Structural and Multidisciplinary Optimization manuscript No.
(will be inserted by the editor)

A compliance based design problem of structures under multiple load cases

Athanasios Makrodimopoulos · Atul Bhaskar · Andy J. Keane

Received: date / Accepted: date

Abstract There are two popular methods concerning the optimal design of structures. The first is the minimization of the volume of the structure under stress constraints. The second is the minimization of the compliance for a given volume. For multiple load cases an arising issue is which energy quantity should be the objective function. Regarding the sizing optimization of trusses, Rozvany proved that the solution of the established compliance based problems leads to results which are awkward and not equivalent to the solutions of minimization of the volume under stress constraints, unlike under single loading [1]. In this paper, we introduce the "envelope strain energy" problem where we minimize the volume integral of the worst case strain energy of each point of the structure. We also prove that in the case of sizing optimization of statically non-indeterminate[2] trusses, this compliance method gives the same optimal design as the stress based design method.

Keywords Structural optimization · Truss optimization · Compliance methods · Envelope strain energy method · Stress based methods

EPSRC grant EP/E004547/1.

A. Makrodimopoulos (corresponding author)
School of Engineering Sciences, University of Southampton
Tel.: ++44-(0)2380-595194
Fax : ++44-(0)2380-594813
E-mail: thanmac2000@yahoo.gr

A. Bhaskar
School of Engineering Sciences, University of Southampton
Tel.: ++44-(0)2380-593825
Fax : ++44-(0)2380-594813
E-mail: a.bhaskar@soton.ac.uk

A.J. Keane
School of Engineering Sciences, University of Southampton
Tel.: ++44-(0)2380-592944
Fax : ++44-(0)2380-594813
E-mail: andy.keane@soton.ac.uk

[1] the layouts would be the same if in the compliance problem the volume is set equal to the result of the first problem

[2] the term non-indeterminate includes both statically determinate trusses and mechanisms

1 Introduction

Consider a structure made of linearly elastic material. Depending on the objective function and constraints, two popular problems of finding the optimal design are (a) the minimization of the weight of the structure under stress constraints - usually the yield restriction, (b) the minimization of compliance i.e. the work of the external loads. For sizing optimization of trusses, interesting simplifications occur for the single loading case. It has been proved that the optimal layout is the same for both problems (a) and (b). Note, however, that the cross sectional areas (apart from non-negativity) should not be constrained.This is discussed in detail by Achtziger (1992, 1996).

For multiple load cases an arising issue for the compliance methods, is which energy quantity should be the objective function. There are two types of compliance methods; the weighted average compliance method and the worst-case compliance. By using very simple examples (Figure 1), Rozvany (2001b) proved that in truss optimization, these compliance methods lead to different optimal layouts when compared to that obtained from stress constrained problem. Moreover, the results are awkward from the aspect of stiffness. E.g. in the case of the structure of Fig. 1b the bar which is subjected to a four times higher load would be expected to have a four times greater cross sectional area than the other bar.

The purpose of this note is to propose a new type of compliance-based objective function for the design of structures, namely the envelope strain energy method. We minimize the integral of the highest strain energy density at every point of a structure. We show that if a truss is statically non-indeterminate, the suggested problem coincides with that of the minimization of the volume under stress constraints. Therefore, this compliance when used as an objective, overcomes the problems raised by the bench-marks of Rozvany (2001b). The general case of trusses is also discussed.

2 The general form of the envelope strain energy problem

Consider a structure V whose optimality depends on the definition of the geometry and material parameters $\boldsymbol{\chi}$. Assume also that the structure is subjected to arbitrarily varying load fields $P(t)$ within a load domain $\mathcal{L}$, where t is a pseudo-time parameter. According to the proposed envelope strain energy method, the following problem has to be solved:

$$
\begin{aligned}
\min \; & \int_V \max_{P(t)\in\mathcal{L}} \boldsymbol{\sigma}(\mathbf{x},t) : \boldsymbol{\varepsilon}(\mathbf{x},t)\,\mathrm{d}V \\
\text{s.t.} \; & \chi \in X \\
& \varepsilon_{ij}(\mathbf{x},t) = \frac{1}{2}\left(\frac{\partial u_i(\mathbf{x},t)}{\partial x_j} + \frac{\partial u_j(\mathbf{x},t)}{\partial x_i}\right) \\
& \sigma_{ij}(\mathbf{x},t) = \sum_{k,l} C_{ijkl}\varepsilon_{kl}(\mathbf{x},t) \\
& \boldsymbol{\sigma} \in S_{\mathrm{eq}}(P(t)),
\end{aligned}
\tag{1}
$$

where $\boldsymbol{\sigma}, \boldsymbol{\varepsilon}$ and $\mathbf{u}$ are the stresses, strains and displacements respectively. Also C_{ijkl} is the elasticity tensor, S_{eq} is the set of stresses which can carry the loads, X is the set of constraints to which the parameters $\boldsymbol{\chi}$ are subjected (including the volume constraint).

In the worst-case compliance, the problem will have the form

$$\begin{aligned}&\min \max_{P(t)\in\mathcal{L}} \int_V \boldsymbol{\sigma}(\mathbf{x},t) : \boldsymbol{\varepsilon}(\mathbf{x},t)\ \mathrm{d}V \\ &\text{s.t. the same constraints as in (1).}\end{aligned} \tag{2}$$

This is different from (1) where the maximum is taken from point to point, whereas in (2) it is taken over the volume integral.

3 Sizing optimization for trusses

Consider a truss structure of NE members and NU degrees of freedom where the only free parameters over which to optimize are the cross sectional areas of the members or, for simplicity, their volumes. Therefore problem (1), when specialised to trusses, after some manipulations and considering the half of compliance reads:

$$\begin{aligned}&\min \sum_{i=1}^{NE} \max_{\mathbf{p}(t)\in\mathcal{L}} \frac{L_i^2}{E_i}\frac{(q_i(t))^2}{2\xi_i}\\ &\text{s.t.} \left(\sum_{i=1}^{NE}\xi_i\mathbf{K}_i\right)\mathbf{u}(t)=\mathbf{p}(t), \quad \forall\mathbf{p}(t)\in\mathcal{L}\\ &\quad q_i(t)=\frac{E_i\xi_i}{L_i^2}\mathbf{b}_i^T\mathbf{u}(t), \quad i=1,\ldots,NE\\ &\quad \sum_{i=1}^{NE}\xi_i\le V\\ &\quad \xi_i\ge 0, \quad i=1,\ldots,NE\end{aligned} \tag{3}$$

where $\mathbf{p},\mathbf{u}\in\Re^{NU}$ are the load and the displacement vectors respectively, ξ_i, L_i and E_i are the volume, the length and the Young's modulus of the ith member. Also $\mathbf{q}(t)$ is the axial forces vector due to the load vector $\mathbf{p}(t)$. The vector $\mathbf{b}_i$ relates the elongation, e_i along the axis of member i to the nodal displacements so that

$$e_i=\mathbf{b}_i^T\mathbf{u}. \tag{4}$$

Also for each member,

$$\mathbf{K}_i=\frac{E_i}{L_i^2}\mathbf{b}_i\mathbf{b}_i^T. \tag{5}$$

We set $r_i=\frac{L_i}{\sqrt{E_i}}\max_{\mathbf{p}(t)\in\mathcal{L}}|q_i(t)|$ and the problem is transformed into

$$\begin{aligned}&\min \sum_{i=1}^{NE}\frac{r_i^2}{2\xi_i}\\ &\text{s.t.} \frac{L_i}{\sqrt{E_i}}\max_{\mathbf{p}(t)\in\mathcal{L}}|q_i(t)|=r_i, \quad i=1,\ldots,NE\\ &\quad \left(\sum_{i=1}^{NE}\xi_i\mathbf{K}_i\right)\mathbf{u}(t)=\mathbf{p}(t), \quad \forall\mathbf{p}(t)\in\mathcal{L}\\ &\quad q_i(t)=\frac{E_i\xi_i}{L_i^2}\mathbf{b}_i^T\mathbf{u}(t), \quad \forall\mathbf{p}(t)\in\mathcal{L}\\ &\quad \sum_{i=1}^{NE}\xi_i\le V\\ &\quad \xi_i\ge 0, \quad i=1,\ldots,NE.\end{aligned} \tag{6}$$

Now assume that the truss is statically non-indeterminate. Therefore we can obtain the axial forces from static equilibrium alone and the problem has the same form as the case of using the principle of complementary energy:

$$\begin{aligned}&\min \sum_{i=1}^{NE}\frac{r_i^2}{2\xi_i}\\ &\text{s.t.}\ r_i=\frac{L_i}{\sqrt{E_i}}\max_{\mathbf{p}(t)\in\mathcal{L}}|q_i(t)|, \quad i=1,\ldots,NE\\ &\quad \mathbf{B}\mathbf{q}(t)=\mathbf{p}(t), \quad \forall\mathbf{p}(t)\in\mathcal{L}\\ &\quad \sum_{i=1}^{NE}\xi_i\le V\\ &\quad \xi_i\ge 0, \quad i=1,\ldots,NE.\end{aligned} \tag{7}$$

Now we apply the technique described by Ben-Tal and Nemirovski (2001), page 130; for a given $\mathbf{q}$ (and consequently $\mathbf{r}$), $\boldsymbol{\xi}$ are the only variables of the problem,

$$\begin{aligned}&\min \sum_{i=1}^{NE}\frac{r_i^2}{2\xi_i}\\ &\text{s.t.} \sum_{i=1}^{NE}\xi_i\le V\\ &\quad \xi_i\ge \mathbf{0}, \quad i=1,\ldots,NE.\end{aligned} \tag{8}$$

Eventually we get

$$\xi_i = V r_i / \sum_{i=1}^{NE} r_i. \tag{9}$$

Then the optimization problem takes the following form

$$\begin{aligned} \min \; & \sum_{i=1}^{NE} r_i \\ \text{s.t.} \; & r_i = \frac{L_i}{\sqrt{E_i}} \max_{\mathbf{p}(t)\in\mathcal{L}} |q_i(t)|, \quad i = 1,\ldots,NE \\ & \mathbf{Bq}(t) = \mathbf{p}(t), \quad \mathbf{p}(t) \in \mathcal{L} \end{aligned} \tag{10}$$

which can be transformed into,

$$\begin{aligned} \min \; & \sum_{i=1}^{NE} r_i \\ \text{s.t.} \; & |q_i(t)| \leq \frac{\sqrt{E_i}}{L_i} r_i, \;\; i = 1,\ldots,NE, \;\; \forall \mathbf{p}(t) \in \mathcal{L} \\ & \mathbf{Bq}(t) = \mathbf{p}(t), \quad \forall \mathbf{p}(t) \in \mathcal{L}. \end{aligned} \tag{11}$$

If we apply the substitution

$$\sqrt{E_i} \rightarrow \sigma_{\mathrm{y},i} \quad \text{and} \quad r_i \rightarrow \xi_i. \tag{12}$$

the problem (10) transforms to

$$\begin{aligned} \min \; & \sum_{i=1}^{NE} \xi_i \\ \text{s.t.} \; & |q_i(t)| \leq \sigma_{y,i}\xi_i/L_i, \;\; i = 1,\ldots,NE, \;\; \forall \mathbf{p}(t) \in \mathcal{L} \\ & \mathbf{Bq}(t) = \mathbf{p}(t), \quad \forall \mathbf{p}(t) \in \mathcal{L} \\ & \xi_i \geq 0, \quad i = 1,\ldots,NE \end{aligned} \tag{13}$$

which is the plastic design problem of trusses and the problem of minimizing the volume of an elastic and statically non-indeterminate truss subjected to stress constraints. Note that this similarity is valid if all members have the same ratio $\sigma_y/\sqrt{E}$, because of the substitution in (12). However the question that remains, is how closely problem (3) matches with the problem

$$\begin{aligned} \min \; & \sum_{i=1}^{NE} \xi_i \\ \text{s.t.} \; & |\mathbf{q}_i(t)| \leq \sigma_{\mathrm{y},i}\xi_i/L_i, \;\; i = 1,\ldots,NE, \;\; \forall \mathbf{p}(t) \in \mathcal{L} \\ & \mathbf{q}_i(t) = \frac{E_i \xi_i}{L_i^2} \mathbf{b}_i^T \mathbf{u}(t), \;\; i = 1,\ldots,NE, \;\; \forall \mathbf{p}(t) \in \mathcal{L} \\ & \left(\sum_{i=1}^{NE} \xi_i \mathbf{K}_i \right) \mathbf{u}(t) = \mathbf{p}(t), \quad \forall \mathbf{p}(t) \in \mathcal{L} \\ & \xi_i \geq \mathbf{0}, \quad i = 1,\ldots,NE \end{aligned} \tag{14}$$

which is the general problem of minimizing the volume of any truss, under yield constraints.

Note that the yield restriction is written in terms of the axial forces and not the stresses. In this way a zero value of the cross sectional area will coincide with the non-existance of the bar. Details on this issue are given by Rozvany (2001a) and the references therein. The restriction has the form

$$A_i(|\sigma_i| - \sigma_{\mathrm{y},i}) \leq 0 \Leftrightarrow E_i A_i(|\varepsilon_i| - \sigma_{\mathrm{y},i}/E_i) \leq 0 \tag{15}$$

where ε_i is the strain and A_i is the cross-sectional area. The restriction can be rewritten as

$$E_i \xi_i (\varepsilon_i^2 - (\sigma_{\mathrm{y},i}/E_i)^2) \leq 0 \tag{16}$$

and after using (4) and (5), this becomes

$$\xi_i \mathbf{u}^T \mathbf{K}_i \mathbf{u} \leq \xi_i (\sigma_{\mathrm{y},i}/\sqrt{E_i})^2. \tag{17}$$

Therefore, problem (14), after using (17) takes the form

$$\begin{aligned} \min \; & \sum_{i=1}^{NE} \xi_i \\ \text{s.t.} \; & \xi_i (\mathbf{u}(t))^T \mathbf{K} \mathbf{u}(t) \leq \xi_i \lambda_i^2, \;\; i = 1,\ldots,NE, \;\; \forall \mathbf{p}(t) \in \mathcal{L} \\ & \left(\sum_{i=1}^{NE} \xi_i \mathbf{K}_i \right) \mathbf{u}(t) = \mathbf{p}(t), \quad \forall \mathbf{p}(t) \in \mathcal{L} \\ & \xi_i \geq 0, \quad i = 1,\ldots,NE. \end{aligned}$$

$$(18)$$

where $\lambda_i = \sigma_{y,i}/\sqrt{E_i}$. The envelope strain energy is

$$\begin{aligned} \Psi &= \sum_{i=1}^{NE} \max_{\mathbf{p}(t)\in\Delta} q_i(t)e_i(t) \\ &= \sum_{i=1}^{NE} \xi_i \max_{\mathbf{p}(t)\in\Delta} (\mathbf{u}(t))^T \mathbf{K}\mathbf{u}(t) \end{aligned} \tag{19}$$

Now, assume that $\lambda_i = \lambda, \quad \forall i$. From the first constraint of (18) we conclude that

$$\Psi \leq \lambda^2 V. \tag{20}$$

Equality occurs when for each member i either $\xi_i = 0$ or there exists a $P(t^*) \in \Delta$ so that $(\mathbf{u}(t^*))^T\mathbf{K}\mathbf{u}(t^*) = \lambda^2$. Of course it still remains a question if this can be the the result of the envelope strain energy problem. That could result to an equivalance with the Fully Stress Design (FSD) method.

In case that $\mathcal{L}$ is a convex hyperpolyhedron which consists of m vertices, all terms as $\mathbf{u}(t), \mathbf{p}(t)$ will be replaced by terms $\mathbf{u}^{(j)}, \mathbf{p}^{(j)}$ with $j = 1, \ldots, m$ as in the paper of Makrodimopoulos et al (2010).

4 Discussion

The worst case compliance is a lower bound of the envelope strain energy. Their values will coincide if in all points of the structure, the highest strain energy is caused for the same load case.

Regarding sizing optimization of trusses; as we can see e.g., in Ben-Tal and Nemirovski (2001) or Bendsøe (1995), the compliance problems (single loading, worst case compliance and weighted average compliance) are initially non-convex. However after some manipulation they can be transformed into convex ones. An interesting alternative is the use of the complementary energy as given by Bendsøe (1995). In this case we only express the equilibrium in terms of stress. The equilibrium in terms of displacements and stiffness matrix arises after applying the Karush-Kuhn-Tucker conditions. However this does not occur if we try to minimize the envelope of complementary energy subjected to the equilibrium constraints in terms of axial forces.

Physically, the problem of the envelope strain energy takes into account all the load cases both in the objective function and the constraints. On the other hand the problem of the worst-case compliance results in a layout which is optimal only for the worst case. This means that the structure will not be optimal if other load cases are applied.

5 Conclusions

The envelope strain energy as an objective function in structural optimization is proposed for the first time. This compliance is believed to be able to give more plausible optimal layouts as the objective function takes into account the influence of all load cases. For the case of sizing optimization of trusses, it has been proved that at least for statically non-indeterminate trusses, the optimal layout would be similar to the one of the stress constrained problem. Therefore, this compliance overcomes the problems raised by the bench-marks of Rozvany (2001b).

The resulting numerical optimization problem is complicated as it is non-convex. Whether this problem could be transformed into a convex one is an interesting open question.

References

Achtziger W (1992) Truss topology design under multiple loading. DFG-Report (FSP Applied Optimization and Control), No 367, University of Bayreuth, FRG

Achtziger W (1996) Truss topology optimization including bar properties different for tension and compression. Structural and Multidisciplinary Optimization 12:63–74

Ben-Tal A, Nemirovski A (2001) Lectures on modern convex optimization. MPS-SIAM Series on Optimization, Philadelphia

Bendsøe M (1995) Optimization of Structural Topology, Shape and Material. Springer-Verlag, Berlin

Makrodimopoulos A, Bhaskar A, Keane A (2010) Second-order cone programming formulations for a class of problems in structural optimization. Structural and Multidisciplinary Optimization 40:365–380

Rozvany G (2001a) On design-dependent constraints and singular topologies. Structural and Multidisciplinary Optimization 21:164–172

Rozvany G (2001b) Stress ratio and compliance based methods in topology optimization–a critical review. Structural and Multidisciplinary Optimization 21:109–119

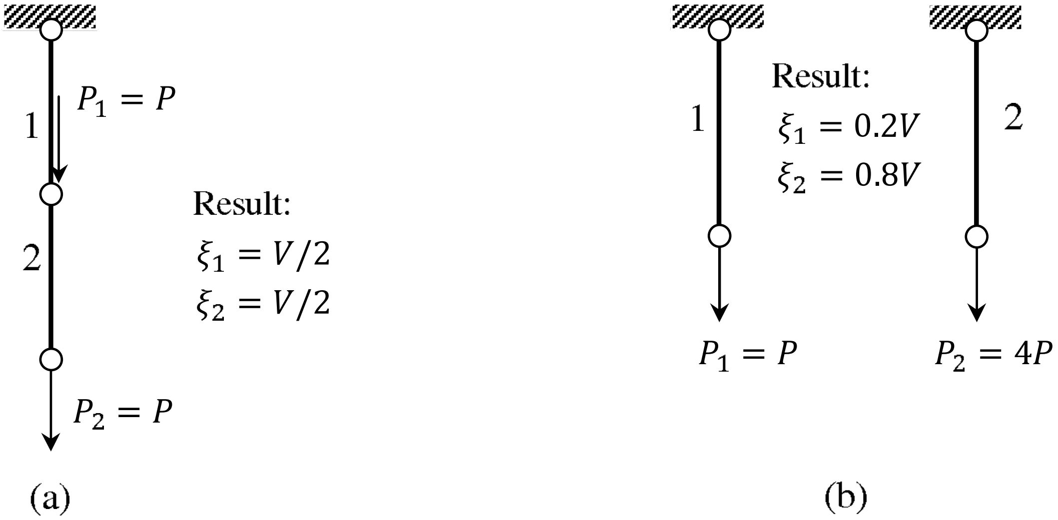

Fig. 1 The two examples from Rozvany (2001b). The structures are subjected to alternate loading. In both examples the two bars have the same properties (Young's modulus, yield strength and length). The envelope strain energy method results in similar optimal layouts to the stress constrained method. According to Rozvany (2001b), for the structure (a) the weighted average compliance results to $\xi_1 = 0586V$ and $\xi_2 = 0.414V$. For the structure (b) the worst-case compliance results to $\xi_1 = 0.0588V$ and $\xi_2 = 0.9412V$.