Some constraints on the physical realizability of a mathematical construction

Francisco Hernández-Quiroz[1] and **Pablo Padilla**[2]

Abstract. Mathematical constructions of abstract entities are normally done disregarding their actual physical realizability. The definition and limits of the physical realizability of these constructions are controversial issues at the moment and the subject of intense debate.

In this paper, we consider a simple and particular case, namely, the physical realizability of the enumeration of rational numbers by Cantor's diagonalization by means of an Ising system.

We contend that uncertainty in determining a particular state in an Ising system renders impossible to have a reliable implementation of Cantor's diagonal method and therefore a stronger physical system is required. We also point out what are the particular limitations of this system from the perspective physical realizability.

1 Introduction

"There is no quantum world. There is only an abstract quantum description. It is wrong to think physics' task is to discover how Nature is. Physics deals with what is possible to say about Nature."

This quote is attributed to Niels Bohr, when he was asked whether the quantum formalism reflected the underlying physical reality. Bohr's, other philosophers' and scientists' opinions aside, a good deal of paper has been used to analyse the possibility of describing and understanding reality by means of formal mathematical tools. Barrow, Chaitin, Hawking and Penrose (among others) have advanced some ideas with varying degrees of formality.

Here we address a reciprocal question: given a mathematical construction and a particular physical system, is the latter adequate to "implement" the former? By implementation we mean an actual physical device that (a) has structural properties that correspond to components of the mathematical entity (some have talked about an isomorphism between physical and mathematical structures [3], but a weaker notion may also do); (b) a physical procedure that can produce experimental results which reflect accurately corresponding properties of the mathematical construction.

These are very intricate and hard questions to be answered definitely in a general case. Our aim is more modest, namely to explore a specific instance of this problem: we take the classical Cantor's diagonalization for the enumeration of the rational numbers [2] and how it can be implemented by an Ising system. We provide a specific implementation and show its limitations deriving from properties of the physical system itself.

This leads us to think that some clearly defined mathematical questions cannot always be posed and answered within the context of a particular physical system. Of course, the more general question of the existence of a physical system realizing a particular mathematical construction is beyond the limits of this work but we hope our example helps to stimulate discussions on this line of thought. The standard interpretation of quantum mechanics regarding physically meaningful questions is that it should be possible to pose them in such a way that they can be answered experimentally.

The reciprocal question is also interesting: to what extent mathematical constructions should be considered valid? One possible approach, would imply that only those mathematical constructions that can actually be implemented by means of a physical system can in fact be used, at least in terms of computation.

In the next section we present—as a reminder—Cantor's diagonalization method for enumerating the rational numbers. The third section deals with Ising systems and its properties.The fourth section presents our implementation of Cantor's method and how to find a specific rational number. In the final section, which is the central part of this paper, we show how our system is unable to perform the task for which it was designed due to intrinsic limitations of Ising systems and other physical principles, and we also discuss some implications.

2 Cantor's diagonalization

In 1878 Cantor defined rigorously when two sets have the same cardinality. Let A and B be two sets. They have the same number of elements if and only if there exists a bijection between them, i.e., a function $f : A \to B$ which is both injective and surjective.

He also proved that the set of natural numbers and the set of rational numbers are equinumerous, even though the former is a proper subset of the latter. His argument introduced an ingenious device to construct a one-to-one correspondence between the two sets. The idea is that rational numbers are not arranged according to the traditional $<$ relation, but rather, by taking advantage of the fact that a rational number (in accordance with the etimology of the name) can be regarded as the ratio of two integers. For example, the number 0.5 is also represented by the fraction $1/2$.

The fractional representation of a number, let us say m/n, can be transformed into the convention that the pair *(m,n)* represents this very number. Now consider the list

$$(1, 1), (1, 2), (2, 1), (1, 3), (2, 2), (3, 1), (1, 4), \ldots$$

where pairs are arranged so that the sum of the two components is increasing; pairs whose sum produces the same value are ordered by the traditional $<$ order applied to the first coordinate of the pairs. By omitting pairs representing the same number (which can always be calculated in a finite number of steps as the list is being produced), this is a bijection between natural and rational numbers, and thus both sets have the same cardinality.

[1] School of Science, UNAM, email: fhq@ciencias.unam.mx
[2] Institute for Applied Mathematics, IIMAS, UNAM, email: pablo@mym.iimas.unam.mx

If we set aside the traditional objections posed by mathematical constructivists to the idea of actual infinite sets, Cantor's argument seems very straightforward and has been regarded as such ever since. However we could take a mathematical constructive perspective and reject Cantor's device (and his whole set theory, for that matter).

But we can also take a different constructive perspective, which we may name *physical constructivism*: What requirements should a particular physical system meet in order to serve as a basis for implementing Cantor's device? At first sight there must be physical systems on which this may not be possible (although the symmetrical question does not seem easy to answer). Specifically, we will analyse the feasibility of Ising models for this task in the next section.

3 Ising models

In the last decades, some models in physics have played a central role in understanding specific connections between mathematical aspects of the theory and experiments. One of such is precisely the Ising model. We use it here for different purposes. We suggest that it can be taken as a real system in which Cantor's diagonal procedure could be implemented and therefore as a starting point from which conclusions can be drawn regarding the limitations that mathematical constructions could have in the physical world. This is due to the fact that, in principle, the physical configurations of the system can be put in correspondence with rational numbers. Moreover, for the Ising model a direct relationship between the physical entropy and the informational entropy can be established, allowing a quantitative comparison .

We briefly recall what the Ising model is about and later on we make a few remarks on the entropy of a discrete physical system. What follows is basically adapted from [5].

We consider a magnetic material in which the electrons determining the magnetic behaviour are localized near the atoms of a lattice and can have only two magnetization states (spin up or down). The spin for a given site in this lattice will be identified with the of the 0's or 1's used in the mathematical construction of the previous section to write down the binary expansion of the rational numbers. Notice that we need only a finite number or 0's or 1's since these expansions will be either finite or periodic. For instance, we might put in a row all numbers (m, n) of a fixed height one after the other with a conventional sequence to denote beginning and end of a number. As mentioned before, the magnetization S_i can take only two values ± 1 that we identify with 0 and 1 respectively. There is a Hamiltonian associated in the presence of an external magnetic force depending on the site, h_i which is given by:

$$H = -J \sum_{i,k} S_i S_j - \sum_i h_i S_i,$$

where the sum over i and k runs over all possible nearest-neighbour pairs of the lattice and J is the so called exchange constant.

The fact that is important to stress is that a possible enumeration of the rationals correspond to a particular physical configuration. Notice that we are desregarding the obvious limitation of size. That is, in Cantor's procedure we need an infinite number of rows and columns, that is an ideal lattice, whereas a physical material will necessarily have finite size. Nevertheless, we will see that even then, there are physical constraints that are imposed by the quantum nature of the system to the entropy, which can be interpreted as informational restrictions on the physical realizability of the mathematical construction.

For a continuous system whose configuration is denoted by C, where the configuration space is assumed to be endowed with a measure μ (for simplicity one may think of $\mathbb{R}^d$, the entropy associated with a specific probability distribution P is given by

$$S[P] = -\int d\mu(C) P(C) ln P(C),$$

that is, the expected value of $-lnP(C)$ with respect to μ.

By dividing the space into cells of size ε^d the entropy of the continuous system can be well approximated by the entropy of the discrete system resulting from the partition:

$$S_{disc} = S_{cont} - d ln(\varepsilon).$$

As a matter of fact, the ε can be taken to be the Planck constant for a quantum system. This observation will be important later on.

4 Implementing Cantor's method

As we mentioned before, we can in principle use the Ising system to physically array and enumerate the rational numbers and locate any of them in this array. In fact the question: "How to find a rational number in the list?" is well defined and would only need a finite number of steps.

In the section devoted to the Ising model, we recalled equation 3 for the entropy of a quantum system. Notice that the second term is positive and independent of the details of the system, only due to the quantum nature of the same. This has an important implication in terms of the possibility of actually determining the state in which the Ising model is. If we relate the information content with the entropy of the system we see that, in order for the state of the system to be completely determined, we would need zero entropy [4]. This is physically impossible. Moreover, a lower bound for the entropy is related not only to the discrete (quantum) nature of the system, but it also depends on the temperature and other parameters. The conclusion is that even when the counting and locating procedure is well defined, there is always an intrinsic error. Of course one might argue that this is probably due to the chosen system, but the reasoning is general enough as to suggest that no matter what physical implementation we choose, there will always exist this limitation.

5 Conclusion: Uncertainty comes in the way or how real is reality?

We have argued that uncertainty in determining a particular state in an Ising system renders impossible to have a reliable implementation of Cantor's diagonal method. There are also other related mathematical constructions that could be analysed in a similar way. For instance, Cantor's proof of the uncountability of the real numbers relies on similar ideas. As a matter of fact, in the usual argument, a contradiction is obtained by producing a real number that cannot be included in a proposed enumeration. This is done by considering the diagonal sequence and taking its negation. Once this is done, it can be shown that if t is the truth value of the element of this sequence intersecting the diagonal, then it would have to satisfy the relation

$$t = 1 - t,$$

which leads to a contradiction *if one assumes the only possible truth values are 0 or 1* (see for instance chapter 2 on diagonalization in[1]). However, this equation does not pose any problem if t is interpreted in a probabilistic way and assigned a value of $1/2$. This opens up a series of even subtler questions such as whether we can actually have a physical model of the real numbers and many others, that from our perspective, are worth addressing.

Many other people have previously addressed this questions either in general terms or for particular mathematical concepts. A pioneering work is [6], which posed the question of realizing an abstract mapping process within the constraints of a physical version of Church's thesis. A very recent case study in the field of control and quantum systems can be found in [7].

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