On Skeletons, Diameters and Volumes of Metric Polyhedra

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Abstract. We survey and present new geometric and combinatorial properties of some polyhedra with application in combinatorial optimization, for example, the max-cut and multicommodity flow problems. Namely we consider the volume, symmetry group, facets, vertices, face lattice, diameter, adjacency and incidence relations and connectivity of the metric polytope and its relatives. In particular, using its large symmetry group, we completely describe all the 13 orbits which form the 275 840 vertices of the 21-dimensional metric polytope on 7 nodes and their incidence and adjacency relations. The edge connectivity, the i-skeletons and a lifting procedure valid for a large class of vertices of the metric polytope are also given. Finally, we present an ordering of the facets of a polytope, based on their adjacency relations, for the enumeration of its vertices by the double description method.

1 Introduction

We first recall the definition of the *metric polytope* m_n and some of its relatives and present some applications to well known optimization problems of those polyhedra. The general references are Bayer and Lee [8] and Ziegler [31] for polytopes and Brouwer, Cohen and Neumaier [9] for graphs. For a complete study of the applications and the combinatorial optimization aspects of those polyhedra, we refer, respectively, to the surveys Deza and Laurent [17] and Poljak and Tuza [29].

For all 3-sets $\{i,j,k\} \subset N=\{1,\ldots,n\}$, we consider the following inequalities:

$$x_{ij} - x_{ik} - x_{jk} \leq 0 \ . \tag{1}$$

The inequalities (1) induce the $3\binom{n}{3}$ facets which define the *metric cone* M_n. Then, bounding the later by the following inequalities:

$$x_{ij} + x_{ik} + x_{jk} \leq 2 \tag{2}$$

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we obtain the *metric polytope* m_n. The $3\binom{n}{3}$ facets defined by (1), which can be seen as triangle inequalities for distance x_{ij} on $\{1,2,\ldots,n\}$, are called *homogeneous triangle facets.* The $\binom{n}{3}$ facets defined by the inequalities (2) are called *non-homogeneous triangle facets*, and by *triangle facet* we denote a facet of either type (1) or (2).

While the *cut cone* C_n is the conic hull of all, up to a multiple, $\{0,1\}$-valued extreme rays of the metric cone, the *cut polytope* c_n is the convex hull of all $\{0,1\}$-valued vertices of the metric polytope. Those two polyhedra can also be defined independently from the metric cone and polytope in the following ways.

Given a subset S of $N = \{1,2,\ldots,n\}$, the *cut* defined by S consists of the pairs (i,j) of elements of N such that exactly one of i, j is in S. By $\delta(S)$ we denote both the cut and its incidence vector in $\mathbb{R}^{\binom{n}{2}}$, that is, $\delta(S)_{ij} = 1$ if exactly one of i, j is in S and 0 otherwise for $1 \leq i < j \leq n$. By abuse of language, we use the term cut for both the cut itself and its incidence vector, so $\delta(S)_{ij}$ are considered as coordinates of a point in $\mathbb{R}^{\binom{n}{2}}$. The cut polytope of the complete graph c_n, which is also called the complete bipartite subgraphs polytope, is the convex hull of all 2^{n-1} cuts, and the cut cone C_n is the conic hull of all $2^{n-1}-1$ nonzero cuts. Those polyhedra were considered by many authors, see for instance [2, 7, 15, 16, 17, 18, 19, 21, 23, 24] and references therein. One of the motivations for the study of these polyhedra comes from their applications in combinatorial optimization, the most important being the max-cut and multicommodity flow problems.

Given a graph $G = (N,E)$ and nonnegative weights w_e, $e \in E$, assigned to its edges, the *max-cut* problem consists in finding a cut $\delta(S)$ whose weight $\sum_{e\in\delta(S)} w_e$ is as large as possible. It is a well-known NP-complete problem. By setting $w_e = 0$ if e is not an edge of G, we can consider without loss of generality the complete graph K_n. Then the max-cut problem can be stated as a linear programming problem over the cut polytope c_n as follows:

$$\begin{aligned} \max \quad & w^T \cdot x \\ \text{subject to} \quad & x \in c_n \ . \end{aligned}$$

Since the metric polytope is a relaxation of the cut polytope, optimizing $w^T \cdot x$ over c_n instead of m_n provides an upper bound for the max-cut problem [7].

With E the set of edges of the complete graph K_n, an instance of the *multicommodity flow problem* is given by two nonnegative vectors indexed by E: a capacity $c(e)$ and a requirement $r(e)$ for each $e \in E$. Let $U = \{e \in E : r(e) > 0\}$. If T denotes the subset of N spanned by the edges in U, then we say that the graph $G = (T,U)$ denotes the *support* of r. For each edge $e = (s,t)$ in the support of r, we seek a flow of $r(e)$ units between s and t in the complete graph. The sum of all flows along any edge $e' \in E$ must not exceed $c(e')$. If such a set of flows exists, we call c, r *feasible.* A necessary and sufficient condition for feasibility is given by the Japanese theorem of Iri [22] and Onaga and Kakusho [26]: a pair c,r is feasible if and only if $(c-r)^T x \geq 0$ is valid over the metric cone. For example, the triangle facet induced by (1) can be seen as an elementary solvable flow problem with $c(ij) = r(ik) = r(jk) = 1$ and $c(e) = r(e) = 0$ otherwise, so the inequalities (1) correspond to $(c-r)^T x \geq 0$ for $x \in M_n$. In other words, the dual metric cone is the cone of all feasible multicommodity flow problems.

2 Skeletons and Diameters

2.1 Previous Results

The polytope c_n is a $\binom{n}{2}$ dimensional 0–1 polyhedron with 2^{n-1} vertices and m_n is a polytope of same dimension with $4\binom{n}{3}$ facets inscribed in the cube $[0,1]^{\binom{n}{2}}$. We have $c_n \subseteq m_n$ with equality only for $n \leq 4$. It is easy to see that the point $\omega_n = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ is the center of gravity of both c_n and m_n and is also the center of the sphere of radius $r = \frac{1}{2}\sqrt{n(n-1)}$ where all the cuts lie. Another two geometric characteristics of the cut polytope c_n are its *width* and *geometric diameter*. We recall that while the width of a polytope P is equal to the minimum distance between a pair of parallel hyperplanes containing P in the slice between them, the geometric diameter of P is the maximum distance between a pair of supporting hyperplanes. The width of c_n is 1 and its geometric diameter is $\frac{n}{2}$ for n even and $\frac{1}{2}\sqrt{n^2-1)}$ for n odd. Any facet, respectively *subfacet* (that is, a face of codimension 2), of the metric polytope contains a facet, respectively a subfacet, of the cut polytope and the vertices of the cut polytope are vertices of the metric polytope, in fact the cuts are precisely the integral vertices of the metric polytope. Actually the metric polytope m_n wraps the cut polytope c_n very tightly since, in addition to the vertices, all edges and 2-faces of c_n are also faces of m_n, for 3-faces it is false for $n \geq 4$, see [14, 19]. In other words, c_n is a *segment of order* 2, but not 3, of m_n and its dual, m_n^*, is a segment of order 1 of c_n^* in terms of [25]: a polytope P is a segment of order s of a polytope Q if they have the same dimension and if every i-face of P is a face of Q for $0 \leq i \leq s$. The polytope c_n is 3-neighbourly, see [19]. Any two cuts are adjacent both on c_n and on m_n [7, 27]; in other words m_n is *quasi-integral* in terms of [30], that is, the skeleton of the convex hull of its integral vertices, i.e. the skeleton of c_n, is an induced subgraph of the skeleton of the metric polytope itself. While the diameter of m_n^* is 2, the diameters of c_n^* and m_n are respectively conjectured to be 4 and 3, see [13, 23]. We recall that the skeleton of a polytope is the graph formed by its vertices and edges.

The metric polytope and the cut polytope share the same symmetry group, that is, the group of isometries preserving a polytope. This group is isomorphic to the automorphism group of the *folded n-cube*: $Aut(\square_n) \approx Is(m_p) = Is(c_p)$, see [15, 23]. We recall that the folded n-cube is the graph whose vertices are the partitions of $N = \{1, \ldots, n\}$ into two subsets, two partitions being adjacent when their common refinement contains a set of size one, see [9]. More precisely, for $n \geq 5$, $Is(m_n) = Is(c_n)$ is induced by permutations on $N = \{1, \ldots, n\}$ and *switching reflections by a cut.* Given a cut $\delta(S)$, the switching reflection $r_{\delta(S)}$ is defined by $y = r_{\delta(S)}(x)$ where $y_{ij} = 1 - x_{ij}$ if $(i,j) \in \delta(S)$ and $y_{ij} = x_{ij}$ otherwise. These symmetries preserve the adjacency relations and the linear independency. Using the partition of the faces of m_n and c_n into orbits of their symmetry group, the face lattice for small dimensions ($d = 3, 6$ and 10) was given in [14].

We finally mention the following link with metrics. There is an evident $1-1$ correspondence between the elements of the metric cone and all the semi-metrics on n points. Moreover the elements of the cut cone correspond precisely to the semi-metrics on n points that are isometrically embeddable into some l_1^m, see [1], it is easy to check that such minimal m is smaller or equal to $\binom{n}{2}$.

Another relative of the metric cone is the *solitaire cone* S_B, that is, the cone generated by all the possibles moves of a Solitaire Peg game played on a board B. This cone shares a lot of similar properties with the metric cone, see [5]. In particular, for a game played on the line graph T_n of the complete graph K_n, the complete solitaire cone S_{T_n} equals the dual metric cone M_n^*, see [5].

2.2 New Results

The Metric Polytope on Seven Nodes. In Table 1 we present the 13 orbits under permutations and switching which form the 275 840 vertices of the metric polytope m_7. For each orbit O_i, we give a representative vertex v_i, the size of the orbit $|O_i|$, its size $|O_i \cap F|$ restricted to a facet and the incidence I_{v_i} and the adjacency A_{v_i} of any vertex belonging to the orbit O_i.

Table 1. The orbits of vertices of the metric polytope on seven nodes

Orbit O_i	Representative vertex v_i	$\|O_i\|$	$\|O_i \cap F\|$	I_{v_i}	A_{v_i}
O_1	$(0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0)$	64	48	105	55 226
O_2	$\frac{2}{3}(1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$	64	16	35	896
O_3	$\frac{2}{3}(1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1)$	1 344	384	40	763
O_4	$\frac{2}{3}(1,1,1,1,0,1,1,1,1,1,0,1,1,1,1,1,1,1,1,1,1)$	6 720	2 160	45	594
O_5	$\frac{2}{3}(1,1,1,1,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,0)$	2 240	784	49	496
O_6	$\frac{1}{4}(1,2,3,1,2,1,1,2,2,1,2,1,1,2,3,2,3,2,1,2,1)$	20 160	4 320	30	96
O_7	$\frac{1}{3}(1,1,1,1,1,1,2,2,1,1,1,2,1,1,1,1,1,1,2,2,2)$	4 480	832	26	76
O_8	$\frac{2}{5}(2,1,1,1,1,2,2,1,1,1,1,2,1,1,1,2,1,1,2,1,2)$	23 040	4 608	28	57
O_9	$\frac{1}{3}(2,2,1,1,1,2,2,1,1,1,1,2,1,1,1,2,1,1,2,1,2)$	40 320	6 336	22	46
O_{10}	$\frac{1}{3}(1,1,1,1,1,1,2,2,1,1,1,2,1,1,1,2,1,1,2,2,2)$	40 320	6 624	23	39
O_{11}	$\frac{2}{7}(1,2,3,2,1,2,1,2,1,2,1,1,2,1,1,1,2,2,1,1,1)$	40 320	7 200	25	30
O_{12}	$\frac{1}{5}(3,2,3,3,1,1,1,2,2,2,2,3,3,3,3,4,4,2,2,4,2)$	16 128	2 880	25	27
O_{13}	$\frac{1}{6}(1,2,4,2,2,2,1,3,3,3,3,2,2,2,4,2,2,2,4,4,4)$	80 640	13 248	23	24
Total		275 840	49 440		

Lemma 1. *For any vertex v_i of m_n, with $|O_i|$ denoting the size of the orbit of v_i, $|O_i \cap F|$ the size of its restriction to a facet and I_{v_i} the incidence of v_i, we have:*

$$|O_i| \cdot I_{v_i} = |O_i \cap F| \cdot 4\binom{n}{3} \tag{3}$$

Proof. Let $\{v_1, \ldots, v_K\}$ and $\{F_1, \ldots, F_L\}$ be respectively an ordering of the orbit O_i and of the triangle facets, and set $\chi_{kl} = 1$ if the vertex v_k belongs to the triangle facet F_l and 0 otherwise. We have:

$$\sum_{k,l} \chi_{kl} = \sum_l (\sum_k \chi_{kl}) = \sum_l (|O_i \cap F|) = |O_i \cap F| \cdot 4\binom{n}{3}$$

and also,

$$\sum_{k,l} \chi_{kl} = \sum_k (\sum_l \chi_{kl}) = \sum_k (I_{v_i}) = |O_i| \cdot I_{v_i} .$$ □

Table 2. Orbit-wise adjacencies relations of a cut in the skeleton of m_7

O_1	O_2	O_3	O_4	O_5	O_6	O_7	O_8	O_9	O_{10}	O_{11}	O_{12}	O_{13}
63	56	945	3 570	980	7 560	1 120	5 400	8 820	6 930	6 930	2 772	10 080

In Table 2 we present orbit-wise the 55 226 neighbours of a vertex belonging to the orbit O_1, that is a cut. For example, 945 in the third column means that a cut is adjacent to 945 vertices belonging to the orbit O_3, see Section 4 for details. Since all the facets incident to the origin $\delta(\emptyset)$ are precisely the $3\binom{n}{3}$ homogeneous triangle facets, to each vertex adjacent to $\delta(\emptyset)$ corresponds an extreme ray of the metric cone. In other words, the adjacency A_{v_1} of a cut equals the number of extreme rays of the metric cone M_n. We recall that the 41 orbits under permutations of the extreme rays of M_7 were previously found by GRISHUKHIN[21]. Table 2 also implies that the cuts form a dominating clique in the skeleton of m_7, that is, every vertex is adjacent to a cut, as conjectured by LAURENT AND POLJAK [24]. We have:

Corollary 2. *The metric cone on seven nodes has exactly 55 226 extreme rays.*

Corollary 3. *The diameter of the metric polytope on seven nodes is $\delta(m_7) = 3$.*

Proof. The cuts forming a dominating clique, we have $\delta(m_7) \leq 3$. Then, v_{13} and its switching by $\delta(3)$ having no common neighbour, see [12] , we have $\delta(m_7) \geq 3$.

Connectivity. A graph is said to be c edge connected provided it has at least $c + 1$ vertices and no two vertices can be separated by removing fewer that c edges. With C such maximal c, let $C(P)$ denote the edge connectivity of the skeleton of a polytope P. We have:

Theorem 4. *The edge connectivity of thc metric and cut polytope is:*

1. $C(m_n^*) = 2^{\frac{(n-3)(n^2-7)}{3}}$ *for* $n \geq 4$ *and* $C(m_3^*) = 3$.

2. $C(m_4) = 7$, $C(m_5) = 10$, $C(m_6) = 35$, $21 \leq C(m_7) \leq 24$.

3. $C(c_n^*) = \binom{n}{2}$.

4. $C(c_n) = 2^{n-1} - 1$.

Proof. We recall the following result of PLESNÍK [28]. The connectivity of a graph of diameter 2 equals its minimum degree. Then, the skeleton of m_n^* being of diameter 2 and with constant degree $k = 2^{\frac{(n-3)(n^2-7)}{3}}$ for $n \geq 4$, it implies *1.* The diameter of m_4, m_5 and m_6 being 2, it also implies *2* for $n \leq 6$. The facet F_n of c_n induced by the following inequality:

$$\sum_{1 \leq i < j \leq n} b_i b_j x_{ij} \leq 2 \quad \text{where} \quad b = (-(n-4), 1, 1, \ldots, 1)$$

is a simplex facet which contains exactly the $\binom{n}{2}$ cuts $\delta(\{i\})$ for $2 \leq i \leq n$ and $\delta(\{i,j\})$ for $2 \leq i < j \leq n$. This implies that $C(c_n^*) \leq \binom{n}{2}$. Then, BALINSKI's theorem [6] stating that the connectivity of the skeleton of a polytope is at least its dimension, we obtain *3.* The skeleton of c_n being the complete graph, *4* is straightforward. □

The i-Skeletons. We consider the following two families of graphs. while $G^i(P)$ denotes the graph which vertices are all the i-faces of a polytope P, two i-faces being adjacent if and only if $f_i^1 \cap f_i^2$ is a $(i-1)$-face of P, $G_i(P)$ is the graph which vertices are all the i-faces of P, two i-faces being adjacent if and only if f_i^1 and f_i^2 belong to the same $(i+1)$-face of P. We have:

Proposition 5.

1. $G_0(c_n) = K_{2^{n-1}}$.

2. $G_1(c_n) = L(K_{2^{n-1}})$.

3. $G_2(c_n)$ *has* $\binom{2^{n-1}}{3}$ *vertices and two vertices* f_2^1 *and* f_2^2 *are adjacent if and only if:*

$$|f_2^1 \cap f_2^2| = 2 \ \text{or} \ |f_2^1 \cup f_2^2| = 4, \ \text{and} \ f_2^1 \cup f_2^2 \ \text{is a face of} \ c_4.$$

4. *The complement of* $G^{\binom{n}{2}-1}(m_n)$ *is locally the bouquet of* $(n-3)$ (3×3)*-grids with common* K_3.

Proof. The cut polytope being 3-neighbourly, *1* and *2* are straightforward. The $\binom{2^{n-1}}{3}$ 2-faces of c_n are partitioned into the orbits respectively represented by

$f_2^{r,s,t} = \{\delta(\emptyset), \delta(1,\ldots,r+s), \delta(r+1,\ldots,r+s+t)\}$ for all triplets of integers $\{r,s,t\}$ such that $1 \le r \le \lfloor \frac{n}{3} \rfloor$, $0 \le s \le r$, $r \le t \le min(\lfloor \frac{n-r}{2} \rfloor, \lfloor \frac{n}{2} \rfloor - s, n-2r-s)$ and their incidence relations follows. For *4*, that is the skeleton of the dual metric polytope, see [13]. □

Volumes. In Table 3 we give the volumes of m_n and c_n for $n \le 6$. Both volumes seam to quickly vanish to 0 and their ratio, which can be consider as a measure of the tightness of the relaxation of c_n by m_n, seams to stay relatively close to 1. For $n \ge 5$, the volumes were computed using the reverse search method for vertex enumeration using lexicographic pivoting, implemented by AVIS. The code used was lrs Version 2.5i, an earlier version of the code is described in [3]. Since all facets of m_n are equivalent under permutation and switching, the volume of m_n equals $4\binom{n}{3}$ times the volume of the pyramid with basis one facet and apex the center of gravity ω_n of m_n. Comparing the volume of this pyramid and of c_n to the volume of the standard $\binom{n}{2}$-simplex of edge length 2, we have:

$$\frac{Vol(m_n)\cdot\binom{n}{2}!}{2^{\binom{n}{2}}4\binom{n}{3}} = 2^{-4},\ 2^{-5},\ \frac{5\cdot 2^{-3}}{3},\ \frac{7\cdot 281}{3^4} \qquad \text{for } n = 3,\ldots,6.$$

$$\frac{Vol(c_n)\cdot\binom{n}{2}!}{2^{\binom{n}{2}}} = 2^{-2},\ 2^{-1},\ 2^{3},\ 11\cdot 149 \qquad \text{for } n = 3,\ldots,6.$$

Table 3. Volumes of small metric and cut polytopes

$\#n$ nodes	Volume (m_n)	Volume (c_n)	Vol(c_n)/Vol(m_n)
3	1/3	1/3	100%
4	2/45	2/45	100%
5	4/1 701	32/14 175	≈ 96%
6	71 936/1 477 701 225	2 384/58 046 625	≈ 84%

2.3 Summary Tables

In Tables 4, 5 and 6 we sum up known and conjectured results concerning the skeletons and diameters of the metric and cut polytopes. In particular, we give the number of vertices $\#V$ and facets $\#F$ of those polytopes, the incidences I_v and I_f of their vertices and facets, the adjacencies A_v and A_f of their vertices and facets, and the diameter and connectivity of m_n and c_n and of their dual polytopes m_n^* and c_n^*. For example, the last value of the column I_f of Table 5 means that a facet of the cut polytope contains at least $\binom{n}{2}$ vertices, that is, is a simplex and at most $3\cdot 2^{n-3}$ vertices, that is $\frac{3}{4}$ of the total number of vertices of c_n, this bound being reached only by the $4\binom{n}{3}$ triangle facets, see [13]. In the last row of Tables 4 and 5, $A_{\delta(S)}$, A_{Tr} and $\#F_{C_n}$ respectively denote the adjacency of a cut in m_n, the adjacency of a triangle facet in c_n and the number of facets of the cut cone.

Table 4. Skeletons and diameters of metric polytopes

#nodes	#V	I_v	A_v	#F	I_f	A_f	$\delta(m_n)$	$\delta(m_n^*)$
3	4	3	3	4	3	3	1	1
4	8	12	7	16	6	6	1	2
5	32	10∼30	10∼25	40	16	24	2	2
6	544	20∼60	35∼296	80	176	58	2	2
7	275 840	22∼105	24∼55 226	140	49 440	112	3	2
n		$\binom{n}{2}? \sim 3\binom{n}{3}$	$\binom{n}{2}? \sim A_{\delta(S)}?$	$4\binom{n}{3}$		$\frac{2(n-3)(n^2-7)}{3}$	3?	2

Table 5. Skeletons and diameters of cut polytopes

#nodes	#V	I_v	A_v	#F	I_f	A_f	$\delta(c_n)$	$\delta(c_n^*)$
3	4	3	3	4	3	3	1	1
4	8	12	7	16	6	6	1	2
5	16	40	15	56	10∼12	10∼28	1	2
6	32	210	31	368	15∼24	15∼142	1	3
7	64	38 780	63	116 764	21∼48	21∼11 432	1	$3 \leq \delta(c_7^*) \leq 4$
8	128	49 604 520	127	217 093 472	28∼96	28∼?	1	?
n	2^{n-1}	$\#F_{C_n}$	$2^{n-1}-1$		$\binom{n}{2} \sim 3 \cdot 2^{n-3}$	$\binom{n}{2} \sim A_{T_r}?$	1	4?

Table 6. Connectivity of the metric and cut polytopes

#nodes	$C(m_n)$	$C(m_n^*)$	$C(c_n)$	$C(c_n^*)$
3	3	3	3	3
4	7	6	7	6
5	10	24	15	10
6	35	58	31	15
7	$21 \leq C(m_7) \leq 24$	112	63	21
n	$\binom{n}{2}?$	$2\frac{(n-3)(n^2-7)}{3}$	$2^{n-1}-1$	$\binom{n}{2}$

Conjecture 6.

1. *The adjacency of a cut, that is, the number of extreme rays of the metric cone, is maximal in the skeleton of m_n. It holds for $n \leq 7$.*
2. *For n large enough, at least one vertex of m_n is simple, (that is, the incidence equals the dimension of the polytope). If true, it would imply that the edge connectivity, the minimal incidence and the minimal adjacency of the skeleton of m_n are equal to $\binom{n}{2}$. It holds for $n = 3$ and 5.*
3. *The adjacency of a triangle facet is maximal in the skeleton of c_n^*. It holds for $n \leq 7$.*

Table 7. Skeletons and diameters of metric cones

#nodes	#R	I_r	A_r	#F	I_f	A_f	$\delta(M_n)$	$\delta(M_n^*)$
3	3	2	2	3	2	2	1	1
4	7	8~9	6	12	5	5	1	2
5	25	9~24	9~20	30	14	19	2	2
6	296	16~50	23~190	60	113	45	2	2
7	55 226	20~90	20~18 502	105	12 821	86	3	2
n	$A^{m_n}_{\delta(S)}$	$\binom{n}{2} - 1? \sim (n-1)\binom{n-1}{2}$	$\binom{n}{2} - 1? \sim A_{\delta(\{1\})}?$	$3\binom{n}{3}$	$A^{m_n}_{\delta(S)/F}$	$\frac{(n-3)(n^2-6)}{2}$	3?	2

Table 8. Skeletons and diameters of cut cones

#nodes	#R	I_r	A_r	#F	I_f	A_f	$\delta(C_n)$	$\delta(C_n^*)$
3	3	2	2	3	2	2	1	1
4	7	8~9	6	12	5	5	1	2
5	15	27~30	14	40	9~11	9~22	1	2
6	31	114~130	30	210	14~23	14~98	1	3
7	63	11 343~16 460	62	38 780	20~47	20~4 928	1	$3 \leq \delta(C_7^*) \leq 4$
8	127	?	126	49 604 520	27~95	27~?	1	?
n	$2^{n-1} - 1$	$I_{\delta(E)}? \sim I_{\delta(\{1\})}?$	$2^{n-1} - 2$	$I^{c_n}_{\delta(S)}$	$\binom{n}{2} - 1 \sim 3 \cdot 2^{n-3} - 1$	$\binom{n}{2} - 1 \sim A_{Tr}?$	1	4?

Table 9. Connectivity of the metric and cut cones

#nodes	$C(M_n)$	$C(M_n^*)$	$C(C_n)$	$C(C_n^*)$
3	2	2	2	2
4	6	5	6	5
5	9	19	14	9
6	23	45	30	14
7	20	86	62	20
n	$\binom{n}{2} - 1?$	$\frac{(n-3)(n^2-6)}{2}$	$2^{n-1} - 2$	$\binom{n}{2} - 1$

In Tables 7, 8 and 9 we give corresponding results concerning the skeletons and diameters of the metric and cut cones. Those results can be almost directly deduced from the ones given in Tables 4, 5 and 6. In the last row of Table 7, $A_{\delta(\{1\})}$, $A^{m_n}_{\delta(S)}$ and $A^{m_n}_{\delta(S)/F}$ respectively denote the adjacency of the cut $\delta(\{1\})$ in M_n, the adjacency of a cut in m_n and its restriction to a facet of m_n. In the last row of Table 8, $I_{\delta(\{1\})}$, $I_{\delta(E)}$, $I^{c_n}_{\delta(S)}$ and A_{Tr} respectively denote the incidence of the cut $\delta(S)$ with $|S| = 1$ and $|S| = \lfloor \frac{n}{2} \rfloor$ in C_n, the incidence of a cut in c_n and the adjacency of a triangle facet in C_n. For example, the column I_r of Table 7 gives that the maximal incidence of the extreme rays of M_n equals the one of a cut $\delta(S)$ with $|S| = 1$, that is, $I_{max} = I_{\delta(\{1\})} = (n-1)\binom{n-1}{2}$.

Remark. The values $\#F$ for $n = 8$ in Tables 5 and 8 are due to CHRISTOF AND REINELT who recently computed the facets of c_8 and C_8, see [10, 11]. The 217 093 472 facets of c_8 form 147 orbits under its symmetry group; for more information about those facets and the 49 604 520 on s of C_8 see the following WWW site: **http://www.iwr.uni-heidelberg.de/iwr/comopt/soft/SMAPO**.

Theorem 7. *The edge connectivity of the metric and cut cone is:*

1. $C(M_n^*) = \frac{(n-3)(n^2-6)}{2}$ *for* $n \geq 4$ *and* $C(M_3^*) = 2$.
2. $C(M_4) = 6$, $C(M_5) = 9$, $C(M_6) = 23$, $C(M_7) = 20$.
3. $C(C_n^*) = \binom{n}{2} - 1$.
4. $C(C_n) = 2^{n-1} - 2$.

Proof. The cuts forming a clique and the skeleton of M_n^* being of diameter 2 with constant degree $k = (n-3)(n^2-6)/2$ for $n \geq 4$, we have *1* and *4*. A switching of the facet F_n given in the proof of Theorem 4 is a simplex facet of C_n, this implies *3*. Applying BALINSKI's theorem [6] to a section of C_n by a bounding hyperplane, we have $C(C_n^*) = \binom{n}{2} - 1$. The same arguments as for the proof of Theorem 4 give item *2*. □

Proposition 8.

1. *A facet of C_n contains at most $3 \cdot 2^{n-3} - 1$ extreme rays; this bound being reached only by the $3\binom{n}{3}$ triangle facets.*
2. *At least one facet of C_n is a simplex. This implies that the minimal incidence and the minimal adjacency of the skeleton of C_n^* are equal to $\binom{n}{2} - 1$.*
3. *An extreme ray of M_n belong to at most $(n-1)\binom{n-1}{2}$ facets; this bound being reached by only the n cuts $\delta(S)$ of size $|S| = 1$.*
4. *The cuts $\delta(S)$ and the extreme rays $\hat{\delta}(S)$ defined for $2 \leq |S| \leq n-2$ by $\hat{\delta}(S) = d(K_{S,\bar{S}})$ (that is $\hat{\delta}(S)_{st} = 1$ if s and t adjacent and 2 otherwise) form a subgraph of diameter 2 in the skeleton of M_n.*

Proof. Item *1* can be easily deduced form the corresponding result for c_n. A switching of the facet F_n given in the proof of Theorem 7 is a simplex facet of C_n stated in *2*. To prove item *3*, we first recall the following property of the vertices of m_n given in [13]. A vertex v of m_n belongs to at most $3\binom{n}{3}$ facets, that is $\frac{3}{4}$ of the total number of facets of m_n, this bound being reached only by the cuts. More precisely, for v a vertex of m_n and any 3-set $\sigma = \{i, j, k\} \subset N$, we have:

1. either v belongs to exactly 3 of the 4 facets supported by σ; and then $\{v_{ij}, v_{ik}, v_{jk}\} \subset \{0,1\}$,
2. or v belongs to exactly 2 of the 4 facets supported by σ; and then, with $0 < \alpha < 1$, we have $\{v_{ij}, v_{ik}, v_{jk}\} = \{0, \alpha, \alpha\}$ or $\{1, \alpha, 1-\alpha\}$,
3. or v belongs to at most 1 of the 4 facets supported by σ; and then we have $\{v_{ij}, v_{ik}, v_{jk}\} \cap \{0,1\} = \emptyset$.

Then, one can easily check that, in M_n, a cut $\delta(S)$ of size $|S| = s$ belongs to exactly $3\binom{n}{3} - (n-s)\binom{s}{2} - s\binom{n-s}{2}$ triangle facets with the convention $\binom{i}{j} = 0$ for $i < j$. This, with above items 1 and 2, implies that the incidence in M_n of a cut is higher than the one of any other extreme rays. A cut of size $|S| = 1$ being of maximal incidence among the cuts, this completes the proof of item *3*. Using the same notation for the extreme rays of M_n and the corresponding vertices of m_n, the relation in m_n: $\delta(\emptyset)$ not adjacent to $\hat{\delta}(S)$ if and only if $|S| \leq 1$ implies the following relation in M_n: $\delta(\{i\})$ not adjacent to $\hat{\delta}(S)$ if and only if $S = \{i\}$ or $\{i,j\}$. Then, for example, a common neighbour of $\hat{\delta}(\{i,j\})$ and $\hat{\delta}(\{k,l\})$ and of $\hat{\delta}(\{i,j\})$ and $\delta(\{i,j\})$ is $\delta(\{r\})$ for any 5-tuple $\{i,j,k,l,r\}$. This implies *4*. □

Conjecture 9.

1. The adjacency of a cut $\delta(S)$ with $|S| = 1$ is maximal in the skeleton of M_n. It holds for $n \leq 7$.

2. For n large enough, at least one extreme ray of M_n is simple, (that is, the incidence plus one equals the dimension of the cone). If true, it would imply that the edge connectivity, the minimal incidence and the minimal adjacency of the skeleton of M_n are equal to $\binom{n}{2} - 1$. It holds for $n = 3, 5$ and 7.

3. The incidence of a cut $\delta(S)$ in C_n is minimal, respectively maximal, for $|S| = \lfloor \frac{n}{2} \rfloor$, respectively for $|S| = 1$. It holds for $n \leq 7$.

4. The adjacency of a triangle facet is maximal in the skeleton of C_n^*. It holds for $n \leq 7$.

3 Lifting Construction

In this section we present a construction which, under given conditions on a vertex v of m_n, maps v to a vertex of a higher dimensional metric polytope. Let v be a point in $\mathbb{R}^{\binom{n}{2}}$, the *diameter* $\delta(v)$ and *radius* $r(v)$ of v are defined by:

$$\delta(v) = 2r(v) = \max_{1 \leq i < j \leq n} v_{ij} \quad . \tag{4}$$

We consider the following mapping:

$$\Lambda_\alpha^m : \quad \mathbb{R}^{\binom{n}{2}} \to \mathbb{R}^{\binom{n+m}{2}}$$

$$\begin{aligned} \Lambda_\alpha^m(v)_{ij} &= v_{ij} && \text{for } 1 \leq i < j \leq n \\ &= \alpha && \text{for } 1 \leq i \leq n < j \leq n+m \\ &= 2\alpha && \text{for } n < i < j \leq n+m \end{aligned}$$

Then, $\Lambda_\alpha^m(v)$ is a vertex of m_{n+m} if and only if $codim(T_{n+m}(\Lambda_\alpha^m(v))) = 0$ where $T_{n+m}(v)$ is the set of all triangle facets of m_{n+m} containing v.

Case $m = 1$. With $T_{ij,k}$ and P_{ijk} respectively denoting the facet induced by (1) and (2), we have by construction:

$$T_{n+1}(\Lambda_\alpha^1(v)) = T_n(v) \cup T \ . \tag{5}$$

Where

$$T = \bigcup_{v_{ij}=2\alpha} \{T_{ij,n+1}\} \bigcup_{v_{ij}=0} \{T_{i\,(n+1),j}\} \bigcup_{v_{ij}=0} \{T_{j\,(n+1),i}\} \bigcup_{v_{ij}=2-2\alpha} \{P_{ij\,(n+1)}\}.$$

The equality (5) clearly implies

$$\Lambda_\alpha^1(v) \in m_{n+1} \Longleftrightarrow r(v) \leq \alpha \leq 1 - r(v) \tag{6}$$

and

$$r(v) < \alpha < 1 - r(v) \Longrightarrow codim(T_{n+1}(\Lambda_\alpha^1(v))) \geq n \ . \tag{7}$$

This means that a necessary condition for $\Lambda_\alpha^1(v)$ to be a vertex of m_{n+1} is $\alpha = r(v)$ or $\alpha = 1 - r(v)$. Since we have $\Lambda_{1-\alpha}^1(v) = r_{\delta(\{n+1\})}(\Lambda_\alpha^1(v))$, we can consider only the case $\alpha = r(v)$ (we recall that $r_{\delta(\{n+1\})}$ is the switching by the cut $\delta(\{n+1\})$, see Sect. 2.1.). We call $\Lambda_{r(v)}^1(v)$ the *radial extension* of v and denote it by $\Lambda^1(v)$.

Before stating the conditions on v to lift it to m_{n+1}, we need the following two definitions. Call a graph $G = (N, E)$ *good*, $N = \{1, 2 \dots, n\}$, if it has a partial subgraph $G' = (N, E')$ with $|E'| = |N|$ which does not admit a non-zero edge-weighting $f\colon E' \to \mathbb{R}$ with $\sum_{v \in e \in E'} f_e = 0$ for each $v \in N$. The graph $\Gamma(v)$ on N is defined by: s and t adjacent if and only if $v_{st} = \delta(v)$. For example, if $v = \frac{1}{3}d(G)$ for a graph G of diameter 2 (that is $v_{st} = \frac{1}{3}$ if s and t adjacent and $\frac{2}{3}$ otherwise), then $\Gamma(v)$ is the complement of G and $\Lambda^1(v) = \frac{1}{3}d(\nabla G)$ where ∇G is the suspension of G, that is, G plus one vertex adjacent to all vertices of G.

Theorem 10. *For any vertex v of m_n such that $\Gamma(v)$ is good, the radial extension $\Lambda^1(v)$ is a vertex of m_{n+1}.*

Proof. Since $\Gamma(v)$ is good, it has a partial subgraph $\Gamma' = (N, E')$ with $|E'| = n$ which does not admit a non-zero edge-weighting. Clearly, any connected graph with n vertices and less than n edges is either a tree, or an *odd cycled tree* or an *even cycled tree*, where an odd cycled tree, respectively even cycled tree, is a tree plus one edge forming with it an odd, respectively even, cycle. Since a tree has $n - 1$ edges and an even cycled tree admits unwanted edge-weighting, they are both not good and therefore Γ' can only be a *odd cycled forest*, that is, contains for each connected components of Γ its spanning odd cycled tree. Now, since v is a vertex of m_n, $T_n(v)$ contains $\binom{n}{2}$ linearly independent triangle facets which form the set $T_n'(v)$. Then, the $\binom{n}{2} + n = \binom{n+1}{2}$ facets of the set $T_n'(v) \cup_{ij \in E'} T_{ij,n+1}$ are linearly independent facets containing $\Lambda^1(v)$, since if not, Γ' admits a non-zero weighting and therefore Γ is not good. This implies $codim(T_{n+1}(\Lambda^1(v))) = 0$ and completes the proof. □

Case $m \geq 2$. As for the case $m = 1$, we need to consider only the case $\alpha = r(v)$. Similarly, $\Lambda^m_{r(v)}(v)$ is called the *radial m-extension* of v and denoted by $\Lambda^m(v)$. By construction, for $m \geq 2$ we have:

$$T_{n+m}(\Lambda^m(v)) = T_n(v) \cup T \ . \tag{8}$$

Where

$$T = \bigcup_{v_{ij}=\delta(v),\, 1\leq i<j\leq n<k\leq m} \{T_{ij,k}\} \bigcup_{v_{ij}=0,\, 1\leq i<j\leq n<k\leq m} \{T_{ik,j}\} \bigcup_{v_{ij}=0,\, 1\leq i<j\leq n<k\leq m} \{T_{jk,i}\}$$

$$\bigcup_{v_{ij}=1,\, 1\leq i<j\leq n<k\leq m} \{P_{ijk}\} \bigcup_{1\leq k\leq n<i<j\leq n+m} \{T_{ij,k}\} \bigcup_{\delta(v)=1,\, 1\leq k\leq n<i<j\leq n+m} \{P_{ijk}\}$$

$$\bigcup_{m\geq 3,\, n<i<j<k\leq n+m,\, \delta(v)=\frac{2}{3}} \{P_{ijk}\}.$$

The equality (8) implies:

$$\Lambda^2(v) \in m_{n+2} \text{ and, for } m \geq 3,\ \Lambda^m(v) \in m_{n+m} \Longleftrightarrow \delta(v) \leq \frac{2}{3} \ . \tag{9}$$

Theorem 11. *For any vertex v of m_n such that $\Gamma(v)$ is good and, for $m \geq 3$, $\delta(v) \leq \frac{2}{3}$, the radial m-extension $\Lambda^m(v)$ is a vertex of m_{n+m}.*

Proof. The proof is similar to the one of Theorem 10. We consider the following set of $\binom{n}{2} + n \cdot m + \binom{m}{2} = \binom{n+m}{2}$ triangle facets containing $\Lambda^m(v)$: $T'_n(v) \cup (\cup_{ij\in E', n<k<n+m} T_{ij,k}) \cup_{1\leq n<i<j} T_{ij,k}$. The graph $\Gamma(v)$ being good, they are linearly independent and therefore we have $codim(T_{n+m}(\Lambda^1(v))) = 0$. □

Remark.

1. The condition that v is a vertex of m_n is not necessary. For example, $v = \frac{2}{3}d(K_4)$ is not a vertex of m_4 but $\Lambda^1(v) = \frac{2}{3}d(K_5)$ is a vertex of m_5.

2. We do not know any vertex of m_n with no good graph $\Gamma(v)$ such that $\Lambda^1(v)$ is a vertex of m_{n+1}.

3. Among the 13 representatives given in Table 1, for $i = 2, 3, 4, 5, 8, 9$ the vertices v_i are both good and satisfy $\delta(v) \leq \frac{2}{3}$. We have $v_2 = \frac{2}{3}d(K_7)$, $v_7 = \frac{1}{3}d(K_7 - C_{2,3,4} - C_{5,6,7})$, $v_8 = \frac{2}{5}d(K_7 - C_7)$, $v_9 = \frac{1}{3}d(K_7 - C_7 - P_{1,3})$ and $v_{10} = \frac{1}{3}d(K_7 - C_{2,3,4} - C_{5,6,7} - P_{4,5})$ where C_s and P_s. respectively denotes the cycle and the path on the subset $s \subset \{1, 2, \ldots, 7\}$, C_7 being the cycle on 7 nodes.

4. For $n \geq 5$, v a vertex of m_n and $\Gamma(v) = \tilde{T}$ for a tree T which is not a star, LAURENT [23] proved that $\Lambda^1(v)$ is a vertex of m_{n+1}.

5. With G an almost complete t-partite graph, AVIS [2] proved that $\frac{1}{3}d(G)$ is a vertex of m_n, Theorem 11 implies that $\Lambda^1(\frac{1}{3}d(G))$ and $\Lambda^2(\frac{1}{3}d(G))$ are vertices of, respectively, m_{n+1} and m_{n+2} as well.

Proposition 12. *For G a complete t-partite graph on 8 nodes, $v = \frac{1}{3}d(G)$ is a vertex of m_8 only for $G = K_{4,3,1}$ and $K_{3,3,2}$. The point $v = \frac{1}{3}d(G_e)$ is also a vertex of m_8 for $G_e = K_{3,3,1,1} - e$, $K_{4,2,2} - e$ and $K_{6,1,1} - e$ where e is an edge of, respectively, the subgraph $K_{3,3}$, $K_{4,2}$ and $K_{1,1}$.*

Proof. Theorem 11 gives that $v = \frac{1}{3}d(G)$ is a vertex of m_8 for $G = K_{4,3,1}, K_{3,3,2}$ and $K_{3,3,1,1} - e$. To check if the others complete t-partite graphs induce a vertex of m_8, we built the set $T(v)$ of triangle facets containing the point $v = \frac{1}{3}d(G)$ and then check by computer if they intersect on a vertex. Considering some subsets of $T(v)$, we found that the graphs $K_{4,2,2} - e$ and $K_{6,1,1} - e$ induce a vertex of m_8.

4 Computational Aspects

All facets of the metric polytopes being equivalent under permutations and switching, it is enough to compute all the vertices belonging to one facet. In [21] GRISHUKHIN used this technique to compute the 41 orbits of extreme rays under permutations of the metric cone on 7 nodes. This *vertex enumeration* problem was solved using the double description method *cdd* implemented by FUKUDA [20]. The algorithm first constructs a simplex starting with a non-degenerate subset of $d+1$ inequalities where d is the dimension, then at each step one inequality is inserted. The efficiency of this algorithm highly depends on the order in which the inequalities are inserted. It is observed that the results seem to be good when the size of the intermediate polytope produced at each step stay as small as possible. For this important ordering issues we refer to AVIS, BREMMER AND SEIDEL [4] where, in particular, worst case behavior polyhedra are constructed.

To obtain the 275 840 vertices of the 21-dimensional polytope m_7 we used the following ordering. The 140 facets were inserted such that $F_1 - F_4$, $F_5 - F_8, \ldots, F_{137} - F_{140}$ form the 35 maximal cocliques of the skeleton of m_7^*, that is, by set of 4 facets with the same support. Then to order those cocliques, we consider the following Hausdorff distance between cocliques of facets. With C and C' two cocliques, we have $d(C, C') = \max d(F, G)$ where F, respectively G, is a facet of C, respectively C' and $d(F, G) = 0$ if $codim(F \cap G) = 2$ and 1 otherwise. The cocliques are then ordered by the maximal cocliques (of cocliques) of the graph which nodes are the cocliques of facets and edges given by the previous Hausdorff distance. The same operation being repeated for cocliques of cocliques of facets and so on.

This ordering gave us much better results that the classical *lexico-graphic*, *min-cut off* and *max-cut off* ordering which respectively selects a facet which cuts off the minimum, respectively maximum, number of vertices of the intermediate polytope, see [20]. This ordering by maximal cocliques of the dual skeleton gave also excellent results for the computation of the Solitaire cone and its relatives, see [5]. In all those cases, including the metric polytope, the maximal size of the intermediate polyhedra was less than twice the size of the final one.

Computation of Table 2. For each representative vertex v_i we computed the cone C_i generated by the set $T(v_i)$ of all triangle facets containing v_i. Clearly, to each extreme ray of this cone pointed on v_i corresponds a neighbour of v_i, in other words, the size of C_i equals the adjacency A_{v_i} of v_i in m_7. Then, by a tedious one by one checking of all the extreme rays of C_i, we listed all rays pointing to a cut. Finally, using the relation $|O_i| \cdot a_{ij} = |O_j| \cdot a_{ji}$ where $|O_i|$ and a_{ij} respectively denotes the size of the orbit O_i and the number of vertices of O_j adjacent to v_i, we filled Table 2. For example, the 30 facets containing v_6 form the cone C_6 which have 96 extreme rays, that is, $A_{v_6} = 96$. Out of those 96 rays, exactly 24 point to a cut. Then, $64 \times a_{1,6} = 20160 \times 24$ implies $a_{1,6} = 7560$.

Remark. Clearly we have $a_{1,1} = 2^{n-1}-1$; the values $a_{2,1} = 2^{n-1}-n-1$ and $a_{3,1} = 2^{n-1}-3n+2$ were given in [13]. So we have $a_{i,1} = 63, 56, 45, 34, 28, 24, 16, 15, 14, 11, 11, 11, 8$ for $i = 1, 2, \ldots 13$. The complete list of cuts adjacent to v_i for $i = 4, \ldots, 13$ is:

- v_4 adjacent to $\delta(S)$ for $S = \{i,j\}$ with $3 \le i < j \le 5$ and for $S = \{i,j,k\}$ with $\{i,j,k\} \cap \{3,4,5\} \ne \emptyset$,
- $v_5 \sim \delta(S)$ for $S = \{i,j\}$ with $2 \le i < j \le 5$, $S = \{1,i,j\}$ with $2 \le i < j \le 5$ and for $S = \{i,j,k\}$ with $2 \le i < j < k \le 7$ and $j \ne 6$.
- $v_6 \sim \delta(S)$ for $S = \emptyset, \{1\}, \{4\}, \{6\}, \{1,2\}, \{1,5\}, \{1,7\}, \{2,6\}, \{3,4\}, \{4,7\}, \{5,6\}, \{6,7\}, \{1,2,3\}, \{1,2,7\}, \{1,3,5\}, \{1,4,7\}, \{1,5,7\}, \{2,3,4\}, \{2,3,6\}, \{2,6,7\}, \{3,4,5\}, \{3,5,6\}, \{4,6,7\}, \{5,6,7\}$,
- $v_7 \sim \delta(S)$ for $S = \emptyset$, $S = \{i\}$ with $i \ne 1$ and for $S = \{i,j\}$ with $i = 2,3,4$ and $j = 5,6,7$,
- $v_8 \sim \delta(S)$ for $S = \emptyset, \{1,3\}, \{1,4\}, \{1,5\}, \{2,5\}, \{2,6\}, \{3,6\}, \{4,7\}, \{1,3,5\}, \{1,3,6\}, \{1,4,6\}, \{2,4,6\}, \{2,4,7\}, \{2,5,7\}, \{3,5,7\}$,
- $v_9 \sim \delta(S)$ for $S = \emptyset, \{1\}, \{3\}, \{1,4\}, \{1,5\}, \{3,6\}, \{3,7\}, \{1,3,5\}, \{1,3,6\}, \{1,4,6\}, \{2,4,6\}, \{2,4,7\}, \{2,5,7\}, \{3,5,7\}$,
- $v_{10} \sim \delta(S)$ for $S = \emptyset, \{4\}, \{5\}, \{2,5\}, \{2,6\}, \{2,7\}, \{3,5\}, \{3,6\}, \{3,7\}, \{4,6\}, \{4,7\}$,
- $v_{11} \sim \delta(S)$ for $S = \emptyset, \{1\}, \{3\}, \{1,2\}, \{1,6\}, \{3,4\}, \{4,5\}, \{2,3,7\}, \{2,5,7\}, \{3,6,7\}, \{5,6,7\}$,
- $v_{12} \sim \delta(S)$ for $S = \{3\}, \{5\}, \{1,3\}, \{4,5\}, \{4,7\}, \{5,6\}, \{1,3,4\}, \{1,4,7\}, \{1,5,6\}, \{1,6,7\}, \{2,3,5\}$,
- $v_{13} \sim \delta(S)$ for $S = \emptyset, \{5\}, \{6\}, \{7\}, \{4,7\}, \{1,2,7\}, \{4,5,7\}, \{4,6,7\}$.

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