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GENERALIZED HYERS–ULAM STABILITY OF AN AQCQ-FUNCTIONAL EQUATION IN NON-ARCHIMEDEAN BANACH SPACES

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ABSTRACT. In this paper, we prove the generalized Hyers–Ulam stability of the following additive-quadratic-cubic-quartic functional equation

$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(2y)+f(-2y)-4f(y)-4f(-y)$

in non-Archimedean Banach spaces.

1. INTRODUCTION AND PRELIMINARY

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \le |r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n|\le 1$ for all $n\in\mathbb{N}$. A trivial example of

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a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. [18] *Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it satisfies the following conditions:*

(*i*) $\|x\| = 0$ *if and only if* $x = 0$;

(*ii*) $\|rx\| = |r|\|x\| \quad (r \in K, x \in X)$;

(*iii*) *the strong triangle inequality* $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ *holds for all* $x, y \in X$.

Then $(X, \|\cdot\|)$ is called a non-Archimedean normed space.

Definition 1.2. (*i*) *Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called Cauchy if for a given $\varepsilon > 0$ there is a positive integer N such that*

$$\|x_n - x_m\| \leq \varepsilon$$

for all $n, m \geq N$.

(*ii*) *Let $\{x_n\}$ be a sequence in a non-Archimedean normed space X. Then the sequence $\{x_n\}$ is called convergent if for a given $\varepsilon > 0$ there are a positive integer N and an $x \in X$ such that*

$$\|x_n - x\| \leq \varepsilon$$

for all $n \geq N$. Then we call $x \in X$ a limit of the sequence $\{x_n\}$, and denote by $\lim_{n\to\infty} x_n = x$.

(*iii*) *If every Cauchy sequence in X converges, then the non-Archimedean normed space X is called a non-Archimedean Banach space.*

The stability problem of functional equations originated from a question of Ulam [37] concerning the stability of group homomorphisms. Hyers [10] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [1] for additive mappings and by Th.M. Rassias [27] for linear mappings by considering an unbounded Cauchy difference. The paper of Th.M. Rassias [27] has provided a lot of influence in the development of what we call the *generalized Hyers–Ulam stability* or the *Hyers–Ulam–Rassias stability* of functional equations. A generalization of the Th.M. Rassias theorem was obtained by Găvruta [9] by replacing the unbounded Cauchy difference by a general control function in the spirit of Th.M. Rassias' approach.

The functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y)$$

is called a *quadratic functional equation*. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. A generalized Hyers–Ulam stability problem for the quadratic functional equation was proved by Skof [36] for mappings $f : X \to Y$, where X is a normed space and Y is a Banach space. Cholewa [2] noticed that the theorem of Skof is still true if the relevant domain X is replaced by an Abelian group. Czerwik [3] proved the generalized Hyers–Ulam stability of the quadratic functional equation.

The stability problems of several functional equations have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [4], [8], [11], [13], [14], [16], [20]–[35]).

In [12], Jun and Kim considered the following cubic functional equation

$$f(2x+y)+f(2x-y)=2f(x+y)+2f(x-y)+12f(x),$$

which is called a *cubic functional equation* and every solution of the cubic functional equation is said to be a *cubic mapping*.

In [15], Lee et al. considered the following quartic functional equation

$$f(2x+y)+f(2x-y)=4f(x+y)+4f(x-y)+24f(x)-6f(y),$$

which is called a *quartic functional equation* and every solution of the quartic functional equation is said to be a *quartic mapping*.

M. S. Moslehian and Th. M. Rassias [17] proved the Hyers–Ulam–Rassias stability of the Cauchy functional equation and the quadratic functional equation in non-Archimedean spaces.

Recently, M. Eshaghi Gordji and M. Bavand Savadkouhi [6] proved the generalized Hyers–Ulam stability of cubic and quartic functional equations in non-Archimedean spaces.

In this paper, we prove the generalized Hyers–Ulam stability of the additive-quadratic-cubic-quartic functional equation (0.1) in non-Archimedean Banach spaces.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|16|=|4|^2=|2|^4\neq 1$ and $|8|=|2|^3$.

2. Generalized Hyers–Ulam stability of the functional equation (0.1)

One can easily show that an odd mapping $f:X\to Y$ satisfies (0.1) if and only if the odd mapping mapping $f:X\to Y$ is an additive-cubic mapping, i.e.,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x).$$

It was shown in Lemma 2.2 of [7] that $g(x):=f(2x)-2f(x)$ and $h(x):=f(2x)-8f(x)$ are cubic and additive, respectively, and that $f(x)=\frac{1}{6}g(x)-\frac{1}{6}h(x)$.

One can easily show that an even mapping $f:X\to Y$ satisfies (0.1) if and only if the even mapping $f:X\to Y$ is a quadratic-quartic mapping, i.e.,

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+2f(2y)-8f(y).$$

It was shown in Lemma 2.1 of [5] that $g(x):=f(2x)-4f(x)$ and $h(x):=f(2x)-16f(x)$ are quartic and quadratic, respectively, and that $f(x)=\frac{1}{12}g(x)-\frac{1}{12}h(x)$.

For a given mapping $f:X\to Y$, we define

$$\begin{aligned}Df(x,y):=\;&f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+6f(x)\\&-f(2y)-f(-2y)+4f(y)+4f(-y)\end{aligned}$$

for all $x,y\in X$.

We prove the generalized Hyers–Ulam stability of the functional equation $Df(x,y)=0$ in non-Archimedean Banach spaces: an odd case.

Theorem 2.1. *Let θ and p be positive real numbers with $p < 3$. Let $f : X \to Y$ be an odd mapping satisfying*

$$\|Df(x,y)\| \le \theta(\|x\|^p + \|y\|^p) \tag{2.1}$$

for all $x, y \in X$. Then there exists a unique cubic mapping $C : X \to Y$ such that

$$\|f(2x) - 2f(x) - C(x)\| \le \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|2|^p}\|x\|^p \tag{2.2}$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.1), we get

$$\|f(3y) - 4f(2y) + 5f(y)\| \le 2\theta\|y\|^p \tag{2.3}$$

for all $y \in X$.

Replacing x by $2y$ in (2.1), we get

$$\|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\| \le (|2|^p + 1)\theta\|y\|^p \tag{2.4}$$

for all $y \in X$.

By (2.3) and (2.4),

$$\begin{aligned} &\|f(4y) - 10f(2y) + 16f(y)\| \\ &\quad \le \max\{\|4(f(3y) - 4f(2y) + 5f(y))\|, \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\|\} \\ &\quad \le \max\{|4| \cdot \|f(3y) - 4f(2y) + 5f(y)\|, \|f(4y) - 4f(3y) + 6f(2y) - 4f(y)\|\} \\ &\quad \le \max\{2 \cdot |4|, |2|^p + 1\}\theta\|y\|^p \end{aligned} \tag{2.5}$$

for all $y \in X$. Letting $y := \frac{x}{2}$ and $g(x) := f(2x) - 2f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 8g\left(\frac{x}{2}\right)\right\| \le \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|2|^p}\|x\|^p \tag{2.6}$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|8^l g\left(\frac{x}{2^l}\right) - 8^m g\left(\frac{x}{2^m}\right)\right\| \\ &\quad \le \max\left\{\left\|8^l g\left(\frac{x}{2^l}\right) - 8^{l+1} g\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|8^{m-1} g\left(\frac{x}{2^{m-1}}\right) - 8^m g\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\quad \le \max\left\{|8|^l \left\|g\left(\frac{x}{2^l}\right) - 8g\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |8|^{m-1}\left\|g\left(\frac{x}{2^{m-1}}\right) - 8g\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\quad \le \max\{2 \cdot |4|, |2|^p + 1\} \cdot \max\left\{\frac{|8|^l}{|2|^{pl+1}}, \cdots, \frac{|8|^{m-1}}{|2|^{p(m-1)+1}}\right\}\theta\|x\|^p \\ &\quad = \max\{2 \cdot |4|, |2|^p + 1\} \cdot |2|^{(3-p)l-1}\theta\|x\|^p \end{aligned} \tag{2.7}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (2.7) that the sequence $\{8^k g(\frac{x}{2^k})\}$ is Cauchy for all $x \in X$. Since Y is a non-Archimedean Banach space, the sequence $\{8^k g(\frac{x}{2^k})\}$ converges. So one can define the mapping $C : X \to Y$ by

$$C(x) := \lim_{k\to\infty} 8^k g\left(\frac{x}{2^k}\right)$$

for all $x \in X$.

By (2.1),

$$\begin{aligned}\|DC(x,y)\| &= \lim_{k\to\infty}\left\|8^k Dg\left(\frac{x}{2^k},\frac{y}{2^k}\right)\right\| \\ &\le \max\left\{\frac{|2|^p\cdot|8|^k}{|2|^{pk}}\theta(\|x\|^p+\|y\|^p), \frac{|2|\cdot|8|^k}{|2|^{pk}}\theta(\|x\|^p+\|y\|^p)\right\} \\ &= \lim_{k\to\infty}\max\{|2|^p,|2|\}|2|^{(3-p)k}\theta(\|x\|^p+\|y\|^p)=0\end{aligned}$$

for all $x,y\in X$. So $DC(x,y)=0$. Since $g:X\to Y$ is odd, $C:X\to Y$ is odd. So the mapping $C:X\to Y$ is cubic. Moreover, letting $l=0$ and passing the limit $m\to\infty$ in (2.7), we get (2.2). So there exists a cubic mapping $C:X\to Y$ satisfying (2.2).

Now, let $C':X\to Y$ be another cubic mapping satisfying (2.2). Then we have

$$\begin{aligned}\|C(x)-C'(x)\| &= \left\|8^qC\left(\frac{x}{2^q}\right)-8^qC'\left(\frac{x}{2^q}\right)\right\| \\ &\le \max\left\{\left\|8^qC\left(\frac{x}{2^q}\right)-8^qg\left(\frac{x}{2^q}\right)\right\|, \left\|8^qC'\left(\frac{x}{2^q}\right)-8^qg\left(\frac{x}{2^q}\right)\right\|\right\} \\ &\le \max\{2\cdot|4|,|2|^p+1\}\frac{|2|^{3q}}{|2|^{pq+1}}\theta\|x\|^p,\end{aligned}$$

which tends to zero as $q\to\infty$ for all $x\in X$. So we can conclude that $C(x)=C'(x)$ for all $x\in X$. This proves the uniqueness of C. □

Theorem 2.2. *Let θ and p be positive real numbers with $p>3$. Let $f:X\to Y$ be an odd mapping satisfying* (2.1). *Then there exists a unique cubic mapping $C:X\to Y$ such that*

$$\|f(2x)-2f(x)-C(x)\|\le\max\{2\cdot|4|,|2|^p+1\}\frac{\theta}{|8|}\|x\|^p$$

for all $x\in X$.

Proof. It follows from (2.6) that

$$\left\|g(x)-\frac{1}{8}g(2x)\right\|\le\max\{2\cdot|4|,|2|^p+1\}\frac{\theta}{|8|}\|x\|^p$$

for all $x\in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.3. *Let θ and p be positive real numbers with $p<1$. Let $f:X\to Y$ be an odd mapping satisfying* (2.1). *Then there exists a unique additive mapping $A:X\to Y$ such that*

$$\|f(2x)-8f(x)-A(x)\|\le\max\{2\cdot|4|,|2|^p+1\}\frac{\theta}{|2|^p}\|x\|^p$$

for all $x\in X$.

Proof. Letting $y:=\frac{x}{2}$ and $g(x):=f(2x)-8f(x)$ in (2.5), we get

$$\left\|g(x)-2g\left(\frac{x}{2}\right)\right\|\le\max\{2\cdot|4|,|2|^p+1\}\frac{\theta}{|2|^p}\|x\|^p \tag{2.8}$$

for all $x\in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.4. *Let θ and p be positive real numbers with $p > 1$. Let $f : X \to Y$ be an odd mapping satisfying* (2.1)*. Then there exists a unique additive mapping $A : X \to Y$ such that*

$$\|f(2x) - 8f(x) - A(x)\| \leq \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|2|}\|x\|^p$$

for all $x \in X$.

Proof. It follows from (2.8) that

$$\left\|g(x) - \frac{1}{2}g\,(2x)\right\| \leq \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|2|}\|x\|^p$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Now we prove the generalized Hyers–Ulam stability of the functional equation $Df(x, y) = 0$ in non-Archimedean Banach spaces: an even case.

Theorem 2.5. *Let θ and p be positive real numbers with $p < 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and* (2.1)*. Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \leq \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|2|^p}\|x\|^p$$

for all $x \in X$.

Proof. Letting $x = y$ in (2.1), we get

$$\|f(3y) - 6f(2y) + 15f(y)\| \leq 2\theta\|y\|^p \tag{2.9}$$

for all $y \in X$.

Replacing x by $2y$ in (2.1), we get

$$\|f(4y) - 4f(3y) + 4f(2y) + 4f(y)\| \leq (|2|^p + 1)\theta\|y\|^p \tag{2.10}$$

for all $y \in X$.

By (2.9) and (2.10),

$$\begin{aligned} &\|f(4x) - 20f(2x) + 64f(x)\| \qquad (2.11)\\ &\quad\leq \max\{\|4(f(3x) - 6f(2x) + 15f(x))\|, \|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\|\}\\ &\quad\leq \max\{|4|\|f(3x) - 6f(2x) + 15f(x)\|, \|f(4x) - 4f(3x) + 4f(2x) + 4f(x)\|\}\\ &\quad\leq \max\{2 \cdot |4|, |2|^p + 1\}\theta\|y\|^p \end{aligned}$$

for all $x \in X$. Letting $g(x) := f(2x) - 4f(x)$ for all $x \in X$, we get

$$\left\|g(x) - 16g\left(\frac{x}{2}\right)\right\| \leq \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|2|^p}\|x\|^p \tag{2.12}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.6. *Let θ and p be positive real numbers with $p > 4$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then there exists a unique quartic mapping $Q : X \to Y$ such that*

$$\|f(2x) - 4f(x) - Q(x)\| \le \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|16|}\|x\|^p$$

for all $x \in X$.

Proof. It follows from (2.12) that

$$\left\|g(x) - \frac{1}{16}g\left(2x\right)\right\| \le \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|16|}\|x\|^p$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.7. *Let θ and p be positive real numbers with $p < 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \le \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|2|^p}\|x\|^p$$

for all $x \in X$.

Proof. Letting $g(x) := f(2x) - 16f(x)$ in (2.11), we get

$$\left\|g(x) - 4g\left(\frac{x}{2}\right)\right\| \le \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|2|^p}\|x\|^p \tag{2.13}$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Theorem 2.8. *Let θ and p be positive real numbers with $p > 2$. Let $f : X \to Y$ be an even mapping satisfying $f(0) = 0$ and (2.1). Then there exists a unique quadratic mapping $T : X \to Y$ such that*

$$\|f(2x) - 16f(x) - T(x)\| \le \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|4|}\|x\|^p$$

for all $x \in X$.

Proof. It follows from (2.13) that

$$\left\|g(x) - \frac{1}{4}g\left(2x\right)\right\| \le \max\{2 \cdot |4|, |2|^p + 1\}\frac{\theta}{|4|}\|x\|^p$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.1. □

Let $f_o(x) := \frac{f(x)-f(-x)}{2}$ and $f_e(x) := \frac{f(x)+f(-x)}{2}$. Then f_o is odd and f_e is even. f_o, f_e satisfy the functional equation (0.1). Let $g_o(x) := f_o(2x) - 2f_o(x)$ and $h_o(x) := f_o(2x) - 8f_o(x)$.

Then $f_o(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x)$. Let $g_e(x) := f_e(2x) - 4f_e(x)$ and $h_e(x) := f_e(2x) - 16f_e(x)$. Then $f_e(x) = \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x)$. Thus

$$f(x) = \frac{1}{6}g_o(x) - \frac{1}{6}h_o(x) + \frac{1}{12}g_e(x) - \frac{1}{12}h_e(x).$$

Theorem 2.9. *Let θ and p be positive real numbers with $p < 1$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.1). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\ &\le \max\{2 \cdot |4|, |2|^p + 1\} \cdot \max\left\{\frac{1}{|6|}, \frac{1}{|12|}\right\} \frac{\theta}{|2|^p}\|x\|^p \\ &= \max\{2 \cdot |4|, |2|^p + 1\} \cdot \frac{\theta}{|12| \cdot |2|^p}\|x\|^p\end{aligned}$$

for all $x \in X$.

Theorem 2.10. *Let θ and p be positive real numbers with $p > 4$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.1). Then there exist an additive mapping $A : X \to Y$, a quadratic mapping $T : X \to Y$, a cubic mapping $C : X \to Y$ and a quartic mapping $Q : X \to Y$ such that*

$$\begin{aligned}\left\| f(x) - \frac{1}{6}A(x) - \frac{1}{12}T(x) - \frac{1}{6}C(x) - \frac{1}{12}Q(x) \right\| \\ &\le \max\{2 \cdot |4|, |2|^p + 1\} \cdot \max\left\{\frac{1}{|6| \cdot |8|}, \frac{1}{|12| \cdot |16|}\right\} \theta\|x\|^p \\ &= \max\{2 \cdot |4|, |2|^p + 1\} \cdot \frac{\theta}{|192|}\|x\|^p\end{aligned}$$

for all $x \in X$.

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