, \beta n)$ -aperiodicity of B^3 in G . Lemma is proved. \square

The formulation of the analogue of Lemma 15.10 [19],[20] is changed as follows. For any finite subgroup \mathcal{G} of $G_\omega(i)$ equations from [19],[20] hold with the exponent

n replaced by $2n$. If, in addition, \mathcal{G} is a finite subgroup of $G_\omega(i)$, conjugate to a subgroup of $\mathcal{K}(A_j)$, $j \leq i$, where the period A_j is special, then the equations from part (a) of Lemma 15.10 [19],[20] hold with exponent n . The proof is retained. In the case the relation of rank j is $A_j^n = 1$, the claim follows from Lemmas 17 and 18 (below) applied in smaller rank: the subgroup $\mathcal{F}(A_j)$ is trivial, there are no $\mathcal{F}(A_j)$ -involutions, and, hence, the equations are trivially satisfied.

The inductive step from rank i to $i + 1$ starts with the following lemmas.

Lemma 15. *If a word B from \mathcal{B} is of infinite order in rank i , then B is simple in rank i .*

Proof. By the choice of the set \mathcal{B} and the definition of simple words, B is simple in rank 0 and therefore it is simple in all ranks $i \leq i_0$ (as in [20], i_0 stands for the maximal rank for which $\|A_{i_0}\| < C$). Let now $i > i_0$. Assume that B is not simple in rank i . By definition of a simple in rank i word, and because B has infinite order in rank i , this means that B is not cyclically reduced in rank i , that is, there exists a word X conjugated to B in rank i such that $\|X\| < \|B\|$. Let Δ be an annular reduced diagram representing conjugacy of B and X in rank i . Denote by p, q contours of $\partial\Delta$ labelled by B, X respectively. Without loss of generality we may assume that the path q is cyclically geodesic in Δ , i.e. geodesic in its homotopy class in Δ . If $r(\Delta) > 0$ then, by Lemma 5.7 [19],[20], there is a cell Π of positive rank in Δ and contiguity subdiagrams Γ_p, Γ_q of Π to p, q respectively such that the sum of contiguity degrees of Π to p, q via Γ_p, Γ_q is greater than θ . However, by Lemma 14, $\|\Gamma_p \wedge \Pi\| < \beta \|\partial\Pi\|$ and, by Lemma 3.3 [19],[20], $\|\Gamma_q \wedge \Pi\| < \alpha \|\partial\Pi\|$. Thus, the inequality $\theta > \alpha + \beta$ implies that Δ does not have cells of positive rank. But this is a contradiction since the word B was chosen to be cyclically reduced in $G = G(0)$. \square

Lemma 16. *Let $l \neq k$ and assume that some words $B_l, B_k \in \mathcal{B}$ are of infinite order in the group $G_\omega(i)$. Then B_l and B_k are not commensurable in $G_\omega(i)$.*

Proof. Let $l > k$. By the choice of the set \mathcal{B} , the words B_l and B_k are not commensurable in $G = G_\omega(0)$. Let now $i \geq 1$, and assume, on the contrary, that the words $B_l, B_k \in \mathcal{B}$ have infinite order and are commensurable in $G_\omega(i)$. Let Δ be an annular reduced diagram of rank i for the conjugacy of some non-trivial powers B_l^s and B_k^t . Denote by p and q contours of Δ so that $lab(p) \equiv B_l^s$, $lab(q) \equiv B_k^t$. The words B_l, B_k are not commensurable in $G_\omega(0)$, hence $r(\Delta) > 0$. By Lemma 5.7 [19],[20], there is a cell Π of positive rank $j \leq i$ and contiguity subdiagrams Γ_p, Γ_q of π to p, q respectively, such that $\|\Gamma_p \wedge \Pi\| + \|\Gamma_q \wedge \Pi\| > \theta \|\partial\Pi\|$.

Note that $\|A_j\| \leq \|B_k\| < n^{-2} \|B_l\|$. Therefore $\|\partial\Pi\| < \|B_l\|$ and $lab(\Gamma_p \wedge p)$ is a subword of $B_l^{\pm 3}$. It follows from Lemma 14 that $\|\Gamma_p \wedge \Pi\| < \beta \|\partial\Pi\|$. By Lemmas 15 and 3.4 [19],[20], $\|\Gamma_q \wedge \Pi\| < \alpha \|\partial\Pi\|$. This is a contradiction since $\theta > \alpha + \beta$. \square

The following lemmas describe the structure of finite subgroups associated to the periods distinguished by the elements from the set \mathcal{B} . They precede Lemma 18.5 [19],[20].

Lemma 17. *Let a non-trivial power B^r of some word $B \in \mathcal{B}$ be conjugate in $G_\omega(i)$ to an element $A_{i+1}^\ell F$, for some $\ell \in \mathbb{Z}, \ell \neq 0, F \in \mathcal{F}(A_{i+1})$. Then $\mathcal{F}(A_{i+1}) = \{1\}$.*

Proof. The word B is of infinite order in rank i and, by Lemma 15 it is simple in rank i . Let K be a finite subgroup of $G_\omega(i)$ normalized by B^r . Our aim is to

show that K is trivial. Assuming the contrary, choose a word F_0 representing a non-trivial element from K . Since K is finite and is normalized by B^r , there exists an integer $s > 0$ such that

$$B^{-s}F_0B^s \stackrel{i}{=} F_0.$$

Choose s such that $|s| > n^2\|F_0\|$ and let Δ be a disk reduced diagram of rank i representing this equation with the standard contour $\partial\Delta = bpcq$, where $lab(b) \equiv F_0, lab(p) \equiv B^s, lab(c) \equiv F_0^{-1}, lab(q) \equiv B^{-s}$. Assuming $r(\Delta) > 0$, by Lemma 5.7[19],[20], one will find a θ -cell Π in Δ . Denote $j = r(\Pi)$ (remark that $\|B\| \geq \|A_j\|$) and let $\Gamma_b, \Gamma_p, \Gamma_c$ and Γ_q be contiguity subdiagrams (some of them may be absent) of Π to b, p, c and q respectively. Sections b and c may be regarded as geodesic sections of $\partial\Delta$. Therefore, by Lemma 3.3 [19],[20],

$$\max\{\|\Gamma_b \wedge \Pi\|, \|\Gamma_c \wedge \Pi\|\} < \alpha\|\partial\Pi\|.$$

Suppose that $\|\Gamma_b \wedge \Pi\| + \|\Gamma_c \wedge \Pi\| > (\alpha + \beta)\|\partial\Pi\|$. In particular, both Γ_b and Γ_c are present, and the contiguity degrees of Π to b and to c via Γ_b and Γ_c are greater than β . Denote the standard contours $\partial\Gamma_b = u_1b_1u_2u_b$ and $\partial\Gamma_c = v_1c_1v_2u_c$, where $b_1 = \Gamma_b \wedge b$, $u_b = \Gamma_b \wedge \Pi$, $c_1 = \Gamma_c \wedge c$, $u_c = \Gamma_c \wedge \Pi$. By Lemma 3.1 [19],[20],

$$\max\{\|u_1\|, \|u_2\|, \|v_1\|, \|v_2\|\} < \gamma\|\partial\Pi\|.$$

It follows that initial and terminal vertices p_- and p_+ of the path p can be joined in Δ by a path of length less than

$$\|F_0\| + \gamma\|\partial\Pi\| + 1/2\|\partial\Pi\| + \gamma\|\partial\Pi\| + \|F_0\| = 2\|F_0\| + (2\gamma + 1/2)n\|A_j\| < \rho\|p\|.$$

This contradicts Lemma 6.1 [19],[20] since p is a smooth section of $\partial\Delta$ and of the boundary of any subdiagram of Δ that contains p in its contour. Consequently, $\|\Gamma_b \wedge \Pi\| + \|\Gamma_c \wedge \Pi\| \leq (\alpha + \beta)\|\partial\Pi\|$.

Let us assume now that Γ_p is present. Denote the standard contour $\partial\Gamma_p = d_1p_1d_2p_2$, where $p_1 = \Gamma_p \wedge p$ and $p_2 = \Gamma_p \wedge \Pi$. If $\|p_2\| \geq \beta\|\partial\Pi\|$, then by Lemma 15 and Lemma 20.2 [19],[20] applied to Γ_p ($\tau(\Gamma_p) < \tau(\Delta)$), one gets $\|p_1\| < (1 + \varepsilon)\|B\|$. Hence, $lab(p_1)$ is a subword of $B^{\pm 3}$, and Lemma 14 yields a contradiction. Therefore, $\|\Gamma_p \wedge \Pi\| < \beta\|\partial\Pi\|$. Arguing in the same way, we obtain that $\|\Gamma_q \wedge \Pi\| < \beta\|\partial\Pi\|$ in the case Γ_q is present. It follows that

$$\|\Gamma_b \wedge \Pi\| + \|\Gamma_p \wedge \Pi\| + \|\Gamma_c \wedge \Pi\| + \|\Gamma_q \wedge \Pi\| < (\alpha + 3\beta)\|\partial\Pi\| < \theta\|\partial\Pi\|,$$

contradicting with the definition of a θ -cell.

Therefore, $r(\Delta) = 0$ and equality $B^{-s}F_0B^s = F_0$ holds in the group $G(0)$. This means that F_0 belongs to the elementary subgroup of B in $G(0)$. Since the elementary subgroup of B is cyclic, F_0 is trivial in $G_\omega(i)$. Consequently, $\mathcal{F}(A_{i+1}) = \{1\}$. \square

Lemma 18. *Let a non-trivial power B^r of some word $B \in \mathcal{B}$ be conjugate in $G_\omega(i)$ to an element $A_{i+1}^\ell F$, for some $\ell \in \mathbb{Z}, \ell \neq 0, F \in \mathcal{F}(A_{i+1})$. Then there are no $\mathcal{F}(A_{i+1})$ -involutions.*

Proof. By Lemma 17, we may assume that $B^r \stackrel{i}{=} X^{-1}A_{i+1}^\ell X$ for some word X . Let J be a $\mathcal{F}(A_{i+1})$ -involution. Then $J_0^{-1}B^rJ_0 \stackrel{i}{=} B^{-r}$, where $J_0 \stackrel{i}{=} X^{-1}JX$. It follows from Lemma 17 and the definition of a $\mathcal{F}(A_{i+1})$ -involution that J (and therefore J_0) is of order two in rank i .

Choose s such that $|s| > n^2\|J_0\|$ and consider a reduced diagram Δ of rank i representing the equation

$$J_0^{-1}B^sJ_0 \stackrel{i}{=} B^{-s}.$$

Arguing as in the proof of Lemma 17, we obtain that $r(\Delta) = 0$. Consequently, $J_0^{-1}B^sJ_0 \stackrel{0}{=} B^{-s}$, and J_0 belongs to the elementary subgroup of B in $G(0)$. Since the order of J_0 in $G_\omega(i)$ is finite while the order of B in $G_\omega(i)$ is infinite, it follows that $J_0 \stackrel{i}{=} 1$ (and therefore $J \stackrel{i}{=} 1$). \square

The inductive proof is completed as in [19], [20]. As a result, we obtain a presentation

$$G_\omega(\infty) = \langle \mathcal{A} \parallel R \in \mathcal{R}_0 \cup \{A_1^{n(1)}, A_2^{n(2)}, \dots\} \rangle.$$

of an infinite group of exponent $2n$. The proof that $G_\omega(\infty)$ has trivial center repeats the argument from § 21 [19].

Now we explain that there is a continuum of pairwise non-isomorphic groups among $\{G_\omega(\infty)\}_\omega$. We notice first that, by Lemmas 10.1 and 10.2 [20], [19], and Lemma 16, the set of special periods is infinite and there is a special period corresponding to each element $B \in \mathcal{B}$. Presentations of groups $G_\omega(\infty)$ are now distinguished by the orders of special periods: if ω and ω' are two different sequences of 0's and 1's, find the first place (say, k) where they differ. By construction and Lemma 10.4 [20], [19], the special period that corresponds to B_k has different orders in $G_\omega(\infty)$ and $G_{\omega'}(\infty)$, meaning that $G_\omega(\infty)$ and $G_{\omega'}(\infty)$ are quotients of G over different normal subgroups. By Remark 2, we see that there is a continuous family of pairwise non-isomorphic infinite centerless quotients of G of exponent $2n$. Theorem 3 is proved.

8. PERIODIC QUOTIENTS OF LARGE GROUPS

This section is devoted to proving Theorem 4 and Corollaries 1,2.

Let \mathcal{C} be a collection of groups. A group G is said to be *residually in \mathcal{C}* if for every $g \in G \setminus \{1\}$ there is a homomorphism $\varphi : G \rightarrow H$ for some group $H \in \mathcal{C}$ such that $\varphi(g) \neq 1$. For a prime number p , denote by \mathcal{C}_p the collection consisting of all finite p -groups. Since every finite p -group is nilpotent we can make

Remark 3. 1) If H is a finite p -group, then there is an integer $n \in \mathbb{N}$ such that $\delta_n^p(H) = \{1\}$. 2) For a finitely generated group G , $\bigcap_{j=0}^{\infty} \delta_j^p(G) = \{1\}$ if and only if G is residually in \mathcal{C}_p .

Since free groups are residually in \mathcal{C}_p for any prime p ([23, 14.2.2]), we have

Lemma 19. *If F is a free group and $p \in \mathbb{N}$ is a prime number then $\bigcap_{j=0}^{\infty} \delta_j^p(F) = \{1\}$.*

Throughout this section F_2 will denote the free group of rank 2. For a group G , we will write $G \twoheadrightarrow F_2$ if there exists an epimorphism from G to F_2 . The lemma below was proved by Olshanskii and Osin in [37]. (We apply it in the special case when $\Pi = (p, p, \dots)$ is a constant sequence.)

Lemma 20. ([37, Lemma 3.2]) *Let G be a finitely generated group, $N \triangleleft G$ be a normal subgroup of finite index and let $p \in \mathbb{N}$ be a prime number. Suppose that $N \twoheadrightarrow F_2$. Then for any element $g \in N$ there is $m > 0$ such that if $g^n \in \delta_m^p(N)$, then $\delta_m^p(N)/\langle\langle g^n \rangle\rangle^G \twoheadrightarrow F_2$.*

(Here we modified the original formulation of [37, Lemma 3.2] by replacing $\delta_r^p(G)$ with an arbitrary finite index normal subgroup N , observing that the proof continues to be valid in this, more general, situation).

Lemma 21. *Suppose that G is a finitely generated group, $N \triangleleft G$ is a finite index normal subgroup, such that $N \twoheadrightarrow F_2$, and p is an arbitrary prime number. Then for each sequence $\omega \in \Omega = \{0, 1\}^{\mathbb{N}}$, there is a group H_{ω} containing a normal subgroup $Q_{\omega} \triangleleft H_{\omega}$ such that*

- (i) H_{ω} is a quotient of G ;
- (ii) $H_{\omega}/Q_{\omega} \cong G/N$;
- (iii) Q_{ω} is a periodic p -group;
- (iv) $\bigcap_{i=0}^{\infty} \delta_i^p(Q_{\omega}) = \{1\}$;
- (v) if $\omega \neq \omega' \in \Omega$ then there is $v \in \mathbb{N}$ such that $|Q_{\omega}/\delta_v^p(Q_{\omega})| \neq |Q_{\omega'}/\delta_v^p(Q_{\omega'})|$; consequently $Q_{\omega} \not\cong Q_{\omega'}$.

Proof. Our argument will be similar to the one used in the proof of [37, Thm. 1.2].

First, enumerate all the elements of N : $N = \{f_1, f_2, \dots\}$. Let $\Omega_i = \{0, 1\}^i$ be the set of sequences of 0's and 1's of length i , $i \in \mathbb{N}$, and let Ω_0 consist of the empty sequence \emptyset . Denote $G_{\emptyset} = G$, $r_{\emptyset} = q_{\emptyset} = 0$, and assume that for some $i \geq 0$ and for every $\iota \in \Omega_i$ we have already constructed the quotient-group G_{ι} of G and integers $r_{\iota}, q_{\iota} \geq 0$ such that the images of f_1, \dots, f_i have finite orders in G_{ι} and $\delta_{r_{\iota}}^p(N_{\iota}) \twoheadrightarrow F_2$, where N_{ι} is the image of N in G_{ι} under the natural homomorphism (the numbers q_{ι} are auxiliary, and will be used during the inductive argument).

Choose any $\zeta = (\zeta_1, \dots, \zeta_{i+1}) \in \Omega_{i+1}$, and take $\iota = (\zeta_1, \dots, \zeta_i) \in \Omega_i$.

Observe that the subgroup $E_{\iota} = \bigcap_{j=0}^{\infty} \delta_j^p(N_{\iota})$ is characteristic in N_{ι} , hence it is normal in G_{ι} . Denote $K_{\iota} = G_{\iota}/E_{\iota}$, $L_{\iota} = N_{\iota}/E_{\iota} \triangleleft K_{\iota}$; then

$$(4) \quad \bigcap_{j=0}^{\infty} \delta_j^p(L_{\iota}) = \{1\}.$$

Since $E_{\iota} = \bigcap_{j=0}^{\infty} \delta_j^p(\delta_{r_{\iota}}^p(N_{\iota}))$, Lemma 19 implies that $\delta_{r_{\iota}}^p(L_{\iota}) \twoheadrightarrow F_2$. Let $l \geq i + 1$ be the smallest index such that the image of f_l has infinite order in K_{ι} . Denote by g the image of $(f_l)^{p^{r_{\iota}}}$ in K_{ι} ; thus $g \in \delta_{r_{\iota}}^p(L_{\iota})$ is an element of infinite order.

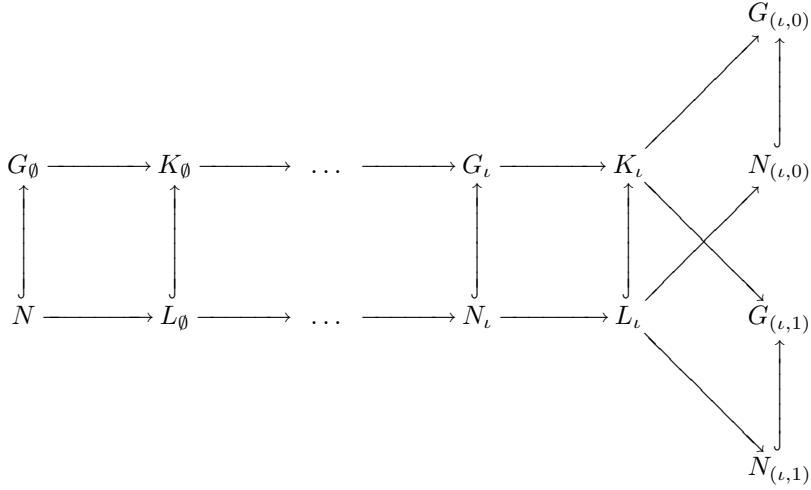
Now we apply Lemma 20 to find $m \in \mathbb{N}$ such that

$$(5) \quad \delta_{r_{\iota}+m}^p(L_{\iota}) / \langle \langle g^n \rangle \rangle^{K_{\iota}} = \delta_m^p(\delta_{r_{\iota}}^p(L_{\iota})) / \langle \langle g^n \rangle \rangle^{K_{\iota}} \twoheadrightarrow F_2,$$

for any integer n such that $g^n \in \delta_{r_{\iota}+m}^p(L_{\iota})$.

Set $s = \max\{r_{\iota} + m, q_{\iota}\}$ so that $g^{p^s} \in \delta_s^p(L_{\iota}) \leq \delta_{r_{\iota}+m}^p(L_{\iota}) \cap \delta_{q_{\iota}}^p(L_{\iota})$. By (4) one can find the smallest integer v_{ι} such that $g^{p^s} \notin \delta_{v_{\iota}}^p(L_{\iota})$. Set $r_{\zeta} = r_{\iota} + m$ and $q_{\zeta} = v_{\iota}$. Note that $q_{\zeta} > s \geq q_{\iota}$, and the integers r_{ζ} and q_{ζ} only depend on the prefix ι of the sequence ζ , and are independent of the value of ζ_{i+1} .

If $\zeta_{i+1} = 0$, define $n_{\zeta} = p^s$ (thus, $g^{n_{\zeta}} \in L_{\iota} \setminus \delta_{v_{\iota}}^p(L_{\iota})$), otherwise, if $\zeta_{i+1} = 1$, define $n_{\zeta} = p^{q_{\zeta}}$ (thus, $g^{n_{\zeta}} \in \delta_{v_{\iota}}^p(L_{\iota})$). Denote $G_{\zeta} = K_{\iota}/\langle \langle g^{n_{\zeta}} \rangle \rangle^{K_{\iota}}$, and let N_{ζ} be the image of L_{ι} in G_{ζ} (see the diagram below).



Observe, that

$$\delta_{r_\zeta}^p(N_\zeta) \twoheadrightarrow F_2 \text{ by (5),}$$

$$(6) \quad N_\zeta/\delta_u^p(N_\zeta) \cong L_\iota/\delta_u^p(L_\iota) \cong N_\iota/\delta_u^p(N_\iota) \text{ for every } u \leq q_\iota, \text{ and}$$

$$(7) \quad |N_{(\iota,0)}/\delta_{v_\iota}^p(N_{(\iota,0)})| < |N_{(\iota,1)}/\delta_{v_\iota}^p(N_{(\iota,1)})| = |L_\iota/\delta_{v_\iota}^p(L_\iota)| = |N_\iota/\delta_{v_\iota}^p(N_\iota)| < \infty.$$

Thus, for each $i \geq 0$ and each $\iota \in \Omega_i$, we have constructed a quotient G_ι of G . Additionally, we have an epimorphism $G_\iota \twoheadrightarrow G_\zeta$ whenever ι is a prefix of ζ . Therefore, for every $\omega \in \Omega$ we can define the group G_ω as a direct limit of the groups G_ι , where ι runs over all prefixes of ω . Let $N_\omega \triangleleft G_\omega$ be the image of N in G_ω , and let H_ω be the quotient of G_ω by $E_\omega = \bigcap_{j=0}^\infty \delta_j^p(N_\omega)$. Finally, let Q_ω be the image of N_ω in H_ω .

By construction, Q_ω is periodic (as a quotient of N), and $\bigcap_{j=0}^\infty \delta_j^p(Q_\omega) = \{1\}$. Since N is finitely generated, then so is Q_ω . Therefore, for every j , $Q_\omega/\delta_j^p(Q_\omega)$ is a finite group whose order divides p^j , hence Q_ω is a p -group.

The second claim of the lemma also follows from the construction. It remains for us to prove the fifth claim. Suppose $\omega \neq \omega' \in \Omega$, and let $\iota \in \Omega_i$, $i \geq 0$, be the longest common prefix of ω and ω' .

By (6) we have

$$Q_\omega/\delta_{v_\iota}^p(Q_\omega) \cong N_\omega/\delta_{v_\iota}^p(N_\omega) \cong N_\zeta/\delta_{v_\iota}^p(N_\zeta) \text{ and}$$

$$Q_{\omega'}/\delta_{v_\iota}^p(Q_{\omega'}) \cong N_{\omega'}/\delta_{v_\iota}^p(N_{\omega'}) \cong N_{\zeta'}/\delta_{v_\iota}^p(N_{\zeta'}),$$

where $\zeta, \zeta' \in \Omega_{i+1}$ are prefixes of length $i+1$ of ω and ω' respectively. Using this together with (7) we get

$$|Q_\omega/\delta_{v_\iota}^p(Q_\omega)| = |N_\zeta/\delta_{v_\iota}^p(N_\zeta)| \neq |N_{\zeta'}/\delta_{v_\iota}^p(N_{\zeta'})| = |Q_{\omega'}/\delta_{v_\iota}^p(Q_{\omega'})|,$$

yielding (v). Thus the lemma is proved. \square

Proof of Theorem 4. Applying Lemma 21 to G , N and p , for each $\omega \in \Omega$ we obtain a group H_ω together with a normal subgroup Q_ω of index $k = |G : N|$ enjoying the properties (i)-(v) from the claim of Lemma 21.

Since Q_ω is finitely generated, property (iv) implies that it is residually finite. And since $|H_\omega : Q_\omega| = |G : N| < \infty$ by (ii), H_ω is residually finite for every $\omega \in \Omega$. Property (iii) ensures that Q_ω is a p -group, and, hence, the group H_ω is periodic.

It remains to observe that for every $\omega \in \Omega$, the finitely generated group G_ω can have only finitely many distinct normal subgroups of index k , hence among $\{G_\omega \mid \omega \in \Omega\}$ there must be $|\Omega| = 2^{\aleph_0}$ of pairwise non-isomorphic groups. \square

Proof of Corollary 1. First, we observe that since G/N is a p -group, there is $n \in \mathbb{N}$ such that $\delta_n^p(G) \leq N$ (by the first part of Remark 3). Since $|G : \delta_n^p(G)| < \infty$ and there is an epimorphism $\varphi : N \twoheadrightarrow F_2$, the subgroup $\varphi(\delta_n^p(G))$ has finite index in $\varphi(N) = F_2$. Hence $\delta_n^p(G) \twoheadrightarrow F_2$. Thus, without loss of generality, we can suppose that $N = \delta_n^p(G)$ for some $n \in \mathbb{N}$.

Now, we apply Lemma 21 to G , $N = \delta_n^p(G)$ and p , to obtain the groups H_ω as in its claim. Since $H_\omega/Q_\omega \cong G/\delta_n^p(G)$ is a finite p -group, the property (iii) of Q_ω implies that H_ω is a periodic p -group. From the proof of Lemma 21 we see that Q_ω is the image of N under the natural homomorphism $G \rightarrow H_\omega$, therefore $Q_\omega = \delta_n^p(H_\omega)$, and for every $j \in \mathbb{N}$, $\delta_j^p(Q_\omega) = \delta_{j+n}^p(H_\omega)$. Hence, property (iv) of Q_ω yields property (b) of H_ω , and properties (ii) and (v) together yield (c) (for $t = v + n$). \square

Proof of Corollary 2. By a theorem of Baumslag and Pride [5], for any prime p there are a normal subgroup $N \triangleleft G$ and $m \in \mathbb{N}$ such that $|G : N| = p^m$ and $N \twoheadrightarrow F_2$. To conclude, it remains to apply Corollary 1 to the pair (G, N) . \square

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