Classifications of recurrence relations via subclasses of (H, m)–quasiseparable matrices

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Abstract. The results on characterization of orthogonal polynomials and Szegö polynomials via tridiagonal matrices and unitary Hessenberg matrices, resp., are classical. In a recent paper we observed that tridiagonal matrices and unitary Hessenberg matrices both belong to a wide class of $(H, 1)$–quasiseparable matrices and derived a complete characterization of the latter class via polynomials satisfying certain EGO–type recurrence relations. We also established a characterization of polynomials satisfying three–term recurrence relations via $(H, 1)$–well–free matrices and of polynomials satisfying the Szegö–type two-term recurrence relations via $(H, 1)$–semiseparable matrices.

In this paper we generalize all of these results from *scalar* (H,1) to the *block* (H,m) case. Specifically, we provide a complete characterization of (H, m)–quasiseparable matrices via polynomials satisfying *block* EGO–type two–term recurrence relations. Further, (H, m)–semiseparable matrices are completely characterized by the polynomials obeying *block* Szegö–type recurrence relations. Finally, we completely characterize polynomials satisfying m–term recurrence relations via a new class of matrices called (H, m)–well–free matrices.

1. Introduction.

1.1. Classical three–term and two–term recurrence relations and their generalizations

It is well–known that real–orthogonal polynomials $\{r_k(x)\}$ satisfy three–term recurrence relations of the form

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k \cdot r_{k-2}(x), \quad \alpha_k \neq 0, \gamma_k > 0. \tag{1.1}$$

It is also well–known that Szegö polynomials $\{\phi_k^{\#}(x)\}$, or polynomials orthogonal not on a real interval but orthogonal on the unit circle, satisfy slightly different three–term recurrence relations of the form

$$\phi_k^{\#}(x) = \left[\frac{1}{\mu_k} \cdot x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k}\right] \phi_{k-1}^{\#}(x) - \frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \cdot \phi_{k-2}^{\#}(x). \tag{1.2}$$

Noting that the essential difference between these two sets of recurrence relations is the presence or absence of the x dependence in the $(k-2)$–th polynomial, it is natural to consider the more general three–term recurrence relations of the form

$$r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x), \tag{1.3}$$

containing both (1.1) and (1.2) as special cases, and to classify the polynomials satisfying such three–term recurrence relations.

Also, in addition to the three–term recurrence relations (1.2), Szegö polynomials satisfy two–term recurrence relations of the form

$$\begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^{\#}(x) \end{bmatrix} \tag{1.4}$$

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for some auxiliary polynomials $\{\phi(x)\}$ (see, for instance, [GS58], [G48]). By relaxing these relations to the more general two–term recurrence relations

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k) r_{k-1}(x) \end{bmatrix}, \tag{1.5}$$

it is again of interest to classify the polynomials satisfying these two–term recurrence relations.

In [BEGO08], these questions were answered, and the desired classifications were given in terms of the classes of matrices $A = [a_{i,j}]_{i,j=1}^n$ related to the polynomials $\{r_k(x)\}$ via

$$r_k(x) = \frac{1}{a_{1,0}a_{2,1}\cdots a_{k+1,k}} \det(xI - A)_{(k\times k)}, \quad k = 0,\ldots,n. \tag{1.6}$$

Note that this relation involves the entries of the matrix A and two additional parameters $a_{1,0}$ and $a_{n+1,n}$ outside the range of parameters of A. In the context of this paper, these parameters not specified by the matrix A can be any nonzero arbitrary numbers[1]. These classifications generalized the well–known facts that real–orthogonal polynomials and Szegö polynomials were related to irreducible tridiagonal matrices and almost unitary Hessenberg matrices, respectively, via (1.6). These facts as well as the classifications of polynomials satisfying (1.3), (1.5), and a third set to be introduced later, respectively, are given in Table 1.

TABLE 1. Correspondence between recurrence relations satisfied by polynomials and related subclasses of quasiseparable matrices, from [BEGO08].

Recurrence relations	**Matrices**
real–orthogonal three–term (1.1)	irreducible tridiagonal matrices
Szegö two–term (1.4)/three–term (1.2)	almost unitary Hessenberg matrices
general three–term (1.3)	$(H,1)$–well–free (Definition 5.1)
Szegö–type two–term (1.5)	$(H,1)$–semiseparable (Definition 4.1)
EGO–type two–term (3.2)	$(H,1)$–quasiseparable (Definition 1.1)

Furthermore, the classes of matrices listed in Table 1 (and formally defined below) were shown in [BEGO08] to be related as is shown in Figure 1.

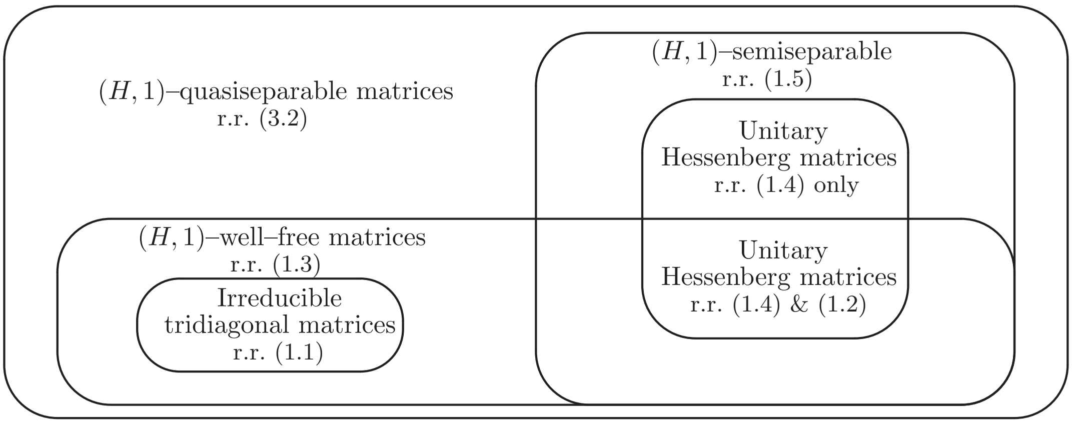


FIGURE 1. Relations between subclasses of $(H,1)$–quasiseparable matrices, from [BEGOT07].

While it is likely the reader is familiar with tridiagonal and unitary Hessenberg matrices, and perhaps quasiseparable and semiseparable matrices, the class of well–free matrices is newer and less well–known. We take a moment to give a brief description of this class (a more rigorous description is provided below in Section 5.1). A matrix is well–free provided it has **no** columns that consist of all zeros above (but not including) the main diagonal, unless that column of zeros lies to the left of a block of **all** zeros. That is, no columns of the form shown in Figure 2 appear in the matrix.

[1]More details on the meaning of these numbers will be provided in Section 2.1 below.

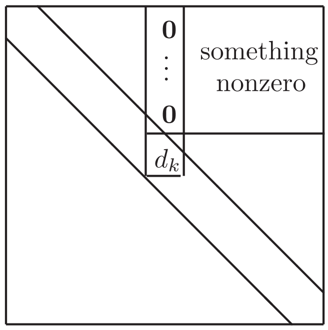


FIGURE 2. Illustration of a well.

As stated in Table 1, it was shown in [BEGO08] that the matrices related to polynomials satisfying recurrence relations of the form (1.3) are not just well–free, but $(H,1)$–well–free; i.e., they are well–free and also have a $(H,1)$–quasiseparable structure, which is defined next.

1.2. Main tool: quasiseparable structure

In this section we give a definition of the structure central to the results of this paper, and explain one of the results shown above in Table 1. We begin with the definition of (H,m)–quasiseparability next.

Definition 1.1 ((H,m)–quasiseparable and weakly (H,m)–quasiseparable matrices). *Let A be a strongly upper Hessenberg matrix (i.e. upper Hessenberg with nonzero subdiagonal elements: $a_{i,j}=0$ for $i>j+1$, and $a_{i+1,i}\neq 0$ for $i=1,\dots,n-1$). Then if over all symmetric partitions of the form*

$$A=\left[\begin{array}{c|c} * & A_{12} \\ \hline * & * \end{array}\right],$$

(i) $\max\operatorname{rank} A_{12}=m$, *then A is (H,m)–quasiseparable, and*
(ii) $\max\operatorname{rank} A_{12}\leqslant m$, *then A is* weakly *(H,m)–quasiseparable.*

For instance, the rank m blocks (respectively rank at most m blocks) of a 5×5 (H,m)–quasiseparable matrix (respectively weakly (H,m)–quasiseparable matrix) would be those shaded below:

$$\left[\begin{array}{c|cccc} \star & \star & \star & \star & \star \\ \hline \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \end{array}\right] \quad \left[\begin{array}{cc|ccc} \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ \hline 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \end{array}\right] \quad \left[\begin{array}{ccc|cc} \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ \hline 0 & 0 & \star & \star & \star \\ 0 & 0 & 0 & \star & \star \end{array}\right] \quad \left[\begin{array}{cccc|c} \star & \star & \star & \star & \star \\ \star & \star & \star & \star & \star \\ 0 & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star \\ \hline 0 & 0 & 0 & \star & \star \end{array}\right]$$

1.3. Motivation to extend beyond the $(H,1)$ case

In this paper, we extend the results of these classifications to include more general recurrence relations. Such generalizations are motivated by several examples for which the results of [BEGO08] are inapplicable as they are not order $(H,1)$; one of the simplest of such is presented next.

Consider the three–term recurrence relations (1.1), one could ask what classes of matrices are related to polynomials satisfying such recurrence relations if more than three terms are included. More specifically, consider recurrence relations of the form

$$x\cdot r_{k-1}(x)=-a_{k,k}r_k(x)-a_{k-1,k}r_{k-1}(x)-\cdots-a_{k-(l-1),k}\cdot r_{k-(l-1)}(x) \tag{1.7}$$

It will be shown that this class of so–called l–recurrent polynomials is related via (1.6) to $(1, l-2)$–banded matrices (i.e., one nonzero subdiagonal and $l-2$ nonzero superdiagonals) of the form

$$A = \begin{bmatrix} a_{0,1} & \cdots & a_{0,l-1} & 0 & \cdots & 0 \\ a_{1,1} & a_{1,2} & \cdots & a_{1,l} & \ddots & \vdots \\ 0 & a_{2,2} & & & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & & a_{n-(l-1),n} \\ \vdots & & \ddots & a_{n-2,n-2} & & \vdots \\ 0 & \cdots & \cdots & 0 & a_{n-1,n-1} & a_{n-1,n} \end{bmatrix}. \tag{1.8}$$

This equivalence cannot follow from the results of [BEGO08] as summarized in Table 1 because those results are limited to the simplest $(H, 1)$–quasiseparable case. As we shall see in a moment, the matrix A of (1.8) is $(H, l-2)$–quasiseparable, as defined next.

Considering the motivating example of the matrix A of (1.8), it is easy to see that the structure forces many zeros into the blocks A_{12} of Definition 1.1 (the shaded blocks above), and hence the ranks of these blocks can be small compared to their size. It can be seen that in the case of a $(1, m)$–banded matrix, the matrices A_{12} have rank at most m, and so are (H, m)–quasiseparable.

This example is only one simple example of the need to extend the results listed in Table 1 from the scalar $(H, 1)$–quasiseparable case to the block (H, m)–quasiseparable case.

1.4. Main results

The main results of this paper can be summarized next by Table 2 and Figure 3, analogues of Table 1 and Figure 1 above, for the most general case considered in this paper.

TABLE 2. Correspondence between polynomial systems and subclasses of (H, m)-quasiseparable matrices

	Recurrence relations	**Matrices**
Classical	real–orthogonal three–term (1.1)	irreducible tridiagonal matrices
	Szegö two–term (1.4)/three–term (1.2)	almost unitary Hessenberg matrices
[BEGO08]	general three–term (1.3)	$(H, 1)$–well–free (Definition 5.1)
	Szegö–type two–term (1.5)	$(H, 1)$–semiseparable (Definition 4.1)
	EGO–type two–term (3.2)	$(H, 1)$–quasiseparable (Definition 1.1)
This paper	general l–term (5.1)	(H, m)–well–free (Definition 5.5)
	Szegö–type two–term (4.4)	(H, m)–semiseparable (Definition 4.1)
	EGO–type two–term (3.1)	(H, m)–quasiseparable (Definition 1.1)

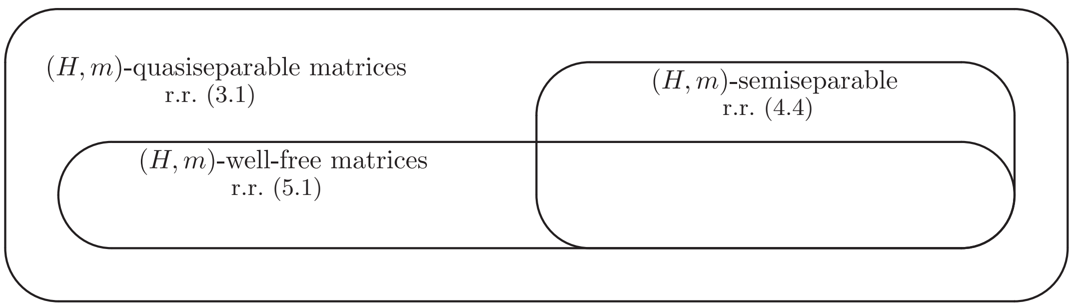
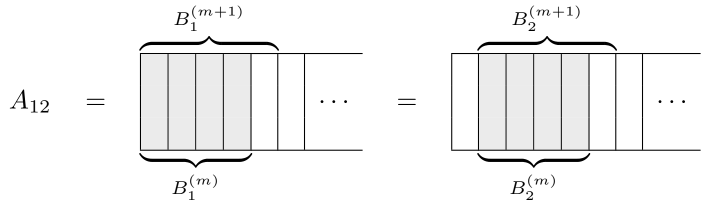


FIGURE 3. Relations between subclasses of (H, m)–quasiseparable matrices.

Table 2 and Figure 3 both reference (H, m)–well–free matrices, the definition of which is not obvious how to obtain from the definition of $(H, 1)$–well–free matrices given above. In Section 5, the details of this extension are given, but we briefly describe the new definition here. A matrix is (H, m)–well–free if

$$\operatorname{rank} B_i^{(m)} = \operatorname{rank} B_i^{(m+1)} \qquad i = 1, 2, \ldots \tag{1.9}$$

where the matrices $B_i^{(m)}$ are formed from the columns of the partition A_{12} of Definition 1.1, as



More details on this definition are given below in Section 5. We show in this paper that (H, m)–well–free matrices and polynomials satisfying

$$r_k(x) = \underbrace{(\delta_{k,k}x + \varepsilon_{k,k})r_{k-1}(x) + \cdots + (\delta_{k+m-2,k}x + \varepsilon_{k+m-2,k})r_{k+m-3}(x)}_{m+1 \text{ terms}}, \tag{1.10}$$

provide a complete characterization of each other.

Next, consider briefly the $m = 1$ case to see that this generalization reduces properly in the $(H, 1)$ case. For $m = 1$, this relation implies that no wells form of width $m = 1$ as illustrated in Figure 2. Similarly, for $m = 1$, (1.10) gives the three–term recurrence relations (1.3), which, as stated in Table 1, provide a characterization of $(H, 1)$–well–free matrices. Similar classification results are obtained for the other classes presented in Table 2.

2. Correspondences between Hessenberg matrices and polynomial systems

In this section we give details of the correspondence between (H, m)–quasiseparable matrices and systems of polynomials defined via (1.6), and explain how this correspondence can be used in classifications of quasiseparable matrices in terms of recurrence relations and vice versa.

2.1. A bijection between invertible triangular matrices and polynomial systems

Let $\mathcal{T}$ be the set of invertible upper triangular matrices and $\mathcal{P}$ be the set of polynomial systems $\{r_k\}$ with $\deg r_k = k$. We next demonstrate that there is a bijection between $\mathcal{T}$ and $\mathcal{P}$. Indeed, given a polynomial system $R = \{r_0(x), r_1(x), \ldots, r_n(x)\} \in \mathcal{P}$ satisfying $\deg(r_k) = k$, there exist unique n–term recurrence relations of the form

$$r_0(x) = a_{0,0}, \quad x \cdot r_{k-1}(x) = a_{k+1,k} \cdot r_k(x) - a_{k,k} \cdot r_{k-1}(x) - \cdots - a_{1,k} \cdot r_0(x), \quad a_{k+1,k} \neq 0, \quad k = 1, \ldots, n \tag{2.1}$$

because this formula represents $x \cdot r_{k-1} \in \mathbb{P}_k$ ($\mathbb{P}_k$ being the space of all polynomials of degree at most k) in terms of $r_k, r_{k-1}, r_{k-2}, \ldots, r_0$, which form a basis in $\mathbb{P}_k$, and hence these coefficients are unique. Forming a matrix $B \in \mathcal{T}$ from these coefficients as $B = [a_{i,j}]_{i,j=0}^n$ (with zeros below the main diagonal), it is clear that there is a bijection between $\mathcal{T}$ and $\mathcal{P}$, as they share the same unique parameters.

It is shown next that this bijection between invertible triangular matrices and polynomials systems (satisfying $\deg r_k(x) = k$) can be viewed as a bijection between strongly Hessenberg matrices together with two free parameters and polynomial systems (satisfying $\deg r_k(x) = k$). Furthermore, the strongly Hessenberg matrices and polynomial systems of this bijection are related via (1.6). Indeed, it was shown in [MB79] that the *confederate matrix* A, the strongly upper Hessenber matrix defined by

$$A = \begin{bmatrix} a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,n} \\ a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & a_{2,2} & a_{2,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{n-2,n} \\ 0 & \cdots & 0 & a_{n-1,n-1} & a_{n-1,n} \end{bmatrix}, \tag{2.2}$$

or in terms of B as

$$B = \left[\begin{array}{c|ccccc} a_{0,0} & a_{0,1} & a_{0,2} & a_{0,3} & \cdots & a_{0,n} \\ 0 & a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n} \\ 0 & 0 & a_{2,2} & a_{2,3} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & a_{n-2,n} \\ \vdots & 0 & \cdots & 0 & a_{n-1,n-1} & a_{n-1,n} \\ \hline 0 & 0 & \cdots & 0 & 0 & a_{n,n} \end{array}\right] = \left[\begin{array}{c|cccc} a_{0,0} & & & & \\ 0 & & & & \\ \vdots & & A & & \\ 0 & & & & \\ \hline 0 & 0 & \cdots & 0 & a_{n,n} \end{array}\right], \tag{2.3}$$

is related to the polynomial system R via (1.6). This shows the desired bijection, with $a_{0,0}$ and $a_{n,n}$ serving as the two free parameters.

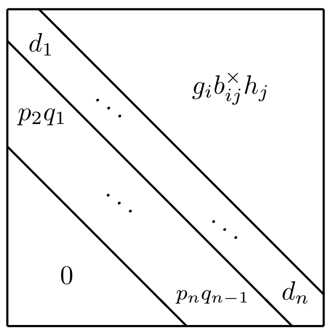
Remark 2.1. Based on this discussion, if $R = \{r_0, r_1, \ldots, r_{n-1}, r_n\}$ is related to a matrix A via (1.6), then $R_{a,b} = \left\{ar_0, \frac{1}{a}r_1, \ldots, \frac{1}{a}r_{n-1}, br_n\right\}$ for any nonzero parameters a and b provides a full characterization of all polynomial systems related to the matrix A.

2.2. Generators of (H, m)–quasiseparable matrices.

It is well known that Definition 2.2, given in terms of ranks is equivalent to another definition in terms of a sparse representation of the elements of the matrix called *generators* of the matrix, see, e.g., [EG99a] and the references therein. Such sparse representations are often used as inputs to fast algorithms involving such matrices. We give next this equivalent definition.

Definition 2.2 (Generator definition for (H, m)–quasiseparable matrices). *A matrix A is called (H, m)–quasiseparable if* **(i)** *it is upper Hessenberg, and* **(ii)** *it can be represented in the form*

$A =$



(2.4)

where $b_{ij}^{\times} = b_{i+1} \cdots b_{j-1}$ for $j > i+1$ and $b_{ij}^{\times} = 1$ for $j = i+1$. The elements

$$\{p_k, q_k, d_k, g_k, b_k, h_k\},$$

called the generators of the matrix A, are matrices of sizes

	$p_{k+1}q_k$	d_k	g_k	b_k	h_k
sizes	1×1	1×1	$1 \times u_k$	$u_{k-1} \times u_k$	$u_{k-1} \times 1$
range	$k \in [1, n-1]$	$k \in [1, n]$	$k \in [1, n-1]$	$k \in [2, n-1]$	$k \in [2, n]$

subject to $\max_k u_k = m$. *The numbers u_k, $k = 1, \ldots, n-1$ are called the* orders *of these upper generators.*

Remark 2.3. The generators of a (H, m)–quasiseparable matrix give us an $\mathcal{O}(nm^2)$ representation of the elements of the matrix. In the $(H, 1)$–quasiseparable case, where all generators can be chosen simply as scalars, this representation is $\mathcal{O}(n)$.

Remark 2.4. The subdiagonal elements, despite being determined by a single value, are written as a product $p_{k+1}q_k$, $k = 1, \ldots, n-1$ to follow standard notations used in the literature for quasiseparable matrices. We emphasize that this product acts as a single parameter in the Hessenberg case to which this paper is devoted.

Remark 2.5. The generators in Definition 2.2 can be always chosen to have sizes $u_k = m$ for all k by padding them with zeros to size m.

Also, the ranks of the submatrices A_{12} of Definition 1.1 represent the smallest possible sizes of the corresponding generators. That is, denoting by[2] $A_{12}^{(k)} = A(1:k, k+1:n)$ the partition A_{12} of the k–th symmetric partition, then

$$\operatorname{rank} A_{12}^{(k)} \leqslant u_k, \qquad k = 1, \dots, n.$$

For details on the existence of minimal size generators, see [EG05].

2.3. A relation between generators of quasiseparable matrices and recurrence relations for polynomials.

One way to establish a bijection (up to scaling as described in Remark 2.1) between subclasses of (H,m)–quasiseparable matrices and polynomial systems specified by recurrence relations is to deduce conversion rules between generators of the classes of matrices and coefficients of the recurrence relations. In this approach, a difficulty is encountered which is described by Figure 4.

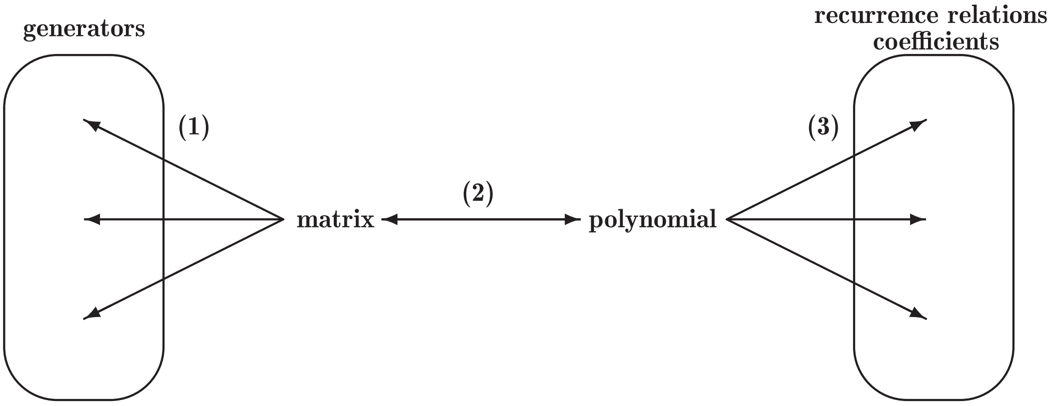


FIGURE 4. Relations between subclasses of (H,m)–quasiseparable matrices and polynomials.

The difficulty is that the relation **(2)** shown in the picture is one–to–one correspondence but **(1)** and **(3)** are not. This fact is illustrated in the next two examples.

Example 2.6 (Nonuniqueness of recurrence relation coefficients). In contrast to the n–term recurrence relations (2.1), other recurrence relations such as the l-term recurrence relations (5.1) corresponding to a given polynomial system are not unique. As a simple example of a system of polynomials satisfying more than one set of recurrence relations of the form (5.1), consider the monomials $R = \{1, x, x^2, \dots, x^n\}$, easily seen to satisfy the recurrence relations

$$r_0(x) = 1, \qquad r_k(x) = x \cdot r_{k-1}(x), \quad k = 1, \dots, n$$

as well as the recurrence relations

$$r_0(x) = 1, \qquad r_1(x) = x \cdot r_{k-1}(x), \qquad r_k(x) = (x+1) \cdot r_{k-1}(x) - x \cdot r_{k-2}(x), \quad k = 2, \dots, n.$$

Hence a given system of polynomials may be expressed using the same recurrence relations but with different coefficients of those recurrence relations.

Example 2.7 (Nonuniqueness of (H,m)–quasiseparable generators). Similarly, given a (H,m)–quasiseparable matrix, there is a freedom in choosing the set of generators of Definition 2.2. As a simple example, consider

[2]The MATLAB notation $A(i:j, k:l)$ denotes the submatrix obtained from rows $i, i+1, \dots, j$ and columns $k, k+1, \dots, l$.

the matrix

$$\begin{bmatrix} 0 & \frac{1}{2} & 0 & \cdots & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \ddots & \vdots \\ 0 & \frac{1}{2} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{1}{2} \\ 0 & \cdots & 0 & \frac{1}{2} & 0 \end{bmatrix}$$

corresponding to a system of Chebyshev polynomials. It is obviously $(H,1)$–quasiseparable and can be defined by different sets of generators, with either $g_k = 1, h_k = \frac{1}{2}$ or $g_k = \frac{1}{2}, h_k = 1$.

Remark 2.8. To overcome the difficulties of the nonuniqueness demonstrated here, we can define equivalence classes of generators describing the same matrix and equivalence classes of recurrence relations describing the same polynomials. Working with representatives of these equivalence classes resolves the difficulty.

We begin classification of recurrence relations of polynomials with considering EGO–type two–term recurrence relations (3.1) in Section 3 and associating the set of all (H,m)–quasiseparable matrices with them. Section 4 covers the correspondence between polynomials satisfying (4.4) and (H,m)–semiseparable matrices. In Section 5 we consider l-term recurrence relations (5.1) and (H,m)–well–free matrices.

3. (H,m)–quasiseparable matrices & EGO–type two–term recurrence relations (3.1)

In order to proceed with the classification of recurrence relations for polynomials corresponding to subclasses of (H,m)–quasiseparable matrices, a first step is the classification of the recurrence relations corresponding to the entire class of (H,m)–quasiseparable matrices. That is, in this section we will provide the proof of the following theorem.

Theorem 3.1. *Suppose A is a strongly upper Hessenberg matrix. Then the following are equivalent.*

(i) *A is (H,m)–quasiseparable.*

(ii) *There exist auxiliary polynomials $\{F_k(x)\}$ for some α_k, β_k, and γ_k of sizes $m \times m$, $m \times 1$ and $1 \times m$, respectively, such that the system of polynomials $\{r_k(x)\}$ related to A via (1.6) satisfies the EGO–type two–term recurrence relations*

$$\left[\begin{array}{c} F_0(x) \\ \hline r_0(x) \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ \cdot \\ \vdots \\ 0 \\ \hline a_{0,0} \end{array}\right], \quad \left[\begin{array}{c} F_k(x) \\ \hline r_k(x) \end{array}\right] = \left[\begin{array}{c|c} \alpha_k & \beta_k \\ \hline \gamma_k & \delta_k x + \theta_k \end{array}\right] \left[\begin{array}{c} F_{k-1}(x) \\ \hline r_{k-1}(x) \end{array}\right]. \tag{3.1}$$

Remark 3.2. Throughout the paper, we will avoid distinguishing between (H,m)–quasiseparable and weakly (H,m)–quasiseparable matrices. The difference is technical; for instance, considering an $(H,2)$–quasiseparable matrix as a weakly $(H,3)$–quasiseparable matrix corresponds to artificially increasing the size of the vectors $F_k(x)$ in (3.1) by one. This additional entry corresponds to a polynomial system that is identically zero, or otherwise has no influence on the other polynomial systems. In a similar way, any results stated for (H,m)–quasiseparable matrices are valid for weakly (H,m)–quasiseparable matrices through such trivial modifications.

This theorem, whose proof will be provided by the lemma and theorems of this section, is easily seen as a generalization of the following result for the $(H,1)$–quasiseparable case from [BEGO08].

Corollary 3.3. *Suppose A is a strongly Hessenberg matrix. Then the following are equivalent.*

(i) *A is $(H,1)$–quasiseparable.*

(ii) *There exist auxiliary polynomials $\{F_k(x)\}$ for some scalars α_k, β_k, and γ_k such that the system of polynomials $\{r_k(x)\}$ related to A via (1.6) satisfies the EGO–type two–term recurrence relations*

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ a_{0,0} \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}. \tag{3.2}$$

In establishing the one–to–one correspondence between the class of polynomials satisfying (3.1) and the class of (H,m)–quasiseparable matrices, we will use the following lemma which was given in [BEGOT07] and is a consequence of Definition 2.2 and [MB79].

Lemma 3.4. *Let A be an (H,m)–quasiseparable matrix specified by its generators as in Definition 2.2. Then a system of polynomials $\{r_k(x)\}$ satisfies the recurrence relations*

$$r_k(x) = \frac{1}{p_{k+1}q_k}\left[(x-d_k)r_{k-1}(x) - \sum_{j=0}^{k-2} g_{j+1}b^{\times}_{j+1,k}h_k r_j(x)\right], \tag{3.3}$$

if and only if $\{r_k(x)\}$ is related to A via (1.6).

Note that we have not specified the sizes of matrices g_k, b_k and h_k in (3.3) explicitly but the careful reader can check that all matrix multiplications are well defined. We will omit explicitly listing the sizes of generators where it is possible.

Theorem 3.5 (EGO–type two–term recurrence relations $\Rightarrow$ (H,m)–quasiseparable matrices). *Let R be a system of polynomials satisfying the EGO–type two–term recurrence relations (3.1). Then the (H,m)–quasiseparable matrix A defined by*

$$\begin{bmatrix} -\frac{\theta_1}{\delta_1} & -\frac{1}{\delta_2}\gamma_2\beta_1 & -\frac{1}{\delta_3}\gamma_3\alpha_2\beta_1 & -\frac{1}{\delta_4}\gamma_4\alpha_3\alpha_2\beta_1 & \cdots & -\frac{1}{\delta_n}\gamma_n\alpha_{n-1}\alpha_{n-2}\cdots\alpha_3\alpha_2\beta_1 \\ \frac{1}{\delta_1} & -\frac{\theta_2}{\delta_2} & -\frac{1}{\delta_3}\gamma_3\beta_2 & -\frac{1}{\delta_4}\gamma_4\alpha_3\beta_2 & \cdots & -\frac{1}{\delta_n}\gamma_n\alpha_{n-1}\alpha_{n-2}\cdots\alpha_3\beta_2 \\ 0 & \frac{1}{\delta_2} & -\frac{\theta_3}{\delta_3} & -\frac{1}{\delta_4}\gamma_4\beta_3 & \ddots & -\frac{1}{\delta_n}\gamma_n\alpha_{n-1}\cdots\alpha_4\beta_3 \\ 0 & 0 & \frac{1}{\delta_3} & -\frac{\theta_4}{\delta_4} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -\frac{1}{\delta_n}\gamma_n\beta_{n-1} \\ 0 & \cdots & 0 & 0 & \frac{1}{\delta_{n-1}} & -\frac{\theta_n}{\delta_n} \end{bmatrix} \tag{3.4}$$

with generators

$$d_k = -\frac{\theta_k}{\delta_k}, \quad k=1,\dots,n, \qquad p_{k+1}q_k = \frac{1}{\delta_k}, \quad k=1,\dots,n-1,$$

$$\boxed{g_k} = \boxed{\beta_k^T}, \quad k=1,\dots,n-1,$$

$$\boxed{b_k} = \boxed{\alpha_k^T}, \quad k=2,\dots,n-1, \qquad \boxed{h_k} = -\frac{1}{\delta_k}\boxed{\gamma_k^T}, \quad k=2,\dots,n$$

corresponds to the system of polynomials R via (1.6).

Proof. Considering EGO-type recurrence relations (3.1) we begin with

$$r_k(x) = (\delta_k x + \theta_k)r_{k-1}(x) + \gamma_k F_{k-1}(x). \tag{3.5}$$

Using the relation $F_{k-1}(x) = \alpha_{k-1}F_{k-2}(x) + \beta_{k-1}r_{k-2}(x)$, (3.5) becomes

$$r_k(x) = (\delta_k x + \theta_k)r_{k-1}(x) + \gamma_k\beta_{k-1}r_{k-2}(x) + \gamma_k\alpha_{k-1}F_{k-2}(x) \tag{3.6}$$

The equation (3.6) contains $F_{k-2}(x)$ which can be eliminated as it was done on the previous step. Using the relation $F_{k-2}(x) = \alpha_{k-2}F_{k-3}(x) + \beta_{k-2}r_{k-3}(x)$ we get

$$r_k(x) = (\delta_k x + \theta_k)r_{k-1}(x) + \gamma_k\beta_{k-1}r_{k-2}(x) + \gamma_k\alpha_{k-1}\beta_{k-2}r_{k-3}(x) + \gamma_k\alpha_{k-1}\alpha_{k-2}F_{k-3}(x).$$

Continue this process and noticing that F_0 is the vector of zeros we will obtain the n–term recurrence relations

$$\begin{aligned} r_k(x) \quad = \quad & (\delta_k x + \theta_k)r_{k-1}(x) + \gamma_k\beta_{k-1}r_{k-2}(x) + \gamma_k\alpha_{k-1}\beta_{k-2}r_{k-3}(x) \\ & + \gamma_k\alpha_{k-1}\alpha_{k-2}\beta_{k-3}r_{k-4}(x) + \cdots + \gamma_k\alpha_{k-1}\cdots\alpha_2\beta_1 r_0(x), \end{aligned} \tag{3.7}$$

which define the matrix (3.4) with the desired generators by using the n–term recurrence relations (3.3). $\square$

Theorem 3.6 (**(H,m)–quasiseparable matrices $\Rightarrow$ EGO–type two–term recurrence relations).** *Let A be a (H,m)–quasiseparable matrix specified by the generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$. Then the polynomial system R corresponding to A satisfies*

$$\left[\begin{array}{c} F_0(x) \\ \hline r_0(x) \end{array}\right] = \left[\begin{array}{c} 0 \\ \vdots \\ \cdot \\ \vdots \\ 0 \\ \hline a_{0,0} \end{array}\right], \quad \left[\begin{array}{c} F_k(x) \\ \hline r_k(x) \end{array}\right] = \left[\begin{array}{c|c} \alpha_k & \beta_k \\ \hline \gamma_k & \delta_k x + \theta_k \end{array}\right] \left[\begin{array}{c} F_{k-1}(x) \\ \hline r_{k-1}(x) \end{array}\right], \tag{3.8}$$

with

$$\alpha_k = \frac{p_k}{p_{k+1}} b_k^T, \quad \beta_k = -\frac{1}{p_{k+1}} g_k^T, \quad \gamma_k = \frac{p_k}{p_{k+1} q_k} h_k^T, \quad \delta_k = \frac{1}{p_{k+1} q_k}, \quad \theta_k = -\frac{d_k}{p_{k+1} q_k}.$$

Proof. It is easy to see that every system of polynomials satisfying $\deg r_k = k$ (e.g. the one defined by (3.8)) satisfy also the n–term recurrence relations

$$r_k(x) = (\alpha_k x - a_{k-1,k}) \cdot r_{k-1}(x) - a_{k-2,k} \cdot r_{k-2}(x) - \ldots - a_{0,k} \cdot r_0(x) \tag{3.9}$$

for some coefficients $\alpha_k, a_{k-1,k}, \ldots, a_{0,k}$. The proof is presented by showing that these n–term recurrence relations in fact coincide exactly with (3.3), so these coefficients coincide with those of the n–term recurrence relations of the polynomials R. Using relations for $r_k(x)$ and $F_{k-1}(x)$ from (3.8), we have

$$r_k(x) = \frac{1}{p_{k+1} q_k} \left[(x - d_k) r_{k-1}(x) - g_{k-1} h_k r_{k-2}(x) + p_{k-1} h_k^T b_{k-1}^T F_{k-2}(x) \right]. \tag{3.10}$$

Notice that again using (3.8) to eliminate $F_{k-2}(x)$ from the equation (3.10) will result in an expression for $r_k(x)$ in terms of $r_{k-1}(x)$, $r_{k-2}(x)$, $r_{k-3}(x)$, $F_{k-3}(x)$, and $r_0(x)$ without modifying the coefficients of $r_{k-1}(x)$, $r_{k-2}(x)$, or $r_0(x)$. Again applying (3.8) to eliminate $F_{k-3}(x)$ results in an expression in terms of $r_{k-1}(x), r_{k-2}(x), r_{k-3}(x), r_{k-4}(x), F_{k-4}(x)$, and $r_0(x)$ without modifying the coefficients of $r_{k-1}(x), r_{k-2}(x)$, $r_{k-3}(x)$, or $r_0(x)$. Continuing in this way, the n–term recurrence relations of the form (3.9) are obtained without modifying the coefficients of the previous ones.

Suppose that for some $0 < j < k-1$ the expression for $r_k(x)$ is of the form

$$r_k(x) = \frac{1}{p_{k+1} q_k} [(x - d_k) r_{k-1}(x) - g_{k-1} h_k r_{k-2}(x) - \cdots \\ - g_{j+1} b_{j+1,k}^{\times} h_k r_j(x) + p_{j+1} h_k^T (b_{j,k}^{\times})^T F_j(x)]. \tag{3.11}$$

Using (3.8) for $F_j(x)$ gives the relation

$$F_j(x) = \frac{1}{p_{j+1} q_j} \left(p_j q_j b_j^T F_{j-1}(x) - q_j g_j^T r_{j-1}(x) \right) \tag{3.12}$$

Inserting (3.12) into (3.11) gives

$$r_k(x) = \frac{1}{p_{k+1} q_k} \left[(x - d_k) r_{k-1}(x) - g_{k-1} h_k r_{k-2}(x) - \cdots - g_j b_{j,k}^{\times} h_k r_{j-1}(x) + p_j h_k^T (b_{j-1,k}^{\times})^T F_{j-1}(x) \right]. \tag{3.13}$$

Therefore since (3.10) is the case of (3.11) for $j = k-2$, (3.11) is true for each $j = k-2, k-3, \ldots, 0$, and for $j = 0$, using the fact that $F_0 = 0$ we have

$$r_k(x) = \frac{1}{p_{k+1} q_k} \left[(x - d_k) r_{k-1}(x) - g_{k-1} h_k r_{k-2}(x) - \cdots - g_1 b_{1,k}^{\times} h_k r_0(x) \right] \tag{3.14}$$

Since these coefficients coincide with (3.3) that are satisfied by the polynomial system R, the polynomials given by (3.8) must coincide with these polynomials. This proves the theorem. □

These last two theorems provide the proof for Theorem 3.1, and complete the discussion of the recurrence relations related to (H,m)–quasiseparable matrices.

4. (H,m)–semiseparable matrices & Szegö–type two–term recurrence relations (4.4)

In this section we consider a class of (H,m)–semiseparable matrices defined next.

Definition 4.1 (**(H,m)–semiseparable matrices**). *A matrix A is called (H,m)–semiseparable if* **(i)** *it is strongly upper Hessenberg, and* **(ii)** *it is of the form*

$$A = B + \operatorname{triu}(A_U, 1)$$

with $\operatorname{rank}(A_U) = m$ *and a lower bidiagonal matrix B, where following the MATLAB command* **triu**, $\operatorname{triu}(A_U,1)$ *denotes the strictly upper triangular portion of the matrix A_U.*

Paraphrased, a (H,m)–semiseparable matrix has arbitrary diagonal entries, arbitrary nonzero subdiagonal entries, and the strictly upper triangular part of a rank m matrix. It is obvious from this definition that a (H,m)–semiseparable matrix is (H,m)–quasiseparable. Indeed, let A be (H,m)–semiseparable and $n\times n$. Then it is clear that, if $A_{12}^{(k)}$ denotes the matrix A_{12} of the k–th partition of Definition 1.1, then

$$\operatorname{rank} A_{12}^{(k)} = \operatorname{rank} A(1:k, k+1:n) = \operatorname{rank} A_U(1:k,k+1:n) \leqslant m, \quad k = 1,\ldots,n-1,$$

and A is (H,m)–quasiseparable by Definition 1.1.

Example 4.2 (Unitary Hessenberg matrices are $(H,1)$–semiseparable). Consider again the unitary Hessenberg matrix

$$H = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & \cdots & -\rho_0^*\mu_1\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & \cdots & -\rho_1^*\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ 0 & \mu_2 & -\rho_2^*\rho_3 & \cdots & -\rho_2^*\mu_3\cdots\mu_{n-1}\rho_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -\rho_{n-1}^*\rho_n \end{bmatrix} \tag{4.1}$$

which corresponds to a system of Szegö polynomials. Its strictly upper triangular part is the same as in the matrix

$$B = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & \cdots & -\rho_0^*\mu_1\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ -\frac{\rho_1\rho_1^*}{\mu_1} & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & \cdots & -\rho_1^*\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ -\frac{\rho_1\rho_2^*}{\mu_1\mu_2} & -\frac{\rho_2\rho_2^*}{\mu_2} & -\rho_2^*\rho_3 & \cdots & -\rho_2^*\mu_3\cdots\mu_{n-1}\rho_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{\rho_1\rho_{n-1}^*}{\mu_1\mu_2\cdots\mu_{n-1}} & -\frac{\rho_2\rho_{n-1}^*}{\mu_2\mu_3\cdots\mu_{n-1}} & -\frac{\rho_3\rho_{n-1}^*}{\mu_3\mu_4\cdots\mu_{n-1}} & \cdots & -\rho_{n-1}^*\rho_n \end{bmatrix}. \tag{4.2}$$

which can be constructed as, by definition[3], $\mu_k \neq 0$, $k = 1,\ldots,n-1$. It is easy to check that the rank of the matrix B is one[4]. Hence the matrix (4.1) is $(H,1)$–semiseparable. Recall that any unitary Hessenberg matrix (4.1) uniquely corresponds to a system of Szegö polynomials satisfying the recurrence relations

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \frac{1}{\mu_0}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_k}\begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix}\begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^{\#}(x) \end{bmatrix}, \quad k = 1,2,\ldots,n. \tag{4.3}$$

The next theorem gives a classification of the class of (H,m)–semiseparable matrices in terms of two–term recurrence relations that naturally generalize the Szegö–type two term recurrence relations. Additionally, it gives a classification in terms of their generators as in Definition 2.2. That is, it summarizes the results to be proven in this section.

Theorem 4.3. *Suppose A is a strongly upper Hessenberg $n\times n$ matrix. Then the following are equivalent.*

(i) *A is (H,m)–semiseparable.*
(ii) *There exists a set of generators of Definition 2.2 corresponding to A such that b_k is invertible for $k = 2,\ldots,n$.*

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[3]The parameters μ_k associated with the Szegö polynomials are defined by $\mu_k = \sqrt{1-|\rho_k|^2}$ for $0 \leqslant |\rho_k| < 1$ and $\mu_k = 1$ for $|\rho_k| = 1$, and since $|\rho_k| \leqslant 1$ for all k, we always have $\mu_k \neq 0$.

[4]Every i-th row of B equals the row number $(i-1)$ times $\rho_{i-1}^*/\rho_{i-2}^*\mu_{i-1}$

(iii) *There exist auxiliary polynomials $\{G_k(x)\}$ for some α_k, β_k, and γ_k of sizes $m \times m$, $m \times 1$ and $1 \times m$, respectively, such that the system of polynomials $\{r_k(x)\}$ related to A via* (1.6) *satisfies the Szegö–type two–term recurrence relations*

$$\left[\begin{array}{c} G_0(x) \\ \hline r_0(x) \end{array}\right] = \left[\begin{array}{c} a_{0,0} \\ \vdots \\ \cdot \\ \vdots \\ a_{0,0} \\ \hline a_{0,0} \end{array}\right], \quad \left[\begin{array}{c} G_k(x) \\ \hline r_k(x) \end{array}\right] = \left[\begin{array}{c|c} \alpha_k & \beta_k \\ \hline \gamma_k & 1 \end{array}\right] \left[\begin{array}{c} G_{k-1}(x) \\ \hline (\delta_k x + \theta_k) r_{k-1}(x) \end{array}\right]. \tag{4.4}$$

This theorem, whose proof follows from the results later in this section, leads to the following corollary, which summarizes the results for the simpler class of $(H,1)$–semiseparable matrices as given in [BEGO08].

Corollary 4.4. *Suppose A is an $(H,1)$–quasiseparable matrix. Then the following are equivalent.*

(i) *A is $(H,1)$–semiseparable.*
(ii) *There exists a set of generators of Definition* 2.2 *corresponding to A such that $b_k \neq 0$ for $k = 2, \ldots, n$.*
(iii) *There exist auxiliary polynomials $\{G_k(x)\}$ for some scalars α_k, β_k, and γ_k such that the system of polynomials $\{r_k(x)\}$ related to A via* (1.6) *satisfies the Szegö–type two–term recurrence relations*

$$\begin{bmatrix} G_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} a_{0,0} \\ a_{0,0} \end{bmatrix}, \quad \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k) r_{k-1}(x) \end{bmatrix}. \tag{4.5}$$

4.1. (H,m)–semiseparable matrices. Generator classification.

We next give a lemma that provides a classification of (H,m)–semiseparable matrices in terms of a condition on the generators of an (H,m)–quasiseparable matrix.

Lemma 4.5. *An (H,m)–quasiseparable matrix is (H,m)–semiseparable if and only if there exists a choice of generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ of the matrix such that matrices b_k are nonsingular*[5] *for all $k = 2, \ldots, n-1$.*

Proof. Let A be (H,m)–semiseparable with $\operatorname{triu}(A,1) = \operatorname{triu}(A_U,1)$, where $\operatorname{rank}(A_U) = m$. The latter statement implies that there exist row vectors g_i and column vectors h_j of sizes m such that $A_U(i,j) = g_i h_j$ for all i,j, and therefore we have $A_{ij} = g_i h_j$, $i < j$ or $A_{ij} = g_i b_{ij}^{\times} h_j$, $i < j$ with $b_k = I_m$.

Conversely, suppose the generators of A are such that b_k are invertible matrices for $k = 2, \ldots, n-1$. Then the matrices

$$A_U = \begin{cases} g_i b_{i,j}^{\times} h_j & \text{if } 1 \leqslant i < j \leqslant n \\ g_i b_i^{-1} h_i & \text{if } 1 < i = j < n \\ g_i (b_{j-1,i+1}^{\times})^{-1} h_j & \text{if } 1 < j < i < n \\ 0 & \text{if } j = 1 \text{ or } i = n \end{cases} \qquad B = \begin{cases} d_i & \text{if } 1 \leqslant i = j \leqslant n \\ p_i q_j & \text{if } 1 \leqslant i+1 = j \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

are well defined, $\operatorname{rank}(A_U) = m$, B is lower bidiagonal, and $A = B + \operatorname{triu}(A_U,1)$. □

Remark 4.6. We emphasize that the previous lemma guarantees the existence of a set of generators of a (H,m)–semiseparable matrix with invertible matrices b_k, and that this condition need not be satisfied by all such generator representations. For example, the following matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 2 & 0 \\ & 1 & 1 & 3 & 3 & 0 \\ & & 1 & 1 & 4 & 0 \\ & & & 1 & 1 & 0 \\ & & & & 1 & 1 \end{bmatrix}$$

is $(H,1)$–semiseparable, however it is obviously possible to choose a set of generators for it with $b_5 = 0$.

[5]The invertibility of b_k implies that all b_k are square $m \times m$ matrices.

4.2. (H, m)**–semiseparable matrices. Recurrence relations classification.**

In this section we present theorems giving the classification of (H, m)–semiseparable matrices as those corresponding to systems of polynomials satisfying the Szegö–type two–term recurrence relations (4.4).

Theorem 4.7 (Szegö-type 2-term recurrence relations $\Rightarrow$ (H, m)**–semiseparable matrices).** *Let* $R = \{r_0(x), \dots, r_{n-1}(x)\}$ *be a system of polynomials satisfying the recurrence relations* (4.4) *with* $rank(\alpha_k^T - \beta_k\gamma_k) = m$. *Then the* (H, m)*–semiseparable matrix* A *defined by*

$$\begin{bmatrix} -\frac{\theta_1+\gamma_1\beta_0}{\delta_1} & -\frac{1}{\delta_2}\gamma_2(\alpha_1-\beta_1\gamma_1)\beta_0 & \cdots & -\frac{1}{\delta_n}\gamma_n(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\cdots(\alpha_1-\beta_1\gamma_1)\beta_0 \\ \frac{1}{\delta_1} & -\frac{\theta_2+\gamma_2\beta_1}{\delta_2} & \ddots & -\frac{1}{\delta_n}\gamma_n(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\cdots(\alpha_2-\beta_2\gamma_2)\beta_1 \\ 0 & \frac{1}{\delta_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & -\frac{\theta_n+\gamma_n\beta_{n-1}}{\delta_n} \\ 0 & \cdots & 0 & \frac{1}{\delta_n} \end{bmatrix} \tag{4.6}$$

with generators

$$d_k = -\frac{\theta_k + \gamma_k\beta_{k-1}}{\delta_k}, \quad k = 1, \dots, n, \qquad p_{k+1}q_k = \frac{1}{\delta_k}, \quad k = 1, \dots, n-1,$$

$$\boxed{g_k} = \boxed{\beta_{k-1}^T}, \quad k = 1, \dots, n-1,$$

$$\boxed{b_k^T} = \boxed{\alpha_{k-1}} - \boxed{\beta_{k-1}}\,\boxed{\gamma_{k-1}}, \quad k = 2, \dots, n-1,$$

$$\boxed{\beta_0} = \begin{bmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{bmatrix}, \quad \boxed{h_k} = -\frac{1}{\delta_k}\boxed{b_k}\,\boxed{\gamma_k^T}, \quad k = 2, \dots, n,$$

corresponds to the R *via* (1.6).

Proof. Let us show that the polynomial system satisfying the Szegö–type two–term recurrence relations (4.4) also satisfies EGO–type two–term recurrence relations (3.1). By applying the given two–term recursion, we have

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k G_{k-1}(x) + \beta_k(\delta_k + \theta_k)r_{k-1}(x) \\ \gamma_k G_{k-1}(x) + (\delta_k + \theta_k)r_{k-1}(x) \end{bmatrix} \tag{4.7}$$

Multiplying the second equation in (4.7) by β_k and subtracting from the first equation we obtain

$$G_k(x) - \beta_k r_k(x) = (\alpha_k - \beta_k\gamma_k)G_{k-1}(x) \tag{4.8}$$

Denoting in (4.8) G_{k-1} by F_k and shifting indices from k to $k-1$ we can get the recurrence relation

$$F_k(x) = (\alpha_{k-1} - \beta_{k-1}\gamma_{k-1})F_{k-1}(x) + \beta_{k-1}r_{k-1}(x) \tag{4.9}$$

In the same manner substituting (4.8) in the second equation of (4.7) and shifting indices one can be seen that

$$r_k(x) = \gamma_k(\alpha_{k-1} - \beta_{k-1}\gamma_{k-1})F_{k-1}(x) + (\delta_k x + \theta_k + \gamma_k\beta_{k-1})r_{k-1}(x) \tag{4.10}$$

Equations (4.9) and (4.10) together give necessary EGO–type two–term recurrence relations for the system of polynomials:

$$\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_{k-1} - \beta_{k-1}\gamma_{k-1} & \beta_{k-1} \\ \gamma_k(\alpha_{k-1} - \beta_{k-1}\gamma_{k-1}) & \delta_k x + \theta_k + \gamma_k\beta_{k-1} \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix} \tag{4.11}$$

Theorem 3.5 together with the recurrence relations (4.11) implies that the (H, m)–semiseparable (4.6) and the special choice of generators are valid. $\square$

Theorem 4.8 (**(H,m)–semiseparable matrices $\Rightarrow$ Szegö–type 2–term recurrence relations).** *Let A be a (H,m)–semiseparable matrix. Then for a set of generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ of A such that each b_k is invertible, the polynomial system R corresponding to A satisfies* (4.4)*; specifically,*

$$\left[\begin{array}{c} \boxed{G_k(x)} \\ \hline r_k(x) \end{array}\right] = \frac{1}{p_{k+1}q_k} \left[\begin{array}{c|c} \boxed{v_k} & \boxed{-g_{k+1}^T} \\ \hline \boxed{h_k^T(b_k^T)^{-1}} & 1 \end{array}\right] \left[\begin{array}{c} \boxed{G_{k-1}(x)} \\ \hline u_k(x) r_{k-1}(x) \end{array}\right] \tag{4.12}$$

$$\textit{with } u_k(x) = x - d_k + g_k b_k^{-1} h_k, \qquad v_k = p_{k+1}q_k b_{k+1}^T - g_{k+1}^T h_k^T (b_k^T)^{-1}.$$

Proof. According to the definition of (H,m)–semiseparable matrices the given polynomial system R must satisfy EGO–type two–term recurrence relations (3.1) with b_k invertible for all k. Let us consider these recurrence relations:

$$\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{p_{k+1}q_k} \begin{bmatrix} p_k q_k b_k^T & -q_k g_k^T \\ p_k h_k^T & x - d_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}. \tag{4.13}$$

Let us denote $p_{k+1}F_k(x)$ in the (4.13) as $G_k(x)$ then we can rewrite these equations as

$$\begin{aligned} G_{k-1}(x) &= b_k^T G_{k-2}(x) - g_k^T r_{k-1}(x), \\ r_k(x) &= \frac{1}{p_{k+1}q_k}\left[h_k^T G_{k-2}(x) + (x - d_k) r_{k-1}(x)\right]. \end{aligned} \tag{4.14}$$

Using the invertibility of b_k we are able to derive the $G_{k-2}(x)$ from the first equation of (4.14) and inserting it in the second equation we obtain new recurrence relation

$$r_k(x) = \frac{1}{p_{k+1}q_k}\left[h_k^T (b_k^T)^{-1} G_{k-1}(x) + (x - d_k + g_k b_k^{-1} h_k) r_{k-1}(x)\right]. \tag{4.15}$$

The second necessary recurrence relation can be obtained by substituting (4.15) in the first equation of (4.14) and shifting indices from $k-1$ to k.

$$G_k(x) = \frac{1}{p_{k+1}q_k}\left[(p_{k+1}q_k b_{k+1}^T - g_{k+1}^T h_k^T (b_k^T)^{-1}) G_{k-1}(x) - g_{k+1}^T (x - d_k + g_k b_k^{-1} h_k) r_{k-1}(x)\right] \tag{4.16}$$

This completes the proof. □

This completes the justification of Theorem 4.3.

5. (H,m)–well–free matrices & recurrence relations (5.1).

In this section, we begin by considering the l–term recurrence relations of the form

$$\begin{aligned} r_0(x) = a_{0,0}, \quad r_k(x) &= \sum_{i=1}^{k} (\delta_{ik}x + \varepsilon_{ik}) r_{i-1}(x), \quad k = 1, 2, \ldots, l-2 \\ r_k(x) &= \sum_{i=k-l+2}^{k} (\delta_{ik}x + \varepsilon_{ik}) r_{i-1}(x), \quad k = l-1, l, \ldots, n \end{aligned} \quad . \tag{5.1}$$

As we shall see below, the matrices that correspond to (5.1) via (1.6) form a new subclass of (H,m)–quasiseparable matrices. As such, we then can also give a generator classification of the resulting class. This problem was addressed in [BEGO08] for the $l = 3$ case; that is, for (1.3)),

$$r_0(x) = a_{0,0}, \quad r_1(x) = (\alpha_1 x - \delta_1) \cdot r_0(x), \quad r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x), \tag{5.2}$$

and was already an involved problem. To explain the results in the general case more clearly, we begin by recalling the results for the special case when $l = 3$.

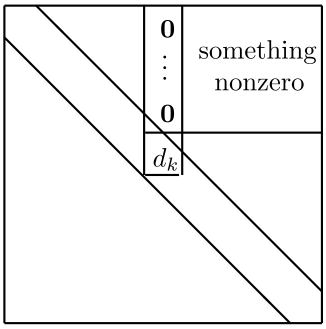
5.1. General three–term recurrence relations (1.3) **&** $(H, 1)$**–well–free matrices**

In [BEGO08], it was proved that polynomials that satisfy the general three–term recurrence relations (5.2) were related to a subclassf $(H, 1)$–quasiseparable matrices denoted $(H, 1)$–well–free matrices. A definition of this class is given next.

Definition 5.1 (($H, 1$)**–well–free matrices).**

- *An $n \times n$ matrix $A = (A_{i,j})$ is said to have a* **well of size one** *in column $1 < k < n$ if $A_{i,k} = 0$ for $1 \leqslant i < k$ and there exists a pair (i, j) with $1 \leqslant i < k$ and $k < j \leqslant n$ such that $A_{i,j} \neq 0$.*
- *A $(H, 1)$–quasiseparable matrix is said to be $(H, 1)$–***well–free** *if none of its columns $k = 2, \ldots, n - 1$ contain wells of size one.*

In words, a matrix has a well in column k if all entries above the main diagonal in the k–th column are zero, **except** if all entries in the upper-right block to the right of these zeros are also zeros, as shown in the following illustration.



The following theorem summarizes the results of [BEGO08] that will be generalized in this section.

Theorem 5.2. *Suppose A is a strongly upper Hessenberg $n \times n$ matrix. Then the following are equivalent.*

(i) *A is $(H, 1)$–well–free.*
(ii) *There exists a set of generators of Definition* 2.2 *corresponding to A such that $h_k \neq 0$ for $k = 2, \ldots, n$.*
(iii) *The system of polynomials related to A via* (1.6) *satisfies the general three–term recurrence relations* (5.2)*.*

Having provided these results, the next goal is, given the l–term recurrence relations (5.1), to provide an analogous classification. A step in this direction can be taken using a formula given by Barnett in [B81] that gives for such recurrence relations a formula for the entries of the related matrix. For the convenience of the reader, a proof of this lemma is given at the end of this section (no proof was given in [B81]).

Lemma 5.3. *Let $R = \{r_0(x), \ldots, r_{n-1}(x)\}$ be a system of polynomials satisfying the recurrence relations* (5.1)*. Then the strongly Hessenberg matrix*

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ \frac{1}{\delta_{11}} & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & \frac{1}{\delta_{22}} & a_{33} & \cdots & a_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \frac{1}{\delta_{n-1,n-1}} & a_{nn} \end{bmatrix} \tag{5.3}$$

with entries

$$\begin{gathered} a_{ij} = -\frac{1}{\delta_{jj}} \left(\frac{\delta_{i-1,j}}{\delta_{i-1,i-1}} + \varepsilon_{ij} + \sum_{s=i}^{j-1} a_{is}\delta_{sj} \right) \\ \frac{\delta_{0j}}{\delta_{00}} = 0, \quad \forall j; \qquad \delta_{ij} = \varepsilon_{ij} = 0, \quad i < j - l + 2 \end{gathered} \tag{5.4}$$

corresponds to R via (1.6)*.*

Remark 5.4. While Lemma 5.3 describes the entries of the matrix A corresponding to polynomials satisfying the l–term recurrence relations (5.1), the structure of A is not explicitly specified by (5.4). Indeed, as surveyed this section, even in the simplest case of generalized three–term recurrence relations (1.3), the latter do not transparently lead to the characteristic quasiseparable and well–free properties of the associated matrices.

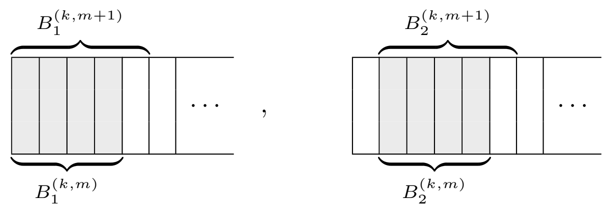
5.2. (H, m)–well–free matrices.

It was recalled in Section 5.1 that in the simplest case of three–term recurrence relations the corresponding matrix was $(H, 1)$–quasiseparable, and moreover, $(H, 1)$–well–free. So , one might expect that in the case of l–term recurrence relations (5.1), the associated matrix might turn out to be $(H, l-2)$–quasiseparable, but how does one generalize the concept of $(H, 1)$–well–free? The answer to this is given in the next definition.

Definition 5.5 (**(H, m)–well–free matrices).**

- *Let A be an $n \times n$ matrix, and fix constants $k, m \in [1, n-1]$. Define the matrices*
$$B_j^{(k,m)} = A(1 : k, j + k : j + k + (m-1)), \quad j = 1, \ldots, n-k-m.$$
Then if for some j,
$$\operatorname{rank}(B_j^{(k,m+1)}) > \operatorname{rank}(B_j^{(k,m)}),$$
the matrix A is said to have a **well of size** m **in partition** k*.*
- *A (H, m)–quasiseparable matrix is said to be (H, m)–***well–free** *if it contains no wells of size m.*

One can understand the matrices $B_j^{(k,m)}$ of the previous definition as, for constant k and m and as j increases, a sliding window consisting of m consecutive columns. Essentially, the definition states that as this window is slid through the partition A_{12} of Definition 1.1, if the ranks of the submatrices increase at any point by adding the next column, this constitutes a well. So a (H, m)–well–free matrix is such that each column of all partitions A_{12} is the linear combination of the m previous columns of A_{12}.



Notice that Definition 5.5 reduces to Definition 5.1 in the case when $m = 1$. Indeed, if $m = 1$, then the sliding windows are single columns, and an increase in rank is the result of adding a nonzero column to a single column of all zeros. This is shown next in (5.5).

$$\begin{array}{c|c|c|c|c} & B_{j-1} & B_j & B_{j+1} & \\ \hline & * & 0 & * & \\ & * & 0 & * & \\ & \vdots & \vdots & \vdots & \\ & * & 0 & * & \\ \hline \end{array} \tag{5.5}$$

In order for a matrix to be $(H, 1)$–quasiseparable, any column of zeros in A_{12} must be the first column of A_{12}; that is, in (5.5), $j = 1$. Thus a well of size one is exactly a column of zeros above the diagonal, and some nonzero entry to the right of that column, exactly as in Definition 5.1.

With the class of (H, m)–well–free matrices defined, we next present a theorem containing the classifications to be proved in this section.

Theorem 5.6. *Suppose A is a strongly upper Hessenberg $n \times n$ matrix. Then the following are equivalent.*

(i) *A is (H, m)–well–free.*
(ii) *There exists a set of generators of Definition 2.2 corresponding to A such that b_k are companion matrices for $k = 2, \ldots, n-1$, and $h_k = e_1$ for $k = 2, \ldots, n$, where e_1 is the first column of the identity matrix of appropriate size.*
(iii) *The system of polynomials related to A via (1.6) satisfies the general three–term recurrence relations (5.1).*

This theorem is an immediate corollary of Theorems 5.7, 5.8, and 5.9.

5.3. (H, m)**–well–free matrices. Generator classification.**

Theorem 5.7. *An (H, m)–quasiseparable matrix is (H, m)–well–free if and only if there exists a choice of generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ of the matrix that are of the form*

$$b_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & \xi_{k,1} \\ 1 & 0 & \ddots & \vdots & \vdots \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \xi_{k,m-1} \\ 0 & \cdots & 0 & 1 & \xi_{k,m} \end{bmatrix}, \quad k = 2, \ldots, n-1, \qquad h_k = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \quad k = 2, \ldots, n. \tag{5.6}$$

Proof. Let $A = (a_{ij})$ be an (H, m)–well–free matrix. Then due to the low rank property of off–diagonal blocks, its entries satisfy

$$a_{ij} = \sum_{s=j-m}^{j-1} a_{is}\alpha_{sj}, \quad \text{if} \quad i < j - m. \tag{5.7}$$

It is easy to see that an (H, m)–well–free matrix B with

$$d_k = a_{kk}, \quad k = 1, \ldots, n, \qquad p_{k+1}q_k = a_{k+1,k}, \quad k = 1, \ldots, n-1,$$
$$g_k = [a_{k,k+1} \cdots a_{k,k+m}], \quad k = 1, \ldots, n-1, \qquad h_k = [1\ 0\ \cdots\ 0]^T, \quad k = 2, \ldots, n,$$
$$b_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & \alpha_{k-m+1,k+1} \\ 1 & 0 & \ddots & \vdots & \vdots \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \alpha_{k-1,k+1} \\ 0 & \cdots & 0 & 1 & \alpha_{k,k+1} \end{bmatrix}, \quad k = 2, \ldots, n-1. \tag{5.8}$$

coincides with A.

Conversely, suppose A is an (H, m)–quasiseparable matrix whose generators satisfy (5.6). Applying (2.4) from Definition 2.2 it follows that

$$a_{ij} = g_i b^{\times}_{i,j} h_j = \begin{cases} \nu_{i,j-i} & i = 1, \ldots, n \quad j = i, \ldots i+m, \\ \sum_{s=j-m}^{j-1} a_{is}\xi_{j-m,s-j+m+1} & i = 1, \ldots, n \quad j = i+m+1, \ldots n. \end{cases} \tag{5.9}$$

This is equivalent to a summation of the form (5.7), demonstrating the low–rank property, and hence the matrix A is (H, m)–well–free according to Definition 5.5. ☐

This result generalizes the generator classification of $(H, 1)$–well–free matrices as given in [BEGO08], stated as a part of Theorem 5.2.

5.4. (H, m)**–well–free matrices. Recurrence relation classification.**

In this section, we will prove that it is exactly the class of (H, m)–well–free matrices that correspond to systems of polynomials satisfying l–term recurrence relations of the form (5.1).

Theorem 5.8 (l–term recurrence relations $\Rightarrow$ $(H, l-2)$–well–free matrices). *Let $A = (a_{ij})_{i,j=1}^n$ be a matrix corresponding to a system of polynomials $R = \{r_0(x), \ldots, r_{n-1}(x)\}$ satisfying (5.1). Then A is (H, m)–well–free.*

Proof. The proof is presented by demonstrating that A has a set of generators of the form (5.6), and hence is (H,m)–well–free. In particular, we show that

$$d_k = a_{kk}, \quad k = 1,\dots,n, \qquad p_{k+1}q_k = \frac{1}{\delta_{kk}}, \quad k = 1,\dots,n-1,$$

$$g_k = [a_{k,k+1} \cdots a_{k,k+l-2}], \quad k = 1,\dots,n-1, \qquad h_k = [\underbrace{1\ 0\ \cdots\ 0\ 0}_{l-2}]^T, \quad k = 2,\dots,n,$$

$$b_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\frac{\delta_{k,k+l-2}}{\delta_{k+l-2,k+l-2}} \\ 1 & 0 & \ddots & \vdots & \vdots \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & -\frac{\delta_{k+l-4,k+l-2}}{\delta_{k+l-2,k+l-2}} \\ 0 & \cdots & 0 & 1 & -\frac{\delta_{k+l-3,k+l-2}}{\delta_{k+l-2,k+l-2}} \end{bmatrix}, \quad k = 2,\dots,n-1, \tag{5.10}$$

$$\text{with} \quad \frac{\delta_{ij}}{\delta_{jj}} = 0 \quad \text{if} \quad i > n-l+2.$$

forms a set of generators of A. We show that with this choice, the entries of the matrix A coincide with those of (5.4). From Definition 2.2, the choice of d_k as the diagonal of A and choice of $p_{k+1}q_k$ as the subdiagonal entries of (5.3) produces the desired result in these locations. We next show that the generators g_k, b_k and h_k define the upper triangular part of the matrix A correctly.

Consider first the product $g_i b_{i+1} b_{i+2} \cdots b_{i+t}$, and note that

$$g_i b_{i+1} b_{i+2} \cdots b_{i+t} = [a_{i,i+t+1} \cdots a_{i,i+t+l-2}]. \tag{5.11}$$

Indeed, for $t = 0$, (5.11) becomes

$$g_i = [a_{i,i+1} \cdots a_{i,i+l-2}],$$

which coincides with the choice in (5.10) for each i, and hence the relation is true for $t = 0$. Suppose next that the relation is true for some t. Then using the lower shift structure of the choice of each b_k of (5.10) and the formula (5.4), we have

$$g_i b_{i+1} b_{i+2} \cdots b_{i+t+1} = [a_{i,i+t+1} \cdots a_{i,i+t+l-2}]\, b_{i+t+1} =$$
$$= \left[a_{i,i+t+2} \cdots a_{i,i+t+l-2} \quad \sum_{p=i+t+1}^{i+t+l-2} \frac{-a_{ip}\delta_{p,i+t+l-1}}{\delta_{i+t+l-1,i+t+l-1}} \right] = [a_{i,i+t+2} \cdots a_{i,i+t+l-1}]. \tag{5.12}$$

And therefore

$$g_i b_{ij}^{\times} h_j = [a_{ij} \cdots a_{i,j+s-1}]\, h_j = a_{ij}, \quad j > i$$

so (5.10) are in fact generators of the matrix A as desired. □

Theorem 5.9 ((H,m)–well–free matrices $\Rightarrow$ $(m+2)$-term recurrence relations). *Let A be an (H,m)–well-free matrix. Then the polynomials system related to A via* (1.6) *satisfies the l–term recurrence relations* (5.1).

Proof. By Theorem 5.7, there exists a choice of generators of A of the form

$$d_k = \nu_{k,0}, \quad k = 1,\dots,n, \qquad p_{k+1}q_k = \mu_k, \quad k = 1,\dots,n-1,$$

$$g_k = [\nu_{k,1} \cdots \nu_{k,m}], \quad k = 1,\dots,n-1, \qquad h_k = [\underbrace{1\ 0\ \cdots\ 0\ 0}_{m}]^T, \quad k = 2,\dots,n,$$

$$b_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & \xi_{k,1} \\ 1 & 0 & \ddots & \vdots & \vdots \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \xi_{k,m-1} \\ 0 & \cdots & 0 & 1 & \xi_{k,m} \end{bmatrix}, \quad k = 2,\dots,n-1. \tag{5.13}$$

We present a procedure to compute from these values the coefficients of (5.1).

1. Take
$$\delta_{ij} = \begin{cases} \frac{1}{\mu_j} & i = j, \\ -\frac{\xi_{j-m,i-j+m+1}}{\mu_{jj}} & j = m+2,\dots,n \quad i = j-m,\dots,j-1. \end{cases} \tag{5.14}$$
2. Calculate ε_{ij} and δ_{ij} for $j = 2,\dots,m+1, \quad i = 1,\dots,j-1$ as any solution of the following system of equations:
$$\begin{cases} \nu_{i,j-i} = -\mu_j \left(\varepsilon_{ij} + \sum_{s=1}^{j-1} \nu_{i,s-i}\delta_{sj}\right) & i = 1, \quad j = 2,\dots,m+1, \\ \nu_{i,j-i} = -\mu_j \left(\delta_{i-1,j}\mu_{i-1} + \varepsilon_{ij} + \sum_{s=1}^{j-1} \nu_{i,s-i}\delta_{sj}\right) & i = 2,\dots,m, \quad j = i+1,\dots,m+1. \end{cases} \tag{5.15}$$
3. Find the remaining ε_{ij}-coefficients using
$$\varepsilon_{ij} = \begin{cases} -\frac{\nu_{1,0}}{\mu_1} & i = j = 1, \\ -\frac{\nu_{j,0}}{\mu_j} - \delta_{j-1,j}\mu_{j-1} & i = j > 1, \\ -\frac{\nu_{i,j-i}}{\mu_j} - \delta_{i-1,j}\mu_{i-1} - \sum_{s=i}^{j-1} \nu_{i,s-i}\delta_{sj} & j = m+2,\dots,n \quad i = j-m,\dots,j-1. \end{cases} \tag{5.16}$$

The proof immediately follows by comparing (5.10), (5.13) and using (5.4). Note that the coefficients of the l–term recurrence relations depend on the solution of the system of equations (5.15), which consists of
$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$$
equations and defines $m(m+1)$ variables. So for the generators (5.13) of an (H, m)–well–free matrix there is a freedom in choosing coefficients of the recurrence relations (5.1) for the corresponding polynomials. □

This completes the justification of Theorem 5.6 stated above. In the $m = 1$ case, this coincides with the result given in [BEGO08], stated as Theorem 5.2.

5.5. Proof of Lemma 5.3

In this section we present a proof of Lemma 5.3, stated without proof by Barnett in [B81].

Proof of Lemma 5.3. The results of [MB79] allow us to observe the bijection between systems of polynomials and dilated strongly Hessenberg matrices. Indeed, given a polynomial system $R = \{r_0(x), \dots, r_{n-1}(x)\}$, there exist unique n–term recurrence relations of the form
$$x \cdot r_{j-1}(x) = a_{j+1,j} \cdot r_j(x) + a_{j,j} \cdot r_{j-1}(x) + \cdots + a_{1,j} \cdot r_0(x), \quad a_{j+1,j} \neq 0, \quad j = 1,\dots,n-1. \tag{5.17}$$
and $a_{1,j},\dots,a_{j+1,j}$ are coefficients of the j-th column of the correspondent strongly Hessenberg matrix A.

Using $\delta_{ij} = \varepsilon_{ij} = 0$, $i < j - l + 2$, we can assume that the given system of polynomials $R = \{r_0(x), \dots, r_{n-1}(x)\}$ satisfies full recurrence relations:
$$r_j(x) = \sum_{i=1}^{j} (\delta_{ij}x + \varepsilon_{ij})r_{i-1}(x), \quad j = 1,\dots,n-1 \tag{5.18}$$
The proof of (5.4) is given by induction on j. For any i, if $j = 1$, it is true that $a_{11} = -\frac{\varepsilon_{11}}{\delta_{11}}$. Next, assuming that (5.4) is true for all $j = 1,\dots,k-1$. Taking $j = k$ in (5.18) we can write that
$$xr_{k-1}(x) = \frac{1}{\delta_{k,k}} r_k(x) - \frac{\varepsilon_{kk}}{\delta_{k,k}} r_{k-1}(x) - \frac{1}{\delta_{k,k}} \sum_{i=1}^{k-1} (\delta_{ik}x + \varepsilon_{ik})r_{i-1}(x). \tag{5.19}$$
From the induction hypothesis and equation (5.17) we can substitute the expression for xr_{i-1} into (5.19) to obtain
$$xr_{k-1}(x) = \frac{1}{\delta_{k,k}} r_k(x) - \frac{\varepsilon_{kk}}{\delta_{k,k}} r_{k-1}(x) - \frac{1}{\delta_{k,k}} \sum_{i=1}^{k-1} \left[\delta_{ik} \sum_{s=1}^{i+1} a_{si} \cdot r_{s-1}(x) + \varepsilon_{ik} r_{i-1}(x) \right]. \tag{5.20}$$
After grouping coefficients in (5.20) we obtain
$$xr_{k-1}(x) = \frac{1}{\delta_{k,k}} r_k(x) - \frac{1}{\delta_{k,k}} \sum_{i=1}^{k} \left[\frac{\delta_{i-1,k}}{\delta_{i-1,i-1}} + \varepsilon_{ik} + \sum_{s=i}^{k-1} a_{is}\delta_{sk} \right] r_{i-1}(x). \tag{5.21}$$
Comparing (5.17) and (5.21) we get (5.4) by induction. □

6. Relationship between these subclasses of (H, m)–quasiseparable matrices

Thus far it has been proved that the classes of (H, m)–semiseparable and (H, m)–well–free matrices are subclasses of the class of (H, m)–quasiseparable matrices. The only unanswered questions to understand the interplay between these classes is whether these two subclasses have common elements or not, and whether either class properly contains the other or not.

It was demonstrated in [BEGO08] that there is indeed a nontrivial intersection of the classes of $(H, 1)$–semiseparable and $(H, 1)$–well–free matrices, and so there is at least some intersection of the (weakly) (H, m) versions of these classes. In the next example it will be shown that such a nontrivial intersection exists in the rank m case; that is, there exist matrices that are both (H, m)–semiseparable and (H, m)–well–free.

Example 6.1. Let A be an (H, m)–quasiseparable matrix whose generators satisfy

$$b_k = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \ddots & \vdots & 1 \\ 0 & 1 & \ddots & 0 & \vdots \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 1 \end{bmatrix} \in \mathbb{C}^{m\times m}, \quad k = 2, \ldots, n-1, \qquad h_k = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{C}^{m}, \quad k = 2, \ldots, n.$$

Regardless of the other choices of generators, one can see that these generators satisfy both Lemma 4.5 and Theorem 5.7, and hence the matrix A is both (H, m)–well–free and (H, m)–semiseparable.

The next example demonstrates that a (H, m)–semiseparable matrix need not be (H, m)–well–free.

Example 6.2. Consider the (H, m)–quasiseparable matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 1 & \cdots & 1 \\ 1 & 0 & 0 & 0 & 1 & \cdots & 1 \\ 0 & 1 & 1 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & 1 & 1 & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 1 & \ddots & 1 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Because of the shaded block of zeros, it can be seen that the matrix is not (H, m)–well–free for any m (provided the matrix is at least 5×5). However, one can observe that $\operatorname{rank}(\operatorname{triu}(A, 1)) = 2$, and hence A is $(H, 2)$–semiseparable. Furthermore, by modifying the 1 elements of the second superdiagonal, this example can be modified to produce an (H, m)–semiseparable matrix for $m = 2, \ldots, n-3$. Thus the class of (H, m)–semiseparable matrices does not contain the class of (H, m)–well–free matrices.

To see that a (H, m)–well–free matrix need not be (H, m)–semiseparable, consider the banded matrix (1.8) from the introduction. It is easily verified to not be (H, m)–semiseparable (for $m < n - l$), however it is $(H, l-2)$–well–free.

This completes the discussion on the interplay of the subclasses of (H, m)–quasiseparable matrices, as it has been shown that there is an intersection, but neither subclass contains the other. Thus the proof of Figure 3 is completed.

7. Conclusion

To conclude, appropriate generalizations of real orthogonal polynomials and Szegö polynomials, as well as several subclasses of $(H, 1)$–quasiseparable polynomials, were used to classify the larger class of (H, m)–quasiseparable matrices for arbitrary m. Classifications were given in terms of recurrence relations satisfied by related polynomial systems, and in terms of special restrictions on the quasiseparable generators.

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