# EXISTENCE OF EQUILIBRIUM WITH UNBOUNDED SHORT SALES: A NEW APPROACH

by

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# Existence of equilibrium with unbounded short sales: A new approach*

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**Abstract**

We introduce a new approach to showing existence of equilibrium in models of economies with unbounded short sales. Inspired by the pioneering works of Hart (1974) on asset market models, Grandmont (1977) on temporary economic equilibrium, and of Werner (1987) on general equilibrium exchange economies, all papers known to us stating conditions for existence of equilibrium with unbounded short sales place conditions on recession cones of agents' preferred sets or, more recently, require compactness of the utility possibilities set.. In contrast, in this paper, we place conditions on the preferred sets themselves. Roughly, our condition is that the sum of the weakly preferred sets is a closed set. We demonstrate that our condition implies existence of equilibrium. In addition to our main theorem, we present two theorems showing cases to which our main theorem can we applied. We also relate our condition to the classic condition of Hart (1974).

*We are pleased to dedicate this paper to Cuong Le Van, a great economist and a wonderful friend. Without the hospitality of the Centre for Economics of the Sorbonne, whose first Director was Professor Le Van, and the hospitality of CERMSEM and Paris 1 the idea of this paper would probably not have happened. It was sitting outside a cafe across the street from the Maison des Sciences Economiques that one the authors of this paper raised the question of "why not assumptions simply on preferred sets" and another author took up the idea. We are honored to have benefitted from Cuong Le Van's intellectual generosity and to advance a line of research to which he has been one of the main contributors. We would also like to thank Nizar Allouch for comments and the participants at the Paris conference in honor of Cuong Le Van, December 2011.

# 1 Introduction

The main contribution of this paper is to demonstrate that the closedness of the sum of weakly preferred sets at individual rational vectors ensures existence of equilibrium in economies with unbounded short sales. This is in contrast to a large number of papers providing existence results based on asymptotic cones of weakly preferred sets or on the set of attainable utility vectors.[1]

When unbounded short sales are allowed, in contrast to Arrow-Debreu-McKenzie general equilibrium models, consumption sets are unbounded below. This problem was studied in the seminal works of Hart (1974) in asset market models[2] and in Green (1973) and Grandmont (1977,1982) for temporary equilibrium models. Arbitrage conditions sufficient to guarantee existence of equilibrium in general equilibrium models of unbounded exchange economies (e.g., asset exchange economies allowing short sales) have been studied by Hammond (1982), Werner (1987), Nielsen (1989), Page and Wooders (1996), and more recently by Page, Wooders, and Monteiro (1999), Allouch, Le Van and Page 2002) among others. All these papers place conditions limiting arbitrage opportunities on the recession cones of the preferred sets of agents, thus ruling out arbitrarily large trades. A significant advance was made by Dana, Le Van and Magnien (1999), who introduced the condition of compactness of the utility possibilities set and show that all prior conditions ensuring existence of equilibrium imply their conditions.[3]

To illustrate the problem created by unbounded short sales for existence of economic equilibrium, suppose that two agents have diametrically opposed preferences. For example, one agent may want to buy arbitrarily large amounts of one commodity and sell another commodity short while the other agent may prefer to do the opposite. In such a situation, there are unbounded arbitrage opportunities and no equilibrium exists. To ensure existence of equilibrium arbitrage opportunities must be limited. Note that the idea that agents do not have diametrically opposed preferences is a statement about relationships of the sets of allocations preferred or weakly preferred to the endowments.

Our condition ensuring existence of equilibrium, introduced in this paper, is simply that the sum of the weakly preferred sets (each assumed to be closed) is closed. Thus, recession cones do not appear in the condition. For the current paper, we relate our results to the classic result of Hart (1974) in more detail. Other conditions equivalent to that of this paper and the relationships of our condition to other conditions in

[1]See Ha-Huy (2011) for a recent survey.
[2]See also Milne (1981), Hammond (1983), Page (1987, 1996), among others.
[3]See also Allouch (2002)

the literature, especially the compactness condition of Dana, Le Van and Magnien (1999), is left to further research.

## 2 Sum of closed sets and existence of equilibrium

We consider an exchange economy $\mathcal{E}$ with commodity space $V$,assumed to be a finite-dimensional vector space. There is a finite set $I$ of consumers. Each consumer $i \in I$ is described by a consumption set $X_i$ (a closed convex non-empty set of $V$), an endowment vector $\omega_i \in X_i$, and a convex preference relation $R_i \subset X_i \times X_i$ (a convex continuous complete transitive and reflexive binary relation $R_i$). For a point $x \in X_i$, we denote by $P_i(x)$ the preferred set of $x$,

$$P_i(x) := \{x' \in X_i \,:\, x' P_i x\}$$

and we denote by $R_i(x)$ the weakly preferred set,

$$R_i(x) := \{x' \in X_i \,:\, x' R_i x\}.$$

Note that by the definition of $R_i(\omega_i)$, for each $i \in I$ , it holds that:

$$R_i(\omega_i) \subset X_i. \tag{1}$$

We will assume $R_i(x)$ is a unbounded closed convex set, and $P_i(x)$ coincides with the set of interior points of $R_i(x)$, $P_i(x) = intR_i(x)$. [4]

An allocation $x_i \in X_i$ for player $i$ is *individually rational* if $x_i \in R_i(\omega_i)$. The *attainable individual rational allocations* constitute the set

$$\mathcal{A} := \{y = (y_i)_{i \in I} \in \prod_i R_i(\omega_i) \,:\, \sum_i y_i = \sum_i \omega_i\}.$$

The dual of $V$ (the set linear functionals on $V$) is denoted by $V^*$ and constitutes the space of prices.

Given prices $p \in V^*$, we let

$$B_i(p) := \{x \in V \,|\, p(x - \omega_i) \leq 0\}$$

denote the *budget set* of the $i$-th agent, $i \in I$. Our definition of equilibrium is standard.

[4] If we assume that endowment vectors belong to the interiors of the consumption sets, that is, $\omega_i \in intX_i$, for all $i$, then the assumption $P_i(x) = intR_i(x)$ need not be made.

**Definition**. An *equilibrium* is a tuple $(x_i)_{i\in I} \in \mathcal{A}$ and prices $p \in V^*$ such that, for each $i \in I$, $x_i \in B_i(p)$ and $P_i(x_i) \cap B_i(p) = \emptyset$.

For an economy with bounded short sales, since the following property always holds, it can be applied to the weakly preferred sets of agents:

$$\text{The sum of finitely many convex sets is a convex set.} \tag{2}$$

For an economy with unbounded short sales, except of some apparently mild technical assumptions on the primitives of the economy, we have to require the following property for the weakly preferred sets:

$$\text{The sum of finitely many closed convex sets is a closed convex set.} \tag{3}$$

Contrary to property (2), property (3) is not generally satisfied (see Example 1 below). Therefore we have either to require validity of (3) for weakly preferred sets or to consider a subclass of convex sets in which (3) is fulfilled. An example of such a subclass is the class of polyhedral sets (see, for example, [14]). More subtle examples are studied in [5].

Recall that the *recession cone* $Rec(A)$ of a set $A \subset V$ is a maximal cone $C$ of $V$ such that $A + C = A$. For a cone $C \subset V$, the cone $C^* = \{x \in V,\ x(c) \leq 0$, for every $c \in C\}$ is a *polar cone* to $C$.

Our existence result requires two technical assumptions on the primitives of the economy.[5] The first is a kind of 'nice boundary condition' on individual preferences. Specifically, preferences of each agent $i \in I$ are required to satisfy the following property

$$\begin{array}{c}\text{For each } p \in (RecR_i(\omega_i))^* \text{ there exists } y_i \in R_i(\omega_i) \text{ satisfying} \\ P_i(y_i) \cap \{x \,:\, p(x - \omega_i) \leq 0\} = \emptyset.\end{array} \tag{4}$$

This condition ensures that, given a price $p$ in a restricted set of prices, either there exists some preferred allocation that is not affordable with the budget constraint determined by those prices or the hyperplane $p(x) = p(\omega_i)$ is an asymptote to the set of allocations $RecR_i(\omega_i)$.[6] For example, (4) holds if there exists $y_i \in V$ such that:

$$P_i(y_i) \subset \omega_i + (-RecR_i(\omega_i))^*.$$

[5] In this paper, we are not pursuing for the strongest form of technical assumptions.

[6] The condition (4) rules out the necesssity of a "no satiation" condition. Such conditions has been much studied in the leterature on unbounded short sales.

Our second technical assumption is:

$$\begin{aligned} \text{There exists } \hat{p} &\in V^* \setminus 0,\ a \in V, \text{ such that for all } i \in I, \\ R_i(\omega_i) &\subseteq \{x : \hat{p}(x) \geq 0\} + a. \end{aligned} \tag{5}$$

This is simply the requirement that there is a hyperplane for which the preferred sets of all agents are on one side of the hyperplane.

**Theorem 1** Assume that the economy $\mathcal{E}$ satisfies (4), (5), and the following property:

$$\begin{gathered} \text{For any individually rational allocation } (x_1, \ldots, x_{|I|}) \text{ the sum} \\ \textstyle\sum_{i \in I} R_i(x_i) \text{ is closed.} \end{gathered} \tag{6}$$

Then there exists and equilibrium.

Condition (6) points us to the question of closedness of the sum of closed sets. This sum is not always closed, contrary, for example, to the property that the sum of convex sets is always convex. The following example also appears in Florenzano and Le Van (2001, p14).

**Example**. The sets $A = \{(x, y) : xy \geq 1, x, y \geq 0\}$ and $B := \{(x, y) : xy \leq -1, x \leq 0,\ y \geq 0\}$ are closed. But the sum $A + B = \{(x, y) : y > 0\}$ is not (see Figure 1).

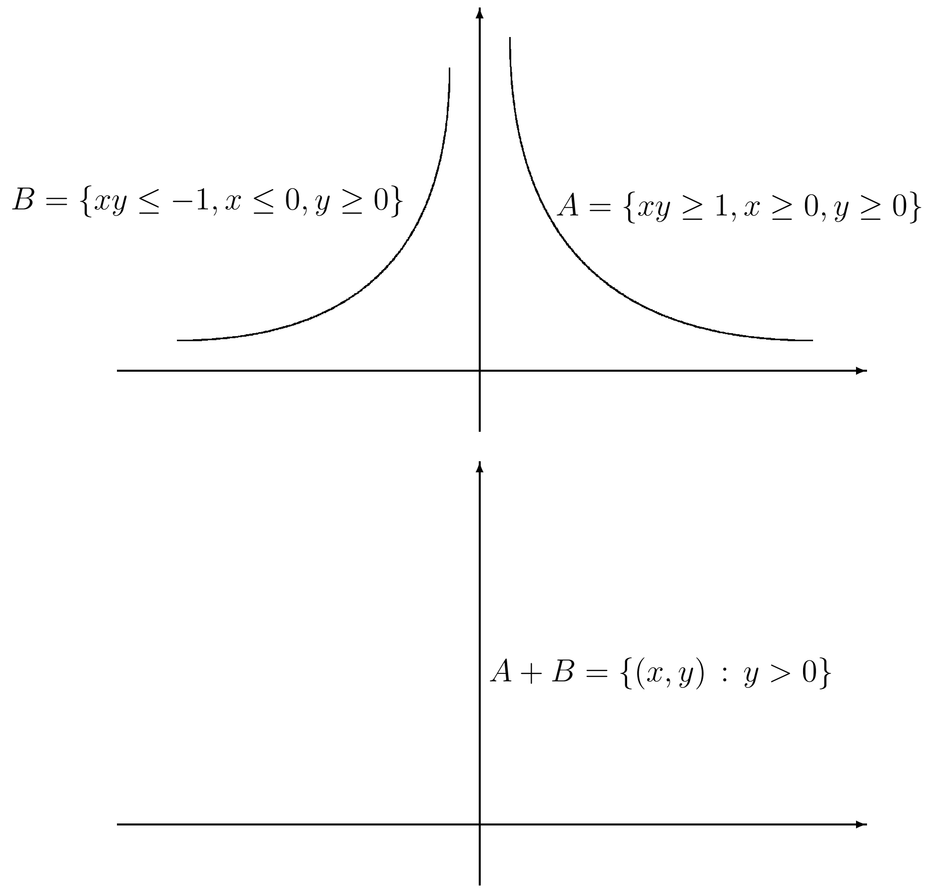



Figure 1.

The following two theorems are corollaries of the main theorem.

**Theorem 2** Assume that the economy $\mathcal{E}$ satisfies (4), (5), and, for each $i \in I$ and each $x_i \in R(\omega_i)$, the set

$$R_i(x_i) \text{ is a polyhedral set.} \tag{7}$$

Then there exists equilibrium.

**Proof**. The sum of a finite set of polyhedral sets is a closed convex set ([14]). Because of this, from (7) follows (6). Thus the conditions of Theorem 1 are met. □

**Theorem 3** Assume that the economy $\mathcal{E}$ satisfies (4), (5) and there exists a pointed cone $C$, such that for each $i \in I$, it holds that

$$R_i(\omega_i) \subset C. \tag{8}$$

Then there exists equilibrium.

**Proof**. We have to check that validity of (8) implies validity (6). This follows from [19]. □

Before going to prove the main theorem, we will relate our result to Hart's (1974) classic result.

## 3 Hart's no arbitrage condition and sums of closed sets

Hart (1974) introduces a version of the condition of Weak No Market Arbitrage (WNMA) on net trades. Recall that the *lineal* $Lin(A)$ of a set $A \subset V$ is a maximal linear subspace $L$ of $V$ such that $A + L = A$.

**Definition 1** *The economy $\mathcal{E}$ satisfies* WNMA

$$\begin{array}{c} \textit{For any tuple } y_i \in Rec(R_i(\omega_i)), i \in I \textit{ such that } \sum_{i \in I} y_i = 0, \\ \textit{it holds that } y_i \in Lin(R_i(\omega_i)) \textit{ for each } i \in I. \end{array} \tag{9}$$

**Proposition 1** *Let the* WNMA *condition hold. Then*

$\sum_{i \in I} R_i(\omega_i)$ *is closed.*

**Proof**. Follows from Corollary 9.1.1 in Rockafellar (1970). Q.E.D.

To prove the existence of an equilibrium, Hart also assumed the validity of a uniformity condition[7]: *Uniformity* implies that

$$Lin(R_i(\omega_i)) = Lin(R_i(x)) \text{ for any } x \in R_i(\omega_i). \tag{10}$$

Uniformity and WNMA together imply condition (6). In fact, since the recession cone is a monotone operator with respect to set inclusion ($RecB \subset RecA$ if $B \subset A$) we have the validity of WNMA for all sets $R_i(x_i)$ with individually rational $x_i$, $i \in I$. Due to Proposition 1, we obtain the validity of (6).

Thus, as a consequence of this and Theorem 1, we obtain a variant of Hart's theorem.

**Corollary 1** *Let the economy $\mathcal{E}$ satisfy (2), WNMA and Uniformity. Then there exists an equilibrium of the economy $\mathcal{E}$.*

[7] A number of papers in the literature have required that arbitrage opportunities be invariant or uniform with respect to endowments; see, for example, Allouch, Le Van and Page (2002).

# 4 A proof of the Main Theorem

First, we define a normalized price set. For that, for each $i \in I$ , we define a maximal cone $\mathbf{R}_i$ with the following property:

$$\text{for any } x \in R_i(\omega_i), \text{ there holds } P_i(x) + \mathbf{R}_i \subset P_i(x).$$

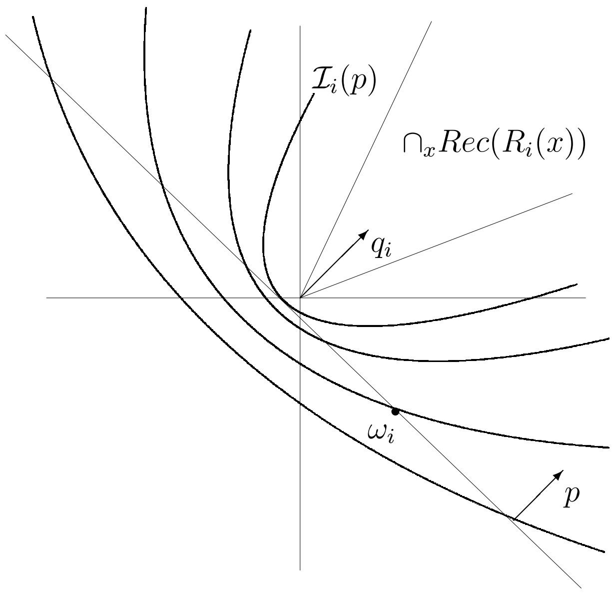
Consider the following set

$$\mathbf{P} := \text{ a base of the cone}\{p \in V^* \,:\, p \in \bigcap_i \mathbf{R}_i^*\}.$$

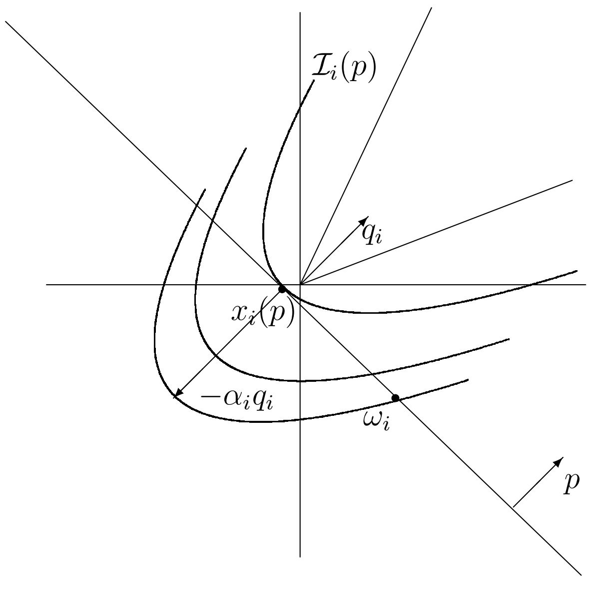
From (5) the set $\mathbf{P}$ is a non-empty convex compact.

We define a correspondence $f : \mathbf{P} \Rightarrow \mathbf{P}$. For $p \in \mathbf{P}$, $f(p)$ is defined using the following auxiliary construction. Specifically, for a given price vector $p$, we modify the economy. In the modified economy, each agent $i \in I$ has the same endowment $\omega_i$ and the same consumption set $X_i$ as in the initial economy. The modified preference relation $R_i(p) \subset X_i \times X_i$ is defined by the following rule: pick the most preferred element $x_i(p)$ with respect to the initial relation $R_i$ within the budget set $B_i(p) = \{x \in X_i \,:\, p(x) \leq p(\omega_i)\}$. Since $p \in \bigcap_i \mathbf{R}_i^*$ and due to assumption (4), such an element either exists or the hyperplane $p(x) = p(\omega_i)$ is an asymptote to some $R_i(p)$. Let $\mathcal{I}_i(p) := \{x \in X_i : xI_ix_i(p)\}$ be the indifference level passing through $x_i(p)$ if $x_i(p)$ exists, and let $\mathcal{I}_i(p)$ be the indifference level $I_i(p)$ otherwise (see Figure 2). Pick a direction $q_i \in int(\cap_{x \in R_i(\omega_i)} \mathrm{Rec}(R_i(x)))$. Then define the indifference levels of the modified preference $R_i(p)$ by parallel translations of $\mathcal{I}_i(p)$ in the direction $q_i$.

Now let $\alpha_i$ be such that the point $x_i(p) - \alpha_i q_i$ belong to an indifferent level of the modified preference, which passes through the endowment $\omega_i$ (see Picture 3).




Picture 2.




Picture 3.

Now, due to the closedness assumption (3), we can aggregate the modified preferences. That is, we define indifference levels of the aggregated preference from

$$\sum_i (\mathcal{I}_i(p) + \alpha_i q_i t), \text{ if } t \in [-1, 0] \text{ and } \sum_i (\mathcal{I}_i(p) + q_i t), \text{ if } t \geq 0.$$

Denote by $\widetilde{I}(\sum_{i\in I} \omega_i)$ the modified indifference level passing through $\sum_{i\in I} \omega_i$. Let $CI(\sum_{i\in I} \omega_i)$ be the convex set bounded by $\widetilde{I}(\sum_{i\in I} \omega_i)$. Denote by $f(p)$ the normal cone[8] to $CI(\sum_{i\in I} \omega_i)$ at $\sum_{i\in I} \omega_i$.

Thus, the auxiliary construction is finished, the mapping $f(p)$ is specified, and the correspondence $f : \mathbf{P} \Rightarrow \mathbf{P}$ is defined.

By standard arguments, the correspondence $f : \mathbf{P} \Rightarrow \mathbf{P}$ is upper semi-continuous. Then there exists a fixed point $p^* \in f(p^*)$. This fixed point provides us with an equilibrium in the initial economy. Q.E.D.

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[8] For $a \in A$, the set $N(A, a) = \{p \in V^* : p(a - a') \geq 0, \text{ for all } a' \in A\}$ denotes the normal cone of the set $A$ at the point $a$.

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