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# Clustering in $N$ -Player Preemption Games <sup>\*</sup>

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## Abstract

We study a complete information preemption game in continuous time. A finite number of firms decide when to make an irreversible, observable investment. Upon investment, a firm receives flow profits which decrease in the number of firms that have invested. The cost of investment declines over time exogenously. We characterize the subgame-perfect equilibrium outcome, which is unique up to a permutation of players. When the preemption race among late investors is sufficiently intense, the preemption incentive for earlier investors disappears, and two or more investments occur at the same time. We identify a sufficient condition in terms of model parameters: clustering of investments occurs if the flow profits from consecutive investments are sufficiently close. This shows how clustering can occur in the absence of coordination failures, informational spillovers or positive payoff externalities.

KEYWORDS: Timing Games, Preemption, Technology Adoption, Dynamic Entry.

JEL CLASSIFICATION: C73, L13, O3.

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# 1 Introduction

Consider a game of timing in which players have to decide when to make an investment. The cost of investing declines over time. A firm earns a positive profit flow upon investment, but profit flows decline in the number of investors. This is a *preemption game*: Delay exogenously increases payoffs through lower investment cost, but each player also has an incentive to invest early, because there is an early mover advantage.

In a preemption game, investment by a player reduces the post-investment flow profit for later investors, and hence the incentive of the remaining players to invest. Therefore, our intuition lets us expect a period of delay until the next investment occurs. This intuition is correct for the case of two players (Fudenberg and Tirole (1985)) but, as we show in this paper, may fail otherwise.

We study a general  $N$ -Player investment preemption game and identify a mechanism that generates clustering of investment times: When the preemption race among late investors is very intense, the preemption incentive for earlier investors is reduced. If this effect is sufficiently strong, two or more investments occur at the same time. This happens when the flow profits of subsequent investments are sufficiently close.

The mechanism we identify is novel. Both the theoretical and the empirical literature on timing games have focused on different factors that can generate clustering of investment times. The theoretical explanations for the presence of clusters include coordination failures, as in Levin and Peck (2003), positive network externalities, as in Mason and Weeds (2010), and informational spillovers (e.g. Chamley and Gale (1994)), where rival investment signals a high profitability of investment. Brunnermeier and Morgan (2010) show that “herding” occurs in a preemptive “clock game”, but attribute this herding effect to private information being (partially) revealed by the first player to act. In our model, clusters are due purely to preemption and backward induction. Coordination failures are ruled out by assumption, rival investment has no informational content, no positive externalities, and lowers the post-investment flow profit for later investors.

Our analysis therefore provides an alternative interpretation of the empirical evidence. A large body of empirical literature has examined how rival adoption or market entry affects the timing of a firm’s own technology adoption or market entry. Several papers have found that adoption by a rival accelerates the adoption by remaining firms (clusters are the

most extreme form of acceleration).<sup>1</sup> This acceleration has been interpreted as evidence of positive payoff externalities or informational effects, but we provide a simpler alternative explanation based purely on preemption.

The mechanism through which clusters arise in our model is the following. Suppose there are three firms: if being the second investor is profitable relative to being the third, the preemption race to be the second investor is intense, and in equilibrium the second investment occurs early. In order to obtain monopoly profits for some time, a firm would have to invest even earlier. If monopoly profits are not much higher than duopoly profits, no firm wants to incur the extra-cost that is necessary to invest strictly before the second investor, and the first and second investments are clustered. In a game with more than three players, a similar mechanism can cause clusters of any size, at any point in the investment sequence.

This simple mechanism, purely based on preemption, has useful implications for the inference that can be drawn about firms profits, when the timing of investments and the pattern of clustering are observed for a given market. Our result implies that if investment times are generated by a preemption game and clustering is observed, a bound on the decline in profits due to rival investment can be calculated.<sup>2</sup>

The remainder of the paper is organized as follows. Immediately below we discuss the related literature. Section 2 introduces the model. Section 3.1 illustrates the benchmark case of a two-player game, in which investments are never clustered in equilibrium. Section 3.2 describes the mechanism that generates clustering, in the context of the three-player game. Section 3.3 characterizes the unique equilibrium outcome of the  $N$ -player game. Section 3.4 derives a sufficient condition on the primitives of the model for the presence of a cluster of two or more investments. Section 4 concludes.

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<sup>1</sup>These papers find that a firm's adoption hazard increases when a rival adopts. Karshenas and Stoneman (1993, p.521) interpret this acceleration of adoption as "epidemic effects" in the adoption of numerically controlled machine tools. Hannan and McDowell (1987, p.186) speak of "spillover effects" in the context of ATM adoption by banks, and Levin, Levin and Meisel (1992, p.347) argue that there is an informational effect via customers in the case of optical scanners in grocery stores.

<sup>2</sup>Identifying clusters empirically may be difficult because data often come in yearly intervals as it is the case in the empirical literature mentioned above. Increasing access to administrative data sources alleviates this problem. For example Kaniowski and Peneder (2008) use social security data to identify the date of hire of the first employee as the day of entry of a firm.

## Related literature

Seminal papers by Reinganum (1981a) and Fudenberg and Tirole (1985) study technology adoption games of unobservable and observable actions with two players respectively. Reinganum (1981b) derives the equilibrium of a game of technology adoption with  $N$  firms and unobservable actions. Anderson and Engers (1994) study a modified game where there is a time window in which players decide to act and payoffs of a static game played only among those who have acted are collected after expiration of this time window. In two related papers, Park and Smith (2008, 2010) analyze a timing game with a general payoff structure and more than two players. See Park and Smith (2008) for the case of unobservable actions, and Park and Smith (2010) for the case of observable actions and a continuum of players. Goetz (1999) discusses the case of a continuum of firms where the preemption motive is absent.

Hoppe (2002) surveys the extensive literature on technology preemption games, changing the assumptions on the information structure or the payoff structure. Hoppe and Lehmann-Grube (2005) study a version of a two-player game with a general deterministic payoff structure. More recently, Hopenhayn and Squintani (2011) consider a game with privately observed payoffs. Bobtcheff and Mariotti (2011) study a game with uncertainty regarding the presence of a competitor.

We rely on the equilibrium property of rent equalization to characterize conditions for clustering. Fudenberg and Tirole (1985) show that in the technology adoption game with observable actions and two players rent equalization must hold. They also illustrate why it may not hold with more than two players. The case of two players is also studied in Gilbert and Harris (1984) when analyzing a game where firms engage in lumpy capacity investments. While positive profits are earned with unobservable actions, they show that there exists a class of subgame perfect equilibria in the game with two players and observable actions where rents are fully dissipated. They conjecture that their arguments extend to the  $N$ -player game.

Mills (1988) shows that rent dissipation in a game of preemptive investment depends crucially on the ability of firms to make costless credible threats. If credible threats are costly, for example because investment must be made in temporarily separated steps, rents are not dissipated and profits almost as high as monopoly profits can be achieved in equilibrium. Mills (1991) analyzes a multiplayer model of lumpy capacity investment very close

to ours. Arguing that rent equalization must hold in equilibrium, he discusses the welfare implications of preemptive investment, and in particular the possibility that it leads to excessive and/or premature entry. Our paper contributes to this literature by establishing the possibility of clusters of investments, investigating the mechanism behind them, and their implications for the interpretation of industry data.

Strategic investment has also been studied in a real options framework. Greater uncertainty over the profitability of investment increases the option value of waiting and thus the tendency to delay investment. For recent examples see Weeds (2002) and references therein, as well as the survey by Hoppe (2002). In independent work, Bouis, Huisman and Kort (2009) study dynamic investment in oligopoly in a real options framework and find comparative statics result that are closely related to ours.<sup>3</sup> The real options approach allows aggregate uncertainty in the payoff process, but is restricted to a specific payoff growth process for payoffs (a Brownian motion with drift).

Bulow and Klemperer (1994) study a model with a seller who has multiple identical objects and multiple buyers with independent private values. They show that if buyers' valuations are not too different, frenzies of simultaneous purchases can occur because a purchase by a buyer increases the remaining buyers' willingness to pay. In our model, investment by a player lowers the flow profit achievable by the next investor. Nonetheless, clusters are possible if this decrease is sufficiently small, and the ensuing preemption race to take the role of the next investor is sufficiently intense.

## 2 Model

### 2.1 The investment game

We analyze an infinite horizon dynamic game in continuous time. At time 0, a new investment opportunity becomes available, and  $N$  identical players (firms) have to decide if, and when, to seize this opportunity. The investment opportunity can be interpreted as adoption of a new technology, or entry into a new market. Investment is observable and irreversible.

The set of firms is denoted by  $\mathbf{N} = \{1, \dots, N\}$  and a single firm is denoted by  $i \in \mathbf{N}$ .

The model corresponds to the one studied by Reinganum (1981*a*, 1981*b*) and Fudenberg and Tirole (1985) except for the following: Until a firm invests, it receives constant flow

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<sup>3</sup>Bouis et al. (2009) provide an explicit argument for existence of equilibrium, and for the existence of a cluster for the case of  $N = 3$ . For the general  $N$ -player game they provide comparative statics results under the assumption that equilibrium exists, and a numerical example.

of profits  $\pi_0$  which we normalize to zero. This assumption that pre-investment payoffs are independent of the number of earlier investments will be essential for obtaining a unique outcome in each subgame, which in turn guarantees rent equalization.<sup>4</sup> Upon investment, a firm earns flow profits of  $\pi(m)$ , where  $m$  is the number of firms that have already invested at a given point in time. Let  $\boldsymbol{\pi} = (\pi(1), \pi(2), \dots, \pi(N))$  denote a flow profit structure.

Let  $c(t)$  be the present value at time zero of the cost of investing at time  $t$ . If the outcome of the game is that the vector of investment times is  $T^j$ , for  $j = 1, \dots, N$ , and firm  $i$  is the  $j$ -th investor, then firm  $i$ 's payoff is the following:

$$V_i^j(T^1, T^2, \dots, T^j, \dots, T^N) = \sum_{m=j}^N \pi(m) \int_{T^m}^{T^{m+1}} e^{-rs} ds - c(T^j) \quad (1)$$

where  $r$  denotes the common discount rate, and  $T^{N+1} \equiv +\infty$ .

We introduce the following assumptions:

### Assumption 1

*Flow profits  $\pi(m)$  are (i) strictly positive for any  $m$  and (ii) strictly decreasing in  $m$ .*

Investing always increases period payoffs for a firm, but the benefits of investing decrease in the total number of investors: as more firms invest, competition among the investors becomes more intense.

### Assumption 2

*The current value cost function  $(c(t) e^{rt})$  is (i) strictly decreasing and (ii) strictly convex.*

The cost of investing declines over time. This may capture upstream process innovations or economies of learning and scale. Moreover, cost declines at a decreasing rate.

### Assumption 3

*(i) At time zero, the investment cost exceeds discounted monopoly payoffs:  $c(0) > \frac{\pi(1)}{r}$ .*

*(ii) Eventually, investment is profitable for all players:  $\exists \tau$  such that  $c(\tau) e^{r\tau} < \frac{\pi(N)}{r}$ .*

Assumption 3(i) guarantees that investing at time zero is too costly. No firm would invest immediately, even if it could thereby preempt all other firms and enjoy monopoly profits  $\pi(1)$  forever. Assumption 3(ii) ensures that the value of investing becomes positive in

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<sup>4</sup>However, the normalization of pre-investment payoffs to zero, rather than any positive constant, will not affect the results of the paper.

finite time: The cost of investing eventually reaches a level sufficiently low, that it becomes profitable to invest, even for a firm facing maximum competition.<sup>5</sup>

In what follows, we will denote by  $t_j$  the  $j$ -th equilibrium investment time.<sup>6</sup> If the  $j$ -th and  $(j + 1)$ -th investments occur at the same instant in time, i.e.  $t_j = t_{j+1}$ , we say that they are **clustered**.

## 2.2 Strategies in continuous-time preemption games

We model strategies in a timing game with observable actions and continuous time adopting the framework introduced by Simon and Stinchcombe (1989).<sup>7</sup> Each player has two actions available, “wait” and “invest.” Players can move at any time in  $[0, \infty)$ . A decision node is a point in time paired with a complete description of past moves, and a pure strategy is defined as a function that assigns an action to each node. An outcome is a complete record of the decisions made throughout the game.

In this framework, the question of how to associate an outcome to a continuous-time strategy profile is addressed in the following way. A continuous-time strategy here is interpreted as “a set of instructions about how to play the game on every conceivable discrete-time grid” (Simon and Stinchcombe (1989), p. 1174). For any continuous-time strategy profile, a sequence of outcomes is generated by restricting play to an arbitrary sequence of increasingly fine discrete-time grids, and the limit of this sequence of outcomes is defined as the continuous-time outcome of the profile. Simon and Stinchcombe (1989) identify conditions for the existence and the uniqueness of this limit. The strategies we consider here will satisfy these conditions.<sup>8</sup>

The Simon and Stinchcombe (1989) framework is defined for pure strategies only.<sup>9</sup> A well-known problem with modeling preemption games in continuous time is that typically games in this class do not have an equilibrium in pure strategies, due to the possibility of

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<sup>5</sup>We will discuss the implications of relaxing this assumption in Section 3.3.

<sup>6</sup>In Proposition 1 we prove that the vector of equilibrium investment times  $(t_1, \dots, t_N)$  is unique.

<sup>7</sup>Hoppe and Lehmann-Grube (2005) first showed how to adopt this framework to model a preemption game.

<sup>8</sup>The conditions required are (F1) an upper bound on the number of moves, (F2) that strategies depend piecewise continuously on time, and (F3) strong right continuity of the strategies with respect to histories. Condition (F1) is naturally satisfied, because investment is a one-time irreversible decision. Condition (F2) is an explicit restriction that we impose on the strategies. We satisfy condition (F3) by considering strategies which depend only on how many firms have already invested but not on the time when they have invested.

<sup>9</sup>For  $N = 2$  our model is a special case of Fudenberg and Tirole (1985). They develop an alternative methodology for modeling games in continuous time, based on an extended definition of mixed strategies. When applied to the two player case, their methodology generates the same equilibrium outcome. We conjecture that this would be true also for the general  $N$ -player case.



coordination failures. We explicitly rule out coordination failures introducing a randomization device as in Katz and Shapiro (1987), Dutta, Lach and Rustichini (1995), and Hoppe and Lehmann-Grube (2005):

**Assumption 4**

*If  $n$  firms invest at the same instant  $t$  (with  $n = 2, 3, \dots, N$ ), then only one firm, each with probability  $\frac{1}{n}$ , succeeds.*

The randomization device is introduced for the purpose of ruling out the possibility that simultaneous investments occur as a consequence of a coordination failure. At the same time, it allows for the presence of simultaneous investments as long as they are optimal for the firms involved.

To illustrate how the randomization device works, consider the case  $N = 2$ . First, suppose that at a given time  $t$  each firm would like to invest, *provided that the rival does not do so*. Suppose that at time  $t$  both firms try to invest. Without the randomization device, they would both be successful, i.e. they would both pay the cost  $c(t)$  and start receiving flow payoffs  $\pi(2)$ . This would constitute a coordination failure: ex-post, each firm would regret having invested. With the randomization device instead, if at time  $t$  both firms try to invest, “the clock stops”. The game proceeds as follows with time standing still. First, only one of the two firms (each with probability  $1/2$ ) successfully invests, i.e. only one actually pays the cost  $c(t)$ . Then, the remaining firm observes that its opponent has invested. It has two options. It can try to invest “consecutively but at the same instant of time,”<sup>10</sup> that is after observing the first investment, but at the same time  $t$ . Alternatively it can let the clock restart and the game continue. Since the firm observes that its opponent has invested at  $t$ , and investment at  $t$  was optimal only provided that the rival did not invest, it selects the second option. Hence, at  $t$  only one firm invests and there is no coordination failure.

Now suppose that at time  $t$  each firm finds it optimal to invest *whether or not the rival does so*. If both firms try invest at  $t$ , the clock stops. Only one firm is successful. The remaining firm has again the option to invest or let the clock restart. Now, however it will choose to invest immediately because it is optimal to do so. Hence there will be a cluster

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<sup>10</sup>See Simon and Stinchcombe (1989), p. 1177.

of two investments, which does not constitute a coordination failure.<sup>11,12</sup>

### 2.3 The optimal “stand-alone” investment times

In this subsection we illustrate the basic trade-off of the investment problem, abstracting from strategic considerations. Consider the hypothetical problem of a firm who acts as a single decision maker and has to select the optimal time to make an investment which has cost  $c(t)$  and guarantees flow payoff of  $\pi(j)$  forever, for  $j \in \{1, \dots, N\}$ , where  $c(t)$  and  $\pi(j)$  satisfy assumptions 1 to 3. This firm would choose  $t$  to maximize the following profit:

$$f_j(t) \equiv \frac{\pi(j)}{r} e^{-rt} - c(t). \quad (2)$$

We denote the solution to this problem as  $T_j^*$ . Adopting the terminology in Katz and Shapiro (1987), we define it as the **stand-alone investment time for  $\pi(j)$** . Observe that  $f_j(t)$  is strictly quasi-concave and that  $T_j^*$  is well-defined<sup>13</sup> for every  $j \in 1, \dots, N$  as the solution to

$$f'_j(t) = 0 \iff -\pi(j) e^{-rt} - c'(t) = 0. \quad (3)$$

At time  $T_j^*$ , the marginal benefit from delaying investment, that is the cost reduction  $c'(t)$ , is exactly equal to the marginal cost, that is the foregone discounted profit flow  $\pi(j) e^{-rt}$ . Before  $T_j^*$ , a player is willing to delay because the cost is decreasing at a speed that more than compensates the foregone profit flow. After  $T_j^*$ , a player would rather invest immediately than delay. It follows from the implicit function theorem that  $T_j^* < T_{j'}^*$  for  $j < j'$ : For a larger foregone profit flow, that is for  $j < j'$ , the stand-alone time is earlier.

## 3 Equilibrium analysis

We now return to the strategic environment, and solve for the equilibria of the game. A feature of any equilibrium of the game that is built into our assumptions is the following:

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<sup>11</sup>For an interpretation of this randomization device, we refer to Dutta, Lach and Rustichini (1995). An alternative assumption is made in Dutta and Rustichini (1993): If two firms stop simultaneously, both receive a convex combination of the payoff from being the only one to stop at that time, and the payoff from stopping later, at the optimal time for a “follower”. This assumption introduces the possibility of clusters of simultaneous investments through a mechanism that is unrelated to the one illustrated in our paper.

<sup>12</sup>An alternative formalization of continuous time strategies has been introduced recently by Murto and Välimäki (2011). To be applied in this model, it also needs to be augmented with a randomization device to avoid non-existence of equilibrium.

<sup>13</sup>For a proof, see Claim 1 in the Appendix.

**Lemma 1** *In any pure-strategy SPNE, no firm invests at  $t = 0$ , all firms invest in finite time, and the last investment takes place at the stand-alone investment time  $T_N^*$ .*

Assumptions 1(ii) and 3(i) guarantee that investment at time zero is too costly.<sup>14</sup> Assumptions 2(i) and 3(ii) guarantee that all firms invest in finite time. The result that the last equilibrium investment time is exactly the stand-alone investment time  $T_N^*$  is not surprising: when only one active firm is left, it maximizes the profit (2) for  $j = N$ .

Next, we introduce our benchmark: the two-player investment game analyzed by Fudenberg and Tirole (1985).

### 3.1 The benchmark case: two firms, no clustering

Suppose  $N = 2$ . The easiest way to capture the intuition for this game is to use a backward induction approach. By Lemma 1, both firms invest no later than the second stand-alone investment time  $T_2^*$ . Therefore, each firm anticipates that if it invests first at some time  $t < T_2^*$ , the opponent will follow at  $T_2^*$ . The payoff from this early investment will then be what Fudenberg and Tirole define as the *Leader Payoff*:

$$L(t) = \pi(1) \int_t^{T_2^*} e^{-rs} ds + \pi(2) \int_{T_2^*}^{\infty} e^{-rs} ds - c(t). \quad (4)$$

Alternatively, a firm could wait until  $T_2^*$  and receive the *Follower Payoff*:

$$F(t) = \pi(2) \int_{T_2^*}^{\infty} e^{-rs} ds - c(T_2^*). \quad (5)$$

The benefit from being the leader, rather than the follower, is that high profits  $\pi(1)$  are earned for some period. The cost is that early investment is more expensive than late investment. The fact that the cost of investment, although initially prohibitive, is decreasing and convex, guarantees that the leader and follower payoff curves have the shape illustrated in Figure 1. Fudenberg and Tirole (1985) prove that the first time at which an investment occurs in equilibrium,  $t_1$ , is the earliest time when the two curves intersect. In equilibrium, firms invest at different points in time<sup>15</sup> and payoffs are the same for both firms (there is *rent equalization*).

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<sup>14</sup>This fact is crucial in proving that all players receive the same payoff in equilibrium. Relaxing assumption 3(i), one could generate an equilibrium in which some players invest at time zero and receive a higher payoff than the remaining players, who would instead invest later and receive all the same, lower payoff.

<sup>15</sup>The assumption that pre-investment payoffs are constant rules out the possibility of what Fudenberg and Tirole (1985) define as “late joint adoption equilibria”.

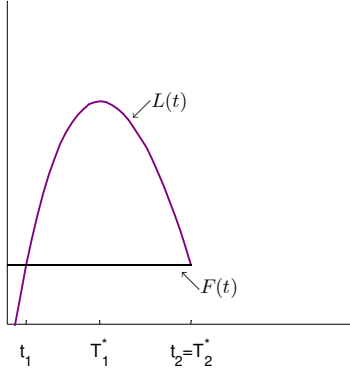


Figure 1: **No clustering in the two-player game.** The LPC brings the first investment forward to  $t_1$ . Earlier preemption is not profitable: before  $t_1$ ,  $F(t)$  exceeds  $L(t)$ . The figure is drawn for cost function  $c(t) = 2 \cdot 10^4 e^{-(\alpha+r)t}$  for  $\alpha = 0.24$ ,  $r = 0.1$ , and flow profits  $\pi = (440, 300)$ .

The mechanism at work is the following: if unconstrained by strategic considerations, a single firm would like to invest at the stand-alone time  $T_1^*$ . Also in the presence of an opponent, each firm would like to invest first, at  $T_1^*$ . The opponent would then follow at  $T_2^*$ . The leader would receive a higher payoff than the follower. This cannot be an equilibrium because the firm who takes the role of the follower could profitably deviate and preempt the opponent by investing at  $T_1^* - \varepsilon$ . The presence of a second player introduces a **Leader Preemption Constraint (LPC)** on the time of the first investment: Leader investment cannot take place at a time when earlier preemption is profitable. As a consequence, the first investment must occur strictly earlier than  $T_1^*$ . In particular, it must occur weakly before the first intersection of the Leader and Follower payoff functions. Since the leader payoff function is increasing in that interval, the first investment will occur at the latest time that satisfies the LPC, i.e. the first intersection of the two curves.

### 3.2 The three-firm game: When are the first two investments clustered?

In this section, we move away from the two-player benchmark and illustrate the possibility of clustering in a the context of a three-player game. As in the two-player game, the first investment must occur strictly earlier than  $T_1^*$ . The key difference from the two-player game is that  $t_1$  is identified by the presence of *two* constraints. One is the LPC constraint discussed above. The second is what we call the **Follower Preemption Constraint (FPC)**. The latter reflects the fact that the first investment is followed by a preemption race among the remaining two players. This race among followers determines an upper bound on the time of the first investment.

The aim of this section is twofold. First, we show that the first and second investments can be clustered or not clustered. Which case occurs depends on which of the two constraints

on  $t_1$ , the FPC or the LPC, is binding in equilibrium. Then, we examine how the model primitives determine which constraint is binding. We proceed by solving the game by backward induction. Figure 2 illustrates.

### The two-firm subgame

Suppose that the first investment has occurred, and consider the ensuing two-firm subgame. It is analogous to the two-firm game of Section 3.1. By Lemma 1, all firms must invest by  $T_3^*$ . Each firm anticipates that if it invests first at some time  $t < T_3^*$ , the opponent will follow at  $T_3^*$ . The *Leader Payoff* and *Follower Payoff* for this subgame are

$$L_2(t) = \pi(2) \int_t^{T_3^*} e^{-rs} ds + \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(t) \text{ and} \quad (6)$$

$$F_2(t) = \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(T_3^*) \quad (7)$$

respectively. The threat of preemption guarantees that the first investment in the subgame must take place at the earliest time when  $L_2(t) = F_2(t)$ . The second investment time in the game,  $t_2$ , coincides with this intersection. The last investment occurs at  $T_3^*$ .

### The Follower Preemption Constraint (FPC)

The conclusion above that the second investment occurs at the earliest intersection of  $L_2(t)$  and  $F_2(t)$  clearly assumes that the first investment must occur weakly before this intersection. We show by contradiction that this must be the case in equilibrium. Suppose that the first investment took place strictly later, at some time  $\tau \leq T_3^*$ . A two-firm subgame would then start at  $\tau$ . Because  $L_2(\tau) > F_2(\tau)$ , both firms would prefer to be leader rather than follower in this subgame. They would both try to invest at  $\tau$ , one would succeed and the other would invest later at  $T_3^*$ . This cannot be an equilibrium because each of the last two investors receives a lottery between  $L_2(\tau)$  and  $F_2(\tau)$  while it could deviate and guarantee itself a payoff arbitrarily close to  $L_2(\tau)$ . Deviating by investing at  $\tau - \varepsilon$ , a firm would be the first investor in the game. It would trigger a two-firm subgame in which one more investment would occur at  $\tau - \varepsilon$  and the last one at  $T_3^*$ . Therefore, the deviator would receive a payoff of  $L_2(\tau - \varepsilon)$ .

We have established that the time of the first investment  $t_1$  is constrained by the presence of a preemption race in the ensuing two-firm subgame: to guarantee that there is rent equalization in this race,  $t_1$  must be no later than the first intersection of the leader and follower payoff curves of the two-firm subgame, namely  $L_2(t)$  and  $F_2(t)$ . We call this the

**Follower Preemption Constraint (FPC).** As the second investment time  $t_2$  coincides with this intersection, we say that the FPC is binding in equilibrium if the first investment occurs exactly at  $t_2$ , and not binding if it occurs strictly earlier than  $t_2$ .<sup>16</sup>

### The Leader Preemption Constraint (LPC)

We now illustrate the LPC for the three-player game. It follows from the analysis above that the second and third investments will occur at  $t_2$ , with  $L_2(t_2) = F_2(t_2)$ , and  $t_3 = T_3^*$  respectively. The FPC requires that the first investment occurs weakly earlier than the first intersection of  $L_2(t)$  and  $F_2(t)$ . Therefore, the Leader payoff in the three-player game, i.e. the payoff of the first investor, is:

$$L_1(t) = \pi(1) \int_t^{t_2} e^{-rs} ds + \pi(2) \int_{t_2}^{T_3^*} e^{-rs} ds + \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(t). \quad (8)$$

The Follower payoff in the three-firm game, i.e. the payoff from being either the second or the third investor, is

$$F_1(t) = \pi(2) \int_{t_2}^{T_3^*} e^{-rs} ds + \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(t_2) = \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(T_3^*). \quad (9)$$

As in the two-player game, there is a LPC: the first investment cannot occur at a time such that earlier preemption is profitable, because otherwise any of the followers would have a profitable deviation. We say that the LPC on  $t_1$  is binding in equilibrium if given the subsequent investment times  $t_2$  and  $t_3$ ,  $L_1(t) > F_1(t)$  for some  $t < t_2$ . Otherwise, preempting the leader is never profitable, and we say that the LPC is not binding.

### The relationship between FPC and LPC

The FPC reflects the intensity of the follower preemption race that starts after the first investment: The more intense this race, the earlier is the first intersection of  $L_2(t)$  and  $F_2(t)$ , i.e. the earlier is  $t_2$ . The LPC instead reflects the intensity of the race to be the first investor. These two constraints are not independent: The intensity of the race to be the first investor is a direct consequence of the intensity of the follower preemption race between the second and the third investors. Hence, the LPC is directly affected by the FPC.

To capture this relationship between the two constraints, notice that the more intense the follower preemption race is, the earlier  $t_2$  is, hence the tighter the constraint on  $t_1$

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<sup>16</sup>Observe that in the two-player game in Section 3.1 this constraint is absent: After the first investment, there is a single follower left, whose investment time is not determined by a preemption race but by a single-agent decision problem.

imposed by the FPC becomes. At the same time, the earlier  $t_2$  is, the shorter the period for which the first investor earns monopoly profits becomes: Early  $t_2$  makes the role of the first investor less desirable. Therefore, the more intense the follower preemption race is, the less intense is the race to be the first investor: the stronger the FPC, the weaker the LPC.

The key observation of our analysis is that for any given set of parameters, only one of the two constraints is binding, and which constraint is binding is equivalent to whether the first two investments are clustered or not. If the follower preemption race is sufficiently intense, only the FPC is binding, and investments are clustered. Otherwise, only the LPC is binding, and investment times are different. We discuss these two cases below, and illustrate them in Figure 2.

First, observe that the leader payoff  $L_1(t)$  is strictly quasiconcave, and maximized at  $T_1^*$ . By construction, it intersects  $F_1(t)$  in  $t_2$ . Which constraint is binding depends on the relative position of  $t_2$  with respect to  $T_1^*$ . The intuition for this is that  $T_1^*$ , being determined by  $\pi(1)$ , reflects the desirability of the role of the first investor, hence the strength of the LPC, while  $t_2$  reflects the strength of the FPC.

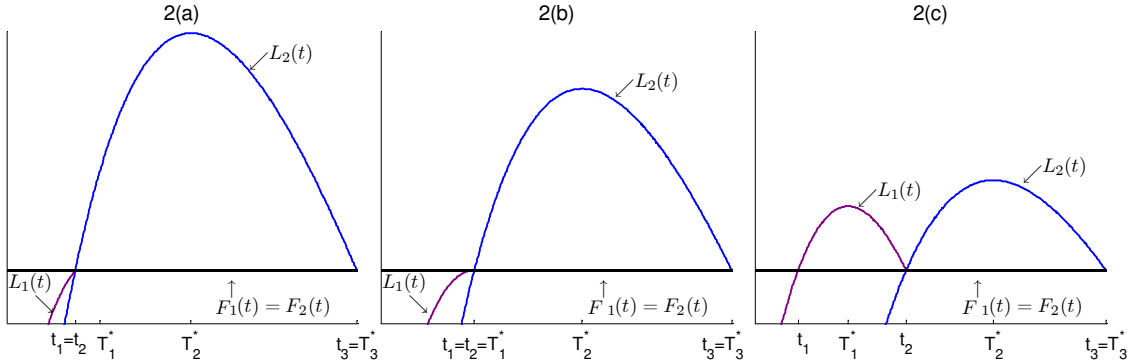


Figure 2: **Clustering vs. No Clustering.** In all three panels, the third investment occurs at  $T_3^*$  and the second at the first intersection of  $L_2(t)$  and  $F_2(t)$ . Duopoly profits decrease from panel 2(a) to 2(c), delaying the second investment time  $t_2$ . In panels (2a) and (2b),  $t_2 \leq T_1^*$ , so in the race to be first,  $F_1(t)$  exceeds  $L_1(t)$  before  $t_2$ . Hence, only the FPC is binding and investments are clustered. Panel (2b) represents the cut-off case with  $t_2 = T_1^*$ . In (2c),  $t_2 > T_1^*$ . Therefore,  $L_1(t)$  exceeds  $F_1(t)$  before  $t_2$  and the LPC alone is binding. The cost function is as in Figure 1. Flow profits  $\pi$  are (440, 300, 150) for (2a), (440, 280.2, 150) for (2b), (440, 240, 150) for (2c).

**Case (i):  $t_2 \leq T_1^*$**  (Figures 2a and 2b).

The LPC is not binding, because the follower payoff  $F_1(t)$  exceeds the leader payoff  $L_1(t)$  at any  $t < t_2$ . The first investment occurs exactly at  $t_2$ : the FPC is binding. The payoffs of all players are equalized. The first two investment times are *clustered*:  $t_1 = t_2$ .

**Case (ii):  $t_2 > T_1^*$**  (Figure 2c).

The LPC is binding: The leader payoff  $L_1(t)$  exceeds the follower payoff  $F_1(t)$  to the left of  $t_2$ . The preemption race to be the first investor brings  $t_1$  forward to the earliest intersection of leader and follower payoffs. The payoffs of all players are equalized. The FPC instead, is not binding. The first two investments are *not clustered*:  $t_1 < T_1^* < t_2$ .

### How model primitives determine the presence of a cluster

As can be seen in Figure 2, whether case (i) or case (ii) will occur is equivalent to whether  $L_2(t) - F_2(t)$ , the incentive to preempt in the 2-player subgame that follows the first investment, is positive or negative when evaluated at  $T_1^*$ .

In the cut-off case of Figure (2b), the preemption incentive evaluated at  $T_1^*$  is zero: the first intersection of  $L_2(t)$  and  $F_2(t)$ , that identifies  $t_2$ , coincides exactly with  $T_1^*$ . The follower preemption race is just intense enough to make the FPC binding and the LPC not binding. If instead the first intersection of  $L_2(t)$  and  $F_2(t)$  occurs to the left of  $T_1^*$ , as in Figure (2a), then the preemption incentive  $L_2(t) - F_2(t)$  evaluated at  $T_1^*$  is strictly positive. Conversely, if it occurs to the right of  $T_1^*$ , as in Figure (2c), then  $L_2(t) - F_2(t)$  evaluated at  $T_1^*$  is strictly negative.

The previous observation allows us to identify the parameter range for which case (i) and case (ii) occur, respectively. Evaluating the preemption incentive  $L_2(t) - F_2(t)$  at  $T_1^*$ :

$$\begin{aligned} L_2(T_1^*) - F_2(T_1^*) &= \pi(2) \int_{T_1^*}^{T_3^*} e^{-rs} ds + \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(T_1^*) - \left[ \pi(3) \int_{T_3^*}^{\infty} e^{-rs} ds - c(T_3^*) \right] \\ &= \frac{\pi(2)}{r} \left[ e^{-rT_1^*} - e^{-rT_3^*} \right] - [c(T_1^*) - c(T_3^*)] \end{aligned} \quad (10)$$

Recalling that  $T_1^*$  is a decreasing function of  $\pi(1)$  and  $T_3^*$  is a decreasing function of  $\pi(3)$ , we observe that the preemption incentive evaluated at  $T_1^*$  is a function of the three profit parameters  $\pi(1)$ ,  $\pi(2)$ ,  $\pi(3)$ . For profit structures such that the preemption incentive (10) is nonnegative, the first two investments are clustered, for all other profit structures, clustering does not occur.

The preemption incentive (10) is monotone in each of the profit parameters  $\pi(1)$ ,  $\pi(2)$  and  $\pi(3)$ . The intuition is captured by looking at how each of them affects the relative strength of the two constraints. First, the preemption incentive (10) is decreasing in  $\pi(1)$ . Hence, starting from the cut-off case, increasing  $\pi(1)$  we fall into case (ii) (no clustering). The intuition is that an increase in  $\pi(1)$  makes the role of leader of the three-player



preemption race more attractive. Hence, the LPC becomes binding.

Next, consider  $\pi(2)$  and  $\pi(3)$ . The preemption incentive (10) is increasing in  $\pi(2)$  and decreasing in  $\pi(3)$ . Hence, starting from the cut-off case, increasing  $\pi(2)$  or decreasing  $\pi(3)$  there continues to be a cluster. The intuition is that an increase in  $\pi(2)$  or a decrease in  $\pi(3)$  makes the role of the second investor more attractive relative to the role of third investor. The preemption race among the followers becomes more intense, and this brings  $t_2$  forward. The FPC constraint becomes stronger. At the same time, earlier  $t_2$  makes the role of the first investor less attractive so the LPC becomes weaker.

In Section 3.4, we will show that the intuition above can be translated into a sufficient condition for the presence of a cluster: given any pair  $(\pi(1), \pi(3))$ , if  $\pi(2)$  is sufficiently close to  $\pi(1)$ , then the first and second investments are clustered.

### 3.3 The general case: $N$ firms

In this section, we formalize and generalize our characterization of the equilibrium outcome of the game with three players to the general case of  $N$  players.

After the first  $j - 1$  investments have taken place, two constraints determine the next investment time  $t_j$ . First, the LPC: preempting the *leader* of the current subgame, i.e. the  $j$ -th investor, by investing earlier than  $t_j$ , must not be profitable. Second, the FPC:  $t_j$  must be weakly earlier than the time of the next investment,  $t_{j+1}$ , which is determined by the preemption race in the subgame played by the *followers* after the  $j$ -th investment.<sup>17</sup>

Proposition 1 below establishes that the equilibrium outcome of the game is unique, that the rent equalization result is preserved even for a general number of players, and allows us to construct a simple recursive algorithm to compute the equilibrium investment times and determine the presence of clusters.

**Proposition 1** *The game admits a unique pure-strategy SPNE outcome, up to a permutation of players. The equilibrium has the following properties:*

- (i) *All players receive the same payoff.*
- (ii) *The  $j$ -th and the  $(j + 1)$ -th investments are clustered if and only if the  $(j + 1)$ -th investment time  $t_{j+1}$  is weakly earlier than the  $j$ -th stand-alone investment time  $T_j^*$ .*

In equilibrium, all players earn a payoff equal to the last investor's:  $\frac{\pi(N)}{r}e^{-rT_N^*} - c(T_N^*)$ .

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<sup>17</sup>The above discussion holds for  $j \leq N - 2$ . In equilibrium, the last two investment times are the equilibrium outcome of a subgame that resembles the two-firm game illustrated in Section 3.1.

The Fudenberg and Tirole (1985) rent-equalization result extends to the  $N$ -player game because we assume that pre-investment payoffs are constant. This in turn implies that in each preemption race the Follower payoff is independent of the exact time of earlier investments. While the game admits a unique equilibrium outcome in terms of investment times and equilibrium payoffs, the role taken by each investor in the investment sequence is not uniquely identified.<sup>18</sup>

Lemma 1 and Proposition 1 suggest the following simple recursive algorithm to compute the equilibrium investment times  $(t_1, t_2, \dots, t_N)$  and determine the presence of clusters:

1. The last investment time is equal to the last stand-alone investment time:  $t_N = T_N^*$ ,
2. For  $j < N$ ,
  - (i) if  $t_{j+1} \leq T_j^*$ , then there is a cluster:  $t_j = t_{j+1}$ .
  - (ii) If  $t_{j+1} > T_j^*$ , then  $t_j < t_{j+1}$ , and  $t_j$  solves the rent equalization condition:

$$L_j(t) - F_j(t) = \pi(j) \int_{t_j}^{t_{j+1}} e^{-rs} ds - c(t_j) + c(t_{j+1}) = 0$$

where  $L_j(t)$  and  $F_j(t)$  are defined analogously to  $L_1(t)$  and  $F_1(t)$ .<sup>19</sup>

Case 2(i) is analogous to Case (i) in Section 3.2: the FPC alone is binding. Case 2(ii) is analogous to Case (ii) in Section 3.2: the LPC alone is binding.

Proposition 1 has two implications that go beyond the features of the three-player example. First, for  $N > 3$  clusters can include more than two simultaneous investments: suppose that the preemption race for the role of  $(j + 1)$ -th investor is sufficiently intense that not only  $t_{j+1} < T_j^*$ , but  $t_{j+1} < T_{j-1}^* < T_j^*$ : in this case, the  $(j - 1)$ -th,  $j$ -th and  $(j + 1)$ -th investments will be clustered. In Section 3.4 we illustrate how model primitives affect the size of a cluster.

Second, clusters can occur not only at the beginning, but at any point of the investment

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<sup>18</sup>In a symmetric equilibrium all firms that have not invested yet try to invest at every equilibrium investment time. The realized order of investments is determined by the randomization device. In an asymmetric equilibrium the order of investments can be pre-determined or partially random. For instance, the randomization device can be used to determine only the first investor. Firms could then coordinate on the identity of later investors based on this outcome. We thank a referee for pointing this out.

<sup>19</sup>For a formal definition, see proof of Proposition 1 in the Appendix.

sequence, except for the last.<sup>20</sup>

### 3.4 The condition for a cluster

In Section 3.2 we have illustrated the mechanism that leads to clustered investments in the special case of  $N = 3$ . We introduced the LPC and the FPC and provided an intuitive discussion of how the model parameters affect these two constraints and determine the presence or absence of a cluster. In this section, we formally investigate how the mechanism illustrated in Section 3.2 relates to the model primitives. More precisely, we ask under which condition on the parameters of the model two or more subsequent investments, at any point in the investment sequence, are clustered.

The answer is that they are clustered if the associated flow profits are sufficiently close. To obtain this result, we first argue that whether two subsequent investments are clustered or not depends only on a subvector of the profit structure  $\boldsymbol{\pi}$ . Second, we present a comparative statics result relating investment times and flow profits. Third, we identify a sufficient condition on the profit structure for a cluster of two or more investments. Figure 3 illustrates the analysis.

We start by observing that the equilibrium characterization in Proposition 1 implies the following:

**Remark 1** *The condition for a cluster of two subsequent investments is independent of the flow-profit parameters associated with earlier investments.*

Proposition 1 states that whether the  $j$ -th and  $(j + 1)$ -th investments are clustered is determined by the comparison of the stand-alone investment time  $T_j^*$  and the  $(j + 1)$ -th equilibrium investment time  $t_{j+1}$ . Hence, the condition for a cluster depends only on the flow-profit parameters affecting  $T_j^*$  and  $t_{j+1}$ . As illustrated in section 2.3,  $T_j^*$  depends only on the flow-profit parameter  $\pi(j)$ . To see which profit parameters determine  $t_{j+1}$ , consider the algorithm identifying the equilibrium investment times. The last investment occurs at  $t_N = T_N^*$ , hence it depends only on one flow-profit parameter,  $\pi(N)$ . The previous

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<sup>20</sup>If investment is not profitable for all players, so that entry is effectively endogenous, then a cluster can also occur at the end of the investment sequence. Suppose we relax Assumption 3(ii) so that only  $M < N$  can profitably invest. After the  $(M - 1)$ -th investment, there is a race among the remaining active players to secure the last profitable investment possibility. By rent equalization, the last investor (as well as all the predecessors) must earn the same equilibrium payoff as the  $(N - M)$  players who do not invest, which is zero. The last investment is thus brought forward to a time  $t_M$  earlier than the stand-alone time  $T_M^*$ . If  $t_M < T_{M-1}^*$ , then there is cluster at  $t_{M-1} = t_M$ . In the baseline model instead, the last investment time  $t_N$  is  $T_N^*$ , therefore it is always strictly later than  $T_{N-1}^*$ .

investment time,  $t_{N-1}$ , depends on  $\pi(N-1)$  and on the flow-profit parameters that affect  $t_N$ , that is on  $\pi(N)$ . Continuing to apply the algorithm, it follows that  $t_{j+1}$  depends only on  $\pi(j+1)$  and on the flow profit parameters that affect later investments, that is on the vector  $(\pi(j+1), \pi(j+2), \dots, \pi(N))$ .

We now present a comparative statics result that will play a key role in the construction of the sufficient condition for a cluster:

**Proposition 2** *Each equilibrium investment time is decreasing in the associated flow profit.*

For expositional purposes consider a profit structure such that the equilibrium investment times  $t_j$  and  $t_{j+1}$  are different. An increase in  $\pi(j)$  makes the role of the  $j$ -th investor more profitable. Rent equalization requires that the  $j$ -th investor receives the same equilibrium payoff as the  $(j+1)$ -th investor. The latter is unaffected by the increase in  $\pi(j)$ . Hence, in equilibrium an increase in  $\pi(j)$  has to be offset by an increase in the investment cost. This implies bringing  $t_j$  forward, because the investment cost is decreasing in time.

The monotonicity of equilibrium investment times in flow profits leads to our main result. An increase in  $\pi(j)$  brings  $t_j$  forward. For sufficiently large  $\pi(j)$ , that is for  $\pi(j)$  sufficiently close to  $\pi(j-1)$ ,  $t_j$  occurs at a time earlier than  $T_{j-1}^*$ : This results in a cluster of the  $j$ -th and  $(j-1)$ -th investment. By the same mechanism, if  $\pi(j)$  is sufficiently close to  $\pi(j-2)$ , the investment time  $t_j$  is brought forward to a time even earlier than  $T_{j-2}^*$ . This results in a cluster of three investments:  $t_{j-2} = t_{j-1} = t_j$ . Figure 3 illustrates this mechanism. More generally, for any  $j \in \{2, \dots, N-1\}$  and  $k \in \{1, \dots, j-1\}$ , the sufficient condition for a cluster of two or more investments is as follows:

**Proposition 3** *If  $\pi(j)$  is sufficiently close to  $\pi(j-k)$ , then the  $j$ -th investment is clustered with the previous  $k$  investments.*

To capture the intuition for this result, consider the simplest case of a cluster of two investments. Suppose that the parameter values are such that the  $j$ -th investment is not clustered with the previous one. Consider an increase in the flow profits  $\pi(j)$ . How does it affect the preemption incentives in the game? The same reasoning illustrated in Section 3.2 and Figure 2 for  $N=3$  and  $j=2$  applies here. The  $(j-1)$ -th investment time is determined by the Follower and Leader Preemption Constraints (FPC and LPC). Everything else equal, an increase in  $\pi(j)$  makes the role of the  $j$ -th investor more attractive. Therefore, the follower preemption race in the subgame starting after the  $(j-1)$ -th investment becomes

more intense. This brings forward the investment time  $t_j$ , which constitutes the upper bound on the investment time  $t_{j-1}$  stemming from the FPC. In turn, an earlier investment time  $t_j$  makes the role of the  $(j-1)$ -th investor less attractive, so the preemption race for the role of the  $(j-1)$ -th investor is less intense, and the LPC becomes weaker. The natural question is whether one can increase  $\pi(j)$  to such an extent that the FPC becomes binding and the LPC not binding. Proposition 3 provides a positive answer: for any value of the remaining primitives of the model, one can always find a  $\pi(j)$  strictly smaller than  $\pi(j-1)$  but sufficiently close to it, such that the FPC is binding, the LPC is not, and the  $j$ -th investment is clustered with the previous one.

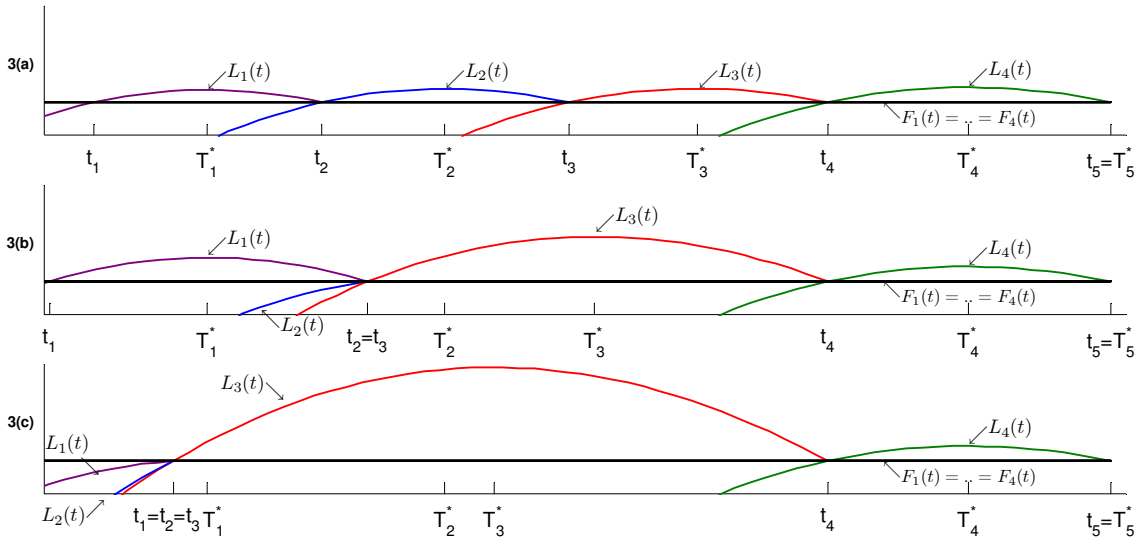


Figure 3: **The Size of a Cluster.** In all three panels,  $N = 5$ , the cost function is  $c(t) = 2 \cdot 10^4 e^{-(\alpha+r)t}$  with  $\alpha = 0.4$  and  $r = 0.1$  and flow profits  $(\pi(1), \pi(2), \pi(4), \pi(5))$  are  $(340, 320, 280, 270)$ . Triopoly profits  $\pi(3)$  increase from 300 in (3a) to 308 in (3b) and to 316 in (3c). The investment times  $t_4$  and  $t_5$  are unaffected by this increase (Remark 1). Also,  $T_1^*$  and  $T_2^*$  are the same in all panels. In (2a),  $T_1^* < T_2^* < t_3$  and no cluster occurs. In (2b) and (2c), the larger  $\pi(3)$  brings the third investment time  $t_3$  forward (Proposition 2). In (2b),  $\pi(3)$  is sufficiently close to  $\pi(2)$  such that  $T_1^* < t_3 < T_2^*$  and the second and third investments are clustered. In (2c) a further increase of  $\pi(3)$  makes it sufficiently close to  $\pi(1)$ , such that  $t_3 < T_1^* < T_2^*$  and the first three investments are clustered (Proposition 3).

We conclude with a remark regarding what can be learned about profits, when data on the timing of investments are observed. Our result says that clustering of entry or adoption times does not imply that payoffs are not declining in rival investment. If investment times are generated by a preemption game and clustering is observed, this implies a bound on the decline in profits due to rival investment. In the case where only time aggregated information is available, for instance in the form of annual data, bounds could nevertheless

be obtained but would be less informative the higher the level of temporal aggregation.

Consider the case  $N = 3$  and fix the investment cost function and the interest rate. Then one can find  $\pi(3)$  as it is the unique solution to the stand-alone problem in (2) given  $T_3^*$ . Knowing  $T_3^*$  and  $\pi(3)$ , one can find  $\pi(2)$  by plugging the observed  $t_2$  into the rent-equalization condition  $L_2(t_2) = F_2(t_2)$ . If  $t_1 < t_2$ , one can repeat the same procedure and use  $L_1(t_1) = F_1(t_1)$  to find  $\pi(1)$ . If instead  $t_1 = t_2$ , i.e. there is a cluster, one can infer that  $\pi(1)$  must be small enough, such that  $T_1^* > t_2$ . Thus, an upper bound  $\bar{\pi}(1)$  for  $\pi(1)$  as a function of  $t_2$  can be obtained. This implies that  $\pi(1)$  must lie in the interval  $[\pi(2), \bar{\pi}(1)]$ .

## 4 Conclusions

This paper analyzed an  $N$ -player preemption game in which players' payoff before investing are constant (and normalized to zero). The game has a unique equilibrium outcome, and the rent equalization result of the two-player game analyzed by Fudenberg and Tirole (1985) is preserved.

We find that clusters of simultaneous investments are possible. When the preemption race among late investors is very intense, the preemption incentive for earlier investors is reduced. If this effect is sufficiently strong, two or more investments occur at the same time. We characterize a sufficient condition on the model primitives for two or more investments to be clustered: The flow profits of subsequent investments must be sufficiently close. Our results imply that the observation of investment clustering in preemptive environments need not reflect informational spillovers, positive externalities, or coordination failures. Instead, clustering implies a bound on the decline in profits due to rival investment.

## Appendix

**Claim 1** For any  $j \in \{1, 2, \dots, N\}$ : (a) the function  $f_j(t)$  is strictly quasi-concave in  $t$ .

(b) It admits a unique global maximum in  $T_j^*$ , defined as the solution to

$$f_j'(t) = 0 \iff -\pi(j) e^{-rt} - c'(t) = 0.$$

(c)  $f_j(T_j^*) > 0$  and  $T_j^* < T_{j'}^*$  for  $j < j'$ .

**Proof of Claim 1. Part (a)** We prove that the function is strictly quasiconcave, by

showing that in every critical point of the function the second derivative is strictly negative. The first derivative  $f'_j(t)$  is equal to  $[-\pi(j)e^{-rt} - c'(t)]$  and the second derivative  $f''_j(t)$  is equal to  $[r\pi(j)e^{-rt} - c''(t)]$ . Using  $f'_j(t) = 0$  we can rewrite  $f''_j$  evaluated at any critical point as

$$f''_j(t) = -c'(t)r - c''(t). \quad (11)$$

By Assumption 2,  $e^{rt}[c'(t) + rc(t)] < 0$  and  $e^{rt}[2c'(t)r + c(t)r^2 + c''(t)] > 0$ . Together, these two inequalities imply that expression (11) is negative.

**Part (b)** We prove that the first order condition  $f'_j(t) = 0$  admits a solution and hence characterizes the unique global maximum of the function by showing that  $f'_j(t)$  is positive at zero and negative for sufficiently large values of  $t$ . Assumptions 1 and 3 guarantee that  $f_j(t)$  is negative at zero and positive at a later time. Quasiconcavity then implies that  $f'_j(0) > 0$ . Moreover, since  $f_j(t)$  is continuous and either always increasing or single peaked, it admits a limit as  $t$  goes to infinity. This limit must be greater than or equal to zero by assumption 3(ii). It must also be smaller than or equal to zero because  $\lim_{t \rightarrow +\infty} f_j(t) = -\lim_{t \rightarrow +\infty} c(t)$ . Hence, the only possible candidate limit is zero. But if that is the case, since the function is positive from some  $\tau$  onwards, as  $t$  goes to infinity it must approach zero from above. Hence it must be decreasing for  $t$  sufficiently large. We therefore conclude that the function  $f_j(t)$  admits a critical point.

**Part (c)** Assumptions 2(i) and 3(ii) imply that  $f_N(t)$  is strictly positive for any  $t \geq T_N^*$ . Thus, Assumption 1(ii) implies that  $f_j(T_j^*) > 0$  for all  $j$ .

Finally, note that by the implicit function theorem

$$\frac{\partial T_j^*}{\partial \pi(j)} = -\frac{-e^{-rT_j^*}}{f''_j(T_j^*)} < 0,$$

where the inequality holds because  $T_j^*$  is a maximum, hence the denominator is negative. Therefore, Assumption 1(ii) implies that  $T_1^* < T_2^* < \dots < T_N^*$ . ■

**Proof of Lemma 1.** Assumptions 1(ii) and 3(i) guarantee that there is no investment at time zero: the cost of investing immediately is higher than the maximum amount of profits a firm can obtain in this game.

The proof of the result that all firms invest in finite time, and the last investment takes place at the stand-alone investment time  $T_N^*$  is split into two parts. First, we show that in equilibrium, at any decision node with one active firm and calendar time  $t$ , the firm plays

“wait” if  $t < T_N^*$  and “invest” otherwise. Then, we show that in any decision node with  $t \geq T_N^*$  any number of active firms play “invest”.

The payoff of a single active firm from investing at time  $t$  is  $f_N(t)$ , defined in equation (2). From Claim 1,  $f_N(t)$  has a strict global maximum in  $T_N^*$  and its maximum value is strictly positive. Therefore, a single active firm will optimally play “wait” if  $t < T_N^*$  and “invest” otherwise.

Next, consider decision nodes with  $t \geq T_N^*$  and two active firms. We show that both firms must invest exactly at  $t$ .

First, suppose that at  $t$  they both play “invest”. By assumption 4 only one of them succeeds and the game enters a subgame with one active firm. As we proved above, this firm invests immediately, so both firms invest at time  $t$  and receive payoff  $f_N(t)$ . No firm has an incentive to deviate from these strategies. With two active firms, deviating and playing “wait” would not change the outcome, nor the deviator’s payoff. The non-deviating firm would invest immediately. The game would therefore enter a subgame with one active firm in which the deviator would optimally invest immediately, as we proved above.

Next, suppose that at time  $t$  only one of the two active firms plays “invest”. The outcome is again that both firms invest at  $t$ , because after one firm invests the game enters a subgame with one active firm in which it is optimal to invest immediately. No firm has an incentive to deviate from these strategies. The firm who plays “wait” has no incentive to deviate because the outcome, hence its payoff, would be unchanged. Now suppose that the firm who plays “invest” deviates. It would get either a payoff of zero, if it never invests, or  $f_N(\tau)$  if it invests at some  $\tau > t$ . Since  $f_N(\cdot)$  is positive and strictly decreasing in the interval considered, the deviation is not profitable.

Finally, suppose that at time  $t$  both firms play “wait”. The argument immediately above shows that each firm would be better off by deviating and playing “invest” at  $t$ .

Repeating the same argument for  $\ell = 3, \dots, N$ , it follows that in any SPNE, at any decision node with  $t \geq T_N^*$  and any number  $\ell$  of active firms, at least one of them plays “invest”, and the claim follows immediately. ■

**Proof of Proposition 1.** Through a series of Lemmata we show that the game admits a unique SPNE outcome, and characterize its properties. The proof is articulated in the following steps:

Denote by  $t_j$  the SPNE investment time of the  $j$ -th investor, for  $j \in \{1, \dots, N\}$ . In



Definition 1, we introduce three functions,  $L_j(t)$ ,  $F_j(t)$ , and their difference  $D_j(t)$ . In Lemma 2 and Lemma 3 we characterize their properties. Over a well-defined subset of their domain,  $L_j(t)$  and  $F_j(t)$  can be interpreted as the payoff of the  $j$ -th investor and the  $(j + 1)$ -th investor, respectively, if the  $j$ -th investment takes place at  $t$  and the following investments take place at  $t_{j+1}, \dots, t_N$  respectively. In the definition, the existence and uniqueness of the SPNE investment times is assumed. In the development of the proof, they will be proved. The existence and uniqueness of  $t_N = T_N^*$  was proved in Lemma 1.

In Lemma 4 we establish that in any subgame with one active firm, it plays “wait” before  $T_N^*$  and “invest” from  $T_N^*$  on. In Lemma 5, we prove that there exists a time  $T_{N-1} < T_N^*$  in which  $L_{N-1}(T_{N-1}) = F_{N-1}(T_{N-1})$  and in Lemma 6 we prove that this is the unique  $(N - 1)$ -th equilibrium investment time. Therefore, the equilibrium payoff of the last two investors is the same.

Finally, in Lemmata 7, 8 and 9 we identify the algorithm for the construction of the equilibrium investment times  $t_j$  for  $j \in \{1, \dots, N-2\}$ , identifying the condition for clustering, and prove that rent equalization holds in equilibrium for all players. The argument is based on the induction principle. Lemma 7 proves that there exists an algorithm to identify the unique  $t_{N-2}$ , given  $t_{N-1}$  and  $t_N$ , and that the equilibrium payoff of the last three investors is the same. Lemma 8 shows that if an analogous algorithm can be used to identify a unique value of  $t_{N-l}$ , given  $t_{N-l+1}, \dots, t_N$ , and rent equalization holds for the last  $l$  players, then the same algorithm identifies a unique value for  $t_{N-l-1}$ , given  $t_{N-l}, \dots, t_N$ , and rent equalization holds for the last  $l + 1$  players. Lemma 9 applies the induction principle to prove that the algorithm can be used to construct the SPNE investment times  $t_1, \dots, t_{N-2}$  and rent equalization holds for all players. This concludes the proof of the Proposition.

**Definition 1** For each  $j \in \{1, \dots, N - 1\}$ , we define the following three functions over the interval  $[0, T_N^*]$ :

$$\begin{aligned} L_j(t) &\equiv \pi(j) \int_t^{t_{j+1}} e^{-rs} ds + \sum_{m=j+1}^N \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(t) \\ F_j(t) &\equiv \sum_{m=j+1}^N \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(t_{j+1}) \\ D_j(t) &\equiv L_j(t) - F_j(t) = \pi(j) \int_t^{t_{j+1}} e^{-rs} ds - c(t) + c(t_{j+1}) \end{aligned}$$

where  $t_{N+1} \equiv +\infty$ .

Notice that  $F_j(t)$  is constant with respect to  $t$ .

**Lemma 2**

(i) The function  $D_j(t)$  attains a unique global maximum in  $T_j^*$ , for  $j \in \{1, \dots, N-1\}$

(ii)  $D_j(T_j^*) \geq 0$  for  $j \in \{1, \dots, N-1\}$ .

**Proof.**

**Part (i).** Notice that

$$D_j(t) = \pi(j) \int_t^{t_{j+1}} e^{-rs} ds - c(t) + c(t_{j+1}) \quad (12)$$

and  $f_j(t)$  as defined in equation (2) differ by a finite constant. By Claim 1,  $f_j(t)$  attains a unique global maximum in  $T_j^*$ , hence the same is true for  $D_j(t)$ .

**Part (ii).** Since  $D_j(t_{j+1}) = 0$  and  $T_j^*$  is the unique global maximizer, it holds that  $D_j(T_j^*) \geq 0$ . ■

**Lemma 3**

(i) If  $T_j^* \leq t_{j+1}$ , then  $\exists T_j \in (0, T_j^*]$ , such that  $D_j(T_j) = 0$ ,

(ii) If  $T_j^* > t_{j+1}$ , then  $D_j(t) < 0$  and  $D_j'(t) > 0 \forall t < t_{j+1}$ .

**Proof.** Since  $D_j(t)$  and  $f_j(t)$  differ by a finite constant, Claim 1 implies that  $D_j(t)$  is strictly quasi-concave. Also,  $D_j(0) < 0$ , since

$$\begin{aligned} L_j(0) &= \pi(j) \int_0^{t_{j+1}} e^{-rs} ds + \sum_{m=j+1}^N \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(0) \\ &< \frac{\pi(1)}{r} - c(0) \\ &< 0 \\ &\leq V^{j+1}(t_1, \dots, t_N) \\ &= \sum_{m=j+1}^N \pi(m) \int_{t_m}^{t_{m+1}} e^{-rs} ds - c(t_{j+1}) \\ &= F_j(0) \end{aligned}$$

Here the second inequality holds by assumption 3(i) and the third because no firm gets a negative payoff in equilibrium as it could always delay investment indefinitely ensuring a payoff of zero. Moreover,  $D_j(t_{j+1}) = 0$ . Therefore, two cases are possible:

(i):  $T_j^* \leq t_{j+1}$ , in which case  $\exists T_j \in (0, T_j^*]$ , such that  $D_j(T_j) = 0$ , and  $D_j(t) > 0$  in the

interval  $t \in (T_j, t_{j+1})$ ,

(ii)  $T_j^* > t_{j+1}$ , in which case  $D_j(t) < 0$  and  $D'_j(t) > 0 \forall t < t_{j+1}$ . ■

In the next Lemma, we analyze decision nodes with one active firm.

**Lemma 4** *In equilibrium, if at time  $t$  there is one active firm, it plays “wait” if  $t < T_N^*$  and “invest” if  $t \geq T_N^*$ .*

**Proof.** The result follows immediately from the proof of Lemma 1. ■

In the next Lemma, we show that for the case  $j = N - 1$ , case (i) of Lemma 3 applies.

**Lemma 5**  $T_{N-1}^* < t_N = T_N^*$  and  $\exists T_{N-1} < T_{N-1}^* < T_N^*$  such that  $D_{N-1}(T_{N-1}) = 0$ .

**Proof.**  $T_{N-1}^* < T_N^*$  by Claim 1 and  $t_N = T_N^*$  by Lemma 1. The rest of the statement follows from Lemma 3. ■

In the next Lemma we show that the  $(N - 1)$ -th equilibrium investment time is  $T_{N-1}$ .

**Lemma 6** *In equilibrium, it holds that:*

(i) *In all subgames starting at  $t \in [T_{N-1}, T_N^*)$ , if there are  $n > 1$  active firms,  $n - 1$  investments take place at  $t$ .*

(ii) *In all subgames starting at  $t \in [0, T_{N-1})$ , if there are  $n = 2$  active firms, each of them plays “wait”.*

(iii)  $t_{N-1} = T_{N-1}$  and the equilibrium payoff of the last two investors is the same.

**Proof.**

**Part (i)** We start from the observation that if  $t$  belongs to the interval  $(T_{N-1}, T_N^*)$ , then  $D_{N-1}(t) > 0$ , while if  $t = T_{N-1}$ , then  $D_{N-1}(t) = 0$ . The proof now consists of three steps: Step (a) identifies two strategy profiles that generate the same outcome and constitute an equilibrium for any subgame with  $n > 1$  active firms starting at  $t \in (T_{N-1}, T_N^*)$ . Step (b) rules out any other candidate equilibrium for subgames with  $n > 1$  active firms starting at  $t \in (T_{N-1}, T_N^*)$ . Step (c) considers subgames with  $n > 1$  active firms starting exactly at  $t = T_{N-1}$ .

(a) We first show that it is an equilibrium for any subgame with  $n > 1$  active firms starting at  $t \in (T_{N-1}, T_N^*)$  that each active firm plays “invest” at all times  $\tau \geq t$ , unless it is the only active firm.<sup>21</sup> The outcome of this candidate equilibrium is that all active firms

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<sup>21</sup>The proof that the above profile is an equilibrium also proves that the following strategy profile, which generates the same outcome, is an equilibrium: *In all subgames with  $n > 1$  active firms starting at  $t \in (T_{N-1}, T_N^*)$  each active firm plays “wait” at  $t$  and “invest” at all times  $\tau > t$ , unless it is the only active one.*

try to invest immediately at  $t$ , until only one is left. The last active firm will then wait to invest until time  $T_N^*$  by Lemma 4. The associated payoff for each firm is a lottery between  $L_{N-1}(t)$  and  $F_{N-1}(t)$  with probability  $\frac{1}{n}$  assigned to  $F_{N-1}(t)$ .

Suppose one firm deviates and plays “wait” at  $t$  although it is not the only active firm. We show that any deviation involving the play of “wait” at  $t$  when one or more other firms are active gives a smaller payoff than the payoff in the candidate equilibrium. We need to distinguish three possible classes of deviations according to the specific circumstances when the deviator plays “wait”.

First, consider the class of deviations in which the firm plays “wait” at  $t$  if at least one other firm is active. At  $t$ , the  $n - 1$  non-deviating firms follow the given strategies, hence  $n - 1$  investments occur at  $t$  and the deviating firm becomes the last active one. By Claim 1, the most profitable deviation in this class is the one in which the deviator later invests at  $T_N^*$ . The associated payoff is  $F_{N-1}(t)$  which is smaller than the lottery between  $L_{N-1}(t)$  and  $F_{N-1}(t)$  given by the candidate equilibrium.

Second, consider the class of deviations in which the firm plays “wait” at  $t$  if the number of active firms is different from  $n - l$ , for a given  $l$  such that  $n - l > 1$ . At  $t$ , the  $n - 1$  non-deviating firms follow the given strategies. The deviation outcome is the following. First, the  $n - 1$  non-deviating firms will play “invest” until  $l$  investments will occur. Then, when there are  $n - l$  active firms left, all of them, including the deviator, will play “invest”. With probability  $\frac{1}{n-l}$ , the deviating firm will successfully invest and receive payoff  $L_{N-1}(t)$ . With the complementary probability, the deviating firm will fail to invest. In the latter case, the game enters a subgame with  $n - l - 1$  active firms in which all firms except for the deviator will continue to play “invest” at  $t$ , until the deviator is the only active firm. By Claim 1, we can again identify the most profitable deviation within this class: If the outcome of the randomization device is such that the deviator is unsuccessful in investing when there are  $n - l$  active firms, the deviator will be the last active firm remaining. In this case, it should then invest at  $T_N^*$ . Hence, the highest payoff in this class of deviations is a lottery between  $L_{N-1}(t)$  and  $F_{N-1}(t)$ , with probability  $\frac{n-l-1}{n-l}$  assigned to  $F_{N-1}(t)$ , which is worse than the lottery deriving from the candidate equilibrium strategies because  $\frac{n-l-1}{n-l} > \frac{1}{n}$ .

Third, an analogous argument shows that the class of deviations in which the firm plays “wait” at  $t$  if the number of active firms is different from  $n - l$ , for a set of at least two integers  $l$ , such that  $n - l > 1$  for every  $l$ , the highest possible payoff is a lottery between  $L_{N-1}(t)$  and  $F_{N-1}(t)$ , which assigns to  $F_{N-1}(t)$  a higher probability than the lottery deriving from

the candidate equilibrium strategies.

(b) Next, we show that if  $t$  belongs to the interval  $(T_{N-1}, T_N^*)$ , no strategy profile different from the ones presented in (a) constitutes an equilibrium. We need to rule out two classes of strategy profiles: one in which at  $t$ , with  $n$  active firms, one or more firm plays “wait” and one or more firms play “invest”, and one in which all active firms play “wait” in an interval with positive measure starting at  $t$ .

We develop the argument by induction. First we show it holds for  $n = 2$  active firms. Then we show that if it holds for  $n = m \geq 2$ , then it holds for  $n = m + 1$ . Finally, we conclude that it holds for any  $n \geq 2$  by the induction principle.

*The statement holds for  $n = 2$ .* We consider two classes of strategy profiles: one in which at  $t$ , with two active firms, one firm plays “wait” and the other plays “invest”, and one in which both active firms play “wait” in an interval with positive measure starting at  $t$ .

No strategy profile in the first class can be an equilibrium, because the firm who plays “wait” ends up being the last active firm in the game, thus receiving a payoff no larger than  $F_{N-1}(t)$ , while it could profitably deviate by playing “invest” and receiving a lottery between  $L_{N-1}(t)$  and  $F_{N-1}(t)$ .

Strategy profiles in the second class imply that the first investment in the subgame occurs strictly later than  $t$  and weakly before  $T_N^*$  (by Lemma 1), at some  $\tau \in (t, T_N^*]$ , and the second at  $T_N^*$  by Lemma 4. The strategy profiles for which both firms play “invest” at  $\tau$  cannot be an equilibrium because both firms get  $\frac{1}{2}[L_{N-1}(\tau) + F_{N-1}(\tau)]$  while each of them could deviate by investing at  $\tau - \varepsilon$  and receiving  $L_{N-1}(\tau - \varepsilon)$ . By continuity,  $\exists \varepsilon > 0$  small enough that this is profitable. The strategies profiles for which only one of the two firms plays “invest” at  $\tau$  cannot be an equilibrium. The firm who does not invest at  $\tau$  must invest at  $T_N^*$  by Lemma 4, thus receiving  $F_{N-1}(\tau)$ . It could profitably deviate by investing at  $\tau - \varepsilon$  and getting  $L_{N-1}(\tau - \varepsilon)$ .

*If the statement holds for  $m$ , it holds for  $m + 1$ .* Again, we need to consider two classes of strategy profiles: one in which at time  $t$ , with  $m + 1$  active firms, only  $\nu < m + 1$  play “invest”, with  $\nu > 0$ , and one in which all active firms play “wait” in an interval with positive measure starting at  $t$ .

No strategy profile in the first class can be an equilibrium. At  $t$ , one of the  $\nu$  firms will successfully invest and the game will enter a subgame with  $m$  active firms and calendar time  $t$ . Given the assumption that the statement holds for  $n = m$ , in such a subgame all  $m$  active

firms invest immediately until only one is left, which invests at  $T_N^*$ . Hence, each of them receives payoff  $L_{N-1}(t)$  with probability  $\frac{m-1}{m}$  and payoff  $F_{N-1}(t)$  with probability  $\frac{1}{m}$ . This implies that any of the  $m+1-\nu$  firms who play “wait” at  $t$  when there are still  $m+1$  active firms receives expected payoff  $\frac{1}{m} [(m-1)L_{N-1}(t) + F_{N-1}(t)]$ . It could profitably deviate by playing “invest” at  $t$  with  $m+1$  active firms. This deviation is profitable because it increases the probability of receiving  $L_{N-1}(t)$  and decreases the probability of receiving  $F_{N-1}(t)$ .

Strategy profiles in the second class imply that the first investment in the subgame occurs strictly later than  $t$  and weakly before  $T_N^*$  (by Lemma 1), at some  $\tau \in (t, T_N^*]$ . Consider any such profile and denote by  $\nu = 1, \dots, m+1$  the number of firms who play “invest” at time  $\tau$  if there are  $m+1$  active firms. At  $\tau$ , one successful investment occurs and the game enters a subgame with  $m$  active firms. Given the assumption that the statement holds for  $n = m$ , at  $\tau$  all the  $m$  remaining active firms invest immediately until only one is left, who invests at  $T_N^*$ . For  $\nu < m+1$ , there is at least one firm who initially plays “wait” at  $\tau$ . This firm receives  $\frac{1}{m} [(m-1)L_{N-1}(\tau) + F_{N-1}(\tau)]$ . It could profitably deviate by investing at  $\tau - \varepsilon$  and receiving  $L_{N-1}(\tau - \varepsilon)$  which is a larger payoff, for  $\varepsilon$  small enough. If instead  $\nu = m+1$ , all firms receive payoff  $\frac{1}{m+1} [mL_{N-1}(\tau) + F_{N-1}(\tau)]$ . Each of them could profitably deviate by preempting the opponents and investing at  $\tau - \varepsilon$ . This would yield payoff  $L_{N-1}(\tau - \varepsilon)$  which is larger than the above, for  $\varepsilon$  small enough.

This completes the induction argument and we can conclude that if  $t$  belongs to the interval  $(T_{N-1}, T_N^*)$ , no strategy profile different from the ones presented in (a) constitutes an equilibrium.

(c) We conclude by considering subgames with  $n > 1$  active firms, starting at  $t = T_{N-1}$ . While  $D_{N-1}(t) > 0$  for  $t \in (T_{N-1}, T_N^*)$ , on the other hand  $D_{N-1}(t) = 0$  for  $t = T_{N-1}$ . This implies that the proof in step (a) also holds for subgames starting at  $t = T_{N-1}$ . Hence the profiles analyzed in (a) also constitute an equilibrium for the subgames with  $n > 1$  active firms, starting at  $t = T_{N-1}$ . Now consider step (b). It shows that for for  $t \in (T_{N-1}, T_N^*)$  strategy profiles in two classes cannot be an equilibrium: profiles in which at  $t$ , with  $n$  active firms, one or more firm plays “wait” and one or more firms play “invest”, and profiles in which all active firms play “wait” in an interval with positive measure starting at  $t$ . For subgames starting at  $t = T_{N-1}$  instead, while it is true that profiles in the second class cannot be an equilibrium, strategy profiles in the first class are an equilibrium. The outcome of such a profile is that  $n-1$  investments occur at  $T_{N-1}$  and one occurs at  $T_N^*$ . All

players receive payoff  $L_{N-1}(T_{N-1}) = F_{N-1}(T_{N-1})$ . Hence, there is no profitable deviation.

**Part (ii).** If  $t$  belongs to the interval  $[0, T_{N-1})$ , it holds that  $L_{N-1}(t) < F_{N-1}(t)$  and that  $L_{N-1}(t)$  is increasing. Given part (i) of this Lemma, and Lemma 4, if firms follow the actions prescribed in part (ii) the expected payoff for each of them is

$$\frac{1}{2} [L_{N-1}(T_{N-1}) + F_{N-1}(T_{N-1})] = L_{N-1}(T_{N-1}) = F_{N-1}(T_{N-1})$$

where the equality comes from the definition of  $T_{N-1}$ . The deviation payoff from investing at some  $\tau$  before  $T_{N-1}$  is  $L_{N-1}(\tau) < L_{N-1}(T_{N-1})$ , hence this is an equilibrium.

Next, we show that there is no other action profile compatible with equilibrium. Suppose that  $\nu \leq 2$  firms play “invest” at  $\tau$ . The equilibrium payoff for any of these early investors is

$$\frac{1}{\nu} [L_{N-1}(\tau) + (\nu - 1)F_{N-1}(\tau)]$$

Each of them could profitably deviate by playing “wait” at  $\tau$ , since  $F_{N-1}(\tau) > L_{N-1}(\tau)$ . This concludes the proof of part (ii).

**Part (iii).** The conclusion that  $t_{N-1} = T_{N-1}$  follows directly from parts (i) and (ii) and from Lemma 4. By construction of  $T_{N-1}$ , this implies rent equalization for the last two investors. ■

We now identify the algorithm for the construction of the equilibrium investment times  $t_j$  for  $j \in \{1, \dots, N - 1\}$ . The argument is based on the induction principle. Lemma 7 contains a statement for  $j = N - 2$ . Lemma 8 shows that if the same statement holds for  $j = N - l$ , then it holds for  $j = N - l - 1$ . Lemma 9 concludes that, by the induction principle, the statement holds for a general  $j$ .

**Lemma 7** Given  $t_N = T_N^*$  and  $t_{N-1} = T_{N-1}$ ,  $t_{N-2}$  can be constructed as follows.

**Part (a)** Suppose  $T_{N-2}^* < T_{N-1}$ . In equilibrium it holds that:

(i) In all subgames starting at  $t \in [T_{N-2}, T_{N-1})$ , if there are  $n > 2$  active firms,  $n - 2$  investments take place at  $t$ .

(ii) In all subgames starting at  $t \in [0, T_{N-2})$ , if there are 3 active firms, each of them plays “wait”

(iii)  $t_{N-2} = T_{N-2}$  and the payoff of the last 3 investors is equalized.

**Part (b)** Suppose  $T_{N-2}^* \geq T_{N-1}$ . In equilibrium it holds that:

(i) In all subgames starting at  $t \in [0, T_{N-1})$ , if there are  $n = 3$  active firms, each of them

plays “wait”

(ii)  $t_{N-2} = T_{N-1}$  and the payoff of the last 3 investors is equalized.

**Proof.**

**Part (a):** By Lemma 3 (i),  $\exists T_{N-2} \in (0 < T_{N-2}^*]$ , such that  $D_{N-2}(T_{N-2}) = 0$ . The proofs of parts (i) and (ii) follow from arguments similar to the proofs of parts (i) and (ii) of Lemma 6, respectively.

The conclusion that  $t_{N-2} = T_{N-2}$  follows directly from parts (i) and (ii) and from Lemmata 6 and 4. By construction of  $T_{N-2}$ , this implies rent equalization for the last three firms.

**Part (b):** By Lemma 3 (ii),  $L_{N-2}(t) < F_{N-2}(t)$  and  $L'_{N-2}(t) > 0 \forall t < T_{N-1}$ . The proof of part (i) follows from arguments analogous to the proof of part (ii) of Lemma 6. The conclusion that  $t_{N-2} = t_{N-1} = T_{N-1}$  follows directly from part (i) and from Lemmata 4 and 6. By construction of  $T_{N-1}$ , this implies rent equalization for the last three firms. ■

**Lemma 8** *If the following statement holds for  $j = N - l$ , with  $l \geq 2$ , then it holds for  $j = N - l - 1$ .*

*Given the last  $N - j$  equilibrium investment times  $(t_{j+1}, \dots, t_N)$ , and given rent equalization for the last  $(N - j)$  investors, the  $j$ -th equilibrium investment time  $t_j$  can be constructed as follows:*

**Part (a)** *Suppose  $T_j^* < t_{j+1}$ . In equilibrium it holds that:*

(i) *In all subgames starting at  $t \in [T_j, t_{j+1})$ , if there are  $n > N - j$  active firms,  $n - N + j$  investments take place at  $t$ .*

(ii) *In all subgames starting at  $t \in [0, T_j)$ , if there are  $n = N - j + 1$  active firms, each of them plays “wait”*

(iii)  *$t_j = T_j$  and the payoff of the last  $N - j + 1$  investors is equalized.*

**Part (b)** *Suppose  $T_j^* \geq t_{j+1}$ . In equilibrium it holds that:*

(i) *In all subgames starting at  $t \in [0, t_{j+1}]$ , if there are  $n = N - j + 1$  active firms, each of them plays “wait”:*

(ii)  *$t_j = t_{j+1}$  and the payoff of the last  $N - j + 1$  investors is equalized.*

**Proof.** Assume that the statement holds for  $j = N - l$ . This implies that either  $t_{N-l} = T_{N-l}$  or  $t_{N-l} = t_{N-l+1}$ , and that in both cases payoffs of the last  $l + 1$  investors are equalized.



Now we need to prove that the statement holds for  $j = N - l - 1$ .

**Part (a):** By Lemma 3 (i),  $\exists T_{N-l-1} \in (0 < T_{N-l-1}^*]$ , such that  $D_{N-l-1}(T_{N-l-1}) = 0$ . The proofs of parts (i) and (ii) follow from arguments similar to the proofs of parts (i) and (ii) of Lemma 6, respectively. For part (iii), the conclusion that  $t_{N-l-1} = T_{N-l-1}$  follows directly from parts (i) and (ii) and from the assumptions. By construction of  $T_{N-l-1}$ , this implies rent equalization for the last  $l + 2$  firms.

**Part (b):** By Lemma 3 (ii),  $D_{N-l-1}(t) < 0$  and  $D'_{N-l-1}(t) > 0 \forall t < t_{N-l}$ . The proof of part (i) follows from arguments analogous to the proof of part (ii) of Lemma 6. The conclusion that  $t_{N-l-1} = T_{N-l}$  follows directly from part (i) and from the assumptions. By construction of  $T_{N-l}$ , this implies rent equalization for the last  $l + 2$  firms. ■

**Lemma 9** *The statement in Lemma 8 holds for any  $j \leq N - 2$ .*

**Proof.** The result follows from Lemmata 7 and 8 by the induction principle. ■

**Proof of Proposition 2.** First, suppose that the profit structure  $\pi$  is such that  $t_j < t_{j+1}$ . By construction,  $t_j$  solves:

$$\frac{\pi(j)}{r} [e^{-rt_j} - e^{-rt_{j+1}}] - [c(t_j) - c(t_{j+1})] = 0.$$

Differentiating implicitly yields:  $\frac{\partial t_j}{\partial \pi(j)} = -\frac{[e^{-rt_j} - e^{-rt_{j+1}}]/r}{-\pi(j)e^{-rt_j} - c'(t_j)}$ . The numerator is positive, because  $t_j < t_{j+1}$ . The denominator is positive as well, because in equilibrium  $t_j < T_{j+1}^*$ . Therefore,  $t_j$  is decreasing in  $\pi(j)$ . Next, suppose instead that the profit structure  $\pi$  is such that  $t_j = t_{j+1}$ . That is, suppose that the profit structure  $\pi$  is such that  $t_{j+1} \leq T_j^*$ . If a marginal change in  $\pi(j)$  does not affect the inequality  $t_{j+1} \leq T_j^*$ ,  $t_j$  is affected only by the flow profit parameters that affect  $t_{j+1}$ , hence it is constant in  $\pi(j)$ . If instead a marginal change in  $\pi(j)$  changes the inequality to  $t_{j+1} > T_j^*$ , we fall into the previous case and  $t_j$  decreases. ■

**Proof of Proposition 3.** Suppose that for a given profit structure  $\pi$ ,  $t_{j-k} < t_j$ . We prove that there exists  $\hat{\pi}(j)$  such that for the modified game with profit structure  $\tilde{\pi}$  with  $\tilde{\pi}(l) = \pi(l)$  for all  $l \leq j - k$  and all  $l > j$ ,  $\tilde{\pi}(j) \in (\hat{\pi}(j), \pi(j - k))$  and  $\tilde{\pi}(l) > \tilde{\pi}(l + 1)$  for all  $l$ , investment times  $t_{j-k}$  and  $t_j$  are clustered.

For the profit structure  $\pi$ , let  $z \geq 1$  be the number of investments occurring jointly in equilibrium at the beginning of the subgame played among the last  $N - j + 1$  players,

so that  $t_j = \dots = t_{j+z-1}$ . Suppose  $z = 1$ . The proof of Proposition 2 shows that in this case  $t_j$  strictly decreases in  $\pi(j)$ . Moreover,  $\lim_{\pi(j) \rightarrow \pi(j-k)} T_j^* = T_{j-k}^*$ . Since  $t_j < T_j^*$ , there exists a  $\hat{\pi}(j)$  large enough that  $t_j < T_{j-k}^*$  in any modified game with profit structure  $\tilde{\pi}$  with  $\tilde{\pi}(l) = \pi(l)$  for all  $l \leq j-k$  and all  $l > j$ ,  $\tilde{\pi}(j) \in (\hat{\pi}(j), \pi(j-k))$  and  $\tilde{\pi}(l) > \tilde{\pi}(l+1)$  for all  $l$ . Now suppose that instead  $z > 1$ . The fact that  $t_{j-k} < t_j$  implies that  $T_{j-k}^* < t_j = t_{j+z-1}$ . Since  $T_j^*$  is strictly decreasing in  $\pi(j)$  and  $\lim_{\pi(j) \rightarrow \pi(j-k)} T_j^* = T_{j-k}^* < t_{j+z-1}$ , there exists a  $\underline{\pi}(j)$  sufficiently close to  $\pi(j-k)$  such that  $T_j^* < t_{j+1} = \dots = t_{j+z-1}$ , and  $t_j < T_j^* < t_{j+1}$  in any modified game with profit structure  $\bar{\pi}$  with  $\bar{\pi}(l) = \pi(l)$  for all  $l \leq j-k$  and all  $l > j$ ,  $\bar{\pi}(j) \in (\underline{\pi}(j), \pi(j-k))$ , and  $\bar{\pi}(l) > \bar{\pi}(l+1)$  for all  $l$ . Then, starting from the game with profit structure  $\bar{\pi}$  the proof for the case  $z = 1$  applies. ■

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