Weak Arithmetical Interpretations for the Logic of Proofs

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Abstract

Artemov established an arithmetical interpretation for the Logics of Proofs $\mathsf{LP}_{\mathsf{CS}}$, which yields a classical provability semantics for the modal logic $\mathsf{S4}$. These Logics of Proofs are parameterized by so-called constant specifications CS that state which axioms can be used in the reasoning process, and the arithmetical interpretation relies on the constant specifications being finite. In this paper, we remove this restriction by introducing weak arithmetical interpretations that are sound and complete for a wide class of constant specifications, including infinite ones. In particular, they interpret the full Logic of Proofs LP.

Keywords: Logic of Proofs, arithmetical interpretation, BHK semantics

1 Introduction

The original motivation for developing the first justification logic, the Logic of Proofs, was to provide intuitionistic logic with an adequate provability semantics. That is a semantics that respects Brouwer's fundamental idea, see, e.g., [21], that

intuitionistic truth means provability. (1)

Heyting and Kolmogorov [13,14,15] gave an explicit (but informal) definition of this notion of intuitionistic truth, which nowadays is known as Brouwer–Heyting–Kolmogorov (BHK) semantics for intuitionistic logic.

BHK semantics is widely accepted as the intended semantics for intuitionistic logic. However, it is purely informal and does not provide a precise definition of intuitionistic truth. This was tackled by Gödel [9] who introduced a modal calculus of classical provability (essentially equivalent to $\mathsf{S4}$) with the intended reading of $\Box F$ as *F is provable*. Gödel defined a translation $t(\cdot)$ from IL to $\mathsf{S4}$ where the translation $t(F)$ of an intuitionistic formula F is given by

prefix each subformula of F with a $\Box$-modality.

He apparently considered this to be an appropriate formalization of Brouwer's thesis (1). Gödel established that

$$\mathsf{IL} \vdash F \quad \text{implies} \quad \mathsf{S4} \vdash t(F) \ .$$

He conjectured that the converse direction also holds, which was later shown by McKinsey and Tarski [17].

However, the ultimate goal of providing a classical provability semantics for IL is not achieved by Gödel's translation because no precise semantics is given for the provability operator $\Box$. The situation can be depicted as follows:

$$\mathsf{IL} \hookrightarrow \mathsf{S4} \hookrightarrow \ldots ??? \ldots \hookrightarrow \text{classical proofs} \ .$$

Artemov [1,2] was able to give a formal provability semantics for $\mathsf{S4}$. He introduced the Logic of Proofs $\mathsf{LP}_{\mathsf{CS}}$, which is a system in the spirit of $\mathsf{S4}$ but with explicit proof terms.

Artemov established a realization theorem, which provides an embedding $r(\cdot)$ of $\mathsf{S4}$ into $\mathsf{LP}_{\mathsf{CS}}$. Further he developed a formal provability semantics for $\mathsf{LP}_{\mathsf{CS}}$, which gives us the following chain of exact embeddings:

$$\mathsf{IL} \hookrightarrow \mathsf{S4} \hookrightarrow \mathsf{LP}_{\mathsf{CS}} \hookrightarrow \text{classical proofs} \ .$$

Hence the Logic of Proofs $\mathsf{LP}_{\mathsf{CS}}$ can be viewed as a formalization of the BHK semantics for intuitionistic propositional logic.

It is important to note that $\mathsf{LP}_{\mathsf{CS}}$ is not one single logic but rather a family of logics that is parameterized by a so-called constant specification CS. The purpose of this constant specification, roughly, is to state which axioms are available for the reasoning process. Artemov's arithmetical semantics only works for finite CS, which in a sense is enough since each proof can only refer to finitely many axioms. However, different proofs may refer to different sets of axioms. So what we actually have is the following statement (where $\mathcal{L}_{\Box}$ is the language of modal logic):

Theorem 1.1 *There exists a realization r such that for each $\mathcal{L}_{\Box}$-formula F*

$$\mathsf{S4} \vdash F \qquad \textit{if and only if} \qquad \begin{array}{l} r(F) \textit{ is arithmetically } \mathsf{CS}\textit{-valid} \\ \textit{for some finite constant specification } \mathsf{CS}. \end{array}$$

In other words, the constant specification and, hence, the notion of validity depend on the formula F. It is the aim of this paper to reverse the order of these two quantifiers: the existential quantifier over CS and the universal quantifier over arithmetical interpretations hidden inside validity. We establish an arithmetical interpretation result where CS does not depend on the formula. Namely, we show the following:

Theorem 1.2 *Let CS be a primitive recursive, axiomatically appropriate, and schematic constant specification. There exists a realization r such that for each $\mathcal{L}_{\Box}$-formula F*

$$\mathsf{S4} \vdash F \qquad \textit{if and only if} \qquad r(F) \textit{ is weakly arithmetically } \mathsf{CS}\textit{-valid.}$$

Of course, there is a price to pay for the independence of the constant specification. Artemov's arithmetical semantics interprets the operations on evidence terms by computable functions on codes for proofs. It seems that this is only possible for finite constant specifications. Hence, we cannot impose this restriction and, consequently, call our semantics *weak*.

2 The Logic of Proofs

Justification terms are built from countably many (justification) constants c_i and countably many (justification) variables x_i according to the following grammar:

$$t ::= c_i \mid x_i \mid (t \cdot t) \mid (t + t) \mid !t \quad .$$

We denote the set of terms by Tm. Formulas are built from countably many atomic propositions p_i according to the following grammar:

$$F ::= p_i \mid \bot \mid (F \to F) \mid t{:}F \quad .$$

Prop denotes the set of atomic propositions and $\mathcal{L}_\mathsf{J}$ denotes the set of formulas. We define negation $\neg$, conjunction $\wedge$, and disjunction $\vee$ as usual.

The axioms of LP consist of all instances of the following schemes:

(i) all propositional tautologies

(ii) $t{:}(A \to B) \to (s{:}A \to t \cdot s{:}B)$ Application

(iii) $s{:}A \vee t{:}A \to s + t{:}A$ Sum

(iv) $t{:}A \to A$ Reflection

(v) $t{:}A \to !t{:}t{:}A$ Positive Introspection

A *constant specification* CS for LP is any subset

$$\mathsf{CS} \subseteq \{(c, A) \mid c \text{ is a constant and } A \text{ is an axiom of } \mathsf{LP}\}.$$

For a constant specification CS the deductive system LP_CS is the Hilbert system given by the axioms above and by the rules modus ponens and axiom necessitation:

$$\frac{A \quad A \to B}{B}\ (\mathrm{MP}) \ , \qquad \frac{(c, A) \in \mathsf{CS}}{c{:}A}\ (\mathrm{AN}) \ .$$

Definition 2.1 A constant specification CS is called

(i) *axiomatically appropriate* if for each axiom A of LP, there is a constant c such that $(c, A) \in \mathsf{CS}$;

(ii) *schematic* if it satisfies the following property: for each constant c, the set of axioms $\{A \mid (c, A) \in \mathsf{CS}\}$ consists of all instances of one or several (possibly zero) axiom schemes of LP;

(iii) *almost schematic* if it is the union of a schematic and a finite constant specification.

3 Decidability for $\mathsf{LP}_{\mathsf{CS}}$

Generated models are models for $\mathsf{LP}_{\mathsf{CS}}$ where the evidence relation is generated by a least fixed point construction. To inductively build-up this least fixed point, we need a monotone operator, which is given as follows.

Definition 3.1 [Evidence closure] Let $\mathcal{B} \subseteq \mathsf{Tm} \times \mathcal{L}_{\mathsf{J}}$. For a set $X \subseteq \mathsf{Tm} \times \mathcal{L}_{\mathsf{J}}$ we define $\mathsf{cl}_{\mathcal{B}}(X) \subseteq \mathsf{Tm} \times \mathcal{L}_{\mathsf{J}}$ by:

(i) if $(t, A) \in \mathcal{B}$, then $(t, A) \in \mathsf{cl}_{\mathcal{B}}(X)$;

(ii) if $(s, A) \in X$ or $(t, A) \in X$, then $(s + t, A) \in \mathsf{cl}_{\mathcal{B}}(X)$;

(iii) if $(s, A) \in X$ and $(t, A \to B) \in X$, then $(t \cdot s, B) \in \mathsf{cl}_{\mathcal{B}}(X)$;

(iv) if $(t, A) \in X$, then $(!t, t{:}A) \in \mathsf{cl}_{\mathcal{B}}(X)$.

Note that $\mathsf{cl}_{\mathcal{B}}$ is a monotone operator on $\mathsf{Tm} \times \mathcal{L}_{\mathsf{J}}$. Hence, it has a least fixed point. We define the *minimal evidence relation* $\mathcal{E}(\mathcal{B})$ over $\mathcal{B}$ to be the least fixed point of $\mathsf{cl}_{\mathcal{B}}$.

Definition 3.2 [Generated Model] A *generated model* is a pair $\mathcal{M} = (\mathsf{val}, \mathcal{B})$ where $\mathsf{val} \subseteq \mathsf{Prop}$ and $\mathcal{B} \subseteq \mathsf{Tm} \times \mathcal{L}_{\mathsf{J}}$. For a constant specification CS, the generated model $\mathcal{M}$ is called a *generated* CS*-model* if $\mathsf{CS} \subseteq \mathcal{B}$.

Definition 3.3 Let $\mathcal{M} = (\mathsf{val}, \mathcal{B})$ be a generated model and D be a formula. We define the relation $\mathcal{M} \Vdash D$ by

(i) $\mathcal{M} \nVdash \bot$

(ii) $\mathcal{M} \Vdash p_i \quad$ iff $\quad p_i \in \mathsf{val}$

(iii) $\mathcal{M} \Vdash A \to B \quad$ iff $\quad \mathcal{M} \nVdash A$ or $\mathcal{M} \Vdash B$

(iv) $\mathcal{M} \Vdash t{:}A \quad$ iff $\quad (t, A) \in \mathcal{E}(\mathcal{B})$ and $\mathcal{M} \Vdash A$.

Definition 3.4 [Finitary model] Let CS be an almost schematic constant specification. Let $\mathcal{C} \subseteq \mathsf{Tm} \times \mathcal{L}_{\mathsf{J}}$ be finite and set $\mathcal{B} = \mathsf{CS} \cup \mathcal{C}$. Further let val be a finite valuation, that is a finite subset of Prop. Then we call the generated CS-model $\mathcal{M} = (\mathsf{val}, \mathcal{B})$ a *finitary* CS*-model*.

A formula D is *valid with respect to finitary* CS*-models* if $\mathcal{M} \Vdash D$ for all finitary CS-models $\mathcal{M}$.

We have soundness and completeness of $\mathsf{LP}_{\mathsf{CS}}$ with respect to finitary CS-models (see [16,6]).

Theorem 3.5 *Let* CS *be an almost schematic constant specification. For each formula* $F \in \mathcal{L}_{\mathsf{J}}$

$$\mathsf{LP}_{\mathsf{CS}} \vdash F \quad \textit{iff} \quad F \textit{ is valid with respect to finitary } \mathsf{CS}\textit{-models.}$$

Remark 3.6 CS is restricted to be almost schematic only because this restriction is present in the definition of finitary models. It is possible to prove soundness and completeness for any CS for an extended class of models.

Finitary models are the key to establishing decidability for Logics of Proofs. In particular, we have the following result.

Theorem 3.7 *Let* CS *be a primitive recursive and almost schematic constant specification. The satisfaction relation for finitary* CS*-models is primitive recursive.*

Proof. By a careful examination of the decision algorithm for the satisfaction relation from [16, Corollary 4.4.8]. □

4 Peano Arithmetic

In this section, we introduce all notions and concepts of Peano Arithmetic PA that will be needed later in order to present arithmetical interpretations for the Logic of Proofs. We employ a formulation of PA that includes symbols for all primitive recursive functions and relations.

The *language* $\mathcal{L}_{\mathsf{PA}}$ *of arithmetic* is the language of first-order logic with countably many (individual) variables, the logical symbols $\bot, \to, \forall$, and the following non-logical symbols:

(i) an n-ary function symbol $\underline{f}$ for each n-ary primitive recursive function f;

(ii) an n-ary relation symbol $\underline{R}$ for each n-ary primitive recursive relation R.

We use $x, y, z, \ldots$ to denote individual variables of $\mathcal{L}_{\mathsf{PA}}$ and hope the reader is able to distinguish them from the justification variables of $\mathcal{L}_{\mathsf{J}}$. Further, we denote formulas of $\mathcal{L}_{\mathsf{PA}}$ by $\phi, \psi, \ldots$. A *sentence* is a formula without free occurrences of variables.

For each natural number n, we use its standard representation $\underbrace{s(s(\ldots s}_{n}(0)\ldots))$ in the language $\mathcal{L}_{\mathsf{PA}}$, call it a *numeral*, and denote it by $\underline{n}$. When working in $\mathcal{L}_{\mathsf{PA}}$, we will often use f for $\underline{f}$, R for $\underline{R}$, and n for $\underline{n}$ whenever the exact typification can be inferred from the context. In particular, we often write $=$ for $\underline{=}$.

Peano Arithmetic PA is given in the language $\mathcal{L}_{\mathsf{PA}}$. It comprises the axioms and rules of first-order predicate logic, equality axioms for the primitive recursive relation $=$, and defining axioms for all primitive recursive functions and relations. As usual, we write $\mathsf{PA} \vdash \phi$ if the formula ϕ is provable in PA.

If ϕ is a sentence, we write $\mathbb{N} \vDash \phi$ to say that ϕ is true in the standard model $\mathbb{N}$ of the natural numbers. In the following, we assume that PA is sound with respect to $\mathbb{N}$: for all $\mathcal{L}_{\mathsf{PA}}$-sentences ϕ,

$$\mathsf{PA} \vdash \phi \quad \text{implies} \quad \mathbb{N} \vDash \phi \ . \tag{2}$$

Definition 4.1 $\mathcal{L}_{\mathsf{PA}}$-formulas ϕ and ψ are called *provably equivalent* if

$$\mathsf{PA} \vdash \phi \leftrightarrow \psi \ .$$

Now we can define several important classes of $\mathcal{L}_{\mathsf{PA}}$-formulas.

Definition 4.2

(i) A *standard primitive recursive formula* is an $\mathcal{L}_{\mathsf{PA}}$-formula of the form

$$\underline{R}(t_0, \ldots, t_{n-1})$$

where R is an n-ary primitive recursive relation.

(ii) A *standard* Σ_1*-formula* is an $\mathcal{L}_{\mathsf{PA}}$-formula of the form

$$\exists x \phi$$

where ϕ is a standard primitive recursive formula.

(iii) An $\mathcal{L}_{\mathsf{PA}}$-formula ϕ is *provably* Σ_1 if there exists a standard Σ_1-formula ψ such that ϕ and ψ are provably equivalent.

(iv) An $\mathcal{L}_{\mathsf{PA}}$-formula ϕ is *provably* Δ_1 if both ϕ and $\neg\phi$ are provably Σ_1.

Provably Δ_1 formulas have a nice closure property.

Lemma 4.3 *The class of provably* Δ_1 *formulas is closed under Boolean connectives and under substitutions of terms for variables. All standard primitive recursive formulas are provably* Δ_1*.*

It is a very important fact that PA is complete for provably Σ_1 sentences.

Lemma 4.4

(i) *Let* ϕ *be a provably* Σ_1 *sentence. If* $\mathbb{N} \vDash \phi$*, then* $\mathsf{PA} \vdash \phi$*.*

(ii) *Let* ϕ *be a sentence such that* $\neg\phi$ *is provably* Σ_1*. If* $\mathbb{N} \nvDash \phi$*, then* $\mathsf{PA} \vdash \neg\phi$*.*

Formulating this lemma for provably Δ_1 sentences yields the following theorem, which we are going to apply often.

Theorem 4.5 *Let* ϕ *be a provably* Δ_1 *sentence.*

(i) *If* $\mathbb{N} \vDash \phi$*, then* $\mathsf{PA} \vdash \phi$*.*

(ii) *If* $\mathbb{N} \nvDash \phi$*, then* $\mathsf{PA} \vdash \neg\phi$*.*

To be able to talk about formulas and proofs of PA within PA, we need a so-called Gödel numbering of $\mathcal{L}_{\mathsf{PA}}$. That is an assignment of a numerical code $\ulcorner\phi\urcorner \in \mathbb{N}$ to each formula $\phi \in \mathcal{L}_{\mathsf{PA}}$.As mentioned above, when working in $\mathcal{L}_{\mathsf{PA}}$, we often use m for $\underline{m}$. Accordingly, whenever $\ulcorner\phi\urcorner$ occurs within an $\mathcal{L}_{\mathsf{PA}}$-formula, what we mean is, of course, the $\mathcal{L}_{\mathsf{PA}}$-term $\underline{\ulcorner\phi\urcorner}$.

Making use of the Gödel numbering, we can state the Diagonalization lemma, which is crucial for defining arithmetical interpretations for $\mathsf{LP}_{\mathsf{CS}}$.

Lemma 4.6 (Diagonalization) *Let* $\psi(y, x_0, \ldots, x_{n-1})$ *be an* $\mathcal{L}_{\mathsf{PA}}$*-formula. There exists an* $\mathcal{L}_{\mathsf{PA}}$*-formula* $\phi(x_0, \ldots, x_{n-1})$ *such that*

$$\mathsf{PA} \vdash \phi(x_0, \ldots, x_{n-1}) \leftrightarrow \psi\Big(\ulcorner\phi(x_0, \ldots, x_{n-1})\urcorner, x_0, \ldots, x_{n-1}\Big)$$

and that is provably Δ_1 *if* ψ *is.*

Last but not least we will use the notion of a proof predicate.

Definition 4.7 A *proof predicate* is a provably Δ_1 formula $\mathsf{Prf}(x, y)$ with no free occurrences of variables other than x and y such that for every $\mathcal{L}_{\mathsf{PA}}$-sentence ϕ we have

$$\begin{aligned} &\mathsf{PA} \vdash \phi \qquad \text{if and only if} \\ &\qquad\qquad \mathbb{N} \vDash \mathsf{Prf}(\underline{n}, \ulcorner\phi\urcorner) \text{ for some natural number } n. \end{aligned} \tag{3}$$

We will not formally establish the existence of proof predicates for PA. A detailed formal construction of a proof predicate is presented, e.g., in [8]. For the rest of this chapter, we simply assume that we are given a proof predicate, which we denote by $\mathsf{Proof}(x, y)$.

5 Weak Arithmetical Interpretation

Definition 5.1 A *weak arithmetical interpretation* is a pair $(*, \mathsf{Prf})$ such that

(i) $*$ maps atomic propositions of $\mathcal{L}_{\mathsf{J}}$ to sentences of $\mathcal{L}_{\mathsf{PA}}$;

(ii) $*$ maps evidence terms of $\mathcal{L}_{\mathsf{J}}$ to numerals of $\mathcal{L}_{\mathsf{PA}}$;

(iii) Prf is a proof predicate;

(iv) for all evidence terms s and t we have:

$$\mathbb{N} \vDash \mathsf{Prf}\Big(s^*, \ulcorner \phi \to \psi \urcorner\Big) \wedge \mathsf{Prf}\Big(t^*, \ulcorner \phi \urcorner\Big) \quad \to \quad \mathsf{Prf}\Big((s \cdot t)^*, \ulcorner \psi \urcorner\Big) \tag{4}$$

$$\mathbb{N} \vDash \mathsf{Prf}\Big(s^*, \ulcorner \phi \urcorner\Big) \vee \mathsf{Prf}\Big(t^*, \ulcorner \phi \urcorner\Big) \quad \to \quad \mathsf{Prf}\Big((s + t)^*, \ulcorner \phi \urcorner\Big) \tag{5}$$

$$\mathbb{N} \vDash \mathsf{Prf}\Big(s^*, \ulcorner \phi \urcorner\Big) \quad \to \quad \mathsf{Prf}\Big((!s)^*, \ulcorner \mathsf{Prf}\big(s^*, \ulcorner \phi \urcorner\big) \urcorner\Big) \ . \tag{6}$$

We extend the mapping $*$ to all formulas of $\mathcal{L}_{\mathsf{J}}$ by setting

$$(t{:}F)^* := \mathsf{Prf}(t^*, \ulcorner F^* \urcorner) \qquad \bot^* := \bot \qquad (F \to G)^* := F^* \to G^* \ .$$

If there is no need to explicitly mention the proof predicate Prf, we denote the weak arithmetical interpretation $(*, \mathsf{Prf})$ by $*$.

A weak arithmetical interpretation $*$ is called a *weak arithmetical* CS*-interpretation* if for each $(c, A) \in \mathsf{CS}$ we have $\mathbb{N} \vDash (c{:}A)^*$. An $\mathcal{L}_{\mathsf{J}}$-formula F is *weakly arithmetically* CS*-valid* if $\mathsf{PA} \vdash F^*$ for all weak arithmetical CS-interpretations $*$.

Lemma 5.2 *For any $\mathcal{L}_{\mathsf{J}}$-formula F and for any weak arithmetical interpretation $(*, \mathsf{Prf})$, the arithmetical formula F^* is a sentence and is provably Δ_1.*

Proof. Follows from Lemma 4.3. □

Theorem 5.3 (Weak Arithmetical Soundness) *Let CS be a constant specification and F be an $\mathcal{L}_{\mathsf{J}}$-formula. Then we have*

$$\mathsf{LP}_{\mathsf{CS}} \vdash F \quad \textit{implies} \quad F \textit{ is weakly arithmetically } \mathsf{CS}\textit{-valid.} \tag{7}$$

Proof. Suppose $*$ is a weak arithmetical CS-interpretation. We show (7) by induction on the $\mathsf{LP}_{\mathsf{CS}}$-derivation of F.

If F is a classical tautology, then so is F^* and we trivially have $\mathsf{PA} \vdash F^*$.

If F is an instance of **j**, **j+**, or **j4**, then $\mathbb{N} \vDash F^*$ follows from (4)–(6). Since F^* is provably Δ_1 by Lemma 5.2, it follows by Theorem 4.5 that $\mathsf{PA} \vdash F^*$.

If F is an axiom $t{:}A \to A$, then $F^* = \mathsf{Prf}(t^*, \ulcorner A^* \urcorner) \to A^*$. We distinguish two cases, depending on whether the sentence $\mathsf{Prf}(t^*, \ulcorner A^* \urcorner)$ is true or false in the standard model:

(i) $\mathbb{N} \vDash \mathsf{Prf}(t^*, \ulcorner A^* \urcorner)$. By (3) we find $\mathsf{PA} \vdash A^*$ and thus

$$\mathsf{PA} \vdash \mathsf{Prf}(t^*, \ulcorner A^* \urcorner) \to A^* \ .$$

(ii) $\mathbb{N} \nvDash \mathsf{Prf}(t^*, \ulcorner A^* \urcorner)$. In this case $\mathsf{Prf}(t^*, \ulcorner A^* \urcorner)$ is a false provably Δ_1 sentence, meaning that $\mathsf{PA} \vdash \neg\mathsf{Prf}(t^*, \ulcorner A^* \urcorner)$ by Theorem 4.5. Therefore, again

$$\mathsf{PA} \vdash \mathsf{Prf}(t^*, \ulcorner A^* \urcorner) \to A^* \ .$$

If F is the conclusion of an instance of axiom necessitation, then F has the form $c{:}A$ where $(c, A) \in \mathsf{CS}$. By assumption, we have $\mathbb{N} \vDash (c{:}A)^*$. Since $(c{:}A)^*$ is a provably Δ_1 sentence by Lemma 5.2, we find by Theorem 4.5 that $\mathsf{PA} \vdash (c{:}A)^*$.

Finally, if F is the conclusion of an instance of modus ponens, the claim follows by the induction hypothesis and the fact that $*$ distributes through implication. □

In the remainder of this section we show completeness of $\mathsf{LP}_{\mathsf{CS}}$ with respect to weak arithmetical CS-interpretations where CS is a primitive recursive and almost schematic constant specification. In order to obtain this result, we will establish the following property:

Lemma 5.4 *For each finitary* CS*-model* $\mathcal{M}_{\mathsf{fin}}$*, there exists a weak arithmetical* CS*-interpretation* $*$ *such that for all* $\mathcal{L}_{\mathsf{J}}$*-formulas* G

$$\mathcal{M}_{\mathsf{fin}} \Vdash G \quad \textit{implies} \quad \mathbb{N} \vDash G^* \ . \tag{8}$$

Weak arithmetical completeness easily follows from Lemma 5.4.

Theorem 5.5 *Let* CS *be a primitive recursive and almost schematic constant specification. For any formula* F *of* $\mathcal{L}_{\mathsf{J}}$ *we have*

$$F \textit{ is weakly arithmetically } \mathsf{CS}\textit{-valid} \quad \textit{implies} \quad \mathsf{LP}_{\mathsf{CS}} \vdash F \ . \tag{9}$$

Proof. Assume that $\mathsf{LP}_{\mathsf{CS}} \nvdash F$. By Theorem 3.5, there exists a finitary CS-model $\mathcal{M}_{\mathsf{fin}}$ with $\mathcal{M}_{\mathsf{fin}} \nVdash F$. Thus $\mathcal{M}_{\mathsf{fin}} \Vdash \neg F$. By Lemma 5.4, there is a weak arithmetical CS-interpretation $*$ such that $\mathbb{N} \vDash (\neg F)^*$, i.e., $\mathbb{N} \vDash \neg(F^*)$. Therefore, $\mathbb{N} \nvDash F^*$, which implies $\mathsf{PA} \nvdash F^*$ by soundness (2) of PA. Hence F is not weakly arithmetically CS-valid. □

The complicated part is to establish (8) from Lemma 5.2. For the rest of this section, we assume that we are given

(i) a primitive recursive and almost schematic constant specification CS and

(ii) a finitary CS-model $\mathcal{M}_{\mathsf{fin}}$.

Further, we assume that the Gödel numbering of the union of $\mathcal{L}_{\mathsf{PA}}$ and $\mathcal{L}_{\mathsf{J}}$ is injective, that is

$$\ulcorner E_1 \urcorner = \ulcorner E_2 \urcorner \quad \text{if and only if} \quad E_1 \equiv E_2$$

for any expressions E_1, E_2.

We first have to decide what objects should serve as 'proofs' in our arithmetical interpretation. There will be two sources of 'proofs':

(i) To begin with, all usual proofs will be 'proofs.' This guarantees that the direction from left to right in (3) is satisfied.

(ii) The second source of 'proofs' are the evidence terms of $\mathcal{L}_{\mathsf{J}}$. Every term t is a 'proof' for all formulas B for which $\mathcal{M}_{\mathsf{fin}} \Vdash t{:}B$.

To take care of the usual proofs, we make use of the usual primitive recursive proof predicate $\mathsf{Proof}(x, y)$ for Peano Arithmetic. Without loss of generality we assume that $\mathbb{N} \nvDash \mathsf{Proof}(\ulcorner s \urcorner, \underline{k})$ for any evidence term s of $\mathcal{L}_{\mathsf{J}}$ and any natural number k.

In order to deal with the evidence terms, we denote by $\mathsf{Prf}(x, y)$ a formula with no free variables other than x and y that will be chosen later based on its desired properties that we are going to discuss now. For any such $\mathsf{Prf}(x, y)$, we can define an auxiliary translation † from $\mathcal{L}_{\mathsf{J}}$-formulas to $\mathcal{L}_{\mathsf{PA}}$-sentences by:

$$
\begin{aligned}
&p^\dagger := \begin{cases} \ulcorner p \urcorner = \ulcorner p \urcorner & \text{if } \mathcal{M}_{\mathsf{fin}} \Vdash p, \\ \neg(\ulcorner p \urcorner = \ulcorner p \urcorner) & \text{otherwise} \end{cases} \qquad \text{for any atomic proposition } p; \\
&(t{:}F)^\dagger := \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner F^\dagger \urcorner) \ ; \\
&\bot^\dagger := \bot \ ; \\
&(F \to G)^\dagger := F^\dagger \to G^\dagger \ .
\end{aligned}
$$

Obviously, atomic propositions that hold in $\mathcal{M}_{\mathsf{fin}}$ are translated to provable sentences and atomic propositions that do not hold in $\mathcal{M}_{\mathsf{fin}}$ are translated to refutable sentences. (We need the translation † to be injective. Therefore, simply putting

$$
p^\dagger := \begin{cases} 0 = 0 & \text{if } \mathcal{M}_{\mathsf{fin}} \Vdash p \\ 0 = 1 & \text{otherwise} \end{cases}
$$

would not be sufficient.)

If the formula $\mathsf{Prf}(x, y)$ contains some relation symbol $\underline{R}$ other than $=$, i.e., some relation symbol not occurring in the †-translation outside of Prf, then this translation is injective, in other words,

$$
F^\dagger \equiv G^\dagger \quad \text{implies} \quad F \equiv G \ . \tag{10}
$$

We assume $F^\dagger \equiv G^\dagger$ and show (10) by induction on the structure of the $\mathcal{L}_{\mathsf{J}}$-formula F.

(i) F is an atomic proposition. By the definition of †, G must also be an atomic proposition and, by the injectivity of the Gödel numbering, G must be the same atomic proposition as F.

(ii) F is $\bot$. By the definition of †, it is clear that $G \equiv \bot$.

(iii) F is a formula $s{:}F_1$. Then G must be of the form $t{:}G_1$. Indeed, suppose towards a contradiction that $G \equiv G_1 \to G_2$. Since $(s{:}F_1)^\dagger = \mathsf{Prf}(\underline{k}, \underline{n})$ for suitable k and n, the sentence $(s{:}F_1)^\dagger \equiv G_1^\dagger \to G_2^\dagger$ would contain the symbol $\underline{R}$, meaning that $G_1^\dagger$ or $G_2^\dagger$ would contain a subformula of the form $\mathsf{Prf}(\underline{k_1}, \underline{n_1})$. It would then remain to count the number of occurrence of $\to$ in $\mathsf{Prf}(x, y)$ to show that $(s{:}F_1)^\dagger \equiv G_1^\dagger \to G_2^\dagger$ is impossible after all. Therefore, $G \equiv t{:}G_1$. By the induction hypothesis and injectivity of the Gödel numbering we conclude $s \equiv t$ and $F_1 \equiv G_1$.

(iv) F is $F_1 \to F_2$. By the same argument as in (iii), we have $G \equiv G_1 \to G_2$. By the induction hypothesis, $F_1 \equiv G_1$ and $F_2 \equiv G_2$.

Thus, (10) is established. For any formula $\mathsf{Prf}(x, y)$ that yields an injective $\dagger$, it can be shown by using the standard techniques for Gödel numbering that binary functions $\mathsf{dag}(x, y)$ and $\mathsf{undag}(x, y)$ such that

$$\mathsf{dag}(\ulcorner B \urcorner, \ulcorner \mathsf{Prf}(x, y) \urcorner) = \ulcorner B^\dagger \urcorner \quad \text{and} \quad \mathsf{undag}(\ulcorner B^\dagger \urcorner, \ulcorner \mathsf{Prf}(x, y) \urcorner) = \ulcorner B \urcorner$$

(it does not matter much how these functions are defined on inputs that are not Gödel numbers of such formulas, e.g., they can be assumed to be constant on all other inputs) are also primitive recursive and our language contains corresponding function symbols $\underline{\mathsf{dag}}$ and $\underline{\mathsf{undag}}$. Note that functions dag and undag are supposed to take the Gödel number of $\mathsf{Prf}(x, y)$ as a parameter. Hence, unlike the translation $\dagger$, these functions do not depend on $\mathsf{Prf}(x, y)$. This means, in particular, that the way undag is defined does not depend on whether $\dagger$ is injective or not. The above property, however, is only guaranteed for injective $\dagger$'s. Note also that dag and undag do depend on the chosen model $\mathcal{M}_{\mathsf{fin}}$.

By Theorem 3.7 the satisfaction relation for $\mathcal{M}_{\mathsf{fin}}$ is primitive recursive. Therefore, there is a binary relation symbol $\underline{\mathsf{Jus}}$ such that

$$\begin{aligned} \mathbb{N} \vDash \underline{\mathsf{Jus}}(\underline{n}, \underline{k}) \quad & \text{if and only if} \\ & \text{there is a term } s \text{ and a formula } F \text{ such that} \\ & \qquad n = \ulcorner s \urcorner, \text{ and } k = \ulcorner F \urcorner, \text{ and } \mathcal{M}_{\mathsf{fin}} \Vdash s{:}F \ . \end{aligned}$$

As mentioned in Lemma 4.3, $\underline{\mathsf{Jus}}(x, y)$ is a provably Δ_1 formula. Using Lemma 4.6, we now define the desired formula $\mathsf{Prf}(x, y)$ to satisfy

$$\mathsf{PA} \vdash \mathsf{Prf}(x, y) \leftrightarrow \mathsf{Proof}(x, y) \vee \underline{\mathsf{Jus}}\Big(x, \underline{\mathsf{undag}}\big(y, \ulcorner \mathsf{Prf}(x, y) \urcorner\big)\Big) \ . \qquad (11)$$

Moreover, since $\mathsf{Proof}(x, y) \vee \underline{\mathsf{Jus}}\big(x, \underline{\mathsf{undag}}(y, z)\big)$ is clearly provably Δ_1, so is our $\mathsf{Prf}(x, y)$.

Thus, by soundness (2) of PA, for the universal closure of (11),

$$\mathbb{N} \vDash \forall x \forall y \left(\mathsf{Prf}(x, y) \leftrightarrow \mathsf{Proof}(x, y) \vee \underline{\mathsf{Jus}}\Big(x, \underline{\mathsf{undag}}\big(y, \ulcorner \mathsf{Prf}(x, y) \urcorner\big)\Big)\right) \ . \qquad (12)$$

Further, (10) holds because the formula $\mathsf{Prf}(x, y)$ contains a relation symbol $\underline{\mathsf{Jus}}$. It follows that undag really performs the inverse translation, so that

informally we have

$$\begin{array}{ll} \mathsf{Prf}(x,y) & \text{if and only if} \\ & \mathsf{Proof}(x,y) \text{ or} \\ & \text{there is a term } s \text{ and a formula } F \text{ such that} \\ & \qquad x = \ulcorner s \urcorner, \text{ and } y = \ulcorner F^\dagger \urcorner, \text{ and } \mathcal{M}_{\mathsf{fin}} \Vdash s{:}F \ . \end{array}$$

The key property of the translation † based on the chosen Prf is that $\mathcal{L}_{\mathsf{J}}$-formulas that hold in $\mathcal{M}_{\mathsf{fin}}$ are translated to true $\mathcal{L}_{\mathsf{PA}}$-sentences and formulas that do not hold in $\mathcal{M}_{\mathsf{fin}}$ are translated to false $\mathcal{L}_{\mathsf{PA}}$-sentences

Lemma 5.6 *For each formula F of $\mathcal{L}_{\mathsf{J}}$ we have:*

(i) $\mathcal{M}_{\mathsf{fin}} \Vdash F$ *implies* $\mathbb{N} \vDash F^\dagger$*;*

(ii) $\mathcal{M}_{\mathsf{fin}} \nVdash F$ *implies* $\mathbb{N} \nvDash F^\dagger$*.*

Proof. By simultaneous induction on the structure of F. We distinguish the following cases:

(i) Let F be an atomic proposition. If $\mathcal{M}_{\mathsf{fin}} \Vdash F$, then $F^\dagger$ is $\ulcorner F \urcorner = \ulcorner F \urcorner$, which clearly is true. If $\mathcal{M}_{\mathsf{fin}} \nVdash F$, then $F^\dagger$ is $\neg(\ulcorner F \urcorner = \ulcorner F \urcorner)$, which clearly is false.

(ii) If $F = \bot$, then trivially we have $\mathcal{M}_{\mathsf{fin}} \nVdash \bot$ and $\mathbb{N} \nvDash \bot$.

(iii) The case of $F = G \to H$ is immediate by induction hypothesis.

(iv) Let $F = s{:}G$. If $\mathcal{M}_{\mathsf{fin}} \Vdash s{:}G$, then $\mathbb{N} \vDash \underline{\mathsf{Jus}}(\ulcorner s \urcorner, \ulcorner G \urcorner)$. Given that

$$\ulcorner G \urcorner = \mathsf{undag}\left(\ulcorner G^\dagger \urcorner, \ulcorner \mathsf{Prf}(x,y) \urcorner\right) \tag{13}$$

we have by (12) that $\mathbb{N} \vDash \mathsf{Prf}\left(\ulcorner s \urcorner, \ulcorner G^\dagger \urcorner\right)$, i.e., $\mathbb{N} \vDash (s{:}G)^\dagger$.

If $\mathcal{M}_{\mathsf{fin}} \nVdash s{:}G$, then $\mathbb{N} \nvDash \underline{\mathsf{Jus}}(\ulcorner s \urcorner, \ulcorner G \urcorner)$. Moreover, $\mathbb{N} \nvDash \mathsf{Proof}\left(\ulcorner s \urcorner, \ulcorner G^\dagger \urcorner\right)$ since by assumption $\mathbb{N} \nvDash \mathsf{Proof}(\ulcorner s \urcorner, \underline{k})$ for any k. Thus, by (12) and (13), we have $\mathbb{N} \nvDash \mathsf{Prf}\left(\ulcorner s \urcorner, \ulcorner G^\dagger \urcorner\right)$, i.e., $\mathbb{N} \nvDash (s{:}G)^\dagger$.

□

Next, we show that $\mathsf{Prf}(x,y)$ is a proof predicate.

Lemma 5.7 *For every $\mathcal{L}_{\mathsf{PA}}$-sentence ϕ we have*

$$\begin{array}{ll} \mathsf{PA} \vdash \phi & \textit{if and only if} \\ & \mathbb{N} \vDash \mathsf{Prf}(\underline{n}, \ulcorner \phi \urcorner) \textit{ for some natural number } n. \end{array}$$

Proof. From left to right. Suppose $\mathsf{PA} \vdash \phi$. Then there is a natural number n such that $\mathbb{N} \vDash \mathsf{Proof}(\underline{n}, \ulcorner \phi \urcorner)$. By (12) we conclude $\mathbb{N} \vDash \mathsf{Prf}(\underline{n}, \ulcorner \phi \urcorner)$.

From right to left. Suppose that $\mathbb{N} \vDash \mathsf{Prf}(\underline{n}, \ulcorner \phi \urcorner)$. Then, by (12), either $\mathbb{N} \vDash \mathsf{Proof}(\underline{n}, \ulcorner \phi \urcorner)$, in which case $\mathsf{PA} \vdash \phi$ follows immediately, or $n = \ulcorner s \urcorner$ for some evidence term s and some $\mathcal{L}_{\mathsf{J}}$-formula F such that $\ulcorner F \urcorner = \mathsf{undag}\left(\ulcorner \phi \urcorner, \ulcorner \mathsf{Prf}(x,y) \urcorner\right)$ and $\mathcal{M}_{\mathsf{fin}} \Vdash s{:}F$. Therefore, $\phi \equiv F^\dagger$ and $\mathcal{M}_{\mathsf{fin}} \Vdash F$. By the previous lemma, $\mathbb{N} \vDash F^\dagger$. Since $F^\dagger$ is a provably Δ_1 sentence, we find $\mathsf{PA} \vdash F^\dagger$, i.e., $\mathsf{PA} \vdash \phi$. □

Now we obtain a weak arithmetical CS-interpretation as follows.

Lemma 5.8 *Let $*$ be a mapping such that $s^* := \underline{\ulcorner s \urcorner}$ for each evidence term s and $P^* := P^\dagger$ for each atomic proposition P. Then the pair $(*, \mathsf{Prf})$ is a weak arithmetical CS-interpretation. Moreover, we have*

$$F^* = F^\dagger \tag{14}$$

for any $\mathcal{L}_\mathsf{J}$-formula F.

Proof. We start with showing (14) by induction on the structure of F. We distinguish the following cases.

(i) If F is an atomic proposition, then $F^* = F^\dagger$ by definition.

(ii) If $F = t{:}G$, then $t^* = \ulcorner t \urcorner$. By induction hypothesis, $G^* = G^\dagger$. Thus,

$$(t{:}G)^* = \mathsf{Prf}(t^*, \ulcorner G^* \urcorner) = \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner G^\dagger \urcorner) = (t{:}G)^\dagger \ .$$

(iii) If $F = \bot$, then $\bot^* = \bot^\dagger$ by definition.

(iv) If $F = G \to H$, then $G^* = G^\dagger$ and $H^* = H^\dagger$ by induction hypothesis. Thus, $(G \to H)^* = G^* \to H^* = G^\dagger \to H^\dagger = (G \to H)^\dagger$.

This finishes the proof of (14).

We show that $(*, \mathsf{Prf})$ is indeed a weak arithmetical CS-interpretation. The mapping $*$ maps atomic propositions of $\mathcal{L}_\mathsf{J}$ to sentences of $\mathcal{L}_\mathsf{PA}$ and evidence terms to numerals. Further, Prf is a proof predicate by the previous lemma. It remains to establish (4)–(6) from Definition 5.1. We only present a proof of (4). The other proofs are similar.

Assume that $\mathbb{N} \vDash \mathsf{Prf}(s^*, \ulcorner \phi \to \psi \urcorner)$ and $\mathbb{N} \vDash \mathsf{Prf}(t^*, \ulcorner \phi \urcorner)$, in other words, $\mathbb{N} \vDash \mathsf{Prf}(\ulcorner s \urcorner, \ulcorner \phi \to \psi \urcorner)$ and $\mathbb{N} \vDash \mathsf{Prf}(\ulcorner t \urcorner, \ulcorner \phi \urcorner)$. By assumption $\mathbb{N} \nvDash \mathsf{Proof}(\ulcorner r \urcorner, \underline{k})$ for any evidence term r and any natural number k. Therefore, by (12) we find $\phi \equiv F^\dagger$ and $\psi \equiv G^\dagger$ for some $\mathcal{L}_\mathsf{J}$-formulas F and G such that

$$\mathcal{M}_\mathsf{fin} \Vdash s{:}(F \to G) \quad \text{and} \quad \mathcal{M}_\mathsf{fin} \Vdash t{:}F \ .$$

Hence, $\mathcal{M}_\mathsf{fin} \Vdash (s \cdot t){:}G$. By (12), we obtain $\mathbb{N} \vDash \mathsf{Prf}(\ulcorner s \cdot t \urcorner, \ulcorner G^\dagger \urcorner)$, which is $\mathbb{N} \vDash \mathsf{Prf}\big((s \cdot t)^*, \ulcorner \psi \urcorner\big)$.

It remains to show that the constant specification is respected. Let $(c, A) \in \mathsf{CS}$. Then $\mathcal{M}_\mathsf{fin} \Vdash c{:}A$. Thus, by Lemma 5.6, we have $\mathbb{N} \vDash (c{:}A)^\dagger$. Hence, by (14), we have $\mathbb{N} \vDash (c{:}A)^*$. □

Now Lemma 5.4 follows easily. First, observe that by Lemma 5.8 the pair $(*, \mathsf{Prf})$ is a weak arithmetical CS-interpretation. Suppose $\mathcal{M}_\mathsf{fin} \Vdash G$. By Lemma 5.6 we find $\mathbb{N} \vDash G^\dagger$, which is $\mathbb{N} \vDash G^*$ by (14). This completes the proof of weak arithmetical completeness.

6 A Semantics of Proofs for Intuitionistic Logic

Definition 6.1 The mapping $\circ : \mathcal{L}_{\mathsf{J}} \to \mathcal{L}_{\Box}$ is defined by:

$$\begin{aligned} P^{\circ} &:= P \text{ for } P \in \mathsf{Prop} \ , \\ \bot^{\circ} &:= \bot \ , \\ (A \to B)^{\circ} &:= A^{\circ} \to B^{\circ} \ , \\ (t{:}A)^{\circ} &:= \Box A^{\circ} \ . \end{aligned}$$

Lemma 6.2 *For any constant specification* CS *and any formula* $F \in \mathcal{L}_{\mathsf{J}}$*,*

$$\mathsf{LP}_{\mathsf{CS}} \vdash F \quad \textit{implies} \quad \mathsf{S4} \vdash F^{\circ} \ .$$

Definition 6.3 [Realization] A *realization* is a mapping $r\colon \mathcal{L}_{\Box} \to \mathcal{L}_{\mathsf{J}}$ such that $\big(r(A)\big)^{\circ} = A$.

The realization theorem [2,7,11] provides an embedding of $\mathsf{S4}$ into $\mathsf{LP}_{\mathsf{CS}}$.

Theorem 6.4 (Realization) *Let* CS *be an axiomatically appropriate and schematic constant specification. There exists a realization* r *such that for each* $\mathcal{L}_{\Box}$*-formula* F

$$\mathsf{S4} \vdash F \quad \textit{implies} \quad \mathsf{LP}_{\mathsf{CS}} \vdash r(F) \ .$$

Theorem 6.5 *Let* CS *be a primitive recursive, axiomatically appropriate, and schematic constant specification. There exists a realization* r *such that for each* $\mathcal{L}_{\Box}$*-formula* F

$$\mathsf{S4} \vdash F \quad \textit{if and only if} \quad r(F) \textit{ is weakly arithmetically } \mathsf{CS}\textit{-valid} \ .$$

Proof. First we show the direction from left to right. By the realization theorem, there exists a realization r such that for each $\mathcal{L}_{\Box}$-formula F

$$\mathsf{S4} \vdash F \quad \text{implies} \quad \mathsf{LP}_{\mathsf{CS}} \vdash r(F) \ .$$

Combining this with Theorem 5.3 we obtain for each $\mathcal{L}_{\Box}$-formula F

$$\mathsf{S4} \vdash F \quad \text{implies} \quad r(F) \text{ is weakly arithmetically } \mathsf{CS}\text{-valid} \ .$$

For the direction from right to left, let r be an arbitrary realization and suppose that $r(F)$ is weakly arithmetically CS-valid. By Theorem 5.5 we obtain $\mathsf{LP}_{\mathsf{CS}} \vdash r(F)$. Hence by Lemma 6.2 we find $\mathsf{S4} \vdash F$. □

To obtain a provability semantics for intuitionistic logic IL, we combine the previous result with the Gödel translation from IL to $\mathsf{S4}$. Let the translation $t(\cdot)$ from IL to $\mathsf{S4}$ be such that for each formula F of the language $\mathcal{L}_{\mathrm{ip}}$ of IL we have

$$\mathsf{IL} \vdash F \quad \text{if and only if} \quad \mathsf{S4} \vdash t(F) \ .$$

Corollary 6.6 *Let* CS *be a primitive recursive, axiomatically appropriate, and schematic constant specification. There exists a realization* r *such that for each* $\mathcal{L}_{ip}$*-formula* F

$$\mathsf{IL} \vdash F \quad \textit{if and only if} \quad r(t(F)) \textit{ is weakly arithmetically } \mathsf{CS}\textit{-valid} \ .$$

7 Related Work

The construction of the proof predicate $\mathsf{Prf}(x, y)$ that we performed is essentially taken from Artemov's original proof of arithmetical completeness for $\mathsf{LP}_{\mathsf{CS}}$, see [1,2,3]. Goris [12] used a similar construction to provide $\mathsf{LP}_{\mathsf{CS}}$ with a semantics of proofs in Buss's system S^1_2.

Our result relies on the fact that $\mathsf{LP}_{\mathsf{CS}}$ is not only decidable for finite constant specifications, but more generally for almost schematic ones. The first general decidability proof for $\mathsf{LP}_{\mathsf{CS}}$ with non-finite CS was done by Mkrtychev [18]. The notion of an *almost schematic* constant specification goes back to Kuznets [16]. For recent presentations of various decidability results, see [6,19,20].

Gödel [10] suggested using a system with explicit proofs for the interpretation of $\mathsf{S4}$ in a lecture already in 1938, but the transcript of the lecture only appeared in 1995. Even before the publication of Gödel's work, Artemov [1] came up with the Logic of Proofs $\mathsf{LP}_{\mathsf{CS}}$ and established the realization theorem as well as completeness with respect to arithmetical interpretations.

The first systems for logics of proofs that feature formulas of the form $t{:}F$, meaning *t is a proof of F*, appear in the work of Artemov and Straßen [4,5] who investigate arithmetical interpretations for these logics. However, these ancestors of $\mathsf{LP}_{\mathsf{CS}}$ had no operations on proof terms and were too weak to capture the $\Box$-modality of $\mathsf{S4}$ in full.

8 Conclusion

What is new in our work is the observation that the construction of the proof predicate for the arithmetical interpretation can be made independent of the constant specification if one considers weak arithmetical interpretations. Our Corollary 6.6 provides a uniform arithmetical provability semantics for intuitionistic logic, a semantics that can be based on any of a wide class of constant specifications. This strengthens the previously known result by Artemov that each intuitionistically valid formula has a constant specification that provides a provability interpretation for this formula. In particular, for the first time, the Logic of Proofs LP itself, i.e., $\mathsf{LP}_{\mathsf{CS}}$ with the total constant specification CS where each constant proves every axiom, provides a provability semantics for intuitionistic logic.

It might be useful to point out exactly how Artemov's semantics was weakened to obtain this result. The obvious change was that the *finiteness of proofs* property had to be dropped. According to Artemov's definition of the proof predicate, each proof can only prove finitely many formulas, whereas in our case the interpretation of a constant might need to prove infinitely many axioms from a given axiom scheme. Secondly, while we define how to interpret arithmetically the $\cdot$, $+$, and $!$ operations on proof terms, we do not extend the corresponding arithmetical functions to the (Gödel numbers of) proofs obtained from the standard Gödel proof predicate Proof.

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