

Dynamics of a Charged Spherically Symmetric Thick Shell

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Abstract: We Consider a spherically symmetric thick shell in two different space times. We have used the equation of motion for thick shell, developed by Khakshournia and Mansouri, to obtain the equation of motion of a charged spherical shell. We Expand the dynamical equation of motion of thick shell, to the first order of its thickness, to compare it with the dynamics of charged thin shell. It is shown that the effect of thickness is to speed up the collapse of the shell.

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1. Introduction

The thin shell formalism of general relativity (GR) has found wide applications in GR and cosmology. This formalism was first developed in [1] and applied to the gravitational collapse problem [2]. Studies on gravitational collapse, dynamics of bubbles and domain walls in inflationary models, wormhole, signature changes, structure and dynamics of voids in the large scale structure of the universe are some of the applications (cf [3] and references their in). Thin shells are considered as zero thickness objects with a δ - function singularity in their energy-momentum and Einstein's tensors.

However the dynamics of a real thick shell has been rarely discussed in the literature because of the complexity to define it within GR and to find its exact dynamical equations. The outstanding paper that modifies the Israel thin shell equations to treat the motion of spherical and Planar thick domain walls is that of Garfinkle and Gregory [4], see also [5].

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According to the results of that paper, the effect of thickness in the first approximation is to reduce effectively the energy density of the wall compared to the corresponding thin domain wall, and therefore to increase the collapse velocity of the wall in vacuum.

I use the formalism developed by Mansouri and Khakshournia(MK) [6] to treat a charged thick shell. The organization of this paper is as follows. In Section 2 I give a brief introduction to MK junction condition of thick shell. I describe the dynamics of charged spherically symmetric thick shell Section 3. A general conclusion is given in Section 4.

2. The Junction Conditions

Consider a spherically symmetric thick shell with two boundaries Σ_1 and Σ_2 dividing the space-time into three regions: M_{in} inside the inner boundary Σ_1 , M_{out} outside the outer boundary Σ_2 , and M for the thick shell having two boundaries Σ_1 and Σ_2 . First of all, write down the appropriate junction condition on each boundary Σ_j ($j = 1, 2$) treated as a three dimensional timelike hypersurface. The continuity of the second fundamental form of Σ_j , or the extrinsic curvature tensor K_{ab} of Σ_j , so that consider $\Sigma_1(\Sigma_2)$ as a boundary surface separating M region from $M_{in}(M_{out})$. This crucial requirement is formulated as

$$[K_{ab}] \stackrel{\Sigma_j}{=} 0 \quad (j = 1, 2), \quad (1)$$

where the square bracket indicates the jump of K_{ab} across Σ_j , ($[K_{ab}] = K_{ab}^+ - K_{ab}^-$). Latin indices range over the intrinsic coordinates of Σ_j denoted by (τ_j, θ, ϕ) , where τ_j is the proper time of Σ_j . In particular, the angular component of equation (1) on each boundary is written as

$$K_{\theta}^{\theta+} |_{\Sigma_1} - K_{\theta}^{\theta-} |_{\Sigma_1} = 0 \quad (2)$$

$$K_{\theta}^{\theta+} |_{\Sigma_2} - K_{\theta}^{\theta-} |_{\Sigma_2} = 0 \quad (3)$$

where the superscript $+(-)$ refers to the side of Σ_j towards which the corresponding unit space like normal vector $n^\alpha(-n^\alpha)$ points. It means that on $\Sigma_1(\Sigma_2)$, the superscript $+$ refers to the region $M(M_{out})$ and the superscript $-$ refers to the region $M_{in}(M)$. Adding equations (2) and (3), to get

$$K_{\theta}^{\theta+} |_{\Sigma_2} - K_{\theta}^{\theta-} |_{\Sigma_1} + K_{\theta}^{\theta+} |_{\Sigma_1} - K_{\theta}^{\theta-} |_{\Sigma_2} = 0. \quad (4)$$

In the next section, this general equation will be applied to the case of a collapsing charged shell.

3. Dynamics of Charged Thick Shell

Consider a spherical thick shell immersed in two different spherically symmetric space-times. The space-time outside the shell is described by Reissner-Nordstrom (RN) metric:

$$ds_o^2 = -f dt^2 + f^{-1} dr^2 + r^2 d\Omega^2 \quad (5)$$

where

$$f = 1 - \frac{2m}{r} + \frac{e^2}{r^2} = 1 - \frac{F(r)}{r} = 1 + \Phi(r) \quad (6)$$

where $F(r)$ or $\Phi(r)$ is an real function, and e is the electric charge.

The space-time inside the shell is described by Minkowski metric:

$$ds_i^2 = -dT^2 + dr^2 + r^2 d\Omega^2 \quad (7)$$

where

$$d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2$$

is the line element on the unit sphere. The induced intrinsic metric on Σ_j may be represented as

$$ds^2|_{\Sigma_j} = -d\tau_j^2 + R_j^2(\tau_j) (d\theta^2 + \sin^2 \theta d\phi^2) \quad (j = 1, 2),$$

where $R_j(\tau_j)$ being the proper radius of Σ_j . Now, define the constant commoving thickness of the shell as follows $2\delta = r_2 - r_1$ where r_1 and r_2 are commoving radii of the boundaries Σ_1 and Σ_2 respectively. Using the metric (5),(7) the relevant extrinsic curvature tensors in the region M are

$$\begin{aligned} K_{\theta}^{\theta+}|_{\Sigma_1} &= \frac{1}{R_1} \sqrt{1 + \dot{R}_1^2 - \frac{F(r_1)}{R_1}} \\ K_{\theta}^{\theta-}|_{\Sigma_2} &= \frac{1}{R_2} \sqrt{1 + \dot{R}_2^2 - \frac{F(r_2)}{R_2}} \end{aligned} \quad (8)$$

and the relevant extrinsic curvature tensors in the regions M_{in} and M_{out} are

$$\begin{aligned} K_{\theta}^{\theta+}|_{\Sigma_2} &= \frac{1}{R_2} \sqrt{1 + \dot{R}_2^2 - \frac{R(r_2)}{R_2}} \\ K_{\theta}^{\theta-}|_{\Sigma_1} &= \frac{1}{R_1} \sqrt{1 + \dot{R}_1^2} \end{aligned} \quad (9)$$

where $R_j \equiv R(r_j, \tau)$ and $R(r_2)$ is the radius of the spherical shell within the commoving surface r_2 .

To obtain the dynamical equation of the thick shell, expand the following quantities in a Taylor series around (r_0) , the mean commoving radius of the thick shell,

$$R(r_j, \tau) = R(r_0, \tau) + \varepsilon_j \delta R'(r_0, \tau) + 0(\delta^2) \quad (10)$$

$$F(r_j) = F(R_0) + \varepsilon_j \delta F'(r_0) + 0(\delta^2) \quad (11)$$

$$R(r_2) = R(r_0) + \delta R'(r_0) + 0(\delta^2) \quad (12)$$

where $\varepsilon_1 = -1$ and $\varepsilon_2 = +1$. Using equations (10), (11), (12) in the expressions (8) and (9), and keeping only terms up to the first order of δ , to get

$$K_{\theta}^{\theta-}|_{\Sigma_1} = \frac{1}{R_0} \sqrt{1 + \dot{R}_0^2} \left(1 + \delta \left(\frac{R'_0}{R_0} - \frac{\dot{R}_0 \dot{R}'_0}{1 + \dot{R}_0^2} \right) \right),$$

$$\begin{aligned}
K_{\theta}^{\theta+} |_{\Sigma_2} &= \frac{1}{R_0} \sqrt{1 + \dot{R}_0^2 - \frac{R(r_0)}{R_0}} \left(1 - \delta \left(\frac{R'_0}{R_0} - \frac{\dot{R}_0 \dot{R}'_0 - \frac{R'(r_0)}{2R_0} + \frac{R'_0 R(r_0)}{2R_0^2}}{1 + \dot{R}_0^2 - \frac{R(r_0)}{R_0}} \right) \right), \\
K_{\theta}^{\theta+} |_{\Sigma_1} &= \frac{1}{R_0} \sqrt{1 + \dot{R}_0^2 - \frac{F(r_0)}{R_0}} \left(1 + \delta \left(\frac{R'_0}{R_0} - \frac{\dot{R}_0 \dot{R}'_0 - \frac{F'(r_0)}{2R_0} + \frac{R'_0 F(r_0)}{2R_0^2}}{1 + \dot{R}_0^2 - \frac{F(r_0)}{R_0}} \right) \right), \\
K_{\theta}^{\theta-} |_{\Sigma_2} &= \frac{1}{R_0} \sqrt{1 + \dot{R}_0^2 - \frac{F(r_0)}{R_0}} \left(1 + \delta \left(\frac{R'_0}{R_0} - \frac{\dot{R}_0 \dot{R}'_0 - \frac{F'(r_0)}{2R_0} + \frac{R'_0 F(r_0)}{2R_0^2}}{1 + \dot{R}_0^2 - \frac{F(r_0)}{R_0}} \right) \right), \quad (13)
\end{aligned}$$

where $R_0 \equiv R(r_0, \tau)$. Substituting equation (13) into equation (4) and noting that $F(r_0) \equiv R(r_0)$, then the thick shell's equation of motion written up to the first order in δ is

$$\alpha - \beta = 2\delta \frac{F'(r_0)}{2\beta R_0} - \delta \left[\frac{R'_0}{R_0} (\alpha - \beta) + \frac{\dot{R}_0 \dot{R}'_0}{\alpha\beta} (\alpha - \beta) + \frac{1}{2\beta R_0} \left(R'(r_0) + \frac{R'_0 F(r_0)}{R_0} \right) \right] \quad (14)$$

where $\alpha = \sqrt{1 + \dot{R}_0^2}$

and $\beta = \sqrt{1 + \dot{R}_0^2 - \frac{R(r_0)}{R_0}} \equiv \sqrt{1 + \dot{R}_0^2 - \frac{F(r_0)}{R_0}}$.

This is the generalization of thin shell dynamical equation up to the first order of the thickness.

To verify the thin shell limit of this thick shell dynamical equation, consider the following definition for the surface energy density of the infinitely thin shell [1],

$$\sigma = \int_{-\varepsilon}^{+\varepsilon} \rho(r, \tau) dn \quad (15)$$

Where n is the proper distance in the direction of the normal n_{μ} and 2ε is the physical thickness of the shell. With the metric (5) equation (15) takes the form

$$\sigma = \int_{-\delta}^{+\delta} \frac{\rho(r, \tau)}{\sqrt{1 + \dot{R}^2 - \frac{F(r)}{R}}} dr \quad (16)$$

Since $\frac{\partial m}{\partial R} = 4\pi R^2 \rho G$, then equation (16) can be written as

$$8\pi G \sigma = \int_{-\delta}^{+\delta} \frac{2 \frac{\partial m}{\partial R} dr}{R^2 \sqrt{1 + \dot{R}^2 - \frac{F(r)}{R}}} = \int_{-\delta}^{+\delta} \frac{F'(r)}{R^2 \sqrt{1 + \dot{R}^2 - \frac{F(r)}{R}}} dr \quad (17)$$

Substituting equations (10) and (11) into equation (17) and integrate it up to the first order in δ to get

$$8\pi G \sigma = 2\delta \frac{F'(r_0)}{R_0^2 \sqrt{1 + \dot{R}_0^2 - \frac{F(r_0)}{R_0}}} + 0(\delta^2) \quad (18)$$

Substituting equations (18) into equation (14) to get

$$\alpha - \beta = 4\pi G\sigma R_0 - \delta \left[\frac{R'_0}{R_0} (\alpha - \beta) + \frac{\dot{R}_0 \dot{R}'_0}{\alpha\beta} (\alpha - \beta) + \frac{1}{2\beta R_0} \left(R'(r_0) + \frac{R'_0 F(r_0)}{R_0} \right) \right] \quad (19)$$

When δ tends to zero the second term on the right hand side goes to zero, then this equation reduced to the equation of motion of charged thin shell.

Rewrite the dynamical equation of thick shell (19) in the form

$$\alpha - \beta = 4\pi G R_0 \tilde{\sigma} \quad (20)$$

where

$$\tilde{\sigma} = \sigma - \frac{\delta}{4\pi G R_0} \left[\frac{R'_0}{R_0} (\alpha - \beta) + \dot{R}_0 \dot{R}'_0 \left(\frac{\alpha - \beta}{\alpha\beta} \right) + \frac{1}{2\beta R_0} \left(R'(r_0) + \frac{R'_0 F(r_0)}{R_0} \right) \right] \quad (21)$$

It has the same form of the equation of thin shell with the effective surface density $\tilde{\sigma}$.

From equation (21) note that the shell starting its collapse at rest when the velocity \dot{R} is negative during the collapse, it becomes more negative with r so that $\dot{R}'_0 < 0$, so $\dot{R}\dot{R}'_0$ must be positive. Also, the radius of the shell layers is increased with r so that $R'(r_0) > 0$. Therefore all terms within the bracket on the right hand side of equation (21) are positive. This leads to the result $\tilde{\sigma} > \sigma$. Solving equation (20) for \dot{R}^2 to get

$$\dot{R}_0^2 = -1 + 4\pi^2 G^2 \tilde{\sigma}^2 R_0^2 + \frac{F(r_0)}{2R_0} + \frac{F^2(r_0)}{64\pi^2 G^2 \tilde{\sigma}^2 R_0^4}. \quad (22)$$

It follows that \dot{R}^2 becomes larger with smaller $\tilde{\sigma}$ and $R_0 > R(r_0)$. Substituting by (6) into equation (22) to get

$$\dot{R}_0^2 = -1 + 4\pi^2 G^2 \tilde{\sigma}^2 R_0^2 + \frac{m}{R_0} + \frac{m^2}{16\pi^2 G^2 \tilde{\sigma}^2 R_0^4} + \frac{e^2}{2R_0^2} \left(-1 - \frac{m}{8\pi^2 G^2 \tilde{\sigma}^2 R_0^3} + \frac{e^2}{32\pi^2 G^2 \tilde{\sigma}^2 R_0^4} \right)$$

Therefore the first order thickness corrections to the Israel thin shell approximation speed up the collapse of the shell.

4. Conclusion

I applied the modified Israel formalism which developed by MK to the case of the collapse of a charged thick shell in RN and obtained the zero thickness limit of the charged thin shell equation, and Israel thin shell equation with $e = 0$.

It has been shown that the effect of thickness up to the first order in the shell thickness, is to speed up the collapse of the shell.

References

- [1] Israel W., *Nuovo Cimento*. 44 B (1966) 1.
- [2] Frolov V.P., *Sov. Phys. JETP* 38 (1974) 393.
- [3] Berezin V.A., Kuzmin V.A. and Tkachev I.I., *Phys. Rev. D* 36 (1987) 2919.
- [4] Garfinkle D. and Gregory R., *phys. Rev. D* 41 (1990) 1889.
- [5] Bonjour F., Charmousis C. and Gregory R., *phys. Rev. D* 62 (2000) 083504.
- [6] Mansouri R. and Khakshournia S., *Gen. Rel. Grav.* 34 (2002) 1847.