TRANSITION FRONTS FOR INHOMOGENEOUS MONOSTABLE REACTION-DIFFUSION EQUATIONS VIA LINEARIZATION AT ZERO

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ABSTRACT. We prove existence of transition fronts for a large class of reaction-diffusion equations in one dimension, with inhomogeneous monostable reactions. We construct these as perturbations of corresponding front-like solutions to the linearization of the PDE at $u = 0$. While a close relationship of the solutions to the two PDEs has been well known and exploited for KPP reactions (and our method is an extension of such ideas from [16]), to the best of our knowledge this is the first time such an approach has been used in the construction and study of fronts for non-KPP monostable reactions.

1. INTRODUCTION

We study transition fronts for the one-dimensional reaction-diffusion equation

$$u_t = u_{xx} + f(x, u), \tag{1.1}$$

with an inhomogeneous non-negative reaction $f \geq 0$ satisfying $f(x, 0) = f(x, 1) = 0$, and with $u \in [0, 1]$. Such PDEs model a host of natural processes such as combustion, chemical reactions, population dynamics and others, with u representing (normalized) temperature, concentration of a reactant, or population density.

Both $u \equiv 0$ and $u \equiv 1$ are equilibrium solutions of (1.1) and one is interested in the study of propagation of reaction in space, that is, invasion of the state $u = 0$ by the state $u = 1$. An important class of solutions modeling the propagation of reaction are transition fronts. A *(right-moving) transition front* is any entire solution $u : \mathbb{R}^2 \to [0, 1]$ of (1.1) which satisfies

$$\lim_{x\to-\infty} u(t, x) = 1 \qquad \text{and} \qquad \lim_{x\to+\infty} u(t, x) = 0 \tag{1.2}$$

for each $t \in \mathbb{R}$. In addition, we also require that for any $\varepsilon > 0$ there exists $L_\varepsilon < \infty$ such that

$$\sup_{t\in\mathbb{R}} \operatorname{diam}\{x \in \mathbb{R} \,|\, \varepsilon \leq u(t, x) \leq 1 - \varepsilon\} \leq L_\varepsilon. \tag{1.3}$$

The definition of a left-moving transition front is similar, with the limits in (1.2) exchanged. We will only study right-moving fronts here because the treatment of both cases is identical, up to a reflection in x. We note that the above definition is from [3, 6, 12].

We will consider here the case of *monostable reactions*, for which $u \equiv 1$ is an asymptotically stable solution while $u \equiv 0$ is unstable. We assume that f is Lipschitz,

$$f(x, 0) = f(x, 1) = 0 \quad \text{for } x \in \mathbb{R}, \tag{1.4}$$

$$a(x) := f_u(x, 0) > 0 \tag{1.5}$$

exists for $x \in \mathbb{R}^d$, and

$$a(x)g_0(u) \le f(x,u) \le a(x)g_1(u) \qquad \text{for } (x,u) \in \mathbb{R} \times [0,1], \tag{1.6}$$

where $g_0, g_1 \in C^1([0,1])$ satisfy

$$g_0(0) = g_0(1) = 0, \qquad g_0'(0) = 1, \qquad g_0(u) > 0 \quad \text{and} \quad g_0'(u) \le 1 \quad \text{for } u \in (0,1), \tag{1.7}$$

$$g_1(0) = 0, \qquad g_1'(0) = 1, \qquad g_1'(u) \ge 1 \quad \text{for } u \in [0,1], \tag{1.8}$$

$$\int_0^1 \frac{g_1(u) - g_0(u)}{u^2} du < \infty. \tag{1.9}$$

Finally, we let

$$a_- := \inf_{x\in\mathbb{R}} a(x) \le \sup_{x\in\mathbb{R}} a(x) =: a_+. \tag{1.10}$$

When the reaction $f(x,u) = f(u) \ge 0$ is homogeneous, a special case of transition fronts are *traveling fronts*. These are of the form $u(t,x) = U(x-ct)$, with some front speed c and front profile U such that $\lim_{s\to-\infty} U(s) = 1$ and $\lim_{s\to\infty} U(s) = 0$, and their study goes back to the seminal works of Kolmogorov, Petrovskii, and Piskunov [5], and Fisher [4]. They considered *KPP reactions*, a special case of monostable reactions with $g_1(u) = u$, and found that for each $c \ge c_0 := 2\sqrt{f'(0)}$ there is a unique traveling front $u(t,x) = U_c(x-ct)$. A simple phase-plane analysis argument (see Aronson and Weinberger [1]) shows that this turns out to be the case for general homogeneous monostable reactions, although with a different $c_0 \ge 2\sqrt{f'(0)}$. In contrast, *ignition reactions*, satisfying $f(u) = 0$ for $u \in [0,\theta] \cup \{1\}$ and $f(u) > 0$ for $u \in (\theta, 1)$ (for some *ignition temperature* $\theta \in (0,1)$), give rise to a single speed $c_0 > 0$ and a single traveling front [1].

Despite many developments for homogeneous and space-periodic reactions in the almost eight decades since [4, 5] (see the reviews [2, 15] and references therein), transition fronts in spatially non-periodic media have only been studied relatively recently. The first existence of transition fronts result, for small perturbations of homogeneous *bistable reactions* (the latter are such that $f(u) < 0$ for $u \in (0,\theta)$ and $f(u) > 0$ for $u \in (\theta,1)$), was obtained by Vakulenko and Volpert [14]. Existence without a hypothesis of closeness to a homogeneous reaction, for ignition reactions of the form $f(x,u) = a(x)g(u)$ with some ignition g, was proved by Mellet, Roquejoffre, and Sire [8], and by Nolen and Ryzhik [11] (see also [7] for uniqueness and stability results for these reactions). Existence, uniqueness, and stability of fronts for general inhomogeneous ignition reactions was proved by Zlatoš [17]. He also proved existence of fronts for some monostable reactions which are in some sense not too far from ignition ones (satisfying, in particular, $\sup_{x\in\mathbb{R}} f_u(x,0) \le \frac{1}{4}c_0^2$, with c_0 the unique speed for some ignition f_0 with $f(x,u) \ge f_0(u)$ for all $(x,u) \in \mathbb{R}^d \times [0,1]$). All these results are based on recovering a front as a locally uniform limit, along a subsequence, of solutions u_n of the Cauchy problem with initial data $u_n(\tau_n, x) \approx \chi_{(-\infty,-n)}(x)$, where $\tau_n \to -\infty$ are such that $u_n(0,0) = \frac{1}{2}$. Existence of a limit u on $\mathbb{R}^2$ is guaranteed by parabolic regularity, and the challenge is to show that u is a transition front. We note that even in the monostable case in [17], when one expects multiple transition fronts, existence of only a single transition front was obtained.

A very different approach has been used by Nolen, Roquejoffre, Ryzhik, and Zlatoš [10], and by Zlatoš [16] to prove existence of multiple transition fronts for inhomogeneous KPP reactions. It is well known that when f is KPP, then there is a close relationship between the solutions of (1.1) and those of its linearization

$$v_t = v_{xx} + a(x)v \tag{1.11}$$

at $u = 0$. The reason for this is that all KPP fronts are *pulled*, with the front speeds determined by the reaction at $u = 0$, which is due to the *reaction strength* $\frac{f(x,u)}{u}$ being largest at $u = 0$ for any fixed $x \in \mathbb{R}$. This is in stark contrast with ignition fronts, which are always *pushed* because they are "driven" by the reaction at intermediate values of u.

One can therefore consider the simpler *front-like solutions* of (1.11), which are of the form

$$v_\lambda(t,x) = e^{\lambda t}\phi_\lambda(x). \tag{1.12}$$

Here $\phi_\lambda > 0$ is a generalized eigenfunction of the operator $\mathcal{L} := \partial_{xx} + a(x)$, satisfying

$$\phi_\lambda'' + a(x)\phi_\lambda = \lambda\phi_\lambda \tag{1.13}$$

on $\mathbb{R}$, which exponentially grows to ∞ as $x \to -\infty$ and exponentially decays to 0 as $x \to \infty$. If we let $\lambda_0 := \sup\sigma(\mathcal{L})$ be the supremum of the spectrum of $\mathcal{L}$, which satisfies $\lambda_0 \in [a_-, a_+]$ because $\sigma(\partial_{xx}) = (-\infty, 0]$, then it is a well known spectral theory result that such ϕ_λ exists precisely when $\lambda > \lambda_0$, and is unique if we also require $\phi_\lambda(0) = 1$.

For KPP reactions one can try to use these solutions to find transition fronts for (1.1) with

$$\lim_{x\to\infty} \frac{u_\lambda(t,x)}{v_\lambda(t,x)} = 1 \tag{1.14}$$

for each $t \in \mathbb{R}$, at least for some $\lambda > \lambda_0$. This has been achieved in [10] for KPP reactions which converge to a homogeneous KPP reaction as $|x| \to \infty$, and for more general KPP reactions in [16]. In both cases one needs $\lambda_0 < 2a_-$ (otherwise it is possible that no transition fronts exist [10]) and $\lambda \in (\lambda_0, 2a_-)$.

In the present paper we show that this linearization approach can be extended to general *non-KPP monostable reactions.* Our method is an extension of the (relatively simple and robust) approach from [16]. There it was discovered that while v_λ is obviously a super-solution of (1.1) when $g_1(u) = u$ (i.e., in the KPP case), one can also use v_λ to find a sub-solution of the form $\tilde{w}_\lambda(t,x) = \tilde{h}_\lambda(v_\lambda(t,x))$, for $\lambda \in (\lambda_0, 2a_-)$ and an appropriate g_0-dependent increasing function $\tilde{h}_\lambda : [0,\infty) \to [0,1)$ with

$$\tilde{h}_\lambda(0) = 0, \qquad \tilde{h}_\lambda'(0) = 1, \qquad \lim_{v\to\infty} \tilde{h}_\lambda(v) = 1, \qquad \tilde{h}_\lambda(v) \le v \quad \text{on } [0,\infty). \tag{1.15}$$

It follows that $\tilde{w}_\lambda \le v_\lambda$, and one can then find a transition front u_λ between the two using parabolic regularity (see below).

Since this $\tilde{w}_\lambda$ depends on g_0 but not on g_1, it remains a sub-solution even for $g_1(u) \ge u$ (the latter follows from (1.8)). On the other hand, v_λ need not be anymore a super-solution. However, we prove here that one can still construct a super-solution of the form $w_\lambda(t,x) = h_\lambda(v_\lambda(t,x))$, for an appropriate increasing $h_\lambda : [0,\infty) \to [0,\infty)$ such that

$$h_\lambda(0) = 0, \qquad h_\lambda'(0) = 1, \qquad h_\lambda''(v) \ge 0 \quad \text{on } h_\lambda^{-1}([0,1]). \tag{1.16}$$

Once again, we then find a transition front u_λ between $\tilde{w}_\lambda$ and $\min\{w_\lambda, 1\}$.

Moreover, a result of Nadin [9] (see also [12]) shows that once some front exists, then also a (time-increasing) *critical front* exists. The latter is a transition front u_C for (1.1) such that if $u \not\equiv u_C$ is any other transition front and $u(t,x) = u_C(t,x)$ for some $(t,x) \in \mathbb{R}^2$, then

$$[u_C(t,y) - u(t,y)](y-x) < 0$$

for all $y \neq x$. That is, a critical front is the (unique up to time translation) "steepest" transition front for (1.1), and is the inhomogeneous version of the minimal speed front for homogeneous reactions. Indeed, if f is homogeneous, then u_C is precisely the traveling front with the minimal speed c_0.

Thus we obtain the following result.

Theorem 1.1. *Assume (1.4)–(1.9), let* $\nu := \sup_{u\in(0,1]} \frac{g_1(u)}{u} \geq 1$, *and let the supremum of the spectrum of* $\mathcal{L} := \partial_{xx} + a(x)$ *be* $\lambda_0 := \sup \sigma(\mathcal{L}) \in [a_-, a_+]$. *If* $\lambda \in (\lambda_0, 2a_-)$ *satisfies*

$$\lambda \leq 2a_- - \frac{2\sqrt{\nu-1}}{\sqrt{\nu}+\sqrt{\nu-1}} a_+, \tag{1.17}$$

then (1.1) *has a transition front* u_λ *with* $(u_\lambda)_t > 0$, *satisfying* (1.14). *In particular, if* λ_0 *is smaller than the right-hand side of* (1.17), *then a critical front* u_C *also exists and* $(u_C)_t > 0$.

Remarks. 1. If $a(x) = f_u(x,0)$ is constant on $\mathbb{R}$, then $\lambda_0 = a_- = a_+$, so the right-hand side of (1.17) is always greater than λ_0. Thus a transition front exists for any g_0, g_1 in this case.

2. The front u_λ does not have a constant speed in general, but when f is stationary ergodic in x, then it almost surely has an *asymptotic speed* $c_\lambda > 0$ in the sense that if $X(t)$ is the rightmost point such that $u(t, X(t)) = \frac{1}{2}$, then

$$\lim_{|t|\to\infty} \frac{X(t)}{t} = c_\lambda.$$

This is because the same claim holds for v_λ [16] and $\tilde{h}_\lambda(v_\lambda) \leq u_\lambda \leq h_\lambda(v_\lambda)$.

3. The result also holds with v_λ replaced by more general solutions of (1.11) of the form $v_\mu(t,x) \equiv \int_{\mathbb{R}} v_\lambda(t,x)d\mu(\lambda)$, with μ a finite non-negative non-zero Borel measure supported on a compact subset of $(\lambda_0, 2a_- - 2\sqrt{\nu-1}(\sqrt{\nu}+\sqrt{\nu-1})^{-1}a_+]$ (or of $(\lambda_0, 2a_-)$ if $\nu = 1$).

4. The result also applies to the more general equation

$$u_t = (A(x)u_x)_x + q(x)u_x + f(x,u)$$

with

$$0 < A_- \leq A(x) \leq A_+ < \infty \qquad \text{and} \qquad |q(x)| \leq q_+ < \infty$$

for all $x \in \mathbb{R}$, provided that $q_+ \leq 2\sqrt{(aA)_-}$ with $(aA)_- := \inf_{x\in\mathbb{R}}[a(x)A(x)]$, where

$$\lambda_0 := \sup_{\psi\in H^1(\mathbb{R})} \frac{\int_{\mathbb{R}}[-A(x)\psi'(x)^2 + q(x)\psi'(x)\psi(x) + a(x)\psi(x)^2]dx}{\int_{\mathbb{R}} \psi(x)^2 dx} \quad (\geq a_-)$$

and $2a_-$ is replaced in (1.17) by

$$\lambda_1 := \inf_{x\in\mathbb{R}} \left\{ a(x) + \sqrt{(aA)_-}\left[\sqrt{(aA)_-} - |q(x)|\right] A(x)^{-1} \right\} \quad (\leq 2a_-).$$

We indicate the proofs of Remarks 2–4 after the proof of the theorem.

Our construction of the super-solution w_λ is of independent interest and extends to more general equations in several dimensions, possibly with time-dependent coefficients. Hence we state it here as a separate result.

Lemma 1.2. *Let the function $f(t,x,u) \geq 0$, positive definite matrix $A(t,x)$, and vector field $q(t,x)$ be all Lipschitz, with $(t,x,u) \in (t_0,t_1) \times \mathbb{R}^d \times [0,1]$ and some $-\infty < t_0 < t_1 \leq \infty$. Assume that $a(t,x) \equiv f_u(t,x,0) > 0$ exists,* (1.4)–(1.9) *hold with $(t,x) \in (t_0,t_1) \times \mathbb{R}^d$ in place of $x \in \mathbb{R}^d$, and define $\nu := \sup_{u\in(0,1]} \frac{g_1(u)}{u} \geq 1$. Let $v > 0$ be a solution of*

$$v_t = \nabla \cdot (A(t,x)\nabla v) + q(t,x) \cdot \nabla v + a(t,x)v$$

on $(t_0,t_1) \times \mathbb{R}^d$. If $\nu > 1$ and for some $\alpha \leq (\sqrt{\nu} - \sqrt{\nu - 1})^2$ (or for some $\alpha < 1$ if $\nu = 1$),

$$\nabla v(t,x) \cdot A(t,x)\nabla v(t,x) \leq \alpha a(t,x)v(t,x)^2 \tag{1.18}$$

holds for all $(t,x) \in (t_0,t_1) \times \mathbb{R}^d$, then there exist increasing functions $\tilde{h}$ satisfying (1.15) *and h satisfying* (1.16) *such that $\tilde{w} := \tilde{h}(v)$ is a sub-solution of*

$$u_t = \nabla \cdot (A(t,x)\nabla u) + q(t,x) \cdot \nabla u + f(t,x,u) \tag{1.19}$$

on $(t_0,t_1) \times \mathbb{R}^d$ and $w := h(v)$ is a super-solution on $[(t_0,t_1) \times \mathbb{R}^d] \cap \{(t,x) \,|\, w(t,x) \leq 1\}$. Therefore, if u solves (1.19) *with*

$$\tilde{w}(t_0,x) \leq u(t_0,x) \leq \min\{w(t_0,x), 1\} \tag{1.20}$$

for all $x \in \mathbb{R}^d$, then for all $(t,x) \in (t_0,t_1) \times \mathbb{R}^d$ we have

$$\tilde{w}(t,x) \leq u(t,x) \leq \min\{w(t,x), 1\}. \tag{1.21}$$

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2. Proof of Theorem 1.1 (using Lemma 1.2)

Let $\lambda \in (\lambda_0, 2a_-)$ and $v = v_\lambda$ be from (1.12), with $\phi = \phi_\lambda > 0$ from (1.13) with $\lim_{x\to\infty} \phi(x) = 0$ and $\phi(0) = 1$. It is proved in [16, Proof of Theorem 1.1] that there exists a unique such ϕ and it satisfies

$$\phi'(x)^2 \leq \alpha a_- \phi(x)^2 \tag{2.1}$$

for $\alpha := 1 - (2a_- - \lambda)a_+^{-1} < 1$ and all $x \in \mathbb{R}$, as well as

$$\phi(x) \geq 2\phi(y) \tag{2.2}$$

for some $L < \infty$ and any $y - x \geq L$.

Since $\alpha \leq (\sqrt{\nu} - \sqrt{\nu - 1})^2$ is, by the definition of α, equivalent to

$$\lambda \leq 2a_- - \left[1 - \left(\sqrt{\nu} - \sqrt{\nu - 1}\right)^2\right] a_+ = 2a_- - \frac{2\sqrt{\nu - 1}}{\sqrt{\nu} + \sqrt{\nu - 1}} a_+$$

(which is (1.17)), Lemma 1.2 applies to v and (1.1). Thus we have (1.21) and a standard limiting argument now recovers an entire solution to (1.1) between $\tilde{w}$ and $\min\{w,1\}$. We let u_n be the solution of (1.1) on $(-n,\infty)\times\mathbb{R}$ with $u_n(-n,x):=\tilde{w}(-n,x)$. Since $\tilde{w}(t,x)\le\min\{w(t,x),1\}$ because $h(v)\ge v$ for $v\in h^{-1}([0,1])$, (1.20) is satisfied with $t_0:=-n$ and we have (1.21) on $(-n,\infty)\times\mathbb{R}$. By parabolic regularity, there is a subsequence of $\{u_n\}$ which converges, locally uniformly on $\mathbb{R}^2$, to an entire solution u of (1.1). We obviously have

$$\tilde{w}\le u\le\min\{w,1\}, \tag{2.3}$$

and (1.14) for $u_\lambda:=u$ follows from $\tilde{h}'(0)=h'(0)=1$. We also have $u_t\ge 0$, because $(u_n)_t\ge 0$ due to $\tilde{w}_t=\tilde{h}'(v)v_t\ge 0$ and the maximum principle for $(u_n)_t$ (which satisfies a linear equation and is non-negative at $t=-n$). The strong maximum principle then gives $u_t>0$ because obviously $u_t\not\equiv 0$. Finally, u is a transition front because the second limit in (1.2) follows from $\lim_{x\to\infty}\phi(x)=0$ and (1.16), and (1.3) holds with

$$L_\varepsilon:=L\left\lceil\log_2\left(\tilde{h}^{-1}(1-\varepsilon)-h^{-1}(\varepsilon)\right)\right\rceil$$

due to (2.3) and (2.2) (with $\tilde{h},h$ from the lemma). The first limit in (1.2) is then obvious from $u\le 1$, and the proof is finished by using the result from [9] for critical fronts.

The claim in Remark 2 is proved as an analogous statement in [16, Theorem 1.2].

The claim in Remark 3 holds because L can be chosen uniformly for all λ in the support of μ [16] and so (2.2) holds with $\phi(\cdot)$ replaced by $v_\mu(t,\cdot)$. Also, v_μ satisfies (2.1) with α corresponding to $\lambda:=\sup\operatorname{supp}\mu$.

The claim in Remark 4 holds because (2.1) and (2.2) continue to hold in that case, albeit with $2a_-$ replaced by λ_1 in the definition of α [16].

3. Proof of Lemma 1.2

Lemma 2.1 in [16] shows that there is an increasing $\tilde{h}=\tilde{h}_\lambda$ as in (1.15) such that $\tilde{w}(t,x):=\tilde{h}(v)$ is a sub-solution of (1.1) with the reaction $\min\{f(x,u),a(x)u\}$ (which is a KPP reaction). Then $\tilde{w}$ is also a sub-solution of (1.1), which yields the first inequality in (1.21).

We will next find an increasing $h=h_\lambda$ as in (1.16) such that $w(t,x):=h(v(t,x))$ will be a super-solution on the space-time domain where $w(t,x)\le 1$, which will yield the second inequality in (1.21) because $u\le 1$ by the hypotheses. Our proof will be a super-solution counterpart to the sub-solution argument in [16, Lemma 2.1]; it was a little surprising to us that such a counterpart argument can be found for non-KPP reactions.

If h is as in (1.16), then (1.18) shows that

$$\begin{aligned} w_t-\nabla\cdot(A\nabla w)-q\cdot\nabla w &= h'(v)[v_t-\nabla\cdot(A\nabla v)-q\cdot\nabla v]-h''(v)\nabla v\cdot A\nabla v \\ &= h'(v)av-h''(v)\nabla v\cdot A\nabla v \\ &\ge a[vh'(v)-\alpha v^2h''(v)] \end{aligned}$$

when $w(t,x)\le 1$. We can then conclude that w is a super-solution of (1.1) where $w(t,x)\le 1$ once we show that on $h^{-1}([0,1])$ we also have

$$vh'(v)-\alpha v^2h''(v)\ge g_1(h(v)). \tag{3.1}$$

It therefore remains to find h satisfying (1.16) and (3.1). We let $c := \alpha^{1/2}+\alpha^{-1/2}$ and notice that since $\gamma+\gamma^{-1} \geq 2\sqrt{\nu}$ for all positive $\gamma \leq \sqrt{\nu}-\sqrt{\nu-1}$, the hypothesis $\alpha \leq (\sqrt{\nu}-\sqrt{\nu-1})^2$ yields $c \geq 2\sqrt{\nu}$. Next let U be the unique solution to the ODE

$$U'' + cU' + g_1(U) = 0 \tag{3.2}$$

on $[s_0, \infty)$, with

$$U(s_0) = 1 \qquad \text{and} \qquad U'(s_0) = -\sqrt{\alpha} g_1(1), \tag{3.3}$$

where $s_0 \in \mathbb{R}$ will be chosen later. (This is the ODE that would be satisfied by the traveling front profile with speed c for the homogeneous reaction $g_1(u)$ if we had $g_1(1) = 0$; this profile would then also satisfy $\lim_{s\to-\infty} U(s) = 1$ and $\lim_{s\to-\infty} U'(s) = 0$ instead of (3.3).)

Notice that $U'(s_0) \geq -\frac{c}{2}$ because $g_1(1) \leq \nu$ and

$$\sqrt{\alpha} \leq (\sqrt{\nu} + \sqrt{\nu-1})^{-1} \leq \nu^{-1/2}.$$

Let $V(s) := U'(s)$, and consider the curve $\theta := \{(U(s), V(s))\}_{s\geq s_0}$. It is easy to see that θ cannot leave the closed triangle T in the (U, V) plane with sides $V = 0$, $U = 1$, and $V = -\frac{c}{2}U$. This is because $(U(s_0), V(s_0)) \in T$ and on ∂T, the vector field $(V, -cV - g_1(U))$ either points inside T or is parallel to ∂T. Here we use $c \geq 2\sqrt{\nu}$ to obtain on the third side

$$\left(\frac{c}{2}, 1\right) \cdot \left(-\frac{c}{2}U, -c\left(-\frac{c}{2}U\right) - g_1(U)\right) = \frac{c^2}{4}U - g_1(U) \geq \nu U - g_1(U) \geq 0. \tag{3.4}$$

It follows that $U'(s) < 0$ on $[s_0, \infty)$, and since $g_1(U(s)) > 0$, $U(s)$ cannot have local minima on $[s_0, \infty)$. Hence $\lim_{s\to\infty} U(s)$ exists and $\lim_{s\to\infty} U'(s) = 0$. Finally, $g_1 > 0$ on $(0, 1]$ yields $\lim_{s\to\infty} U(s) = 0$.

We now define $h(0) := 0$ and

$$h(v) := U(-\alpha^{-1/2} \ln v) \tag{3.5}$$

for $v \in (0, e^{-\sqrt{\alpha}s_0}]$, so h is increasing and continuous at 0, with $h(e^{-\sqrt{\alpha}s_0}) = 1$ (we then extend h onto $[0, \infty)$ arbitrarily, only requiring that it be increasing). Since $c > 2\sqrt{g_1'(0)} = 2$ and

$$\int_0^1 \frac{g_1(u) - u}{u^2} du < \infty$$

by (1.7) and (1.9), a result of Uchiyama [13, Lemma 2.1] shows $\lim_{s\to\infty} U(s)e^{\sqrt{\alpha}s} \in (0, \infty)$. (This result assumes $g_1(1) = 0$ but we can extend g_1, U to [0,2] so that $g_1(2) = 0$ and U satisfies (3.2), and then apply [13] to $\tilde{g}(u) := \frac{1}{2}g_1(2u)$ and the function $\tilde{U}(s) := \frac{1}{2}U(s)$.)

If we now pick the unique s_0 in (3.3) such that $\lim_{s\to\infty} U(s)e^{\sqrt{\alpha}s} = 1$ (notice that (3.2) is an autonomous ODE), we obtain $h'(0) = 1$. We also have (3.1) on $[0, e^{-\sqrt{\alpha}s_0}]$ because on that interval, (3.2) immediately yields

$$\alpha v^2 h''(v) - vh'(v) + g_1(h(v)) = 0. \tag{3.6}$$

It therefore remains to show that $h''(v) \geq 0$ on $[0, e^{-\sqrt{\alpha}s_0}]$. Due to (3.6) and (3.5), this is equivalent to

$$-U'(s) \geq \sqrt{\alpha}\, g_1(U(s)) \tag{3.7}$$

for $s \ge s_0$. Thus we need to show that θ stays at or below $\psi := \{(U(s), -\sqrt{\alpha}\, g_1(U(s)))\}_{s\ge s_0}$. This is true at $s = s_0$ by the definition of U, so it is sufficient to show that on ψ, the vector field $(V, -cV - g_1(U))$ points either below or is parallel to ψ. This holds because the normal vector to ψ pointing down is $(\sqrt{\alpha}\, g_1'(U)V, V)$, so on ψ we have

$$\begin{aligned}(V, -cV - g_1(U)) \cdot (\sqrt{\alpha}\, g_1'(U)V, V) &= \alpha^{1/2} g_1'(U)V^2 - (\alpha^{1/2} + \alpha^{-1/2})V^2 - \alpha^{1/2} g_1(U)\alpha^{-1/2} V \\ &= \alpha^{1/2}(g_1'(U) - 1)V^2,\end{aligned}$$

which is non-negative due to (1.8) and $U(s) \le 1$ for $s \ge s_0$. It follows that $h''(v) \ge 0$ on $[0, e^{-\sqrt{\alpha}s_0}]$ and so h satisfies (1.16) and (3.1). The proof is finished.

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