FORCING, DAMPING AND DETUNING FOR SINGLE AND COUPLED VAN DER POL OSCILLATORS

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Abstract. We use Melnikov function techniques together with geometric methods of bifurcation theory to study the interactions of forcing, damping and detuning on resonant periodic orbits for single and coupled forced Van Der Pol oscillators. For a coupled pair the local bifurcation geometry relates to singularities in line congruences.

Key words. forced Van Der Pol, coupled oscillators, detuning, bifurcation, line congruence

AMS subject classifications. 34C15, 34C23, 34C25, 34K18, 37G15, 58K05

1. Introduction.

The search for harmonic and subharmonic periodic orbits of forced nonlinear oscillators such as the Van Der Pol equation and the Duffing equation has a long and distinguished pedigree. We do not attempt to survey the literature here, but refer to standard texts such as [18, 17] for background. See also [11, 12, 13, 16, 8] for some more recent studies of local and global dynamical phenomena. In this paper we use the harmonic Melnikov function approach, as refined and developed in a series of papers by Chicone [2, 3, 4], to investigate the interactions of small-amplitude forcing, damping and detuning on bifurcations of certain harmonic periodic orbits for one and then a coupled pair of Van Der Pol oscillators. The new contribution that we make to this study is the use of geometric methods to understand the picture of bifurcations for the multi-parameter families of Melnikov functions that arise. The techniques fall in to the general framework of multi-parameter bifurcation from a manifold as described in [5], and take their inspiration from Chicone's methods and from earlier work of Hale and Taboas [14] (see also [6]) on the interaction of forcing and damping for nonlinear oscillators.

The general technique is as follows. Let γ be a periodic orbit of period $T > 0$ for an autonomous system of ordinary differential equations in $\mathbf{R}^n$, assuming sufficiently regularity so that solutions are unique and that they are defined for all $t \in \mathbf{R}$. Then each point of γ is a fixed point for the map $\phi_T : \mathbf{R}^n \to \mathbf{R}^n$ where $\{\phi_t\}$ is the flow generated by the solutions of the equations. If we now perturb the system by applying a small-amplitude forcing of period T' close to T, while possibly at the same time changing the original equations slightly through the introduction of damping, for example, then the map $F : \mathbf{R}^n \to \mathbf{R}^n$ defined by following solutions of the nonautonomous perturbed system for time T' is close to the original map ϕ_T. The problem is to find the fixed points (if any) of F that are close to γ or, to express it another way, to find which points of γ *persist* nearby as periodic points with period T' for the perturbed system.

A standard method, which presumably goes back at least to Poincaré, is to consider the perturbation as depending on a single parameter $\varepsilon \in \mathbf{R}$ (with no perturbation when $\varepsilon = 0$) and to expand the corresponding map $F = F_\varepsilon$ in powers of ε on a neighbourhood U of γ. The *displacement map*, that is the difference

$$P_\varepsilon = F_\varepsilon - id$$

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between F_ε and the identity map, vanishes at least on γ when $\varepsilon = 0$, but may vanish on a larger set: indeed, in a wide class of examples (including ours) in which $n = 2$ and the unperturbed system is a simple harmonic oscillator, the map P_0 vanishes on the whole of $\mathbf{R}^2$. Let $V \subset U$ denote the zero set of P_0. Then at points $x \in V$ the expansion of F_ε in powers of ε has ε as a factor. After removal of this factor, the remaining term without ε is the first derivative

$$P' := \frac{\partial P_\varepsilon}{\partial \varepsilon}|_{\varepsilon=0} : V \to \mathbf{R}^n,$$

called the *bifurcation map.* Finally, by the implicit function theorem (IFT), provided the zeros of P' are nondegenerate (the Jacobian matrix of P' there is nonsingular), these zeros will persist nearby as zeros of P_ε for small $\varepsilon \neq 0$, corresponding to periodic orbits of the perturbed system with period T'.

Further questions arise if (a) there is more than one independent perturbation parameter, and/or (b) the map P' has degenerate zeros. In fact if (a) applies then we can expect there to be parameter values where (b) applies also. A class of ODE problems where this happens is studied by Hale and Taboas [14]: see also [6]. A general setting for the study of multi-parameter bifurcation problems of this type is described in [5]. In the present paper we use these techniques to analyze the particular examples of a forced Van Der Pol equation and a pair of coupled forced Van Der Pol equations, giving a new geometric perspective on an old problem. We are able to do this in view of the explicit expression for P' that is available when $n = 2$. This formula, due to Diliberto [7], is (with slight correction) exploited to great effect by Chicone in the papers [2, 3, 4] which are the main inspiration for the present work.

2. The general formula for planar systems. We first consider the general system

$$\dot{x} = f(x) + \varepsilon g(x, t; \varepsilon) \qquad x \in \mathbf{R}^2, \quad \varepsilon \in \mathbf{R} \tag{$\mathcal{S}_\varepsilon$}$$

in a neighbourhood U of a periodic orbit γ of period $T > 0$ for the system $\mathcal{S}_0$ (that is, $\varepsilon = 0$), where the maps f and g are assumed to be C^∞ and there are no zeros of f in U. At each point $\xi \in U$ we have a basis for $\mathbf{R}^2$

$$\mathcal{B}(\xi) = \{f(\xi), f^\perp(\xi)\}$$

where if $u = (v, w) \in \mathbf{R}^2$ then $u^\perp$ denotes the vector $(-w, v)$. Let $V \subset U$ be the set of zeros of the displacement map P_0 in U, that is the set of T-periodic orbits in U. For $\xi \in V$ let $\gamma(\xi)$ denote the (T-periodic) orbit of ξ for the unperturbed system $\mathcal{S}_0$.

First we suppose that the perturbing vector field g also has period T in t. Then we have Diliberto's result [7] (see also [2, 3]):

THEOREM 2.1. *The bifurcation map $P' : V \to \mathbf{R}^2$ has the form*

$$P'(\xi) = (\mathcal{N}(\xi), \mathcal{M}(\xi)) \tag{2.1}$$

with respect to the basis $\mathcal{B}(\xi)$, where $\mathcal{N}, \mathcal{M} : V \to \mathbf{R}$ are given by

$$\mathcal{N}(\xi) = \int_0^T \|f\|^{-2}\Big\{f.g - \frac{\alpha(t)}{\beta(t)} f^\perp.g\Big\}dt \tag{2.2}$$

$$\mathcal{M}(\xi) = \int_0^T \|f\|^{-2}\Big\{\frac{1}{\beta(t)} f^\perp.g\Big\}dt, \tag{2.3}$$

with f, g evaluated at $\phi_t(\xi)$: thus the integrals take place along $\gamma(\xi)$. Here the functions $\alpha(t)$ and $\ln\beta(t)$ correspond to the first-order variation in (respectively) time and displacement transverse to the orbit $\gamma(\xi)$ for orbits of $\mathcal{S}_0$ close to $\gamma(\xi)$. Explicit formulae for $\alpha(t)$ and $\beta(t)$ can be found in [7] and [2, 3]. However, in the example that we study the system $\mathcal{S}_0$ is just simple harmonic motion and so $\alpha(t) = 0$ and $\beta(t) = 1$ for all t, and the expressions (2.2),(2.3) become

$$\mathcal{N}(\xi) = \int_0^T \|f\|^{-2}\Big\{f.g\Big\}dt \tag{2.4}$$

$$\mathcal{M}(\xi) = \int_0^T \|f\|^{-2}\Big\{f^{\perp}.g\Big\}dt. \tag{2.5}$$

In cases when $\alpha \neq 0$ or $\beta \neq 1$ then $\gamma(\xi)$ is called *normally nondegenerate* (see [2, 3]), and in this case $\gamma(\xi)$ is an isolated periodic orbit (limit cycle). Finding zeros of P' at points of $\gamma(\xi)$ and checking their nondegeneracy then reduces (via Liapunov-Schmidt reduction) to studying the zeros of a single function $C : \gamma(\xi) \to \mathbf{R}$ called the *bifurcation function.* In our applications we are not able to make this reduction but, as we next see, the introduction of more than one parameter gives a different structure to the problem and allows this difficulty to be bypassed.

2.1. The effect of detuning. Suppose now, in contrast to the above, that the t-period T' of $g(x, t; \varepsilon)$ is not exactly T when $\varepsilon \neq 0$. More precisely, suppose that

$$T' = T + k\varepsilon + O(\varepsilon^2).$$

Then we call k the *detuning* parameter. It is easy to verify (see [2, end of Section 4]) the following simple but crucial fact:

Proposition 2.2. *The effect of detuning on the map P' is to replace the component $\mathcal{N}$ by $\mathcal{N} + k$.* As we shall see, it is the introduction of nonzero detuning that allows us to describe appropriate generic bifurcation behaviour.

3. More than one parameter. Consider next a system in $\mathbf{R}^n$ of the form

$$\dot{x} = f(x) + G(x, t; \varepsilon)\varepsilon \tag{$\widetilde{\mathcal{S}}_\varepsilon$}$$

where now $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)^{\mathtt{t}} \in \mathbf{R}^d$ and $G(x, y; \varepsilon)$ is an $n \times d$ matrix varying smoothly with its arguments. Here we regard x, ε as column vectors and the superscript $^{\mathtt{t}}$ denotes transposition. By setting all except one of the ε_i equal to zero we obtain a 1-parameter system of type $(\mathcal{S}_\varepsilon)$ and a corresponding displacement map with its ε_i-derivative defining the bifurcation map

$$P'_i : V \to \mathbf{R}^n.$$

We then assemble these to form a map $\mathcal{P}$ from V to the space $L(d, n)$ of $n \times d$ real matrices:

Definition 3.1. *The* bifurcation matrix $\mathcal{P}(\xi)$ *is the $n \times d$ matrix whose i^{th} column is $P'_i(\xi) \in \mathbf{R}^n$ for $i = 1, \ldots, d$. The map $\mathcal{P} : V \to L(d, n)$ is the* bifurcation map.

With the displacement map $P_\varepsilon = F_\varepsilon - id : U \to \mathbf{R}^n$ defined as before but now with $\varepsilon \in \mathbf{R}^d$ we have

$$P_\varepsilon(\xi) = \mathcal{P}(\xi)\varepsilon + O(|\varepsilon|^2). \tag{3.1}$$

Taking polar coordinates in $\mathbf{R}^d$ by writing $\varepsilon = \rho s$ where $\rho > 0$ and s belongs to the unit sphere S^{d-1} in $\mathbf{R}^d$, the expression (3.1) becomes

$$P_\varepsilon(\xi) = \rho\mathcal{P}(\xi)s + O(\rho^2) \tag{3.2}$$

from which it is clear that if $s_0 \notin \ker\mathcal{P}(\xi_0)$ then for sufficiently small $\rho > 0$ there are no solutions to $P_\varepsilon(\xi) = 0$ for any ξ in a neighbourhood of ξ_0 and any ε in a cone about the s_0-axis in $\mathbf{R}^d$. Hence we are able to state the following result:

PROPOSITION 3.2. *A necessary condition for a solution branch to emanate from $\xi_0 \in V$ as ε moves away from the origin in $\mathbf{R}^d$ in the direction of $s_0 \in S^{d-1}$ is that $s_0 \in \ker\mathcal{P}(\xi_0)$.* □ That this necessary condition is not sufficient will become clear from examples that we consider below. However, the condition is sufficient if $s_0 \in \ker\mathcal{P}(\xi_0)$ in a way which is nondegenerate with respect to the family of kernels $\ker\mathcal{P}(\xi)$ as ξ varies near ξ_0. The precise meaning of this statement varies depending on the relative sizes of n and d, and we refer to [5] for a fuller discussion of this issue. In the case of a single forced Van Der Pol system we shall be concerned with $n = d = 2$. Here, a typical 2×2 matrix is nonsingular and so has zero kernel, and it is a codimension-1 occurrence for the matrix to have kernel of dimension 1. This is the geometry we exploit in order to locate resonant periodic orbits. For a coupled pair of Van Der Pol systems we shall instead be led to study the 2-dimensional kernels of a certain family of 2×4 matrices of maximal rank.

4. Single Van Der Pol equation. In this section we consider a single forced Van Der Pol system:

$$\begin{aligned} \dot{x} &= \omega_0 y \\ \dot{y} &= -\omega_0 x - \delta(x^2 - 1)y + \varepsilon\cos(\omega t) \end{aligned} \tag{4.1}$$

where δ, ε are small constants and where the angular frequency ω of the forcing term is close to the natural frequency ω_0. Moreover, we allow ω itself to vary with ε so that

$$\frac{2\pi}{\omega} = \frac{2\pi}{\omega_0} + k\varepsilon + O(\varepsilon^2), \tag{4.2}$$

where k is the detuning parameter.

Evaluation of the integrals that appear in $\mathcal{N}$ and $\mathcal{M}$ in (2.4) and (2.5) is in this case very straightforward, since $\phi_t(\xi) = R_{-t}\xi$ where R_t is the 2×2 rotation matrix

$$R_t = \begin{pmatrix} \cos\omega_0 t & -\sin\omega_0 t \\ \sin\omega_0 t & \cos\omega_0 t \end{pmatrix}.$$

Taking $(\varepsilon_1, \varepsilon_2) = (\delta, \varepsilon)$ and $\xi = (r\cos\theta, r\sin\theta)$ for $r > 0$ in (3.1) we find using (2.4),(2.5) (with $T = 2\pi/\omega_0$) that the bifurcation matrix is

$$\mathcal{P}(r,\theta) = \frac{\pi}{\omega_0^2}\begin{pmatrix} 0 & k\pi^{-1}\omega_0^2 + r^{-1}\cos\theta \\ (1 - \frac{1}{4}r^2) & r^{-1}\sin\theta \end{pmatrix}. \tag{4.3}$$

The following results are now easy to check:

PROPOSITION 4.1.

1. *If $\pi\cos\theta + kr\omega_0^2 \neq 0$ then $\mathcal{P}(r,\theta)$ has zero kernel, except when $r = 2$ in which case $\ker\mathcal{P}(r,\theta)$ is spanned by the vector $(1,0)^{\mathsf{t}}$.*

2. *If* $\pi\cos\theta + kr\omega_0^2 = 0$ *and* $r \neq 2$ *then*

$$\ker\mathcal{P}(r,\theta) = span\{(-4\sin\theta, r(4-r^2))^{\mathtt{t}}\}.$$

Let C and E denote the circles in $\mathbf{R}^2$ given respectively by the equations $r = 2$ and $r = -a\cos\theta$ where $a = \pi(k\omega_0^2)^{-1}$.

COROLLARY 4.2. *Given any neighbourhood* W *of* $C \cup E$ *in* $\mathbf{R}^2$*, then for sufficiently small* $(\delta,\varepsilon) \neq (0,0)$ *and detuning* $k \neq 0$ *as in (4.2), every* $\frac{2\pi}{\omega}$*-periodic point for the system (4.1) lies in* W*.* Consider now the kernel map

$$\kappa : C \cup E \to \mathbf{R}P^2 : \xi \mapsto \ker\mathcal{P}(\xi).$$

For $(\delta,\varepsilon) \neq (0,0) \in \mathbf{R}^2$ we denote the line through the origin and (δ,ε) by $\mathbf{R}(\delta,\varepsilon)$ and write $[(\delta,\varepsilon)]$ for the element of $\mathbf{R}P^2$ that this represents.

For all $\xi \in C$ we have $\kappa(\xi) = [(1,0)]$, corresponding to perturbations with $\varepsilon = 0$ and with δ nonzero. The associated bifurcation map $P'_\delta : \mathbf{R}^2 \to \mathbf{R}^2$ is a nonzero constant scalar multiple of $\xi \mapsto (0, (4-r^2))^{\mathtt{t}}$ where $r = |\xi|$, which is highly degenerate (having 1-dimensional image) and from which *a priori* we can deduce little about the zero set of the displacement map P_δ itself: in particular, it may be empty. However, when $\varepsilon = 0$ the system (4.1) is the standard autonomous Van Der Pol system (see [17] for example) which is well known to have a unique periodic orbit that tends to the circle C and whose period tends to $2\pi/\omega_0$ as $\delta \to 0$.

We therefore turn now to the case of $\xi \in E$, where it is the interactions of δ and ε with nonzero detuning k that are important.

Points $\xi \in \tilde{E} := E \setminus \{(0,0)\}$ can be parametrised as

$$\xi = (r\cos\theta, r\sin\theta) = -\frac{a}{2}(1 + \cos 2\theta, \sin 2\theta) \tag{4.4}$$

where $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$. From Proposition 4.1 the image $\kappa(\tilde{E})$ in $\mathbf{R}P^2$ can be realised by the closed curve $B = B^+ \cup B^-$ in $\mathbf{R}^2$ with $B^\pm$ given by the parametrisation

$$\psi_\pm : \theta \mapsto \pm\big(4\sin\theta, a\cos\theta(4 - a^2\cos^2\theta)\big)$$

so that $[\psi_\pm(\theta)] = \kappa(\xi)$ for $\xi \in \tilde{E}$ given by (4.4). This is a convenient way to represent the map κ geometrically, although of course it is only the polar angle of $\psi_\pm(\theta)$ that is significant while $|\psi_\pm(\theta)|$ is irrelevant. Thus for $\xi \in \tilde{E}$ and $(\delta,\varepsilon) \in \mathbf{R}^2$ we have $(\delta,\varepsilon) \in \ker\mathcal{P}(\xi)$ precisely when the line $\mathbf{R}(\delta,\varepsilon)$ intersects B at $\psi_\pm(\theta)$ for the value of θ corresponding to ξ on $\tilde{E}$. Provided this intersection is in general position (transverse) we can deduce the existence of zeros of $P_{\delta,\varepsilon}$ close to ξ.

THEOREM 4.3. *Suppose* $(\delta_0,\varepsilon_0) \neq (0,0) \in \mathbf{R}^2$ *and the line* $\mathbf{R}(\delta_0,\varepsilon_0)$ *intersects* B *transversely at the point* $\psi(\theta_0)$ *where* $\theta_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ *represents the point* $\xi_0 \in E$*. Then for sufficiently small* $\rho > 0$ *and* $(\delta,\varepsilon) = \rho(\delta_0,\varepsilon_0)$ *the system* (4.1) *has a* $\frac{2\pi}{\omega}$*-periodic point close to* ξ_0 *in* $\mathbf{R}^2$*, with orbit tending uniformly to that of* ξ_0 *in all derivatives as* $\rho \to 0$*.* *Proof.* From (3.2) the solutions to $P_{(\delta,\varepsilon)}(\xi) = 0$ with $\rho > 0$ are given by

$$Q(\xi, s, \rho) := \mathcal{P}(\xi)s + O(\rho) = 0 \tag{4.5}$$

where $(\delta,\varepsilon) = \rho s$ with $s \in S^1$. If $s_0 \in \ker\mathcal{P}(\xi_0)$ then $Q(\xi_0, s_0, 0) = 0$. The transversality assumption means that $\frac{\partial Q}{\partial s}(\xi_0, s_0, 0) \neq 0$, and so the IFT implies that (4.5) has a unique C^∞ solution $s = s(\xi,\rho)$ with ξ close to ξ_0 for small ρ and with $s_0 = s(\xi_0, 0)$. □

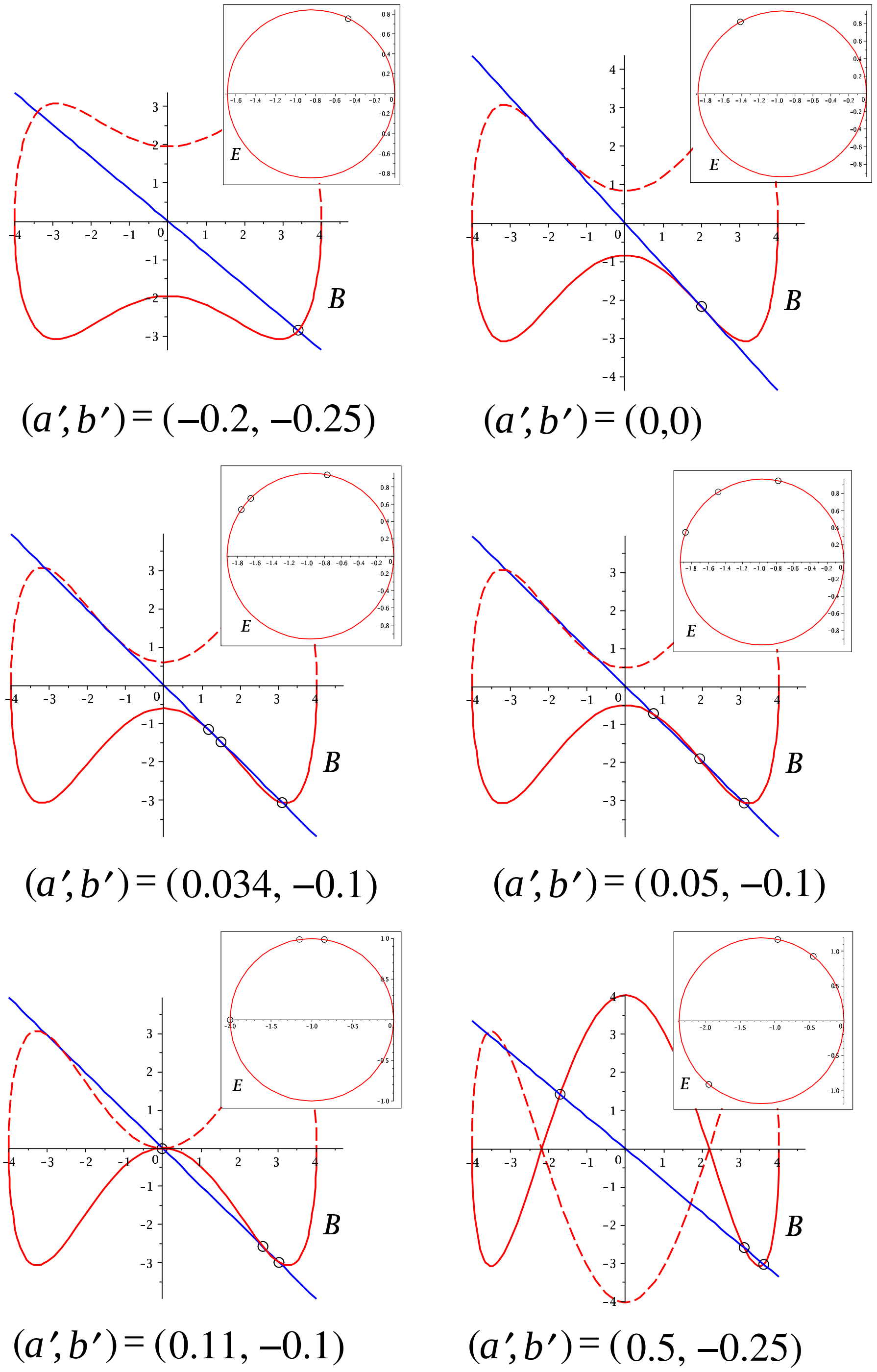

FIG. 4.1. *The bifurcation curve B for various choices of a, showing its intersections with a line $\varepsilon = b\delta$ and the corresponding periodic points on the circle E. Here $(a, b) = (a_2, b_2) + (a', b')$.*

Following the ideas first set out by Hale and Taboas [14] we can say more. As we vary (δ, ε) in $\mathbf{R}^2$ the number of intersections of the line $\mathbf{R}(\delta, \varepsilon)$ with the curve B may vary: typically it will change by 2 as the line passes through a nondegenerate (quadratic) tangency with the curve.

PROPOSITION 4.4. *Suppose the line* $L_1 := \mathbf{R}(\delta_1, \varepsilon_1)$ *has a nondegenerate tangency with the curve* B *at the point* $\psi_\pm(\theta_1)$. *Then there is a* C^1 *curve* Γ_1 *in* $\mathbf{R}^2$, *tangent to* L_1 *at the origin, such that pairs of fixed points of* $P_{(\delta,\varepsilon)}$ *(that is, pairs of* $\frac{2\pi}{\omega}$*-periodic orbits of (4.1)) are created as* (δ, ε) *crosses* Γ_1 *sufficiently close to the origin.*

Proof. The statement clearly holds for solutions to $Q(\xi, s, 0) = 0$, in the notation of Theorem 4.3, with $\Gamma_1 = L_1$. The fact that this behaviour persists for $Q(\xi, s, \rho) = 0$ for small $\rho > 0$ is a consequence of the local persistence (stability) of nondegenerate tangency. For details of this argument see [14], [6] or [5]. □

In Figure 4.1 we show plots of the curve B for various choices of the parameter a (proportional to k^{-1} as in Proposition 4.1), here chosen > 0, together with certain choices of lines $\varepsilon = b\delta$ through the origin that intersect B transversely. We also show the points on E associated to these intersections, that is the corresponding points on the relevant periodic orbits of the unperturbed system from which period-$\frac{2\pi}{\omega}$ periodic orbits of the perturbed system will bifurcate. As the figures suggest, radial lines through the origin in $\mathbf{R}^2$ are typically transverse to or have nondegenerate tangency with B, although there is precisely one value a_2 of the parameter a at which three intersections are created from one via a pitchfork bifurcation at a degenerate (cubic) tangency.

We now describe this behaviour more precisely, in terms of k which we first assume to be positive. Let $L(\alpha)$ denote the line through the origin in $\mathbf{R}^2$ with polar angle $\alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, that is $b = \tan\alpha$.

PROPOSITION 4.5. *There are three ranges of k in which behaviour differs.*

(1) $0 < k < k_1 = \frac{\pi}{2\omega_0^2}$ *(that is $a > 2$).*

Each line $L(\alpha)$ *with* $-frac\pi 2 \leq \alpha \leq 0$ *intersects* B^+ *transversely at a point* $\psi_+(\theta_0)$ *with* $\pi \leq \theta_0 \leq \theta(a) \leq \frac{3\pi}{2}$ *where* $\cos\theta(a) = -2/a$. *As* $k \to k_1$ *we have* $\theta_0 \to \pi$ *since* $\theta(a) \to \pi$.

Moreover, there is a value $\alpha_1(k) < 0$ *such that* $L(\alpha_1)$ *is tangent to* B^+ *at* $\psi_+(\theta_1)$ *where* $\frac{\pi}{2} < \theta_1 < \pi$, *the tangency being nondegenerate. If* $\alpha_1(k) < \alpha < 0$ *there are two transverse intersections of* $L(\alpha)$ *with* B^+ *at points* $\psi_+(\theta)$ *for* $\theta = \theta_2, \theta_3$ *where* $\frac{\pi}{2} < \theta_2 < \theta_1 < \theta_3 < 2\pi - \theta(a)$, *while if* $\alpha_1(k) < \alpha$ *there are none.*

Note: when $k = k_1$ *($a = 2$) the curve B passes through the origin and the analysis in terms of* $\ker \mathcal{P}(\xi)$ *breaks down.*

(2) $k_1 < k < k_2 = \frac{3\pi}{4\sqrt{2}\omega_0^2}$ *(that is* $2 > a > a_2 = \frac{4\sqrt{2}}{3} \sim 1{\cdot}89$*).*

There are precisely two values $-\frac{\pi}{2} < \alpha_1(k) < \alpha_1'(k) < 0$ *of α for which $L(\alpha)$ is tangent to B^+, the tangencies being nondegenerate. The tangency points are* $\psi_+(\theta)$ *for* $\theta = \theta_1, \theta_1'$ *where* $\frac{\pi}{2}\pi < \theta_1 < \theta_1' < \pi$. *If* $\alpha_1(k) < \alpha < \alpha_1'(k)$ *then* $L(\alpha)$ *intersects* B^+ *transversely at three points, while if* $-\frac{\pi}{2} < \alpha < \alpha_1(k)$ *or* $\alpha_1'(k) < \alpha < 0$ *it does so at just one point.*

When $k = k_2$ *there is a cubic tangency of* $L(\alpha_2)$ *with* B^+, *where* $\tan(\alpha_2) = b_2 = -\frac{4\sqrt{2}}{3\sqrt{3}}$.

(3) $k_2 < k$ *(that is $a < a_2$).*

For all $-\frac{\pi}{2} < \alpha < 0$ *there is precisely one intersection of* $L(\alpha)$ *with* B^+ *at* $\psi_+(\theta)$ *where* $\frac{\pi}{2} < \theta < \pi$.

Proof. For $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$ the condition that $\psi(\theta)$ and $\psi'(\theta)$ be linearly dependent is easily verified to be

$$2a^2x^2 - 3a^2x + 4 = 0$$

where $x = \cos^2\theta$. For $0 < a < a_2$ this equation has no real solutions. As a increases through a_2 a pair of solutions is created at $x = \frac{3}{4}$ (corresponding to $\theta = \pi \pm \frac{\pi}{6}$): one of these solutions increases through $x = 1$ (that is $\theta = \pi$) as a increases through 2, and the other solution remains in the interval $(0, \frac{3}{4})$ for all a. Reinterpreting these statements in terms of $k = \frac{\pi}{a\omega_0^2}$ gives the results as described. □

COROLLARY 4.6. *For each* k *with* $0 < k < k_1$ *there is a unique* C^1 *curve* $\Gamma_0(k)$ *through the origin in the* (δ, ε)*-plane such that saddle-node bifurcations of periodic orbits with angular frequency* ω *close to* ω_0 *occur as* (δ, ε) *crosses* $\Gamma_0(k)$ *sufficiently closely to the origin. For* $k_1 < k < k_2$ *there are two such curves* $\Gamma_1(k), \Gamma_1'(k)$*, which approach each other and mutually annihilate as* $|k|$ *increases through* k_2. □ Points (δ, ε) close to the origin and lying in the region between $\Gamma_1(k)$ and $\Gamma_1'(k)$ correspond to *three* points on the circle E, while those outside that region correspond to just one point on E. In the (a, α)-parameter space this behaviour is organised by a cusp bifurcation at (a_2, α_2). See Figure 4.2.

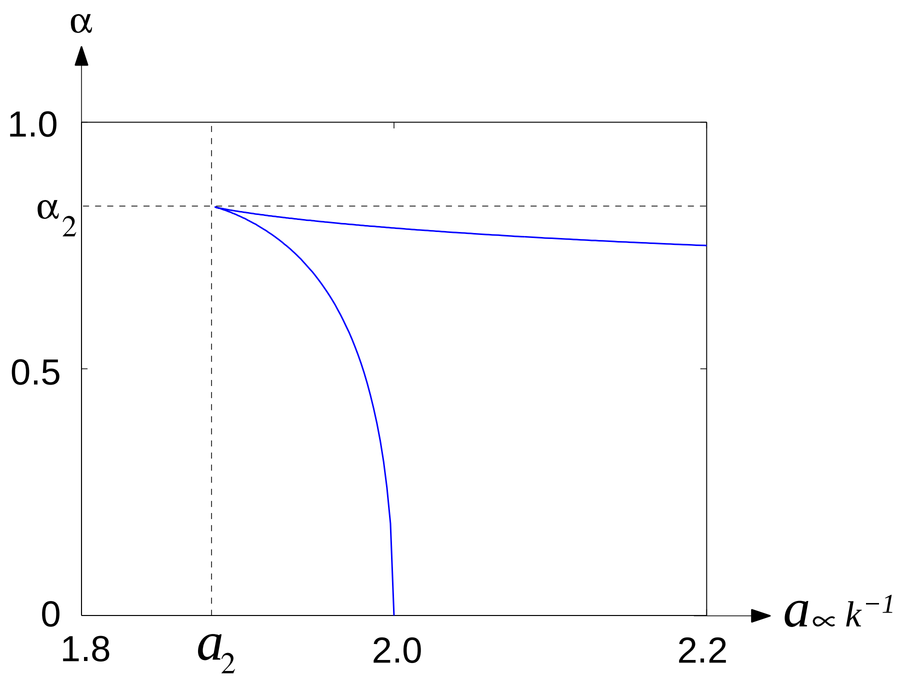

FIG. 4.2. *Bifurcation diagram for the linear problem associated to the system (4.1) where* $a = \pi(k\omega_0^2)^{-1}$ *and* $\varepsilon = \delta\tan\alpha$*: the curve corresponds to saddle-node bifurcations of* $\frac{2\pi}{\omega}$*-periodic orbits, with a cusp bifurcation point at* (a_2, α_2)*. For small* δ, ε *this approximates the true bifurcation behaviour.*

Finally we note that if we take $0 \leq \alpha \leq \frac{\pi}{2}$ the results are repeated using B^- in place of B^+ and replacing θ by $2\pi - \theta$. Moreover, if we choose $k < 0$ throughout then the results are analogous with now $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ rather than $[\frac{\pi}{2}, \frac{3\pi}{2}]$, the circle E lying to the right of the y-axis in this case.

We now turn to apply similar ideas to a coupled pair of oscillators of this same type.

5. A coupled pair of forced Van Der Pol equations. In this section we study a coupled system of equations of the form

$$\begin{aligned} \dot{u}_1 &= v_1 \\ \dot{v}_1 &= -u_1 - c_1 u_2 + h_1 \\ \dot{u}_2 &= v_2 \\ \dot{v}_2 &= -c_2 u_1 - u_2 + h_2 \end{aligned} \tag{5.1}$$

where c_1, c_2 are positive constants and the terms h_1, h_2 are as yet unspecified but considered to be smooth functions of t with small amplitude. After a suitable change of coordinates we derive a bifurcation map as in Section 3, and then specialise to the case where for $i = 1, 2$

$$h_i = \delta_i v_i (1 - u_i^2) + \varepsilon_i \cos(\tilde{\omega}_i t)$$

that characterises the forced Van Der Pol systems. Here the angular frequency $\tilde{\omega}_i$ is close to ω_i, so that

$$\frac{2\pi}{\tilde{\omega}_i} = \frac{2\pi}{\omega_i} + k_i \varepsilon_i + O(\varepsilon_i^2)$$

with k_i the detuning parameter for $i = 1, 2$.

Putting $h_1 = h_2 = 0$ leaves (5.1) as a linear system $\dot{x} = Ax$ with

$$x = (u_1, v_1, u_2, v_2)^{\mathtt{t}} \in \mathbf{R}^4$$

in which the matrix A has eigenvalues $\lambda_j = \pm i\omega_j$ for $i = 1, 2$, where

$$1 - \omega_2^2 = \sqrt{c_1 c_2} = \omega_1^2 - 1.$$

We assume that ω_1, ω_2 are rationally related, so that $n\omega_1 = m\omega_2$ for integers n, m. Constructing the 'eigenvector' matrix

$$R = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & \omega_1 & 0 & \omega_2 \\ l & 0 & -l & 0 \\ 0 & \omega_1 l & 0 & -\omega_2 l \end{pmatrix} \tag{5.2}$$

where $l = \sqrt{\frac{c_2}{c_1}}$ we find that in the new coordinates $(y_1, z_1, y_2, z_2)^{\mathtt{t}} = R^{-1}x$ the system (5.1) becomes

$$\dot{y}_1 = \omega_1 z_1 \tag{5.3}$$

$$\dot{z}_1 = -\omega_1 y_1 + \frac{1}{2\omega_1}(h_1 + kh_2) \tag{5.4}$$

$$\dot{y}_2 = \omega_2 z_2 \tag{5.5}$$

$$\dot{z}_2 = -\omega_2 y_2 + \frac{1}{2\omega_2}(h_1 - kh_2) \tag{5.6}$$

where $kl = 1$ and where h_i is shorthand for $h_i \circ R$ for $i = 1, 2$.

Now suppose that h_i depends smoothly on a parameter $\varepsilon \in \mathbf{R}$ so that $h_i = 0$ when $\varepsilon = 0$, and write

$$g_i = \frac{\partial h_i}{\partial \varepsilon}\Big|_{\varepsilon=0}.$$

Following the general theory set out by Chicone [2, 3, 4] and outlined above in Section 3, we have an associated bifurcation map

$$P' = (\mathcal{N}_1, \mathcal{N}_2, \mathcal{M}_1, \mathcal{M}_2)^{\mathsf{t}} : V \to L(4, 4) \tag{5.7}$$

where

$$\mathcal{N}_1 = \int_0^T \frac{-y_1}{2\|f_1\|^2}(g_1 + kg_2)dt \tag{5.8}$$

$$\mathcal{N}_2 = \int_0^T \frac{-y_2}{2\|f_2\|^2}(g_1 - kg_2)dt \tag{5.9}$$

$$\mathcal{M}_1 = \int_0^T \frac{z_1}{2\|f_1\|^2}(g_1 + kg_2)dt \tag{5.10}$$

$$\mathcal{M}_2 = \int_0^T \frac{z_2}{2\|f_2\|^2}(g_1 - kg_2)dt. \tag{5.11}$$

Here $T = m\frac{2\pi}{\omega_1} = n\frac{2\pi}{\omega_2}$, and the integrals are taken along a T-periodic solution curve γ given by

$$(y_i(t), z_i(t))^{\mathsf{t}} = (r_i \cos(\omega_i t + \theta_i), r_i \sin(\omega_i t + \theta_i))^{\mathsf{t}}, \qquad i = 1, 2$$

of the system $(\dot{y}_i, \dot{z}_i) = f_i(y_i, z_i) = \omega_i(z_i, -y_i)$ passing through the point $\xi = (\xi_1, \xi_2)$ where $\xi_i = (y_i(0), z_i(0)$ for $i = 1, 2$.

We now insert the particular choices of g_1, g_2 that correspond to Van Der Pol systems; we take four cases corresponding to $\varepsilon = \delta_1, \delta_2, \varepsilon_1, \varepsilon_2$ in turn.

Case 1.

$$\left.\begin{aligned} g_1 &= v_1(1 - u_1^2) = (\omega_1 z_1 + \omega_2 z_2)\big(1 - (y_1 + y_2)^2\big) \\ g_2 &= 0 \end{aligned}\right\}.$$

Here we find $\mathcal{N}_1 = \mathcal{N}_2 = 0$ while

$$\mathcal{M}_i = \frac{T}{16\omega_i}(p + r_i^2) \tag{5.12}$$

for $i = 1, 2$, where $p = 4 - 2(r_1^2 + r_2^2)$.

Case2.

$$\left.\begin{aligned} g_1 &= 0 \\ g_2 &= v_2(1 - u_2^2) = l(\omega_1 z_1 - \omega_2 z_2)\big(1 - (ly_1 - ly_2)^2\big) \end{aligned}\right\}.$$

Again $\mathcal{N}_1 = \mathcal{N}_2 = 0$ and now

$$\mathcal{M}_i = \frac{T}{16\omega_i}(q + l^2 r_i^2) \tag{5.13}$$

for $i = 1, 2$, where $q = 4 - 2l^2(r_1^2 + r_2^2)$.

Case 3.

$$g_1 = \cos \tilde{\omega}_1 t, \quad g_2 = 0.$$

Here using Proposition 2.2 we find

$$\mathcal{N}_1 = -\frac{T}{4\omega_1^2 r_1}\cos\theta_1 + mk_1, \qquad \mathcal{M}_1 = \frac{T}{4\omega_1^2 r_1}\sin\theta_1 \tag{5.14}$$

while $\mathcal{N}_2 = \mathcal{M}_2 = 0$.

Case 4.

$$g_1 = 0, \quad g_2 = \cos \tilde{\omega}_2 t.$$

Here

$$\mathcal{N}_2 = \frac{kT}{4\omega_2^2 r_2}\cos\theta_2 + nk_2, \qquad \mathcal{M}_2 = -\frac{kT}{4\omega_2^2 r_2}\sin\theta_2 \tag{5.15}$$

while $\mathcal{N}_1 = \mathcal{M}_1 = 0$.

Assembling the 4×4 bifurcation matrix whose columns correspond to these four cases we obtain

$$\mathcal{P}(\xi) = \frac{T}{16}\begin{pmatrix} 0 & 0 & -a_1\cos\theta_1 + d_1 & 0 \\ 0 & 0 & 0 & a_2 k\cos\theta_2 + d_2 \\ \omega_1^{-1}(p + r_1^2) & \omega_1^{-1}(q + l^2 r_1^2) & a_1 \sin\theta_1 & 0 \\ \omega_2^{-1}(p + r_2^2) & \omega_2^{-1}(q + l^2 r_2^2) & 0 & -a_2 k \sin\theta_2 \end{pmatrix} \tag{5.16}$$

where $a_i = \frac{4}{r_i \omega_i^2}$ and where $d_i = \frac{8}{\pi}\omega_i k_i$ for $i = 1, 2$. Then we find

$$\det \mathcal{P}(\xi) = K\Delta(-a_1\cos\theta_1 + d_1)(a_2 k\cos\theta_2 + d_2) \tag{5.17}$$

where $\Delta = (1 - l^2)(r_1^2 - r_2^2)$ and K is a positive constant.

Write $p_i = (p + r_i^2)$ and $q_i = (q + l^2 r_i^2)$ for $i = 1, 2$, and let

$$b_1 = -a_1\cos\theta_1 + d_1 \tag{5.18}$$

$$b_2 = a_2 k\cos\theta_2 + d_2. \tag{5.19}$$

The following facts are easily verified:

Proposition 5.1.

(1) If $b_1 = 0$ and $b_2\Delta \neq 0$ then $\mathcal{P}(\xi)$ has rank 3 and $\ker \mathcal{P}(\xi)$ *is spanned by the vector*

$$\mu_1(\xi) := (q_2 a_1 \sin\theta_1, -p_2 a_1 \sin\theta_1, -4\omega_1^{-1}\Delta, 0).$$

(2) If $b_2 = 0$ and $b_1\Delta \neq 0$ then $\mathcal{P}(\xi)$ has rank 3 and $\ker \mathcal{P}(\xi)$ *is spanned by the vector*

$$\mu_2(\xi) := (-kq_1 a_2 \sin\theta_2, kp_1 a_2 \sin\theta_2, 0, 4\omega_2^{-1}\Delta).$$

(3) If $b_1 b_2 \neq 0$ and $\Delta = 0$ then $\mathcal{P}(\xi)$ has rank 3 and $\ker \mathcal{P}(\xi)$ *is spanned by the vector*

$$\nu(\xi) := (q_1, -p_1, 0, 0).$$

(4) If $b_1 = b_2 = 0$ and $\Delta \neq 0$ then $\mathcal{P}(\xi)$ has rank 2 and $\ker \mathcal{P}(\xi)$ is spanned by the vectors $\{\mu_1(\xi), \mu_2(\xi)\}$.

Let the subsets of $\mathbf{R}^4$ on which the conditions (1)–(4) hold be denoted by N_1, N_2, N_3, N_4 respectively. Since in case (1) the vector $\mu_1(\xi)$ is independent of θ_2 the map $\kappa : N_1 \to \mathbf{R}P^3$ is degenerate; likewise for the map $\kappa : N_2 \to \mathbf{R}P^3$. In case (3) the vector $\nu(\xi)$ depends on neither θ_1 nor θ_2 and so κ is even more degenerate. However, in case (4), even though $\dim \ker \mathcal{P}(\xi) = 2$, we have a more amenable situation, as we shall now show.

5.1. The linear bifurcation problem. To build a framework for describing the configuration of solutions to $P_\varepsilon = 0$ in case (4) we set out the following terminology:

$$M : = \{(\xi, \varepsilon) \in \mathbf{R}^4 \times \mathbf{R}^4 : P_\varepsilon(\xi) = 0\},$$
$$M^L : = \{(\xi, \varepsilon) \in \mathbf{R}^4 \times \mathbf{R}^4 : \mathcal{P}(\xi)\varepsilon = 0\}.$$

Restricting these to the 3-sphere of radius ρ in parameter space $\mathbf{R}^4$ we write

$$M_\rho : = M \cap (\mathbf{R}^4 \times \rho S^3),$$
$$M_\rho^L : = M^L \cap (\mathbf{R}^4 \times \rho S^3)$$

where $S^3 \subset \mathbf{R}^4$ is the unit 3-sphere.

Let $pr : \mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}^4$ denote projection to the second (that is, the ε) factor. The overall bifurcation behaviour of the displacement map P_ε is determined by the structure of the *solution locus* M and in particular by the geometry of the projection $pr|M : M \to \mathbf{R}^4$. Specifically, if Σ denotes the set of singular points of $pr|M$ then the *bifurcation set* is defined as $\Gamma := pr(\Sigma) \subset \mathbf{R}^4$ since in any given bounded region of M the solution set $\{\xi \in \mathbf{R}^4 : P_\varepsilon(\xi) = 0\}$ can change topologically only when ε crosses Γ.

A first approximation to M for small parameter ε is given by M^L. This set is invariant under scalar multiplication in the parameter space and so M^L is the cone from the origin on M_ρ^L for any $\rho > 0$ and in particular $\rho = 1$. Let Σ^L be the singular set for $pr|M^L$ and write $\Sigma_\rho, \Sigma_\rho^L$ for the intersections of these with $\mathbf{R}^4 \times \rho S^3$. If we note features of $pr(\Sigma_1^L)$ that are robust, i.e. they persist under sufficiently small perturbations, then we can expect that these features (rescaled) will also be present in the projection $pr(\Sigma_\rho)$ for sufficiently small $\rho > 0$ and thus in the bifurcation set Γ sufficiently close to the origin in parameter space $\mathbf{R}^4$. In order to make these ideas precise we shall use results from singularity theory.

The degeneracies in cases (1)–(3) above imply that we cannot expect robustness in the structure of the projection $M_1^L \to S^3$ near points (ξ, ε) where $\xi \in N_1 \cup N_2 \cup N_3$. We therefore focus attention on case (4) where $\xi \in N = N_4$. This is a torus parametrised in polar coordinates in $\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$ as

$$(r_1, r_2) = (2t_1 \cos\theta_1, -2t_2 \cos\theta_2), \quad -\tfrac{\pi}{2} \leq \theta_1, \theta_2 \leq \tfrac{\pi}{2}$$

where

$$4t_i = \pi \omega_i^{-3} k_i^{-1} \chi_i \tag{5.20}$$

for $\chi_1 = 1$ and $\chi_2 = k$. For $\xi \in N$ the kernel of $\mathcal{P}(\xi)$ is the kernel of the 2×4 matrix $A(\xi)$ of rank 2 :

$$A(\xi) = \begin{pmatrix} \omega_1^{-1} p_1 & \omega_1^{-1} q_1 & a_1 \sin\theta_1 & 0 \\ \omega_2^{-1} p_2 & \omega_2^{-1} q_2 & 0 & -a_2 k \sin\theta_2 \end{pmatrix}. \tag{5.21}$$

Writing $\Psi(\xi,\varepsilon) := A(\xi)\varepsilon$ we have $(\xi,\varepsilon) \in M^L$ precisely when $\Psi(\xi,\varepsilon) = 0$. The tangent space to M^L at (ξ,ε) is the kernel of $D\Psi(\xi,\varepsilon)$, given by

$$\ker D\Psi(\xi,\varepsilon) = \{(w,\nu) \in T_\xi N \times \mathbf{R}^4 : (DA(\xi)w)\varepsilon + A(\xi)\nu = 0\}.$$

When $\varepsilon \neq 0 \in \mathbf{R}^4$ the projection $pr|M^L : M^L \to \mathbf{R}^4$ is a local diffeomorphism at $(\xi,\varepsilon) \in M^L$ if and only if

$$\ker D\Psi(\xi,\varepsilon) \cap \ker \pi = \{0\}$$

which means that the only solution w to $(DA(\xi)w)\varepsilon = 0$ is $w = 0$, or in other words the linear map

$$L_{(\xi,\varepsilon)} : T_\xi N \to \mathbf{R}^2 : w \mapsto (DA(\xi)w)\varepsilon$$

is an isomorphism, where $\varepsilon \in \ker A(\xi)$. Denoting the rows of $A(\xi)$ by row vectors $\rho_1^T(\xi), \rho_2^T(\xi)$ we have

$$L_{(\xi,\varepsilon)}w = \begin{pmatrix} \left(D\rho_1^T(\xi)w\right)\varepsilon \\ \left(D\rho_2^T(\xi)w\right)\varepsilon \end{pmatrix} \tag{5.22}$$

$$= \begin{pmatrix} \varepsilon^T D\rho_1(\xi) \\ \varepsilon^T D\rho_2(\xi) \end{pmatrix} w \quad \in \mathbf{R}^2. \tag{5.23}$$

With t_1, t_2 fixed and the torus N now parametrised by $\xi = (\theta_1,\theta_2) \in \mathbf{R}^2$ up to integer multiples of π we can regard $L_{(\xi,\varepsilon)}$ as a linear map $\mathbf{R}^2 \to \mathbf{R}^2$ and we have therefore obtained the following result.

Proposition 5.2. *For $(\xi,\varepsilon) \in M^L$ and $\xi = (\theta_1,\theta_2) \in N$ the linear map $L_{(\xi,\varepsilon)}$ is an isomorphism if and only if* $\det Q(\xi,\varepsilon) \neq 0$ *where*

$$Q(\xi,\varepsilon) = \begin{pmatrix} q_{11}(\xi,\varepsilon) & q_{12}(\xi,\varepsilon) \\ q_{21}(\xi,\varepsilon) & q_{22}(\xi,\varepsilon) \end{pmatrix}$$

with $q_{ij}(\xi,\varepsilon) = \varepsilon^T \partial\rho_i/\partial\theta_j$. □

Corollary 5.3. *Let $\xi_0 \in N$ and $\varepsilon_0 \in \ker A(\xi_0)$ such that* $\det Q(\xi_0,\varepsilon_0) \neq 0$. *Then the projection $M^L \to \mathbf{R}^4$ is a local diffeomorphism at (ξ_0,ε_0). Thus for all $\varepsilon \in \mathbf{R}^4$ sufficiently close to ε_0 there is a unique $\xi(\varepsilon) \in N$ close to ξ_0 such that $(\xi(\varepsilon),\varepsilon) \in M^L$, that is $\varepsilon \in \ker A(\xi(\varepsilon))$; moreover $\xi(\varepsilon)$ varies C^∞ with $\varepsilon \in \mathbf{R}^4$.* □ We can choose a basis $\{\mu_1(\xi),\mu_2(\xi)\}$ for $\ker A(\xi)$ varying smoothly with $\xi \in N$ and write $\varepsilon \in \ker A(\xi)$ as $\varepsilon = \lambda_1\mu_1(\xi) + \lambda_2\mu_2(\xi)$. Then $\det Q(\xi,\varepsilon)$ becomes a quadratic form

$$\det Q(\xi,\varepsilon) = \mathcal{Q}(\xi)(\lambda_1,\lambda_2)$$

in $(\lambda_1,\lambda_2) \in \mathbf{R}^2$. If $\mathcal{Q}(\xi_0)$ is a (positive or negative) definite form then the condition of Corollary 5.3 holds for all $\varepsilon \in \ker A(\xi_0)$. On the other hand, if $\mathcal{Q}(\xi_0)$ indefinite (and nondegenerate) then there exists a pair of lines $\{L_1(\xi_0), L_2(\xi_0)\} \subset \ker A(\xi_0)$ such that

$$L_1(\xi_0) \cup L_2(\xi_0) = \{\varepsilon \in \ker A(\xi_0) : \det Q(\xi_0,\varepsilon) = 0\},$$

and so then $L_1(\xi_0) \cup L_2(\xi_0)$ lies in the singular set Σ^L.

For each $\xi \in N$ the kernel of $A(\xi)$ intersects S^3 in a great 2-sphere, and so $\mathcal{A} = \{\ker A(\xi) : \xi \in N\}$ is a 2-parameter family of 2-spheres in S^3. To visualise this locally we may take radial projection from the origin in $\mathbf{R}^4$ to a fixed affine 3-space

Π in $\mathbf{R}^4$, so that $\mathcal{A}$ becomes a 2-parameter family of lines in Π, also called a *line congruence* in Π.

The local geometry of line congruences has been investigated using methods of singularity theory by Izumiya *et al.* [15]. In that paper the authors show that the singularities that arise in generic line congruences are the same as those that arise in generic maps $\mathbf{R}^3 \to \mathbf{R}^3$, namely *folds*, *cusps* and *swallowtail* points. These are also stable singularities: their local structure persists (up to smooth coordinate change in source and target) under sufficiently small perturbations.

In view of these results we expect Σ_1^L to consist of fold surfaces, possibly with cusp ridges and isolated swallowtail points, together determining the bifurcation structure for the linear problem. To give an explicit proof that this is the case it would be necessary to show that the genericity assumptions used in [15] do in fact hold in our differential equations context. We shall not attempt to do this; instead we characterise explicitly the fold points, and give a numerical illustration showing that cusp ridges also occur.

5.2. Characterisation of fold points. Let $\xi_1 \in N$ be such that $\mathcal{Q}(\xi_1)$ is nondegenerate and indefinite. Each of the lines $L_i(\xi_1)$ in $\ker A(\xi_1) \subset \mathbf{R}^4$ meets S^3 in a pair of points $\ell_i^{\pm}(\xi_1)\,, i = 1, 2$, and so $\xi_1 \mapsto \ell_i^{\pm}(\xi_1)$ gives a smooth parametrisation of a 2-manifold $L_i^{\pm}$ in M_1^L, part of the singular set Σ_1^L. The condition for a singular point to be a *fold* is that the map when restricted to the singular set be nonsingular (see for example [1, 9, 10]), that is that the kernel of the derivative of the map be transverse to the singular set. In our context this condition for $pr|M_1^L$ is that $\ker Q(\xi_1, l_i^{\pm}(\xi_1))$ be transverse to $L_i^{\pm}$, that is that

$$h_i^{\pm}(\xi_1) := D_\xi \det Q(\xi_1, \varepsilon).w(\xi_1, \varepsilon) \neq 0 \tag{5.24}$$

where $\varepsilon = l_i^{\pm}(\xi_1)$ and where $\ker Q(\xi_1, \varepsilon) = span\{w(\xi_1, \varepsilon)\}$. We can therefore state the following bifurcation result.

PROPOSITION 5.4. *Let $\xi_1 \in N$ be such that the quadratic form $\mathcal{Q}(\xi_1)$ is indefinite and nondegenerate. Let $\varepsilon_1 = \ell_i^{\pm}(\xi_1) \in \ker A(\xi_1) \cap S^3$ for $i = 1$ or $i = 2$ and a given sign choice, and suppose $h_i^{\pm}(\xi_1) \neq 0$. Then $pr|M_1^L$ has a fold singularity at $(\xi_1, \varepsilon_1) \in \Sigma_1^L$. Consequently there is a smooth 2-dimensional submanifold $\Gamma_1 = pr(\Sigma_1)$ in S^3 through ε_1 such that as $\varepsilon \in S^3$ crosses Γ_1 the number of solutions $\xi \in N$ to $A(\xi)\varepsilon = 0$ changes by two in a fold (saddle-node, Morse) bifurcation at ξ_1.* By radial projection, the same result holds if S^3 is replaced by an affine hyperspace Π avoiding the origin in $\mathbf{R}^4$.

Generically we expect the locus $h_i^{\pm} = 0$ to be a smooth 1-manifold in N corresponding to a *cusp ridge* in the bifurcation set in S^3 or Π. In Figure 5.1 we give a numerical illustration with the help of MAPLE that confirms this cusp ridge geometry.

5.3. Specific calculations. In general we might expect there to be nonempty open sets in N where $\mathcal{Q}(\xi)$ is definite and where it is indefinite. However, we show that in this system the quadratic form $\mathcal{Q}(\xi)(\lambda_1, \lambda_2)$ is never definite, and the conditions of Proposition 5.4 hold for almost all $\xi_1 \in N$.

Recall that the torus N is defined by $b_1 = b_2 = 0$ and is therefore parametrised in polar coordinates in $\mathbf{R}^4 = \mathbf{R}^2 \times \mathbf{R}^2$ as

$$(r_1, r_2) = (2t_1 \cos\theta_1, -2t_2 \cos\theta_2), \quad -\tfrac{\pi}{2} \leq \theta_1, \theta_2 \leq \tfrac{\pi}{2}$$

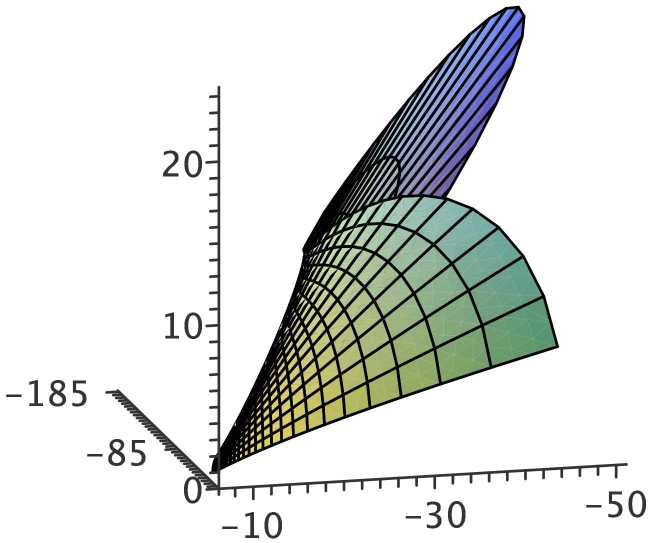

Fig. 5.1. *Cusp ridge for* $1 \leq \theta_1 \leq 1.4, 0 \leq \theta_2 \leq 0.5$ *with* $(t_1, t_2) = (1, 1.5), (\omega_1, \omega_2) = (1, 2.7)$ *and* $k = 1.5$

which correspond in cartesian coordinates to

$$(y_i, z_i) = (-1)^{i+1}(1 + \cos 2\theta_i, \sin 2\theta_i) = (-1)^{i+1} 2t_i(\cos^2\theta_i, \sin\theta_i \cos\theta_i)$$

with t_1, t_2 as in (5.20) for $i = 1, 2$. As a basis $\{v_1, v_2\}$ of $T_\xi N$ we may take

$$v_1 = (\sin 2\theta_1, -\cos 2\theta_1, 0, 0) \tag{5.25}$$
$$v_2 = (0, 0, \sin 2\theta_2, -\cos 2\theta_2). \tag{5.26}$$

We then find, after a little calculation, that $\partial\rho_i/\partial\theta_j = m_{ij}$ where

$$\begin{aligned}
m_{11} &= 8t_1^2\omega_1^{-1}\sin\theta_1\cos\theta_1(1,l^2,0,0)^T + 2t_1^{-1}(\omega_1\cos\theta_1)^{-2}(0,0,1,0)^T \\
m_{12} &= 16t_2^2\omega_1^{-1}\sin\theta_2\cos\theta_2(1,l^2,0,0)^T \\
m_{21} &= 16t_1^2\omega_2^{-1}\sin\theta_1\cos\theta_1(1,l^2,0,0)^T \\
m_{22} &= 8t_2^2\omega_2^{-1}\sin\theta_2\cos\theta_2(1,l^2,0,0)^T + 2kt_2^{-1}(\omega_2\cos\theta_2)^{-2}(0,0,0,1)^T.
\end{aligned}$$

Now taking the basis $\{\mu_1(\xi),\mu_2(\xi)\}$ for $\ker A(\xi)$ as in Proposition 5.1, on substituting $\varepsilon = \mu_1$ and $\varepsilon = \mu_2$ in turn into $q_{ij} = \varepsilon^T m_{ij}$ we obtain two 2×2 matrices $Q_1(\xi), Q_2(\xi)$ where

$$Q_1 = -32t_1(l^2-1)\omega_1^{-3}\begin{pmatrix} 2\sin^2\theta_1 - 1 + t^2C^2 & 4t^2C\sin\theta_1\sin\theta_2 \\ 4\frac{\omega_1}{\omega_2}\sin^2\theta_1 & 2\frac{\omega_1}{\omega_2}t^2C\sin\theta_1\sin\theta_2 \end{pmatrix}, \quad (5.27)$$

$$Q_2 = -32kt_2(l^2-1)\omega_2^{-3}\begin{pmatrix} 2\frac{\omega_2}{\omega_1}t^{-2}C^{-1}\sin\theta_1\sin\theta_2 & 4\frac{\omega_2}{\omega_1}\sin^2\theta_2 \\ 4t^{-2}C^{-1}\sin\theta_1\sin\theta_2 & 2\sin^2\theta_2 - 1 + t^{-2}C^{-2} \end{pmatrix} \quad (5.28)$$

where $t = \frac{t_2}{t_1} = k\frac{k_1}{k_2}\frac{\omega_1^3}{\omega_2^3}$ (see (5.20)) and $C = \frac{\cos\theta_2}{\cos\theta_1}$. Note the symmetry: Q_2 is obtained from Q_1 by interchanging suffices 1 and 2 apart from the factor k. This is of course a consequence of the symmetry of the original system (5.1) and the slight asymmetry in the choice of the 'eigenvector' matrix (5.2).

We next need to consider the quadratic form

$$\mathcal{Q}(\xi)(\lambda_1,\lambda_2) = \det(\lambda_1 Q_1(\xi) + \lambda_2 Q_2(\xi)) \quad (5.29)$$

$$= \sum_{i,j=1,2} D_{ij}\lambda_i\lambda_j \quad (5.30)$$

where D_{ij} is the determinant of the 2×2 matrix whose first column is that of Q_i and second column is that of Q_j. To determine whether $\mathcal{Q}(\xi)$ is a definite form or not we calculate its discriminant

$$D = (D_{12} + D_{21})^2 - 4D_{11}D_{22}.$$

After some algebra in which several agreeable cancellations occur, we arrive at the following result.

PROPOSITION 5.5. *Up to multiplication by a positive constant we have*

$$D = \left(A_1s_1^2 + A_2s_2^2 + \frac{1}{2}A_1A_2\right)^2 - 16A_1A_2s_1^2s_2^2$$

where $s_i = \sin\theta_i$ *(*$i = 1,2$*) and* $A_1 = t^{-2}C^{-2} - 1, A_2 = t^2C^2 - 1$*; observe that* $A_1A_2 = -(A_1 + A_2)$. □ It can easily be checked that $A_1A_2 \le 0$ with equality if and only if $A_1 = A_2 = 0$, that is

$$t_1\cos\theta_1 = t_2\cos\theta_2. \quad (5.31)$$

It follows that $D \ge 0$ for all (θ_1,θ_2) and $D = 0$ when (5.31) holds or when θ_1, θ_2 satisfy one or other of the two conditions

$$\begin{aligned}
s_1 &= 0, \quad 2c_2^4 - c_2^2 = t^{-2} \qquad \text{or} \\
s_2 &= 0, \quad 2c_1^4 - c_1^2 = t^2
\end{aligned}$$

where $c_i = \cos\theta_i$ for $i = 1, 2$. Let $\widetilde{N}$ be the open dense subset on N where none of these equations is satisfied, so that $D > 0$ on $\widetilde{N}$.

Corollary 5.6. *For all $\xi \in \widetilde{N}$ the quadratic form $\mathcal{Q}(\xi)(\lambda_1, \lambda_2)$ is nondegenerate and indefinite.* □ It follows that for *every* point $\xi \in \widetilde{N}$ the description of Proposition 5.4 applies: there exist four points $\ell_i^{\pm}(\xi)$ $(i = 1, 2)$ in the singular set Σ_1^L. If the condition 5.24 holds then these are fold singularities, corresponding to the creation and annihilation of pairs of solutions to the linear problem as ε crosses a smooth 2-manifold in S^3.

5.4. The full nonlinear problem. With the notation of Section 1 and writing $\varepsilon \in \mathbf{R}^4$ as $\varepsilon = \rho s$ with $s \in S^3$ and $\rho \geq 0$ we have as in (3.2)

$$P_\varepsilon(\xi) = \rho\mathcal{P}(\xi)s + O(\rho^2) = \rho\big(\mathcal{P}(\xi) + O(\rho)\big)$$

so that solutions to $P_\varepsilon(\xi) = 0$ with $\varepsilon \neq 0$ and $|\varepsilon|$ small are obtained from the set M_1^L of solutions to $\mathcal{P}(\xi)s = 0$ in $\widetilde{N} \times S^3$ by a perturbation of $O(\rho)$ followed by rescaling by the factor ρ. We have shown above that for every $\xi \in \widetilde{N}$ there are four points of M_1^L where $pr : M_1^L \to S^3$ has a singularity, and that generically at such points this singularity is a fold. The stability of folds under C^∞ perturbations implies that for sufficiently small $\rho > 0$ the projection $pr : \rho^{-1}M_\rho \to S^3$ has a corresponding fold obtained from that of $pr : \rho^{-1}M_\rho^L = M_1^L \to S^3$ by a local diffeomorphism close to the identity and tending to the identity as $\rho \to 0$. The local bifurcation set Γ^L for the linear problem $\mathcal{P}(\xi)\varepsilon = 0$ is a cone from the origin in $\mathbf{R}^4$ to the local fold surface Σ_1^L in S^3, and this corresponds to a 'curved' cone structure for the bifurcation set Γ for the nonlinear problem $P_\varepsilon(\xi) = 0$ obtained from Γ^L by applying a C^1 diffeomorphism at the origin in $\mathbf{R}^4$ with derivative the identity.

Likewise, cusp ridges in Γ_1^L determine in Γ^L cones from the origin on cusp ridges, giving rise to 'curved' cones on cusp ridges in the bifurcation set Γ close to the origin. Any swallowtail points (should they exist) in $\Sigma_1^L \to S^3$ correspond to arcs of swallowtail bifurcation points in Γ emanating from the origin in $\mathbf{R}^4$.

5.5. Interpretation of the results. We conclude by recalling how the bifurcation geometry we have just described relates to the original problem of persistence of harmonic periodic solutions to a pair of coupled Van Der Pol oscillators.

As $\varepsilon = (\delta_1, \delta_2, \varepsilon_1, \varepsilon_2)$ traces out a smooth path close to (but not passing through) the origin in $\mathbf{R}^4$ there exists a corresponding finite set of solutions $\xi \in \widetilde{N}$ to the equation $P_\varepsilon(\xi) = 0$, that is fixed points of the map F_ε. These correspond to periodic solutions with initial data $\xi \in \widetilde{N} \in \mathbf{R}^4$ of the non-autonomous perturbed coupled Van Der Pol system (5.1). A typical path in $\mathbf{R}^4$ will intersect fold cones transversely at discrete points ε which correspond to the creation or annihilation of pairs of periodic orbits in saddle-node bifurcations; for any $\xi \in \widetilde{N}$ and small $\rho > 0$ there will be four points $\varepsilon_i^{\pm}$ close to $\rho\ell_i^{\pm} \in \mathbf{R}^4, i = 1, 2$ as described in Section 5.1 such that a saddle-node bifurcation occurs close to ξ when ε passes through $\varepsilon_i^{\pm}$. Paths in bifurcation space $\mathbf{R}^4$ will typically avoid the 2-dimensional cones on cusp ridges and any isolated swallowtail arcs, but 2-parameter families in $\mathbf{R}^4$ will typically encounter them.

Observe that a path $\varepsilon(\tau)$ in $\mathbf{R}^4$ with $\varepsilon(\tau) \to 0 \in \mathbf{R}^4$ as $\tau \to \tau_0$ need not correspond to a path of solutions $\xi(\tau)$ to $P_\varepsilon(\xi) = 0$ approaching a point $(\xi, 0)$ in the solution locus

M. For this to be the case we require $\varepsilon(\tau)/|\varepsilon(\tau)|$ to tend to a limit in S^3 as $\tau \to \tau_0$. See [14] for further discussion of this important point.

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