Numerical algebraic intersection using regeneration

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Abstract

In numerical algebraic geometry, algebraic sets are represented by witness sets. This paper presents an algorithm, based on the regeneration technique, that solves the following problem: given a witness set for a pure-dimensional algebraic set Z, along with a system of polynomial equations $f : Z \to \mathbb{C}^n$, compute a numerical irreducible decomposition of $V = Z \cap \mathcal{V}(f)$. An important special case is when $Z = A \times B$ for irreducible sets A and B and $f(x, y) = x - y$ for $x \in A$, $y \in B$, in which case V is isomorphic to $A \cap B$. In this way, the current contribution is a generalization of existing diagonal intersection techniques. Another important special case is when $Z = A \times \mathbb{C}^k$, so that the projection of V dropping the last k coordinates consists of the points $x \in A$ where there exists some y in a new set of variables such that $f(x, y) = 0$. This arises in many contexts, such as finding the singularities of A, in which case $f(x, y)$ can be a set of singularity conditions that involve new variables associated to the tangent space of A. The combining of multiple intersection scenarios into one common scheme brings new capabilities and organizational simplification to numerical algebraic geometry.

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1 Introduction

Numerical algebraic geometry concerns the solution of systems of polynomial equations using numerical methods, principally homotopy methods, also known as polynomial continuation. The ground field is assumed to be $\mathbb{C}$ so that continuity applies. One of the main accomplishments of the field of numerical algebraic geometry is the capability to compute a numerical irreducible decomposition of the solution set of a polynomial

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system, first accomplished in [25]. That is, given a polynomial system $F : \mathbb{C}^N \to \mathbb{C}^n$:

$$F(z_1, \ldots, z_N) = \begin{bmatrix} F_1(z_1, \ldots, z_N) \\ \vdots \\ F_n(z_1, \ldots, z_N) \end{bmatrix}, \tag{1}$$

one wishes to decompose its solution set

$$\mathcal{V}(F) := \{z \in \mathbb{C}^N \mid F(z) = 0\}$$

into a union of its irreducible components[1]. Any algebraic set has such a decomposition.

A pure-dimensional algebraic set has the same dimension at every point of the set. Such a set, say $A \subset \mathbb{C}^N$, is represented in numerical algebraic geometry by a finite set of data called a *witness set*. A witness set consists of three entries:

1. a polynomial system, f_A, such that A is a pure-dimensional component of $\mathcal{V}(f_A)$;
2. a generic affine linear polynomial system, $L_A : \mathbb{C}^N \to \mathbb{C}^{\dim A}$; and
3. numerical approximations to the witness point set $W_A = A \cap \mathcal{V}(L_A)$, which is a set of $\deg A$ points.

Accordingly, we write a witness set for A as the ordered set $\mathcal{A} = \{f_A, L_A, W_A\}$. The adjective "generic" in Item 2 is key: there exist linear polynomial systems whose solution sets do not intersect A as required in Item 3, but these are a proper algebraic subset of all possible linear mappings from $\mathbb{C}^N$ to $\mathbb{C}^{\dim A}$. "Generic" means that L_A is not one of these; it is a member of the dense, Zariski-open subset of linear systems whose solution sets intersect A in $\deg A$ points. Clearly, this definition of a witness set also applies to irreducible algebraic sets. A *numerical irreducible decomposition* of the solution set $\mathcal{V}(F)$ consists of one witness set for each of its irreducible components.

For any $a \in W_A$, we know that a is an isolated point in $\mathcal{V}(f_A, L_A)$. If $A \subset \mathcal{V}(f_A)$ is irreducible, then the multiplicity of A with respect to f_A is equal to the multiplicity of a with respect to $\{f_A, L_A\}$. If the multiplicity of A is 1 with respect to f_A, A is said to be *generically reduced*. Otherwise, A is said to be *generically nonreduced*. The focus is on the generically reduced case with §3.1 showing that problems involving generically nonreduced algebraic sets can be solved using the generically reduced case.

Given an irreducible decomposition, one may wish to investigate certain irreducible components further, and in particular, one might wish to find various kinds of intersections. For example, if A is an irreducible component of $\mathcal{V}(f_A)$ and B is an irreducible component of $\mathcal{V}(f_B)$, one may wish to find their common points, $A \cap B$. A special case of this is membership testing, in which one wishes to determine if a given point $x^* \in \mathbb{C}^N$ is in A. This is equivalent to deciding if $A \cap \{x^*\} = \{x^*\}$ or $A \cap \{x^*\} = \emptyset$. Another possibility is that one might wish to extend the analysis of A to find $(A \times \mathbb{C}^k) \cap \mathcal{V}(g(x, y))$,

[1] An irreducible algebraic set is an algebraic set that cannot be expressed as the union of a finite number of its proper algebraic subsets.

where $y \in \mathbb{C}^k$ is some set of newly introduced variables. This may occur, for example, if one wishes to find the singularities of A and $g(x,y) = 0$ is a set of singularity conditions that includes new variables associated to tangent directions in A. The system g could also define other "critical" conditions on A, such as computing critical points of the distance to a fixed point, e.g., [11], and critical points of a projection, e.g., [7, 19].

In this article, we place all these intersection questions into a common setting. Given an irreducible set Z and a polynomial system $f : Z \to \mathbb{C}^n$, we wish to find $V = Z \cap \mathcal{V}(f)$. The examples of the previous paragraph fit this format as follows.

1. For irreducible algebraic sets $A, B \subset \mathbb{C}^N$, the set $A \cap B$ is isomorphic to

$$\{(x,y) \in \mathbb{C}^{2N} \mid x \in A, y \in B, x = y\} = (A \times B) \cap \mathcal{V}(x - y),$$

so we take $Z = A \times B$ and $f = x - y$.

2. Testing if $x^* \in A$ is a special case of the former with $Z = A \times \{x^*\}$ and $f = x - x^*$.

3. For the set $(A \times \mathbb{C}^k) \cap \mathcal{V}(g(x,y))$, we take $Z = A \times \mathbb{C}^k$ and $f = g$.

The treatment of $A \cap B$ as in Item 1 is called "reduction to the diagonal." This was used in [23], where a cascade of homotopies produces witness sets for $V = (A \times B) \cap \mathcal{V}(x - y)$. Although V is isomorphic to $A \cap B$ using the maps $(x,y) \mapsto (x)$ and $(x) \mapsto (x,x)$, it turns out that extra work can be required to build witness sets for the components of $A \cap B$ from those found for V. This was described in [17] using the theory of isosingular sets from [16]. We discuss relevant details in §4.

Like the earlier cascade algorithm of [30] or the dimension-by-dimension algorithm of [29, Chap. 13], the regenerative cascade algorithm of [15] generates witness point supersets for the pure-dimensional components of $\mathcal{V}(F)$. A witness point superset for the pure i-dimensional algebraic set $V_i \subset \mathcal{V}(F)$ is of the form $\widehat{W}_i = W_i \cup J_i$, where W_i is a witness point set for V_i and J_i consists of a finite number of points contained in the union of components of $\mathcal{V}(F)$ of dimension greater than i. The full process of numerical irreducible decomposition consists of the following three steps:

- find a witness point superset $\widehat{W}_i$ for each pure i-dimensional component of $\mathcal{V}(F)$;
- eliminate the "junk sets", J_i, from $\widehat{W}_i$ to obtain W_i; and
- partition W_i into witness point sets for the irreducible components of dimension i.

The elimination of junk for $\mathcal{V}(F)$ can be accomplished using a local dimension test [4] or, if the multiplicity depth is too large for that to be practical, by using the homotopy membership test [26] against all the higher-dimensional solution components. The break-up into irreducible components is done with monodromy [28] backed up by linear trace tests [27]. A good general reference to these techniques is [29].

All of the pre-existing methods for generating witness point supersets apply only to $\mathcal{V}(F)$ or, in the case of [23], to $(A \times B) \cap \mathcal{V}(x - y)$. The main contribution of this article is to treat the more general case of $Z \cap \mathcal{V}(f)$ and to illustrate its usefulness on several examples in §6. In particular, an algorithm patterned after the regenerative cascade algorithm is presented in §3 for computing witness point supersets for the $Z \cap \mathcal{V}(f)$ setting. The last two steps of the decomposition are discussed in §4. Also, as mentioned above, this section shows that the theory of isosingular sets from [16] can be used to finish the construction of the witness sets from the witness point supersets.

2 Cross products of irreducible sets

As noted above, it is useful to apply the capability to compute $Z \cap \mathcal{V}(f)$ to situations where $Z = A \times B$ for irreducible algebraic sets A and B. One could just as well consider the case where A and B are pure-dimensional but not necessarily irreducible, in which case Z is the union of the cross products of all pairs of irreducible components taking one from A and one from B. We shall concentrate on the case of irreducible sets, but one may see that the same results hold for pure-dimensional sets by applying the arguments to each pair.

Let $A \subset \mathbb{C}^{N_A}$ and $B \subset \mathbb{C}^{N_B}$ be irreducible algebraic sets and let $Z = A \times B$. Let z be a set of coordinates on $\mathbb{C}^{N_A+N_B}$ and let $\bar{z} = \begin{bmatrix} z \\ 1 \end{bmatrix}$. Then, we have the following facts:

1. Z is an irreducible algebraic set;
2. $\dim Z = \dim A + \dim B$;
3. $\deg Z = \deg A \cdot \deg B$; and
4. there exists a dense Zariski open subset U of $\mathbb{C}^{\dim Z \times (N_A+N_B+1)}$ such that for all $P \in U$ the set $W_Z := Z \cap \mathcal{V}(P \cdot \bar{z})$ consists of $\deg Z$ isolated points.

The fact that Z is an algebraic set of dimension $\dim A + \dim B$ follows directly from observing that the set of polynomials which vanish on Z is generated by the polynomials which vanish on $A \times \mathbb{C}^{N_B}$ and the polynomials which vanish on $\mathbb{C}^{N_A} \times B$, e.g., [9, Exercise 13.13]. This observation regarding the vanishing polynomials also provides that $\deg(A \times \mathbb{C}^{N_B}) = \deg A$ and $\deg(\mathbb{C}^{N_A} \times B) = \deg B$.

The irreducibility of Z follows directly from the irreducibility of A and B as follows. Suppose that Z_1 and Z_2 are algebraic sets with $Z = Z_1 \cup Z_2$. For every $a \in A$, consider the irreducible set $B_a = \{a\} \times B$. Since $B_a = (B_a \cap Z_1) \cup (B_a \cap Z_2)$, the irreducibility of B_a implies that $B_a \subset Z_1$ or $B_a \subset Z_2$. Therefore, $A = A_1 \cup A_2$ where

$$A_i = \{a \in A \mid B_a \subset Z_i\}.$$

Since A is irreducible, it follows that $A_1 = A$ or $A_2 = A$ so that $Z_1 = Z$ or $Z_2 = Z$.

Suppose that $\mathcal{L}_A \subset \mathbb{C}^{N_A}$ and $\mathcal{L}_B \subset \mathbb{C}^{N_B}$ are general linear spaces of codimension $\dim A$ and $\dim B$, respectively. Then, $\mathcal{L} = \mathcal{L}_A \times \mathcal{L}_B$ is a linear space of codimension

$\dim A + \dim B = \dim Z$ with

$$Z \cap \mathcal{L} = (A \cap \mathcal{L}_A) \times (B \cap \mathcal{L}_B)$$

which consists of $\deg A \cdot \deg B$ points. This shows $\deg Z \geqslant \deg A \cdot \deg B$. Bézout's Theorem provides the other inequality, namely

$$\deg(A \times B) = \deg((A \times \mathbb{C}^{N_B}) \cap (\mathbb{C}^{N_A} \times B)) \leqslant \deg(A \times \mathbb{C}^{N_B}) \cdot \deg(\mathbb{C}^{N_A} \times B) = \deg A \cdot \deg B.$$

The fourth follows from Facts 2 and 3 and the Slicing Theorems [29, Thms. 13.2.1,13.2.2].

In particular, this proof of the third fact shows that intersecting with a linear space constructed by the cross product general linear spaces in each set of variables produces the same number of points as with a general linear space in all variables. Hence, a witness set for $Z = A \times B$ can be easily constructed from witness sets for A and B as shown in the following lemma.

Lemma 2.1 *Let $\{f_A, L_A, W_A\}$ and $\{f_B, L_b, W_B\}$ be witness sets for irreducible and generically reduced algebraic sets $A \subset \mathbb{C}^{N_A}$ and $B \subset \mathbb{C}^{N_B}$, and let $Z = A \times B$. Also, let $\bar{z} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$, where x and y are coordinates on $\mathbb{C}^{N_A}$ and $\mathbb{C}^{N_B}$, respectively. Then, for P chosen at random from $\mathbb{C}^{(\dim A + \dim B) \times (N_A + N_B + 1)}$, the homotopy*

$$h_0(x, y, t) = Z \cap \mathcal{V}\left(t \begin{bmatrix} L_A(x) \\ L_B(y) \end{bmatrix} + (1-t)(P \cdot \bar{z})\right) \tag{2}$$

starting with the $\deg A \cdot \deg B$ *points $W_A \times W_B$ at $t = 1$ is, with probability one, a* complete homotopy *(in the sense defined in [14]) for $Z \cap \mathcal{V}(P \cdot \bar{z})$. Moreover, letting W_Z be the set of endpoints of this homotopy at $t = 0$, $\{(f_A(x), f_B(y)), P \cdot \bar{z}, W_Z\}$ is a witness set for Z.*

Proof. A *complete homotopy for Y* means that the paths are all trackable (exist, are continuous, and advance strictly monotonically with respect to $t \in (0, 1]$; see [14]) and their set of endpoints (the limits as $t \to 0$ of the paths) include all the isolated points in Y. In the lemma, $Y = Z \cap \mathcal{V}(P \cdot \bar{z}) = W_Z$. But from the facts above, we know that W_Z is a set of $\deg Z = \deg A \cdot \deg B$ points for generic $P \in \mathbb{C}^{(\dim A + \dim B) \times (N_A + N_B + 1)}$. By the parameter homotopy theorem [29, Thm. 7.1.1] (also [21]) and Lemma 7.1.2 of [29], homotopy h_0 defines $\deg Z$ trackable solution paths for $t \in [0, 1)$ starting at W_Z with endpoints that include all isolated roots of $H(x, y, 1) = 0$. But at $t = 1$, $h_0(x, y, 1) = 0$ has the $\deg Z$ isolated roots $W_A \times W_B$, so these must be the endpoints of the homotopy paths going from $t = 0$ to $t = 1$. As stated, the homotopy runs in the opposite direction, from $t = 1$ to $t = 0$, but the genericity of L_A and L_B along with the assumption that A and B are generically reduced imply that the points $W_A \times W_B$ are nonsingular roots of $h_0(x, y, 1) = 0$, hence the homotopy paths are nonsingular for

$t \in [0,1]$. The final assertion that we have a witness set for Z is merely an observation that $\{(f_A(x), f_B(y)), P \cdot \bar{z}, W_Z\}$ fits the definition of a witness set. $\square$

In practice, one reduces to the case that $f_A : \mathbb{C}^{N_A} \to \mathbb{C}^{N_A - \dim A}$ and $f_B : \mathbb{C}^{N_B} \to \mathbb{C}^{N_B - \dim B}$ via randomization with Bertini's Theorem. Then, h_0 from (2) is defined by the square homotopy

$$h_0(x, y, t) = \begin{bmatrix} f_A(x) \\ f_B(y) \\ t \begin{bmatrix} L_A(x) \\ L_B(y) \end{bmatrix} + (1-t) P \cdot \bar{z} \end{bmatrix} \tag{3}$$

Remark 2.2 If the requirement that A and B be generically reduced is dropped, the homotopy paths still exist by the parameter homotopy theorem for isolated roots [29, Thm. 7.1.6]. After deflating the paths as in [14], they can be tracked using a standard nonsingular path tracker. While this is a feasible approach, if either of A or B is generally nonreduced, we generally prefer to deflate before applying Lemma 2.1 (see §3.1).

Remark 2.3 For several irreducible algebraic sets, say $A_1, \ldots, A_k$, it is obvious that Lemma 2.1 can be applied $(k-1)$ times in succession to obtain a witness set of $Z = A_1 \times \cdots \times A_k$. This is not necessary since $\dim Z = \dim A_1 + \cdots + \dim A_k$ and $\deg Z = \deg A_1 \cdots \deg A_k$ provides that a single homotopy of the form

$$h_0'(x_1, \ldots, x_k, t) = Z \cap \mathcal{V}\big(t\{L_{A_1}(x_1), \ldots, L_{A_k}(x_k)\} + (1-t)(P \cdot \bar{z})\big) \tag{4}$$

suffices where the dimension of P is adjusted appropriately and the rest of the notation should be clear from context.

Remark 2.4 If one of the factors in the cross product is Euclidean, say $B = \mathbb{C}^k$, then f_B is empty and W_B is the unique root of an arbitrary full-rank square linear system, $L_B : \mathbb{C}^k \to \mathbb{C}^k$. Everything proceeds in the same way as for a more general B.

3 Regenerative cascade

The objective of this section is to solve the following problem.

Problem 1 *Let $Z \subset \mathbb{C}^N$ be a pure-dimensional algebraic set, with witness set $\mathcal{Z} = \{f_Z, L_Z, W_Z\}$, and let $f : \mathbb{C}^N \to \mathbb{C}^n$ be a polynomial system. For each $i = 0, \ldots, \dim Z$, compute a witness point superset, $\widehat{W}_i$, for the pure i-dimensional component of $Z \cap \mathcal{V}(f)$.*

First, let us review some terminology. For $x \in \mathbb{C}^N$, we say that x is a nonsolution of f if $f(x) \neq 0$. Recall that rank$f$ is defined as the rank of the Jacobian matrix of partial derivatives of f evaluated at a generic point of $\mathbb{C}^N$.

With these terms, we may state a slight generalization of the regenerative cascade method of [15] as follows. Let us denote $L_Z = \{L_1, \ldots, L_{\dim Z}\}$, where each of $L_1, \ldots, L_{\dim Z}$ is a single linear function $\mathbb{C}^N \to \mathbb{C}$.

Begin Algorithm 1

- Compute $r = \min(\operatorname{rank} f, \dim Z)$.
- Choose a generic $r \times n$ matrix Λ and define $F = \{F_1, \ldots, F_r\}$ as the polynomials formed as $F = \Lambda \cdot f$, where F and f are treated as column matrices. See Note 1 below for options.
- Set $m = 0$ and $U_0 = W_Z$.
- While true:
  1. Sort U_m into $U_m = \widehat{W}_{\dim Z - m} \cup X_m$, where $\widehat{W}_{\dim Z - m}$ are solutions and X_m are nonsolutions of f.
  2. If $m = r$, exit loop.
  3. Set $m = m + 1$.
  4. Form generic linear functions $L_{m,1}, \ldots, L_{m,d_m}$, each a map from $\mathbb{C}^N$ to $\mathbb{C}$, where $d_m = \deg F_m$. See Note 2 below.
  5. For $i = 1, \ldots, d_m$:
     - Track the solution paths of
     $$\begin{aligned} h_{m,i}(z,t) = \{ & f_Z(z), F_1(z), \ldots, F_{m-1}(z), \\ & tL_m(z) + (1-t)L_{m,i}(z), \\ & L_{m+1}(z), \ldots, L_{\dim Z}(z)\} \end{aligned} \tag{5}$$
     from $t = 1$ towards $t = 0$, starting at the points X_{m-1}. Use a singular endgame, if necessary, to compute the endpoints (the limit as $t \to 0$) of each path.
     - Let $T_{m,i}$ be the endpoints of the homotopy paths for $h_{m,i}$.
  6. End for.
  7. Let $T_m = \cup_{i=1}^{d_m} T_{m,i}$.
  8. Track the solution paths of
  $$\begin{aligned} h_m(z,t) = \{ & f_Z(z), F_1(z), \ldots, F_{m-1}(z), \\ & t\prod_{i=1}^{d_m} L_{m,i}(z) + (1-t)F_m(z), \\ & L_{m+1}(z), \ldots, L_{\dim Z}(z)\} \end{aligned} \tag{6}$$
  from $t = 1$ towards $t = 0$, starting at the points T_m. Use a singular endgame, if necessary, to compute the endpoints.

9. Let U_m be the finite endpoints of the homotopy paths for h_m.

- End while.
- If $r < \dim Z$, then $\widehat{W}_i = \emptyset$ for all $i < \dim Z - r$.
- Return $\{\widehat{W}_0, \ldots, \widehat{W}_{\dim Z}\}$.

End Algorithm 1

Notes:

1. The matrix Λ can be chosen with all entries below the main diagonal as zero. In addition, the entries in f can be reordered arbitrarily, and it is generally advantageous to place them in descending order by degree, in which case the entries in F are also ordered by descending degree.

2. Since $\{L_1, \ldots, L_{\dim Z}\}$ are generic, it is acceptable to choose in every case $L_{i,1} = L_i$. In that case, homotopy $h_{m,1}$ is trivial, and no computation is required to find $T_{m,1}$: it is exactly X_{m-1}.

We will say that Algorithm 1 solves Problem 1 if every homotopy path in the algorithm is trackable and the output is a valid set of witness point supersets for $Z \cap \mathcal{V}(f)$.

Theorem 3.1 *For random choices of all its generic coefficients, Algorithm 1 solves Problem 1 with probability one.*

Proof. The regenerative cascade method of [15] solves Problem 1 for the case that $Z = \mathbb{C}^N$ for some N. However, examination of the proof of that procedure with reference to the Simple Bertini Theorem for Systems [29, Thm. A.8.7] shows that it also holds if Z is any irreducible quasiprojective algebraic set. Furthermore, if Z is the union of irreducible quasiprojective sets of the same dimension, then there is no difference in carrying out the algorithm on one of these components at a time versus carrying it out on the whole set Z, that is, we do not need the witness point set W_Z to be decomposed into witness point sets for the irreducible components of Z. $\square$

3.1 Generically nonreduced sets

The statement of Problem 1 did not require Z to consist of the union of generically reduced irreducible sets $\mathcal{V}(f_Z)$, that is, Z might be a union of several irreducible solution components of $\mathcal{V}(f_Z)$ where some of these might have multiplicity larger than one. If each component of Z is generically reduced, then we have the desirable property that every nonsolution point in X_m and every point in T_m produced by the algorithm is nonsingular with respect to the corresponding polynomial systems. Then, the solution paths of every $h_{m,i}(z,t) = 0$ and $h_m(z,t) = 0$ are also nonsingular for $t \in (0,1]$, so a nonsingular path tracker suffices.

In the case that Z has one or more irreducible components that is generically nonreduced, we can only say that the points X_m and T_m are isolated, and that the homotopy paths exist and are trackable in the sense defined in [14]. Some of the paths will be singular, that is, the Jacobian matrix of partial derivatives of the homotopy function with respect to z is not full-rank. As detailed in [14], trackable paths can be desingularized using a deflation operation, after which a nonsingular path tracker can be applied.

When faced with a generically nonreduced component of Z, an alternative is to deflate it first before applying the regenerative cascade algorithm. Methods for deflating a generically nonreduced irreducible set can be found in [16]. Since different irreducible components of Z may have different deflation sequences (equivalent to Thom-Boardman singularity sequences [2, 8] and summarized in §4), it is required to partially decompose Z based on identical deflation sequences. Irreducible components with the same deflation sequence can be simultaneously deflated. The determinantal deflation procedure of [16], which does not introduce any auxiliary variables, is directly compatible with the regenerative cascade.

The other deflation procedures summarized in [16] introduce new variables associated to tangent directions on the set. For these, the regenerative cascade requires minor modifications. If we suppose that Z is irreducible and generically nonreduced, then such a deflation produces a polynomial system g, an irreducible and generically reduced algebraic set $Y \subset \mathcal{V}(g)$, and a projection map π which is generically one-to-one on Y with $Z = \overline{\pi(Y)}$. Thus, one may attempt to perform computations on Y in place of Z. Following the approach of [12], one needs to change from considering paths which converge in Y to considering the paths for which the image under π converges. For example, for each $z \in Z \backslash \pi(Y)$, there is a path $\alpha : (0, 1] \to Y$ such that $z = \lim_{t \to 0} \pi(\alpha(t))$. However, $\alpha(t)$ must diverge as t approaches 0 since $z \notin \pi(Y)$. Here, only the endpoint of the image under π of the path defined by α was outside of $\pi(Y)$. By genericity, this is true for all paths arising in the regenerative cascade method as well.

3.2 Extrinsic and intrinsic homotopy

In both the homotopies $h_{m,i}$ of (5) and h_m of (6), the linear functions $L_{m+1}, \ldots, L_{\dim Z}$ stand unperturbed as t varies. The same is true for any linear functions in the system F_Z or among the functions $F_1, \ldots, F_{m-1}$ at stage m of the cascade. Gathering all the unchanging linear functions into one linear subsystem of the homotopy, one may use linear algebra to compute the kernel of this subsystem before path tracking commences and then restrict computation in the path tracker to that subspace. Such an approach is said to be working intrinsically on the linear subspace. In contrast, an extrinsic method treats the linear functions just like any other polynomial in the system and, in essence, re-solves the linear part at each step of the path tracker. If there are enough linear functions present, then the intrinsic approach is more efficient, but if there are only a few unchanging linear functions, the extrinsic approach wins. When extrinsic wins, it is because the number of elements in the representation of a basis for the kernel is large, which raises the expense of working intrinsically. Intrinsic implementations of homotopies are discussed in [24, 14, 15] and specifics on assessing the trade-off between extrinsic and intrinsic formulations can be found in [17].

4 Completing the decomposition

The output of Algorithm 1 is a witness point superset, $\widehat{W}_i$, for each pure i-dimensional component of $Z \cap \mathcal{V}(f)$. As described in the introduction, two steps remain to produce a numerical irreducible decomposition, namely:

- eliminate the "junk sets", J_i, from $\widehat{W}_i$ to obtain W_i; and
- partition W_i into witness point sets for the irreducible components of dimension i.

If Z is an irreducible component of $\mathcal{V}(g)$ such that each irreducible component of $Z \cap \mathcal{V}(f)$ is an irreducible component of $\mathcal{V}(g, f)$, then all of the standard numerical algebraic geometric methods apply. That is, the local dimension test [4] and the homotopy membership test [26] can be used to identify J_i, the irreducible components can be deflated at each witness point using positive-dimensional deflation techniques first described in [29, §13.3.2, §15.2.2], and monodromy [28] backed up by linear trace tests [27] can be used to decompose W_i. The following example, motivated by [17, Ex. 2.0.2], highlights some shortcomings of these standard techniques in the context of intersection.

Example 4.1 *Consider $g(x, y, z) = (x+y+z)y$ with $Z = \mathcal{V}(x+y+z)$ and $f(x, y, z) = y$. Clearly, $Z \cap \mathcal{V}(f) = \mathcal{V}(x+z, y)$ is a line but $\mathcal{V}(g, f) = \mathcal{V}(y)$ is a plane. In particular, for each $a \in \mathbb{C}$, the local dimension of $(a, 0, -a)$ with respect to $Z \cap \mathcal{V}(f)$ is 1, but is 2 with respect to $\mathcal{V}(g, f)$. Thus, the local dimension test of [4] can not be used for computing the "junk sets" J_i. Also, each $(a, 0, -a)$ is a smooth point on the plane $\mathcal{V}(g, f) = \mathcal{V}(y)$ showing that standard deflation techniques applied to $\{g, f\}$ at $(a, 0, -a)$ will fail to provide a polynomial system with the requisite irreducible component. Without such a system, the homotopy membership test, monodromy, and linear trace tests can not be employed.*

The key to overcoming these shortcomings and enabling the computation of a numerical irreducible decomposition for $Z \cap \mathcal{V}(f)$ is the theory of isosingular sets [16]. Let g be the polynomial system given at the outset in the witness set for Z; that is, Z is an irreducible component of $\mathcal{V}(g)$. After completing the regeneration cascade, we have witness point supersets $\widehat{W}_i = W_i \cup J_i$. For each witness point $w \in W_i$ with corresponding irreducible component $X \subset Z \cap \mathcal{V}(f)$, we describe below how to use isosingular theory to construct a polynomial system f_w from w, g, and f such that X is an irreducible component of $\mathcal{V}(f_w)$. At the largest k such that $\widehat{W}_k \neq \emptyset$, we know that $\widehat{W}_k = W_k$; that is, $J_k = \emptyset$. So we know that every point in W_k is a witness point, not junk, and therefore once f_w is in hand, monodromy, and linear trace tests discover which of the witness points form a witness set for X and likewise for all of the components at dimension k. Via the homotopy membership test, we can use these witness sets to eliminate from the lower-dimensional witness point supersets any junk points lying on sets of dimension k, and in particular, eliminate all junk from $\widehat{W}_{k-1}$. This leaves us in position to decompose W_{k-1} and to proceed in like fashion down through all the dimensions.

The key step of constructing system f_w depends on isosingular deflation, for which we provide only the key concepts here. We refer the interested reader to [16] for a general overview with [17] providing details related to diagonal intersection.

For a polynomial system G and a point $z \in \mathcal{V}(G) \subset \mathbb{C}^N$, $\mathrm{Iso}_G(z)$ is an irreducible algebraic subset of $\mathcal{V}(G)$ containing z. Since the definition of $\mathrm{Iso}_G(z)$ depends on the deflation sequence of G at z, we define this first and then construct $\mathrm{Iso}_G(z)$. Let

$$\mathrm{dnull}(G, z) := \dim \mathrm{null} JG(z) = N - \mathrm{rank} JG(z)$$

where $JG(z)$ is the Jacobian matrix of G evaluated at z and $\mathrm{null}JG(z)$ is the right null space of $JG(z)$. Let $D(G, z)$ be the polynomial system consisting of G and the $(\mathrm{rank}JG(z)+1) \times (\mathrm{rank}JG(z)+1)$ minors of JG. Define $D^0(G, z) := G$ and, for $k \geqslant 1$, let $D^k(G, z)$ be the polynomial system obtain in this fashion after iterating k times. In particular, $D^k(G, z)$ is the polynomial system for the k^{th} isosingular deflation of G at z. With this setup, the deflation sequence of G at z is

$$d_k(G, z) := \mathrm{dnull}(D^k(G, z), z), \qquad k = 0, 1, \ldots$$

The algebraic closure of the set of points in $\mathcal{V}(G)$ which have the same deflation sequence as z with respect to G is an algebraic set that may decompose into several irreducible components. The irreducible set $\mathrm{Iso}_G(z)$ is the unique such irreducible component that contains z. In particular, the deflation sequence of points in $\mathrm{Iso}_G(z)$ is constant on a nonempty Zariski open subset. Such statement is true for any irreducible algebraic subset of $\mathcal{V}(G)$.

The following theorem, which is a slight generalization of [17, Thm. 5.1.1], describes deflating the irreducible components of $Z \cap \mathcal{V}(f)$.

Theorem 4.2 *Let $Z \subset \mathcal{V}(g) \subset \mathbb{C}^N$ be a union of irreducible components and f be a polynomial system defined on Z. Let $F(x, y) = \{g(x), f(y)\}$, $\Delta = \{(x, x) \mid x \in \mathbb{C}^N\}$, and $\pi(x, y) = x$. If $A \subset Z \cap \mathcal{V}(f)$ is an irreducible component, there exists a nonempty Zariski open set $U \subset A$ such that for all $a \in U$, A is an irreducible component of $\pi(\mathrm{Iso}_F((a, a)) \cap \Delta)$. In particular, for a linear space $\mathcal{L} \subset \mathbb{C}^N$ of dimension $N - \dim A$ chosen randomly, then, with probability one, the set of witness points $A \cap \mathcal{L}$ is contained in U.*

Proof. Since $\mathcal{A} = (A \times A) \cap \Delta$ is an irreducible algebraic set contained in $\mathcal{V}(F)$, there is a nonempty Zariski open set $\mathcal{U} \subset \mathcal{A}$ such that the deflation sequence with respect to F is constant on $\mathcal{U}$. Thus, $U = \pi(\mathcal{U})$ is a nonempty Zariski open subset of A with $\mathcal{U} = (U \times U) \cap \Delta$. Fix $a \in U \subset A$. Since $\mathcal{A} \subset \mathrm{Iso}_F((a, a)) \cap \Delta$, there is an irreducible component $\mathcal{W} \subset \mathrm{Iso}_F((a, a)) \cap \Delta$ with $W = \pi(\mathcal{W})$ such that $\mathcal{A} \subset \mathcal{W}$ and

$A = \pi(\mathcal{A}) \subset \pi(\mathcal{W}) = W$. Thus, W is an irreducible component of $\pi\left(\mathrm{Iso}_F((a,a)) \cap \Delta\right)$ containing A. The first part of the theorem will follow by simply showing that $A = W$.

Since $A \subset Z$ and Z is a union of irreducible components of $\mathcal{V}(g)$, there is an irreducible algebraic set Z' which an irreducible component of $\mathcal{V}(g)$ such that $A \subset Z' \subset Z$. Similarly, since $A \subset \mathcal{V}(f)$, there is an irreducible component $V' \subset \mathcal{V}(f)$ such that $A \subset V'$. Hence, $Z' \times V'$ is an irreducible component of $\mathcal{V}(F)$ with $(a,a) \in Z' \times V'$. In particular, $\mathrm{Iso}_F((a,a)) \subset Z' \times V' \subset Z \times \mathcal{V}(f)$. Therefore,

$$A \subset W \subset \pi\left(\mathrm{Iso}_F((a,a)) \cap \Delta\right) \subset \pi\left((Z \times \mathcal{V}(f)) \cap \Delta\right) = Z \cap \mathcal{V}(f).$$

Since A is an irreducible component of $Z \cap \mathcal{V}(f)$, it follows that $A = W$.

If $\dim A = i$, the last statement follows from the the fact that, for a general linear subspace $\mathcal{L}_i$ of codimension i, $A \cap \mathcal{L}_i = U \cap \mathcal{L}_i$. □

Since each isosingular set is deflatable, Theorem 4.2 provides a prescription for constructing polynomial systems that can be used to complete the last two steps of computing a numerical irreducible decomposition of $Z \cap \mathcal{V}(f)$. Although we have no example, the corresponding irreducible component may be generically nonreduced with respect to the polynomial system constructed in this fashion. However, one may simply use finitely many more steps of isosingular deflation or another deflation procedure for irreducible components, e.g., [29, §13.3.2, §15.2.2], to simplify to the generically reduced case.

The following example demonstrates using Theorem 4.2 on Example 4.1.

Example 4.3 *For g, f, and Z as described in Example 4.1, the algebraic set $A = Z \cap \mathcal{V}(f) = \mathcal{V}(x+z, y)$ is irreducible. Theorem 4.2 allows us to derive this result algorithmically as follows. Suppose we have the witness point $w = (a, 0, -a) \in A$ and the polynomials g and f. We wish to find a polynomial system f_w such that A is an irreducible component of $\mathcal{V}(f_w)$. Following the setup in Theorem 4.2, we first form*

$$F(x,y,z,x',y',z') = \begin{bmatrix} g(x,y,z) \\ f(x',y',z') \end{bmatrix} = \begin{bmatrix} (x+y+z)y \\ y' \end{bmatrix}.$$

For general $a \in \mathbb{C}$, one can verify that the deflation sequence of $(a,0,-a,a,0,-a)$ with respect to F is $5, 3, 3, \ldots$ with the corresponding 3-dimensional set

$$\mathrm{Iso}_F((a,0,-a,a,0,-a)) = \{(b,0,-b,\alpha,0,\beta) \mid b,\alpha,\beta \in \mathbb{C}\}.$$

This isosingular set is defined by adding the 2×2 minors of JF to F. That is, $\mathrm{Iso}_F((a,0,-a,a,0,-a))$ is an irreducible component of $\mathcal{V}(G)$ where

$$G(x,y,z,x',y',z') = \begin{bmatrix} (x+y+z)y \\ y' \\ y \\ x+2y+z \\ y \end{bmatrix}.$$

Theorem 4.2 implies that A is an irreducible component of $\mathcal{V}(G(x,y,z,x,y,z))$; that is, $f_w(x,y,z) = G(x,y,z,x,y,z)$ suffices to form a witness set for A. It is easily seen that this can be simplified to just $f_w(x,y,z) = \{x+z, y\}$.

5 Special cases

In the introduction, several special cases that are amenable to the techniques of Lemma 2.1 and Algorithm 1 were discussed. We now return to these to elucidate some details.

5.1 Diagonal intersection

To compute $A \cap B \subset \mathbb{C}^N$ for irreducible algebraic sets A and B described via witness sets, one may first apply Lemma 2.1 to obtain a witness set for $A \times B$ and then find $(A \times B) \cap \mathcal{V}(x-y)$ using Algorithm 1. There are special items to note about this case.

First, the system $f = x - y$ is linear and so is $F = \Lambda \cdot f$ in Algorithm 1. This has two implications:

- the homotopies $h_{m,i}$ of (5) are always trivial if we choose the option of Note 2 of Algorithm 1; and
- the intrinsic approach discussed in §3.2 is very effective for solving h_m in (6). In fact, only one linear function is perturbed in each h_m.

Second, since $(A \times B) \cap \mathcal{V}(x-y)$ and $A \cap B$ are isomorphic, their pure-dimensional components exist at the same dimensions. The largest possible dimension is $D = \min(\dim A, \dim B)$. Since Algorithm 1 starts at $\dim Z = \dim A + \dim B$ and descends, this means that the witness point supersets for dimension $D+1, D+2, \ldots, \dim Z$ generated by the algorithm are empty. Instead of starting with Lemma 2.1 and proceeding through several unproductive stages of Algorithm 1, one can target directly the dimension D to avoid wasted computation. Let $d = N - D = \max(\dim A, \dim B)$. A suitable homotopy is for targeting dimension D is

$$h_0''(x,y,t) = \begin{bmatrix} f_A(x) \\ f_B(y) \\ \gamma t \begin{bmatrix} L_A(x) \\ L_B(y) \end{bmatrix} + (1-t) \begin{bmatrix} F_1(x,y) \\ \vdots \\ F_d(x,y) \\ L_{d+1}(x,y) \\ \vdots \\ L_{\dim A + \dim B}(x,y) \end{bmatrix} \end{bmatrix}, \tag{7}$$

where γ is chosen at random from the unit circle in $\mathbb{C}$. This homotopy starts from the points $W_A \times W_B$. In effect, this combines Lemma 2.1 with d levels of Algorithm 1—each of which would have $\deg A \cdot \deg B$ homotopy paths—into just one homotopy with that number of paths. Since the target functions $F_1, \ldots, L_{\dim A+\dim B}$ are all linear and since the linear system $\{L_A, L_B\}$ cuts out $\deg Z$ isolated points on Z at $t = 0$, the homotopy is seen to succeed with probability one using the Gamma Trick [29, Lemma 7.1.3]. (The gamma trick is a generalization of a technique first introduced in [20].)

5.2 Membership testing

Another special case is testing whether a given point, say x^*, is a member of an irreducible set A. Membership is true if $A \cap \{x^*\} = \{x^*\}$, else $A \cap \{x^*\} = \emptyset$. So this is a special case of the diagonal intersection: we wish to compute $(A \times \{x^*\}) \cap \mathcal{V}(x - x^*)$. Since $\dim\{x^*\} = 0$, one sees that (7) directly targets a system $\{F_1, \ldots, F_{\dim A}\}$. But this system is just $\Lambda \cdot (x - x^*)$ for generic $\Lambda \in \mathbb{C}^{\dim A \times N}$, so the homotopy (7) becomes

$$h_0''(x, t) = \begin{bmatrix} f_A(x) \\ tL_A(x) + (1 - t)\Lambda(x - x^*) \end{bmatrix}, \tag{8}$$

where the gamma trick is no longer needed since $L_A(x)$ is fully generic in the only variables appearing, namely x. Note that this one homotopy completes the entire procedure, as there are no more dimensions left for the cascade.

One can see that this test is exactly the homotopy membership test introduced in [28] and discussed in [29, §15.4]. While the approach of this paper does not improve on the existing membership test, it is unifying to see the test arise naturally as a special case of our treatment of diagonal intersection.

6 Examples

Let us look at several examples putting these techniques to work using the tools implemented in Bertini [5].

6.1 Critical curve of a surface

In algorithms to compute cell decompositions of the real points contained in a complex algebraic curve [19] or surface [7], a key computation is to find the critical set of a real projection of the given algebraic set. In the case of a curve, critical points mark where the real projection of the real curve reverses direction, hence these points become the endpoints of cells that map bijectively to intervals of the projection coordinate. Similarly, the critical curve of a surface marks where the surface folds over, as viewed from a projection onto a plane. To be more precise, assume $A \subset \mathbb{C}^N$ is an irreducible and reduced algebraic surface, an irreducible component of $\mathcal{V}(f_A)$. Without loss of generality, we may assume that f_A consists of $(N - 2)$ polynomials; if not, it can be randomized down to that by Bertini's Theorem. Also, let $\pi : A \to \mathbb{C}^2$ be a real projection, given as

$\pi(z) = R \cdot z$ for some $R \in \mathbb{R}^{2\times N}$. Denote the Jacobian matrix of partial derivatives of f_A as Jf_A, a $(N-2) \times N$ matrix. Define

$$M(z) = \begin{bmatrix} Jf_A(z) \\ R \end{bmatrix}.$$

Then, the critical curve we seek is $K = A \cap \mathcal{V}(\det(M(z)))$. Note that $\mathcal{V}(f_A)$ might consist of several irreducible components, in which case it would be wasteful to find the entire solution set $\mathcal{V}(f_A, \det(M(z))) = \mathcal{V}(f_A) \cap \mathcal{V}(\det(M(z)))$ when we are only interested in K.

Clearly, if we have precomputed a witness set for A, then Algorithm 1 applies to finding its critical curve. An alternative formulation of the critical condition is $M(z)\xi = 0$ for $\xi \in \mathbb{P}^{N-1}$. Thus, we may compute $(A\times\mathbb{P}^{N-1})\cap\mathcal{V}(M(z)\xi)$, where $\mathbb{P}^{N-1}$ can be treated in $\mathbb{C}^N$ by choosing a random projective patch. In this form, one applies Lemma 2.1 to compute a witness set for $A \times \mathbb{P}^{N-1}$ before initiating the regenerative cascade.

Consider in particular the surface $\mathcal{V}(g)$ from [7], where

$$g = [(x + 0.35)^2(1 - x^2) - y^2]^2 + z^2 - 0.00531441,$$

which is an irreducible and reduced surface of degree eight. Then, $\det(M)$ is a single degree seven polynomial, and Algorithm 1 executes six homotopies (5)—we are using Note 2 of Algorithm 1—each with eight paths, followed by a single homotopy (6) having 56 paths. Twenty of these paths have finite endpoints, which form a witness point set, W_K, for the critical curve. So a witness set for the critical curve is $\{(g, \det M), L_2, W_K\}$, where L_2 is the second linear function in the witness set for A, still standing unperturbed after Algorithm 1 finishes. In the methodology of [7], the cell decomposition of the surface also requires the computation of the critical points of K, which can be accomplished with another application of Algorithm 1.

6.2 Dual variety

For each $v \in \mathbb{P}^N$, one can consider the hyperplane in $\mathbb{P}^N$ defined by $\mathcal{V}(x \cdot v)$. Let $\check{\mathbb{P}}^N$ be the set of all hyperplanes in $\mathbb{P}^N$, which is the dual variety to $\mathbb{P}^N$ and isomorphic to $\mathbb{P}^N$. For an irreducible variety $V \subset \mathbb{P}^N$, consider the tangent space incidence correspondence

$$T(V) = \overline{\{(P, H) \mid P \in V, H \text{ is tangent to } V \text{ at } P\}} \subset \mathbb{P}^N \times \check{\mathbb{P}}^N$$

with projection maps $\pi_1(P, H) = P$ and $\pi_2(P, H) = H$. Clearly, $\overline{\pi_1(T(V))} = V$. In fact, there is a unique irreducible component $U \subset T(V)$ such that $\overline{\pi_1(U)} = V$. With this setup, the *dual variety* of V is the variety $\overline{\pi_2(U)} \subset \check{\mathbb{P}}^N$. That is, the dual variety of V is the closure of the set of all hyperplanes which are tangent to V at a smooth point of V. The other irreducible components of $T(V)$ map under π_1 to various subsets of the singular points of V. See [10, Ex. 15.22] for more details on the tangent space incidence correspondence and dual varieties.

As in §6.1, we may perform our computations on $\mathbb{P}^N$ on a random projective patch in $\mathbb{C}^{N+1}$. In particular, given a witness set for $V \subset \mathbb{P}^N$, Algorithm 1 applies to finding a numerical irreducible decomposition of $T(V)$. One may then produce a pseudo-witness set (as defined in [12]) using the approach of [13]. In particular, one is able to compute the degree of the dual variety and perform membership tests on it.

To demonstrate, consider the homogeneous polynomial system adapted from [22]:

$$g(x,y,z,w) = \begin{bmatrix} g_1(x,y,z,w) \\ g_2(x,y,z,w) \end{bmatrix} = \begin{bmatrix} (x^2+y^2+z^2)^2 - w^4 \\ xyz - w^3 \end{bmatrix}.$$

Clearly, $\mathcal{V}(g)$ decomposes into two irreducible sextic curves in $\mathbb{P}^3$ and we compute a witness set for one the curves, namely

$$V = \{(x,y,z,w) \in \mathbb{P}^3 \mid x^2+y^2+z^2+w^2 = xyz - w^3 = 0\}.$$

We setup the regenerative cascade framework as follows. For $P = (x,y,z,w) \in V \subset \mathbb{P}^3$ and $H = (X,Y,Z,W) \in \check{\mathbb{P}}^3$, we will write that H is tangent to V at P if $P \in H$, i.e., $xX + yY + zZ + wW = 0$, and

$$\operatorname{rank} \begin{bmatrix} H \\ \nabla g_1(P)^T \\ \nabla g_2(P)^T \end{bmatrix} = 2.$$

where $\nabla g_i(P)$ is the gradient vector of g_i evaluated at P. By enforcing this rank condition using a null space formulation, we take

$$f(x,y,z,w,X,Y,Z,W,\ell_0,\ell_1,\ell_2) = \begin{bmatrix} xX+yY+zZ+wW \\ \ell_0 \begin{bmatrix} X \\ Y \\ Z \\ W \end{bmatrix} + \ell_1 \nabla g_1(x,y,z,w) + \ell_2 \nabla g_2(x,y,z,w) \end{bmatrix}.$$

and $Z = V \times \check{\mathbb{P}}^N \times \mathbb{P}^2$. Then, $T(V)$ is simply the image of $Z \cap \mathcal{V}(f)$ obtained by projecting onto (x,y,z,w,X,Y,Z,W), i.e., eliminating ℓ_i, which can be numerically computed using [13].

By working on a random projective patch in each of the three projective spaces, the regenerative cascade approach produced witness sets for the seven irreducible surfaces of $Z \cap \mathcal{V}(f)$. Six of these surfaces are planes which arise from the six points of intersection of the two irreducible sextic curves in $\mathcal{V}(g)$, namely

$$(\pm 1, i, 0, 0), \quad (\pm 1, 0, i, 0), \quad (0, \pm 1, i, 0)$$

where $i = \sqrt{-1}$. With respect to $\{g, f\}$, each of these planes are generically nonreduced but can be deflated. The remaining surface, $S \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3 \times \mathbb{P}^2$, has degree 72 whose projection is a degree 24 surface $U \subset T(V) \subset \mathbb{P}^3 \times \check{\mathbb{P}}^3$ with $\overline{\pi_1(U)} = V$. The dual variety of V, namely $\overline{\pi_2(U)}$, is a degree 18 surface in $\check{\mathbb{P}}^3$.

6.3 Sphere packings

A sphere packing is an assemblage of rigid spheres in tangential contact such that no internal motion of the assembly is possible without breaking at least one contact. (Rigid motion of the whole assembly is allowed.) These are of interest, for example, in designing colloids formed by suspensions of microspheres. By coating the spheres with DNA, one can promote adhesion between them in patterns that bias the formation of certain packings, thereby controlling the properties of the resulting colloid [18]. One approach to predicting which clusters will form starts by determining all possible sphere packings of up to N spheres, with $N = 10$ being the largest case completed so far [1]. It is assumed that all spheres have the same diameter, which we take to be unit length.

A packing can be specified by numbering the spheres 1 to N and listing which ones are in contact. One representation of the contact list is an adjacency matrix, an $N \times N$ matrix with a 1 in the (i, j)-th element if spheres i and j touch, and 0 elsewhere. Renumbering the spheres while maintaining the same contacts produces an isomorphism between adjacency matrices. Finding all packings of N spheres requires sifting through all possible isomorphic groups to find those which are realizable (the solution is real and no two spheres intersect in more than a single point) and rigid.

Rigidity requires at least $3N - 6$ contacts, since each sphere center has three coordinates, each contact exerts one algebraic constraint, and the rigid assembly retains six degrees of freedom of rigid motion. Some packings have greater than $3N - 6$ contacts; the smallest N where this occurs is $N = 10$ [1]. Also, it is possible for a packing to be singular in the sense that the assembly is a solution of multiplicity greater than one to the system of contact constraint equations. The smallest such packing occurs at $N = 9$ [1]. Arkus et al. call this packing "non-rigid," which is reasonable considering that the singularity will allow a physical system, which can violate the contact conditions slightly, to vibrate in the direction associated to the null-space of the system Jacobian matrix. Thus, by their usage of the term, rigidity requires multiplicity one, while packings are assemblages that are isolated solutions of the system of contact equations.

Let us consider a single $N = 9$ case defined by the following contact pairs:

$$\{1\text{-}2,\ 1\text{-}3,\ 1\text{-}4,\ 1\text{-}5,\ 2\text{-}3,\ 2\text{-}4,\ 2\text{-}6,\ 3\text{-}5,\ 3\text{-}6,\ 3\text{-}7,\ 3\text{-}8,\\ 4\text{-}5,\ 4\text{-}6,\ 4\text{-}9,\ 5\text{-}7,\ 5\text{-}9,\ 6\text{-}8,\ 6\text{-}9,\ 7\text{-}8,\ 7\text{-}9,\ 8\text{-}9\},$$

which is a graph with 9 vertices and 21 edges which, if realized as a polyhedron, has 14 triangular faces. With $3N - 6 = 21$ contacts, this case has the possibility of producing minimally constrained rigid packings. We can break this into two pieces by cutting along the closed path 3-5-4-6-3 to obtain subgraph 1 with vertices $1, \ldots, 6$ and edges

$$\{1\text{-}2,\ 1\text{-}3,\ 1\text{-}4,\ 1\text{-}5,\ 2\text{-}3,\ 2\text{-}4,\ 2\text{-}6,\ 3\text{-}5,\ 3\text{-}6,\ 4\text{-}5,\ 4\text{-}6\},$$

and subgraph 2 with vertices $3, \ldots, 9$ and edges

$$\{3\text{-}5,\ 3\text{-}6,\ 3\text{-}7,\ 3\text{-}8,\ 4\text{-}5,\ 4\text{-}6,\ 4\text{-}9,\ 5\text{-}7,\ 5\text{-}9,\ 6\text{-}8,\ 6\text{-}9,\ 7\text{-}8,\ 7\text{-}9,\ 8\text{-}9\}.$$

Each of these has $3N-7$ edges, so solution sets must be at least one-dimensional.

To solve subgraph 1, we take vertices 1,2,3 to be a unit equilateral triangle with known vertices, and solve for the remaining three. Letting v_i be the 3-vector of coordinates for vertex i, we have eight edge equations of the form

$$D_{i,j} := (v_i - v_j)^T (v_i - v_j) - 1 = 0.$$

But since v_1, v_2, and v_3 are known, it is advantageous to rewrite these as

$$D_{2,4} - D_{1,4} = 0,\ D_{3,5} - D_{1,5} = 0,\ D_{3,6} - D_{2,6} = 0,$$
$$D_{2,4} = D_{3,5} = D_{3,6} = D_{4,5} = D_{4,6} = 0,$$

so that we have a system of 3 linear and 5 quadratic equations. A numerical irreducible decomposition of this system gives 5 irreducible components:

- a two-dimensional component of degree 4 with $v_3 = v_4$;
- a one-dimensional component of degree 2 with $v_2 = v_5$ and $v_1 = v_6$;
- a one-dimensional component of degree 4 with $v_2 = v_5$;
- a one-dimensional component of degree 4 with $v_1 = v_6$; and
- a nondegenerate one-dimensional component, call it A, of degree 4.

Only component A is of interest, because the others all have at least two spheres that occupy the same position.

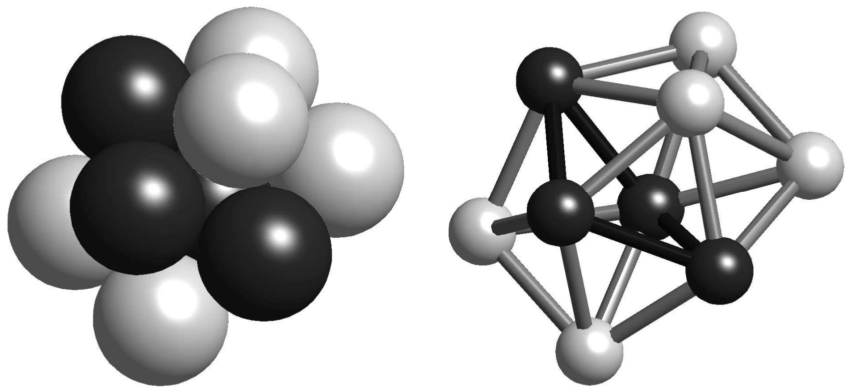
Subgraph 2 can be treated similarly, this time treating the triangle 3-7-8 as known. For clarity of presentation, rename vertices 3,4,5,6 as $3', 4', 5', 6'$, as these are clones of the ones in subgraph 1. So the system of equations to be solved is

$$D_{5',7} - D_{3',5'} = D_{6',8} - D_{3',6} = D_{7,9} - D_{8,9} = 0,$$
$$D_{4',5'} = D_{4',6'} = D_{4',9} = D_{5',7} = D_{5',9} = D_{6',8} = D_{6',9} = D_{7,9} = 0,$$

with $v_{4'}$, $v_{5'}$, $v_{6'}$, and v_9 as unknowns. The solution set of this system has 9 degenerate irreducible components: one of dimension 2, degree 32; four of dimension 1, degree 2; and four of dimension 1, degree 4. None of these is realizable for similar reasons as we saw for solution components of subgraph 1. (The subgraph formed by vertices 3,5,6,7,8,9 is isomorphic to subgraph 1, so this should be expected.) There is one final nondegenerate irreducible solution component that is dimension 1, degree 12. Call it B.

To assemble the two pieces into one packing, it is required that the relative positions of vertices 3,4,5,6 be the same for both. But by construction, the edges along the loops 3-5-4-6-3 and $3'$-$5'$-$4'$-$6'$-$3'$ are already unit length, so all that remains is to make the diagonals equal. That is, we wish to solve

$$(A \times B) \cap \mathcal{V}(D_{3,4} - D_{3',4'},\ D_{5,6} - D_{5',6'}).$$



(a) (b)

Figure 1: A sphere packing for $N = 9$ showing (a) spheres in contact, and (b) smaller spheres with rods indicating contact pairs. Dark spheres indicate the cutting loop 3-5-4-6-3.

This problem consists of solving two quadratic equations defined on $A \times B$, a two-dimensional set of degree $4 \cdot 12 = 48$. This is a considerably smaller problem than if we were to naively solve the original system having 21 edges, which after fixing one triangle and taking differences as in subgraph 1 becomes a system of 3 linear and 15 quadratic equations. Thus, the total degree of the original system is $2^{15} = 32{,}768$ compared to only $48 \cdot 2^2 = 192$ for the incremental approach.

Using Lemma 2.1 to get a witness set for $A \times B$ followed by Algorithm 1 to intersect it with the two quadratics, we obtain 28 points, of which 14 are real. Three final checks of the real solutions are necessary. First, testing that the spheres must only intersect in a point eliminates 10 of the 14 real solutions. Second, a congruence check is necessary since having all six distance pairs equal to one for vertices (3,4,5,6) and for $(3', 4', 5', 6')$ does not necessarily mean that these are congruent: one tetrahedron could be the mirror image of the other. This test eliminates 2 of the remaining 4 solutions. Finally, testing for "non-rigidity" shows that the remaining 2 solutions are indeed rigid. When the parts are assembled to form one packing, holding spheres 1,2,3 in place, these two solutions are mirror images through the plane of spheres 1,2,3. However, as can be seen in Figure 1, the packing is achiral (mirror self-symmetric), so these two solutions are both instances of the same arrangement. In summary, this particular set of contact pairs for $N = 9$ has exactly 1 realizable and rigid sphere packing, which is presented in Figure 1. For better visibility, the same packing is also shown using smaller spheres with the addition of rods that indicate the pairwise contacts. Spheres 3, 4, 5, and 6 have been darkened in the figure to show how the full assembly was cut into two pieces for solving incrementally.

Finally, we note that for clarity we have shown subgraphs 1 and 2 each being solved from scratch all in one blow. But the methods of this paper could have been used more extensively to also solve these by adding new vertices one at a time. Also, since subgraph 2 has a subgraph that is isomorphic to subgraph 1, the interesting solution component B of subgraph 2 could have been derived by adding one vertex to solution component A of subgraph 1. Such maneuvers could be used extensively in a program to find all sphere packings, as the graphs for larger N typically contain subgraphs that are common to graphs for smaller N.

7 Conclusions

Although Lemma 2.1, Theorem 3.1, and Theorem 4.2 represent only mild advances in the theory of numerical algebraic geometry, they have significant practical implications. With these techniques, after first computing numerical irreducible decompositions of two systems f_A and f_B to get, say, irreducible components $A_1, \ldots, A_{n_A}$ of $\mathcal{V}(f_A)$ and $B_1, \ldots, B_{n_B}$ of $\mathcal{V}(f_B)$, one can pick out desired components and find witness supersets for the components of, say, $(A_i \times B_j) \cap \mathcal{V}(f_C)$, where f_C is a third polynomial system defined on $A_i \times B_j$. By avoiding cross products with any other solution components of $\mathcal{V}(f_A)$ or $\mathcal{V}(f_B)$ other than just A_i and B_k, computation can be greatly reduced as compared to the naive approach of analyzing $\mathcal{V}(f_A(x), f_B(y), f_C(x,y))$. The technique generalizes naturally to work on cross products of any number of irreducible algebraic sets.

A useful special case is solving $(A \times \mathbb{C}^k) \cap \mathcal{V}(f_C(x,y))$ where x and y are coordinates on A and $\mathbb{C}^k$, respectively. As the underlying solution technique is regeneration, we call this special case "regeneration extension," since it extends a solution from irreducible component A of $\mathcal{V}(f_A)$ to find the components of $\mathcal{V}(f_A(x), f_C(x,y))$ that project into A under the natural projection: $(x,y) \mapsto (x)$.

It is also unifying to observe that $(A \times B) \cap \mathcal{V}(x-y)$ gives the diagonal intersection technique of [23], while $(A \times \{x^*\}) \cap \mathcal{V}(x-x^*)$ gives the homotopy membership test of [28].

Once a witness superset is obtained at each possible dimension, standard techniques—isosingular deflation, membership testing, monodromy, and traces—can be used to produce a numerical irreducible decomposition of the targeted set.

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