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Homology of Hom Complexes

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Abstract. The hom complex $\mathrm{Hom}(G, K)$ is the order complex of the poset composed of the graph multihomomorphisms from G to K. We use homology to provide conditions under which the hom complex is not contractible and derive a lower bound on the rank of its homology groups.

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1 Introduction

In 1978, László Lovász proved the Kneser Conjecture [8]. His proof involves associating to a graph G its neighborhood complex $\mathcal{N}(G)$ and using its topological properties to locate obstructions to graph colorings. This proof illustrates the fruitful interaction between graph theory, combinatorics and topology. The hom complex $\mathrm{Hom}(G, K)$ is another graph complex, introduced by Lovász to study chromatic numbers of graphs. For any graph G there is a homotopy equivalence $\mathcal{N}(G) \simeq \mathrm{Hom}(K_2, G)$, so this suggests studying the topology of $\mathrm{Hom}(G, K)$ in general to locate obstructions to graph homomorphisms.

Not much is known about $\mathrm{Hom}(G, K)$ in general. The main cases studied so far are $G = K_m$, the complete graph with m vertices, and $G = C_m$, the cycle with m vertices. Babson and Kozlov prove in [2] that $\mathrm{Hom}(K_m, K_n)$ is homotopy equivalent to a wedge of spheres while Čukić and Kozlov proved in [6] that the components of $\mathrm{Hom}(C_m, C_n)$ are points or homotopy equivalent to a circle. In either case, the hom complex is not contractible. In general, if a finite dimensional CW complex X is freely acted upon by $\mathbb{Z}/n\mathbb{Z}$, then the Euler characteristic $\chi(X)$ is divisible by n. In particular, if G has a $\mathbb{Z}/2\mathbb{Z}$ action that flips an edge and K has no loops, then $\mathbb{Z}/2\mathbb{Z}$ acts freely on $\mathrm{Hom}(G, K)$. It follows that $\mathrm{Hom}(G, K)$ is not contractible in this case since $\chi(\mathrm{Hom}(G, K))$ is even. The graphs K_m and C_m have this property, so with much less work we can at least conclude that these hom complexes have an interesting geometric structure. In this paper, we shall approach hom complexes by way of homology theory and give conditions under which $\mathrm{Hom}(G, K)$ has nonzero homology groups in sufficiently high dimensions.

This paper is organized as follows. In Section 2, we introduce the category of graphs and multihomomorphisms and record the relevant results about simplicial complexes and simplicial homology. In Section 3, we give two different quotient graph constructions, and in Section 4, we derive a lower bound on the ranks of homology on $\mathrm{Hom}(G, K)$ and provide examples of hom complexes in which this lower bound is non-trivial.

2 Preliminaries

2.1 Graphs and Multihomomorphisms

A graph G is a pair

$$G = (V(G), E(G))$$

where V is set of points and E is a set of edges connecting points in V. We allow our graphs to contain loops. A multihomomorphism $G \longrightarrow K$ is a function

$$\varphi : V(G) \longrightarrow \mathcal{P}(V(K)) \setminus \emptyset$$

with the property that if $(x, y) \in E(G)$, then $\varphi(x) \times \varphi(y) \in E(K)$. Here $\mathcal{P}$ denotes the power set. In other words, if $\varphi : G \longrightarrow K$ is a multihomomorphism, then if $(x, y) \in E(G)$, there is a complete bipartite graph between $\varphi(x)$ and $\varphi(y)$ in K. If $\varphi : G \longrightarrow K$ and

$\psi : K \longrightarrow L$ are graph multihomomorphisms, then there is a multihomomorphism $\psi \circ \varphi : G \longrightarrow L$ defined by

$$\psi \circ \varphi (x) = \bigcup_{y \in \varphi(x)} \psi (y) .$$

Moreover, each graph G has an identity map 1_G defined by $1_G (x) = \{x\}$ that satisfies $\varphi \circ 1_G = \varphi$ and $1_G \circ \psi = \psi$ whenever either of the compositions is defined. We therefore obtain the category $\mathbf{G}$ of graphs and multihomomorphisms.

An important subcategory of $\mathbf{G}$ is the category $\mathbf{G}_0$ of graphs and graph homomorphisms, where a graph homomorphism $G \longrightarrow K$ is a multihomomorphism in which $|\varphi (x)| = 1$ for all x. We denote by $\mathrm{Hom}_0 (G, K) \subseteq \mathrm{Hom} (G, K)$ the set of all graph homomorphisms from G to H. Observe that $\mathrm{Hom}_0 (G, K)$ is just the set of minimal elements in $\mathrm{Hom} (G, K)$.

Proposition 1. *If $\varphi : G \longrightarrow K$ is an isomorphism in $\mathbf{G}$, then $\varphi \in \mathrm{Hom}_0 (G, K)$.*

Proof. Let ψ be an inverse for φ. Let $v \in V(G)$ and let $\varphi (v) = \{u_1, \ldots, u_k\}$, say. Then $\{v\} = \psi \circ \varphi (x) = \bigcup_i \psi (u_i)$. Consequently, $\psi (u_i) = \{v\}$ for all i. Then

$$\begin{aligned} \{u_i\} &= \varphi \circ \psi (u_i) \\ &= \bigcup_{y \in \psi(u_i)} \varphi (y) \\ &= \varphi (v) . \end{aligned}$$

Therefore $|\varphi (v)| = 1$ and $\varphi \in \mathrm{Hom}_0 (G, K)$ as claimed. □

Finally, notice that if $\varphi : G \cong K$, then $|V(G)| = |V(K)|$ and $(x, y) \in E(G)$ if and only if $(\varphi (x) , \varphi (y)) \in E(K)$. Thus, isomorphic graphs are essentially the same.

2.2 Hom Complexes

A finite oriented abstract simplicial complex K is an ordered set

$$V(K) = \{v_0, \ldots, v_n\}$$

of vertices and a collection of subsets of $V(K)$ called simplices such that

1. Each $\{v_i\}$ is a simplex,
2. If F is a simplex and $E \subseteq F$, then E is a simplex.

The simplex G is called an n-simplex if $|G| = n + 1$ and the greatest n for which there is an n-simplex in K is called its dimension. An abstract simplicial complex is also called a vertex scheme because every abstract simplicial complex is the vertex scheme for a triangulated polyhedron, called its geometric realization. Observe that a graph with out loops is a simplicial complex of dimension at most 1.

If P is a partially ordered set, there is a simplicial complex ΔP, called the order complex of P, whose vertices are the elements of P and whose simplices are those subsets that are linearly ordered. The set $\mathrm{Hom}\,(G, H)$ of all graph multihomomorphisms is an ordered set where $\varphi \leq \psi$ if and only if always

$$\varphi\,(x) \subseteq \psi\,(x)\,.$$

The geometric realization of $\Delta\mathrm{Hom}\,(G, K)$ is called the hom complex and is our main object of study. We will use the notation $\mathrm{Hom}\,(G, K)$ for both the poset and the geometric realization of its order complex, since it is convenient to confuse these two objects.

Remark. The hom complex was originally defined slightly differently as follows. Let $H\,(G, K)$ be the CW complex with $H\,(G, K)_0 = \mathrm{Hom}_0\,(G, K)$ and with one cell for each graph multihomomorphism φ. The dimension and attach maps for $H\,(G, K)$ are determined by the requirement that the closure of the cell corresponding to φ consist of all cells corresponding to the multihomomorphisms ψ for which $\psi \leq \varphi$. Then $\mathrm{Hom}\,(G, K)$ as defined here is a barycentric subdivision of $H\,(G, K)$.

The topology of $\mathrm{Hom}\,(G, K)$ often reflects some combinatorial information about G and K. The following results which can be found in [8, 3] illustrate this idea. Recall that a topological space X is k-connected if X is path connected and $\pi_i\,(X) = 0$ for all $i \leq k$ where π_i denotes the i^{th} homotopy group of X. The connectivity of a space, $\mathrm{conn}\,(X)$ is the greatest k for which X is k-connected. We denote by Γ, the chromatic number of a graph since we are using the usual notation χ for the Euler characteristic of a space.

Theorem 2 (Lovász [8])**.** *If K is a graph, then*

$$\Gamma\,(K) - \Gamma\,(K_2) \geq \mathrm{conn}\,(\mathrm{Hom}\,(K_2, K))$$

Theorem 3 (Babson, Kozlov [3])**.** *If K is a graph, then*

$$\Gamma\,(K) - \Gamma\,(C_{2r+1}) \geq \mathrm{conn}\,(\mathrm{Hom}\,(C_{2r+1}, K))\,.$$

Remark. Lovász originally proved Theorem 2 for the neighborhood complex $\mathcal{N}\,(G)$.

The connectivity of $\mathrm{Hom}\,(G, K)$ is therefore bounded above if $G = K_2$ or $G = C_{2r+1}$, and K is finite with no loops. Consequently, $\mathrm{Hom}\,(G, K)$ has interesting homotopy groups in sufficiently high dimensions. We are investigating when $\mathrm{Hom}\,(G, K)$ has interesting homology for general graphs G and K.

2.3 Homology

The following exposition of homology with complex coefficients is adapted from the algebraic topology books [1, 9]. A graded vector space V is a doubly infinite sequence of vector spaces (V_i), and a morphism of graded vector spaces $\alpha : V \longrightarrow W$ of degree m is a sequence of linear transformations $\alpha_i : V_i \longrightarrow W_{i+m}$. A chain complex (V, d) is a graded vector space and a morphism $d : V \longrightarrow V$ of degree -1 such that $d \circ d = 0$. The map d is called the differential of the complex. Visually, a chain complex is a sequence of the form

$$\cdots \xrightarrow{d^{i+2}} V_{i+1} \xrightarrow{d^{i+1}} V_i \xrightarrow{d^{i}} V_{i-1} \xrightarrow{d^{i-1}} \cdots$$

in which $d^i \circ d^{i+1} = 0$ for all i. Let $B_i(V) = \text{im } d^{i+1}$ and $Z_i(V) = \ker d^i$. Then elements of $B_i(V)$ and $Z_i(V)$ are called the i-dimensional boundaries and i-dimensional cycles, respectively. The requirement $d^i \circ d^{i+1} = 0$ is equivalent to the condition $B_i(V) \subseteq Z_i(V)$. We define the homology of the chain complex to be the graded vector space

$$H_*(V) = (H_i(V))$$

where $H_i(V)$ is the quotient space $Z_i(V)/B_i(V)$, called the i^{th} homology group of the complex. Our next objective is to define the homology of a simplicial complex and use it study the hom complex, $\text{Hom}(G, K)$.

Let X be a finite oriented simplicial complex. To form a chain complex, let $S_*(X) = (S_i(X))$ be the graded complex vector space where $S_i(X)$ is the free complex vector space generated by the set of all i-simplices in X for $0 \leq i \leq \dim X$, and $S_i(X) = 0$ for all $i > \dim X$ and all $i < 0$. If F is a simplex in X, then F inherits an order from the vertices of X and F becomes an oriented simplex. If F is an i-simplex, then F contains $i+1$ oriented i-simplices, obtained by removing a vertex from the simplex. Explicitly, if we let

$$F = [v_0 \ldots v_i]$$

denote an oriented i-simplex, then we obtain $i+1$ oriented $i-1$ simplices of the form

$$d^i_j F = [v_0 \ldots \hat{v}_j \ldots v_i].$$

We thus have j boundary operators $d^i_j : S_i(X) \longrightarrow S_{i-1}(X)$ defined by

$$d^i_j \left(\sum c_k F_k \right) = \sum c_k d^i_j F_k$$

and we define the differential $d : S_*(X) \longrightarrow S_*(X)$ to be the alternating sum

$$d^i = \sum_{j=0}^{i} (-1)^j \, d^i_j.$$

Then $(S_*(X), d)$ is a chain complex; in fact, for an i-simplex F, we have

$$d^{i-1} d^i F = \sum_{j=0}^{i} \sum_{k=0}^{i-1} (-1)^{j+k} \, d^{i-1}_k d^i_j F \qquad (1)$$

and for $k < j$ we have $d^{i-1}_k d^i_j = d^{i-1}_{j-1} d^i_k$. Therefore, the (k, j) and $(j-1, k)$ terms in (3.1) cancel. We define the homology of the simplicial complex to be the homology of this chain complex,

$$H_*(X) = H_*(S_*(X)).$$

If (V, d) and (W, d') are chain complexes, then a chain map is a morphism $\alpha : V \longrightarrow W$ of degree 0 for which

$$\alpha_{i-1} \circ d^i = d'^i \circ \alpha_i$$

for all i. Consequently, if $\alpha : V \longrightarrow W$ is a chain map, then $\alpha_i (B_i (V)) \subseteq B_i (W)$ and $\alpha_i (Z_i (V)) \subseteq Z_i (W)$; therefore, α induces a morphism $H_* (\alpha) : H_* (V) \longrightarrow H_* (W)$ on homology (of degree 0). Thus $H_* (\alpha)$ is a sequence of linear transformations $H_i (\alpha) : H_i (V) \longrightarrow H_i (W)$. If $f : X \longrightarrow Y$ is a simplicial map, that is, a map that takes simplices to simplices, then f induces a chain map $S_* (f) : S_* (X) \longrightarrow S_* (Y)$ defined by

$$S_i (f) \left(\sum c_k \sigma_k \right) = \sum c_k f (\sigma_k) .$$

Therefore H_* is a functor from the category of simplicial complexes to graded vector spaces. It is well known that homology of simplicial complexes satisfies the axioms for a homology theory. The functors H_q from simplicial complexes to complex vector spaces therefore satisfy the following axioms (along with others that we will not be needing).

1. If X is contractible (has the homotopy type of a point), then $H_0 (X) \cong \mathbb{C}$ and $H_n (X) \cong 0$ for all $n \neq 0$.

2. If $X = \coprod X_i$, then $H_n (X) = \bigoplus_i H_n (X_i)$.

3. If $f, g : X \longrightarrow Y$ satisfy $f \simeq g$, then $H_* (f) = H_* (g)$.

Given a simplicial map $f : X \longrightarrow X$, each $H_i (f) : H_i (X) \longrightarrow H_i (X)$ is a linear transformation. The Lefshetz number of f is the number

$$\Lambda_f = \sum_{i=0}^{\dim X} (-1)^i \operatorname{tr} (H_i (f))$$

and the Euler characteristic of a simplicial complex is the number

$$\chi (X) = \sum_{i=0}^{n} (-1)^i \beta_i (X) .$$

where $\beta_i (X) = \dim H_i (X)$ is the i^{th} Betti number of X. Here we record a result from [7, Exercise 3, pg. 181] that we shall rely on throughout this paper.

Theorem 4 (Lefshetz Fixed Point Theorem)**.** *Let $f : X \longrightarrow X$ be a simplicial isomorphism and let* Fixed (f) *be the set of points fixed by f. Then*

$$\chi (\text{Fixed} (f)) = \Lambda_f .$$

By axiom (2) for homology, $H_* (\emptyset) = \{0\}$ and thus $\chi (\emptyset) = 0$. It follows that if $\Lambda_f \neq 0$, then f has a fixed point. Often times, $\text{Hom} (G, K) = \emptyset$; for example, if $\chi (H) < \chi (G)$, then a graph homomorphism $G \longrightarrow H$ composed with an inclusion map $H \longrightarrow K_{\chi(H)}$ provides a coloring of G with $\chi (H)$ colors, and this is impossible. Our main concern is when $\text{Hom} (G, K)$ is non-empty.

Lemma 5. *If X is a non-empty simplicial complex and X is path-connected, then* $\mathrm{tr}H_0(f) = 1$.

Proof. Let $v_0 \in X$. Since X is path-connected, there is a path from v_0 to $f(v_0)$ that we can view as a 1 chain $\sum c_k F_k \in S_1(X)$ such that $f(v_0) - v_0 = d^1(\sum c_k F_k)$. Therefore,

$$\begin{aligned} [f(v_0)] - [v_0] &= [f(v_0) - v_0] \\ &= \left[d^1\left(\sum c_k F_k\right)\right] \\ &= [0] \end{aligned}$$

since $d^1(\sum c_k F_k) \in B_1(X)$. Therefore $H_0(f) = 1_{H_0(X)}$ and $\mathrm{tr}H_0(f) = 1$. □

The next result from [4] due to Bjorner and Baclawski insures that the fixed points of an order preserving map $P \longrightarrow P$ correspond to the fixed points of the induced map on the realization of ΔP.

Theorem 6 (Bjorner and Baclawski)**.** *If P is an ordered set and $f : P \longrightarrow P$ is order preserving and fixed point free, then the induced map Δf on the order complex ΔP is fixed point free.*

Corollary 7. *If $f : \mathrm{Hom}(G, K) \longrightarrow \mathrm{Hom}(G, K)$ is an order preserving map that is fixed point free, then the realization of* $\mathrm{Hom}(G, K)$ *is not contractible.*

Proof. If $\mathrm{Hom}(G, K)$ is empty or disconnected, then $\mathrm{Hom}(G, K)$ is not contractible. Otherwise, $\Lambda_f = 0$ and by Lemma 4, we have $\mathrm{tr}H_0(f) = 1$. Therefore, $\mathrm{tr}H_n(f) \neq 0$ for some $n > 0$ and $H_n(\mathrm{Hom}(G, K)) \neq 0$. By the axioms for homology, we conclude that $\mathrm{Hom}(G, K)$ is not contractible. □

3 Quotient Graphs

In this section, we construct order preserving maps $\mathrm{Hom}(G, K) \longrightarrow \mathrm{Hom}(G, K)$ and characterize their fixed point sets. Let $\mathbf{C}$ be any category. Then the functor $\mathrm{Hom}(a, b)$ is a bifunctor $\mathbf{C} \times \mathbf{C}^{\mathrm{op}} \longrightarrow \mathbf{Set}$, covariant in b and contravariant in a. The map induced by an arrow $\varphi : b \longrightarrow b'$ is the function $\varphi_* : \mathrm{Hom}(a, b) \longrightarrow \mathrm{Hom}(a, b')$ defined by $\varphi_*(\psi) = \varphi \circ \psi$. Similarly, the map induced by an arrow $\varphi : a \longrightarrow a'$ is the function $\varphi^* : \mathrm{Hom}(a', b) \longrightarrow \mathrm{Hom}(a, b)$ defined by $\varphi^*(\psi) = \psi \circ \varphi$. In the case of $\mathbf{G}$, it is well known that the maps φ_* and φ^* are order preserving. The proof is included here for completeness. Let $\mathbf{P}$ denote the category of partially ordered sets and order preserving maps.

Theorem 8. *The hom complex* $\mathrm{Hom}(G, K)$ *is a bifunctor $\mathbf{G}^{op} \times \mathbf{G} \longrightarrow \mathbf{P}$, covariant in K and contravariant in G.*

Proof. We shall show that φ_* is order preserving. The proof that φ^* is order preserving is similar. Let G be a graph and suppose that $\varphi : K \longrightarrow K'$ is a graph multihomomorphism. Let $\psi, \psi' : G \longrightarrow K$ be multihomomorphisms and suppose that $\psi \leq \psi'$. Then

$$\begin{aligned} y \in \varphi_*(\psi)(x) &\implies y \in \varphi \circ \psi(x) \\ &\implies \text{there is a } z \in \psi(x) \text{ such that } y \in \varphi(z) \\ &\implies \text{there is a } z \in \psi'(x) \text{ such that } y \in \varphi(z) \\ &\implies y \in \varphi \circ \psi'(x). \end{aligned}$$

Thus $\varphi_*(\psi)(x) \subseteq \varphi_*(\psi')(x)$ for all $x \in V(G)$ and φ_* is order preserving, as required. □

In particular, an isomorphism $\sigma : G \longrightarrow G$ (which is just an ordinary graph isomorphism by Proposition 1) in **G** induces order isomorphisms $\sigma^* : \mathrm{Hom}(G, K) \longrightarrow \mathrm{Hom}(G, K)$ and $\sigma_* : \mathrm{Hom}(K, G) \longrightarrow \mathrm{Hom}(K, G)$.

Let G be a graph and $\sim$ an equivalence relation on $V(G)$. A quotient graph $G^\sim$ is defined by

$$V(G^\sim) = V(G) / \sim$$

and $([x], [y]) \in E(G')$ if and only if $(u, v) \in E(G)$ for *some* $u \in [x]$ and $v \in [y]$. We now arrive at our first main result.

Theorem 9. *Let A be the set of all $\varphi \in \mathrm{Hom}(G, K)$ for which $x \sim y$ implies that $\varphi(x) = \varphi(y)$. Then*

$$\mathrm{Hom}(G^\sim, K) \cong A.$$

Proof. The projection map $\pi : V(G) \longrightarrow V(G^\sim)$ is a graph homomorphism and induces an order preserving map

$$\pi^* : \mathrm{Hom}(G^\sim, K) \longrightarrow \mathrm{Hom}(G, K).$$

We claim that π^* is an isomorphism onto A. Define $\rho : A \longrightarrow \mathrm{Hom}(G^\sim, K)$ by letting $\rho(\varphi)$ be the unique map $\bar{\varphi} : V(G^\sim) \longrightarrow \mathcal{P}(V(K))$ for which $\bar{\varphi} \circ \pi = \varphi$. Then $\bar{\varphi}$ is a multihomomorphism since π is surjective and π and φ are multihomomorphisms. To see that ρ is order preserving, let $\varphi \leq \psi$ and let $[x] \in V(G^\sim)$. Then if $u \in \bar{\varphi}([x])$, we have $u \in (\bar{\varphi} \circ \pi)(x) = \varphi(x)$. Therefore, $u \in \psi(x)$ and $u \in \bar{\psi}([x])$. Clearly π^* factors as $\pi^* = i_A \circ q$ where $i_A : A \longrightarrow \mathrm{Hom}(G, K)$ is the inclusion map and $q : \mathrm{Hom}(G^\sim, K) \longrightarrow A$ is the map $\varphi \overset{q}{\mapsto} \varphi \circ \pi$. Thus q is an order preserving map, and $q = \rho^{-1}$. □

Corollary 10. *Let $\sigma : G \longrightarrow G$ be an isomorphism. Define $x \sim y$ if and only if $x = \sigma^n(y)$ for some n. Then*

$$\mathrm{Fixed}(\sigma^*) \cong \mathrm{Hom}(G^\sim, K).$$

Proof. We observe that $\sigma^*(\varphi) = \varphi$ if and only if $\varphi(\sigma^n(x)) = \varphi(x)$ for all x and for all n, that is, if and only if $x \sim y$ implies $\varphi(x) = \varphi(y)$. The corollary then follows from Theorem 9. □

Corollary 11. *Under the hypotheses of Corollary 9, if $\sigma^* (\varphi) = \varphi$ for some $\varphi \in \mathrm{Hom}(G, K)$ and K has no loops, then for each $v \in G$, the equivalence class $[v]$ is an independent set in G.*

Proof. Let $v \in V(G)$ and suppose that $u \in [v]$. If $(u, v) \in E(G)$, then $([v], [v]) \in E(G^{\sim})$. Consequently, $\mathrm{Hom}(G^{\sim}, K) = \emptyset$ since K is assumed to not have loops. But this is impossible by Corollary 9. □

In particular, if there is an automorphism σ of G that maps a vertex to one of its neighbors, then σ^* is fixed point free. Notice that the graphs K_2 and C_{2r+1} from Theorem 2 have such automorphisms and we have found another proof that for any finite loop free graph K the hom complexes, $\mathrm{Hom}(K_2, K)$ and $\mathrm{Hom}(C_{2r+1}, K)$ are not contractible.

Now let $\sim$ be an equivalence relation on $V(K)$ and let $K_\sim$ be the graph with vertices $V(K_\sim) = V(K)/\sim$ and $([x], [y]) \in E(G')$ if and only if $(u, v) \in E(G)$ for *all* $u \in [x]$ and $v \in [u]$.

Theorem 12. *Let A be the set of all those $\varphi \in \mathrm{Hom}(G, K)$ for which each $\varphi(x)$ is a union of equivalence classes in K. Then*

$$A \cong \mathrm{Hom}(G, K_\sim).$$

Proof. Let $\pi^\vee : K_\sim \longrightarrow K$ be the multihomomorphism

$$[v] \xrightarrow{\pi^\vee} [v]$$

where we view $[v]$ as a vertex in $K_\sim$ on the left and as a set of vertices in K on the right. Then $\pi^\vee$ induces an order preserving map $\pi^\vee_* : \mathrm{Hom}(G, K_\sim) \longrightarrow \mathrm{Hom}(G, K)$. We claim that $\pi^\vee_*$ is an isomorphism onto A. Define $\rho : A \longrightarrow \mathrm{Hom}(G, K_\sim)$ by letting $\rho(\varphi)$ be the multihomomorphism

$$\rho(\varphi)(x) = \{[u_1], \ldots, [u_n]\}$$

where $\varphi(x) = \bigcup_{i=1}^n [u_i]$. Since there is a complete bipartite graph in K between $[u_i]$ and $[u_j]$ whenever $(u_i, u_j) \in E(K_\sim)$, it follows that $\rho(\varphi)$ is a multihomomorphism. To see that ρ is order preserving, let $\varphi \le \psi$ in A. Let $x \in V(G)$ and suppose that $[u] \in \rho(\varphi)(x)$. Then $[u] \subseteq \varphi(x)$. Thus $[u] \subseteq \psi(x)$ and $[u] \in \rho(\psi)(x)$. Therefore $\rho(\varphi) \subseteq \rho(\psi)$. Finally $\pi^\vee_*$ factors as $\pi^\vee_* = i_A \circ q$ for an order preserving $q : \mathrm{Hom}(G, K_\sim) \longrightarrow A$, and one checks that $q = \rho^{-1}$. □

Corollary 13. *Let $\sigma : K \longrightarrow K$ be an isomorphism and define $x \sim y$ if and only if $x = \sigma^n(y)$ for some n. Then*

$$\mathrm{Fixed}(\sigma_*) \cong \mathrm{Hom}(G, K_\sim).$$

Proof. We show that $\sigma_G(\varphi) = \varphi$ if and only if each $\varphi(x)$ is a union of equivalence classes. Observe that for each n and for each x, we have $\sigma^n(\varphi(x)) = \varphi(x)$. Therefore the group $\langle \sigma \rangle$ acts on $\varphi(x)$ and $\varphi(x)$ is a union of equivalence classes in K. The result follows from Theorem 12. □

Corollary 14. *Let σ be an automorphism of K and let k be the number of orbits of σ. If $\Gamma(G) > \Gamma(K_\sim)$, then $\sigma_G : \mathrm{Hom}(G,K) \longrightarrow \mathrm{Hom}(G,K)$ does not have a fixed point. In particular, if G is not k colorable, then σ_* does not have a fixed point, and therefore $\mathrm{Hom}(G,K)$ is not contractible.*

Proof. Suppose that σ_G has a fixed point, φ. Recall that a k coloring of G corresponds to a graph homomorphism $G \longrightarrow K_k$. For each $x \in V(G)$ there is an equivalence class $O_x \subseteq \varphi(x)$. By construction, the map $x \overset{\psi}{\mapsto} O_x$ is a homomorphism $G \to K_\sim$. Choose a homomorphism $\rho : K_\sim \longrightarrow K_{\Gamma(K_\sim)}$. Then $\rho \circ \psi$ is a homomorphism $G \longrightarrow K_{\Gamma(K_\sim)}$. Thus $\Gamma(G) \le \Gamma(K_\sim)$. □

Remark. The main utility of the quotient graphs is to show that the fixed points of the induced maps σ^* and σ_* are actually new hom complexes. Since the constructions $G^\sim$ and $K_\sim$ are sort of dual to one another, it is not surprising that the fixed points of σ^* correspond to multihomomorphisms *out* of $G^\sim$, while the fixed points of σ_* correspond to multihomomorphisms *in* to $K_\sim$.

4 Lower Bound on the Dimensions of Homology Groups

In this section, we derive a lower bound on the dimensions of the homology groups of $\mathrm{Hom}(G,K)$ and provide the promised examples of $\mathrm{Hom}(G,K)$ where this bound is non-trivial.

Lemma 15. *Let $T : V \longrightarrow V$ be a linear transformation where V is a complex vector space. If λ is an eigenvalue of T, then for all $k > 0$, λ^k is an eigenvalue of T^k.*

Proof. We have $T(v) = \lambda v$ for some vector v. Inductively, if $T^k(v) = \lambda^k v$, then $T^{k+1}(v) = T(\lambda^k v) = \lambda^k T(v) = \lambda^{k+1} v$. □

Theorem 16. *Let $\sigma : G \longrightarrow G$ be an isomorphism and $G^\sim$ be the quotient graph induced by σ. Then*

$$|\chi(\mathrm{Hom}(G^\sim, K))| \le \sum_{i=0}^{\dim(\mathrm{Hom}(G,K))} \beta_i(\mathrm{Hom}(G,K)).$$

Proof. Since G is finite, we have $\sigma^k = 1_G$ for some k and therefore, $(\sigma^*)^k = 1_{\mathrm{Hom}(G,K)}$. Thus, for $0 \le i \le \dim(\mathrm{Hom}(G,K))$, we have $H_i(\sigma^*)^k = 1_{H_i(\mathrm{Hom}(G,K))}$. By Lemma 11, if λ is an eigenvalue of $H_i(\sigma^*)$, then λ^k is an eigenvalue of $1_{H_i(\mathrm{Hom}(G,K))}$. Thus, $\lambda^k = 1$ and $|\lambda| = 1$. Recall that the trace of a linear transformation is the sum of its eigenvalues. Then

$$|\mathrm{tr}(H_i(\sigma^*))| = \left|\sum \lambda_j\right|$$

where the λ_j are the eigenvalues of $H_i(\sigma^*)$. Consequently

$$\begin{aligned}|\mathrm{tr}\left(H_i\left(\sigma^*\right)\right)| &\leq \sum |\lambda_j| \\ &= \beta_i\left(\mathrm{Hom}\left(G,K\right)\right).\end{aligned}$$

By the Lefshetz Fixed Point Theorem and Theorem 10, we have

$$\begin{aligned}|\chi\left(\mathrm{Hom}\left(G^{\sim},K\right)\right)| &= |\chi\left(\mathrm{Fixed}\left(\sigma^*\right)\right)| \\ &= |\Lambda_{\sigma^*}| \\ &\leq \sum |\mathrm{tr}\left(H_i\left(\sigma^*\right)\right)| \\ &\leq \sum_{i=0}^{\dim(\mathrm{Hom}(G,K))} \beta_i\left(\mathrm{Hom}\left(G,K\right)\right). \quad \square\end{aligned}$$

The arguments in Theorem 16 only rely on the properties of homology. Similar reasoning therefore applies *mutatis mutandis* for the hom complex $\mathrm{Hom}\left(G, K_{\sim}\right)$ where $K_{\sim}$ is the quotient graph induced by an automorphism.

Theorem 16 allows us to produce examples of hom complexes with non-zero homology. For example, if $\mathrm{Hom}\left(G,K\right)$ is connected, has $|\chi\left(\mathrm{Hom}\left(G,K\right)\right)| \geq 2$ and G' is a graph with $G'^{\sim} \cong G$, then $\mathrm{Hom}\left(G',K\right)$ has non zero homology in some dimension other than dimension 0. To illustrate this idea we record the following result from [6].

Theorem 17. *Let C_r denote a cycle with r vertices. Then*

$$\chi\left(\mathrm{Hom}\left(C_m,C_n\right)\right) = \begin{cases} 2 & \text{if } n=4 \text{ and } m \text{ is even,} \\ 2n & \text{if } n \mid m \text{ and } n \neq 4, \\ 0 & \text{otherwise.}\end{cases} \tag{2}$$

If $G_1,\ldots,G_n$ are based graphs, let

$$\bigvee G_i$$

denote the wedge of the graphs G_i. In particular, for each based graph G we have the wedge $\bigvee G$, for which an automorphism is specified by cycling through the copies of G. In this case, we see that

$$\left(\bigvee G\right)' \cong G.$$

Then if either of the first two conditions in (5.1) is satisfied, we conclude that $\mathrm{Hom}\left(\bigvee C_m, C_n\right)$ has interesting homology in some dimension.

For an example where $\mathrm{Hom}\left(G',K\right)$ is connected and not contractible, we record several known results. The following is from [2].

Theorem 18. *The hom complex $\mathrm{Hom}\left(K_m,K_n\right)$ has the homotopy type of a wedge of $f\left(m,n\right)$ spheres of dimension $n-m$, where the number $f\left(m,n\right)$ is given by the formula*

$$f\left(m,n\right) = \sum_{k=1}^{m-1} \left(-1\right)^{m+k+1} \binom{m}{k+1} k^n.$$

The next result is taken from [5].

Theorem 19. *Let* $\max\text{val}(G)$ *denote the highest degree of a vertex in* G*. Then*

$$\text{conn}(\text{Hom}(G, K_n)) \geq (n - \max\text{val}(G)) - 2.$$

Next, observe that $\max\text{val}\left(\bigvee_{i=1}^{h} K_m\right) = h(m-1)$. Therefore if $n \geq hm + 2$, then $\text{Hom}\left(\bigvee_{i=1}^{h} K_m, K_n\right)$ is path connected and $\pi_i\left(\text{Hom}\left(\bigvee_{i=1}^{h} K_m, K_n\right)\right) = 0$ for all $i \leq n - (hm+2)$. By the Hurewicz theorem and axiom 1 for homology, we have $H_0\left(\text{Hom}\left(\bigvee_{i=1}^{h} K_m, K_n\right)\right) \cong \mathbb{C}$ and $H_i\left(\text{Hom}\left(\bigvee_{i=1}^{h} K_m, K_n\right)\right) \cong 0$ for all $1 \leq i \leq n - (hm+2)$. Next, we calculate

$$\begin{aligned} |\chi(\text{Hom}(K_m, K_n))| &= \left|1 + (-1)^{n-m} f(m,n)\right| \\ &\geq |1 - f(m,n)| \end{aligned}$$

and observe that by Theorem 16, we have

$$\begin{aligned} |1 - f(m,n)| &\leq \sum_{i=0}^{\dim\left(\text{Hom}\left(\bigvee_{i=1}^{h} K_m, K_n\right)\right)} \beta_i\left(\text{Hom}\left(\bigvee_{i=1}^{h} K_m, K_n\right)\right) \\ &= 1 + \sum_{i=n-(hm+2)+1}^{\dim\left(\text{Hom}\left(\bigvee_{i=1}^{h} K_m, K_n\right)\right)} \beta_i\left(\text{Hom}\left(\bigvee_{i=1}^{h} K_m, K_n\right)\right). \end{aligned}$$

Thus, we can make $\text{Hom}\left(\bigvee_{i=1}^{h} K_m, K_n\right)$ path connected with non-zero homology in dimension as high we please by choosing n sufficiently large. Moreover, this calculation shows that the non-zero (reduced) homology groups appear only in dimensions at least $n-(hm+2)+1$.

5 Conclusion

The main result of this paper is Theorem 16 which relates the homology of a hom complex $\text{Hom}(G, K)$ to the Euler characteristic of another hom complex, obtained by replacing either G or K with a quotient graph. This result allows one to build new hom complexes with desired properties by recognizing symmetries of certain graphs, as illustrated in Section 4. These quotient graphs are also of independent interest as illustrated by Theorems 10 and 13. The fact that

$$\text{Fixed}(\sigma^*) \cong \text{Hom}(G^\sim, K)$$

and

$$\text{Fixed}(\sigma_*) \cong \text{Hom}(G, K_\sim)$$

suggests that the two quotient constructions $G^\sim$ are $K_\sim$ are sort of dual.

For future research, one could look for more examples of graphs for which these results apply. The original motivation for the neighborhood complex was to locate obstructions to graph homomorphisms. It would be interesting if these results could also be used to locate such obstructions and, in particular, obtain information about chromatic numbers.

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