Paul Erdős (1913–1996)

1. Prologue

On October 18, 1996, hundreds of people, including many mathematicians, gathered at Kerepesi Cemetery in Budapest to pay their last respects to Paul Erdős. If there was one theme suggested by the farewell orations, it was that the world of mathematics had lost a legend, one of its great representatives. On October 21, 1996, in accordance with his last wishes, Paul Erdős' ashes were buried in his parents' grave at the Jewish cemetery on Kozma street in Budapest.

Paul Erdős was one of this century's greatest and most prolific mathematicians. He is said to have written about 1500 papers, with almost 500 co-authors. He made fundamental contributions in numerous areas of mathematics.

There is a Hungarian saying to the effect that one can forget everything but one's first love. When considering Erdős and his mathematics, we cannot speak of "first love", but of "first loves", and approximation theory was among them. Paul Erdős wrote more than 100 papers that are connected, in one way or another, with the approximation of functions. In these two short reviews, we try to present some of Paul's fundamental contributions to approximation theory.

A list of Paul's papers in approximation theory is given at the end of this article. These are referenced in this article in the form [ab.n], indicating the n-th item in the year 19ab. This list is a sublist of the official list, of publications by Erdős, in [GN], with a list of additions and corrections available at the website `www.acs.oakland.edu/~grossman/erdoshp.html`. Other references in this article (such as the reference [GN] just used) are listed just prior to that list of Erdős' approximation theory papers.

Numerous articles and obituaries on Erdős have appeared (see, e.g., the web page `www.math.ohio-state.edu/~nevai/ERDOS/`), and more will undoubtedly appear. The interested reader might wish to look at the article by L. Babai which appeared in [Ba].

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2. PAUL ERDŐS AND POLYNOMIALS

Tamás Erdélyi

I will discuss some of Erdős' results related to polynomials that attracted me most. This list reflects my personal taste and is far from complete even within the subdomains I focus on most, namely polynomial inequalities, Müntz polynomials, and the geometry of polynomials.

The two inequalities below (and their various extensions) play a key role in proving inverse theorems of approximation. Let $\mathcal{P}_n$ denote the set of all algebraic polynomials of degree at most n with real coefficients.

Markov's Inequality. *The inequality*

$$\|p'\|_{L^\infty[-1,1]} \leq n^2 \|p\|_{L^\infty[-1,1]}$$

holds for every $p \in \mathcal{P}_n$.

Bernstein Inequality. *The inequality*

$$|p'(y)| \leq \frac{n}{\sqrt{1-y^2}} \|p\|_{L^\infty[-1,1]}$$

holds for every $p \in \mathcal{P}_n$ and $y \in (-1,1)$.

For Erdős, Markov- and Bernstein-type inequalities had their own intrinsic interest and he explored what happens when the polynomials are restricted in certain ways. It had been observed by Bernstein that Markov's inequality for monotone polynomials is not essentially better than for arbitrary polynomials. Bernstein proved that *if n is odd, then*

$$\sup_p \frac{\|p'\|_{L^\infty[-1,1]}}{\|p\|_{L^\infty[-1,1]}} = \left(\frac{n+1}{2}\right)^2,$$

where the supremum is taken over all $0 \neq p \in \mathcal{P}_n$ that are monotone on $[-1,1]$. This is surprising, since one would expect that if a polynomial is this far away from the "equioscillating" property of the Chebyshev polynomial, then there should be a more significant improvement in the Markov inequality. In the short paper [40.04], Erdős gave a class of restricted polynomials for which the Markov factor n^2 improves to cn. He proved that there is an absolute constant c such that

$$|p'(y)| \le \min\left\{ \frac{c\sqrt{n}}{(1-y^2)^2}\,,\ \frac{en}{2} \right\} \|p\|_{L^\infty[-1,1]}\,, \qquad y \in [-1,1]\,,$$

for every polynomial of degree at most n that has all its zeros in $\mathbb{R} \setminus (-1,1)$. This result motivated several people to study Markov- and Bernstein-type inequalities for polynomials with restricted zeros and under some other constraints. Generalizations of the above Markov- and Bernstein-type inequality of Erdős have been extended in many directions by many people including Lorentz, Scheick, Szabados, Varma, Máté, Rahman, Govil, and others. Many of these results are contained in the following, due to P. Borwein and T. Erdélyi [BE]: *there is an absolute constant c such that*

$$|p'(y)| \le c\,\min\left\{ \sqrt{\frac{n(k+1)}{1-y^2}}\,,\ n(k+1) \right\} \|p\|_{L^\infty[-1,1]}\,, \qquad y \in [-1,1]\,,$$

for every polynomial p of degree at most n with real coefficients that has at most k zeros in the open unit disk.

Clarkson and Erdős wrote a seminal paper on the density of Müntz polynomials. Müntz's classical theorem characterizes sequences $\Lambda := (\lambda_i)_{i=0}^\infty$ with

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{1}$$

for which the Müntz space $M(\Lambda) := \operatorname{span}\{x^{\lambda_0}, x^{\lambda_1}, \dots\}$ is dense in $C[0,1]$. Here, $M(\Lambda)$ is the collection of all finite linear combinations of the functions $x^{\lambda_0}, x^{\lambda_1}, \dots$ with real coefficients, and $C(A)$ is the space of all real-valued continuous functions on $A \subset [0,\infty)$ equipped with the uniform norm. If $A := [a,b]$ is a finite closed interval, then the notation $C[a,b] := C([a,b])$ is used.

Müntz's Theorem. *Suppose $\Lambda := (\lambda_i)_{i=0}^\infty$ is a sequence satisfying* (1)*. Then $M(\Lambda)$ is dense in $C[0,1]$ if and only if $\sum_{i=1}^\infty 1/\lambda_i = \infty$.*

The point 0 is special in the study of Müntz spaces. Even replacing $[0,1]$ by an interval $[a,b] \subset [0,\infty)$ in Müntz's Theorem is a non-trivial issue. Such an extension is, in large measure, due to Clarkson and Erdős [43.02] and L. Schwartz [Sc]. In [43.02], Clarkson and Erdős showed that Müntz's Theorem holds on any interval $[a,b]$ with $a > 0$. That is, *for any increasing nonnegative sequence $\Lambda := (\lambda_i)_{i=0}^\infty$ and any $0 < a < b$, $M(\Lambda)$ is dense in $C[a,b]$ if and only if $\sum_{i=1}^\infty 1/\lambda_i = \infty$.* Moreover, they described what kind of functions are in the uniform closure of the span on $[a,b]$ assuming $\sum_{i=1}^\infty 1/\lambda_i < \infty$. Further, they showed that under the assumption $\sum_{i=1}^\infty 1/\lambda_i < \infty$ every function $f \in C[a,b]$ from the uniform closure of $M(\Lambda)$ on $[a,b]$ is of the form

$$f(x) = \sum_{i=0}^\infty a_i x^{\lambda_i}\,, \qquad x \in [a,b)\,. \tag{2}$$

In particular, f can be extended analytically throughout the open disk centered at 0 with radius b.

Erdős considered this result his best contribution to complex analysis. Later, by different methods, L. Schwartz extended some of the Clarkson-Erdős results to the case when the exponents λ_i are arbitrary distinct nonnegative numbers. For example, in that case, under the assumption $\sum_{i=1}^{\infty} 1/\lambda_i < \infty$ every function $f \in C[a,b]$ from the uniform closure of $M(\Lambda)$ on $[a,b]$ can still be extended analytically throughout the region

$$\{z \in \mathbb{C} \setminus (-\infty, 0] : |z| < b\},$$

although such an analytic extension does not necessarily have a representation given by (2). The Clarkson-Erdős results were further extended by Peter Borwein and the author, from the interval $[0,1]$ to subsets of $[0,\infty)$ with positive Lebesgue measure. That is, *if* $\Lambda := (\lambda_i)_{i=0}^{\infty}$ *is an increasing sequence of nonnegative real numbers with* $\lambda_0 = 0$ *and* $A \subset [0,\infty)$ *is a compact set with positive Lebesgue measure, then* $M(\Lambda)$ *is dense in* $C(A)$ *if and only if* $\sum_{i=1}^{\infty} 1/\lambda_i = \infty$. This result had been expected by Erdős and others for a long time.

I find the following result of Erdős and Turán [50.08] especially attractive.

Theorem. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *has* m *positive real zeros, then*

$$m^2 \le 2n \log\left(\frac{|a_0| + |a_1| + \cdots + |a_n|}{\sqrt{|a_0 a_n|}}\right).$$

This result was originally due to Schur. Erdős and Turán rediscovered it with a short proof.

In [39.02], Erdős proved that the arc length from 0 to 2π of a real trigonometric polynomial f of degree at most n satisfying $|f(\vartheta)| \le 1$ is maximal for $\cos n\vartheta$. An interesting question he posed quite often is the following: *Let* $0 < a < b < 2\pi$. *Is it still true that the variation and arc-length in* $[a,b]$ *is maximal for* $\cos(n\vartheta + \alpha)$ *for a suitable* α*?* The following related conjecture of Erdős was open for quite a long time: *Is it true that the arc length from* -1 *to* 1 *of a real algebraic polynomial of degree at most* n *is maximal for the Chebyshev polynomial* T_n*?* This was proved independently by Kristiansen [Kr2] and by Bojanov [Boj].

A well-known theorem of Chebyshev states that *if* p *is a real algebraic polynomial of degree at most* n *and* $z_0 \in \mathbb{R} \setminus [-1,1]$, *then* $|p(z_0)| \le |T_n(z_0)| \cdot \|p\|_{L^\infty[-1,1]}$, *where* T_n *is the Chebyshev polynomial of degree* n. The standard proof of this is based on zero counting which can no longer be applied if z_0 is not real. By letting $z_0 \in \mathbb{C}$ tend to a point in $(-1,1)$, it is fairly obvious that this result cannot be extended to all $z_0 \in \mathbb{C}$. However, a surprising result of Erdős [47.08] shows that Chebyshev's inequality can be extended to all $z_0 \in \mathbb{C}$ outside the open unit disk.

Erdős and Turán were probably the first to discover the power and applicability of an almost forgotten result of Remez. The so-called Remez inequality is not only attractive and interesting in its own right, but it also plays a fundamental role in proving various other things about polynomials. For a fixed $s \in (0,2)$, let

$$\mathcal{P}_n(s) := \{p \in \mathcal{P}_n : m(\{x \in [-1,1] : |p(x)| \le 1\}) \ge 2 - s\},$$

where $m(\cdot)$ denotes linear Lebesgue measure. The Remez inequality concerns the problem of bounding the uniform norm of a polynomial $p \in \mathcal{P}_n$ on $[-1,1]$ given that its modulus is bounded by 1 on a subset of $[-1,1]$ of Lebesgue measure at least $2-s$. That is, *how large can* $\|p\|_{L^\infty[-1,1]}$ *(the uniform norm of* p *on* $[-1,1]$*) be if* $p \in \mathcal{P}_n(s)$*?* The answer is given in terms of the Chebyshev polynomials. The extremal polynomials for the above problem are the Chebyshev polynomials $\pm T_n(x) := \pm\cos(n \arccos h(x))$, where h is a linear function which maps $[-1, 1-s]$ or $[-1+s, 1]$ onto $[-1,1]$.

One of the applications of the Remez inequality by Erdős and Turán [40.05] deals with orthogonal polynomials. Let w be an integrable weight function on $[-1,1]$ that is positive almost everywhere. Denote the sequence of the associated orthonormal polynomials by $(p_n)_{n=0}^{\infty}$. Then a theorem of Erdős and Turán [40.05] states that

$$\lim_{n\to\infty} [p_n(z)]^{1/n} = z + \sqrt{z^2-1}$$

holds uniformly on every closed subset of $\mathbb{C} \setminus [-1,1]$*.*

Erdős and Turán [38.05] established a number of results on the spacing of zeros of orthogonal polynomials. One of these is the following. *Let* w *be an integrable weight function on* $[-1,1]$ *with* $\int_{-1}^{1} (w(x))^{-1}\, dx =: M < \infty$, *and let*

$$(1 >)\, x_{1,n} > x_{2,n} > \cdots > x_{n,n}\, (> -1)$$

be the zeros of the associated orthonormal polynomials p_n *in decreasing order. Let*

$$x_{\nu,n} = \cos\vartheta_{\nu,n}\,, \qquad 0 < \vartheta_{\nu,n} < \pi\,, \qquad \nu = 1, 2, \ldots, n\,.$$

Let $\vartheta_{0,n} := 0$ *and* $\vartheta_{n+1,n} := \pi$*. Then there is a constant* K *depending only on* M *such that*

$$\vartheta_{\nu+1,n} - \vartheta_{\nu,n} < \frac{K\log n}{n}\,, \qquad \nu = 0, 1, \ldots, n\,.$$

This result has been extended by various people in many directions.

Erdős and Freud [74.13] worked together on orthogonal polynomials with regularly distributed zeros. Let α be a nonnegative measure on $(-\infty,\infty)$ for which all the moments

$$\mu_m := \int_{-\infty}^{\infty} x^m\, d\alpha(x)\,, \qquad m = 0, 1, \ldots$$

exist and are finite. Denote the sequence of the associated orthonormal polynomials by $(p_n)_{n=0}^{\infty}$. Let $x_{1,n} > x_{2,n} > \cdots > x_{n,n}$ be the zeros of of p_n in decreasing order. Let $N(\alpha, t)$ denote the number of positive integers k for which

$$x_{k,n} - x_{n,n} \geq t(x_{1,n} - x_{n,n}) .$$

The distribution function β of the zeros is defined, when it exists, as

$$\beta(t) = \lim_{n\to\infty} n^{-1} N_n(\alpha, t) , \qquad 0 \leq t \leq 1 .$$

Let

$$\beta_0(t) = \frac{1}{2} - \frac{1}{\pi} \arcsin(2t-1) .$$

A nonnegative measure α for which the array $x_{k,n}$ has the distribution function $\beta_0(t)$ is called an arc-sine measure. If $d\alpha(x) = w(x)\, dx$ is absolutely continuous and α is an arc-sine measure, then w is called an arc-sine weight. One of the theorems of Erdős and Freud [74.13] states that *the condition*

$$\limsup_{n\to\infty} (\gamma_{n-1})^{1/(n-1)} (x_{1,n} - x_{n,n}) \leq 4$$

implies that α is arc-sine and

$$\lim_{n\to\infty} (\gamma_{n-1})^{1/(n-1)} (x_{1,n} - x_{n,n}) = 4 . \tag{3}$$

They also show that the weights $w_a(x) := \exp(-|x|^a)$, $a > 0$, are not arc-sine. It is further proved by a counter-example that even the stronger sufficient condition (3) in the above-quoted result is not necessary in general to characterize arc-sine measures. As the next result of their paper shows, the case is different if w has compact support. Namely they show that *a weight w, the support of which is contained in $[-1, 1]$, is arc-sine on $[-1, 1]$ if and only if*

$$\limsup_{n\to\infty} (\gamma_n)^{1/n} \leq 2 .$$

A set $A \subset [-1, 1]$ is called a determining set if all weights w, the restricted support $\{x : w(x) > 0\}$ of which contain A, are arc-sine on $[-1, 1]$. A set $A \subset [-1, 1]$ is said to have minimal capacity c if for every $\epsilon > 0$ there exists a $\delta(\epsilon) > 0$ such that for every $B \subset [-1, 1]$ having Lebesgue measure less than $\delta(\epsilon)$ we have $\mathrm{cap}(A \setminus B) > c - \epsilon$. Another remarkable result of this paper by Erdős and Freud is that *a measurable set $A \subset [-1, 1]$ is a determining set if and only if it has minimal capacity* $1/2$.

Erdős' paper [58.05] with Herzog and Piranian on the geometry of polynomials is seminal. In this paper, they proved a number of interesting results and raised many challenging questions. Although quite a few of these have been solved by Pommerenke and others, many of them are still open. Erdős liked this paper very much. In his talks about polynomials, he often revisited these topics and mentioned the unsolved problems again and again. A taste of this paper is given by the following results and still unsolved problems from it. As before, associated with a monic polynomial

$$f(z) = \prod_{j=1}^{n} (z - z_j), \qquad z_j \in \mathbb{C}, \tag{4}$$

let

$$E = E(f) = E_n(f) := \{z \in \mathbb{C} : |f(z)| \le 1\}.$$

One of the results of Erdős, Herzog, and Piranian tells us that *the infimum of* $m(E(f))$ *is* 0, *where the infimum is taken over all polynomials* f *of the form (4) with all their zeros in the closed unit disk (*n *varies and* m *denotes the two-dimensional Lebesgue measure).* Another result is the following. *Let* F *be a closed set of transfinite diameter less than* 1. *Then there exists a positive number* $\rho(F)$ *such that, for every polynomial of the form (4) whose zeros lie in* F*, the set* $E(f)$ *contains a disk of radius* $\rho(F)$. There are results on the number of components of E, the sum of the diameters of the components of E, some implications of the connectedness of E, some necessary assumptions that imply the convexity of E. An interesting conjecture of Erdős states that *the length of the boundary of* $E_n(f)$ *for a polynomial* f *of the form (4) is* $2n + O(1)$. This problem seems almost impossible to settle. The best result in this direction is $O(n)$ by P. Borwein [Bor] that improves an earlier upper bound $74n^2$ given by Pommerenke.

One of the papers where Erdős revisits this topic is [73.01], written jointly with Netanyahu. The result of this paper states that *if the zeros* $z_j \in \mathbb{C}$ *are in a bounded, closed, and connected set whose transfinite diameter is* $1 - c$ $(0 < c < 1)$, *then* $E(f)$ *contains a disk of positive radius* ρ *depending only on* c.

Erőd attributes the following interesting result to Erdős and Turán and presents its proof in his paper. *If*

$$f(z) = \pm \prod_{j=1}^{n} (x - x_j), \qquad -1 \le x_1 \le x_2 \le \cdots \le x_n \le 1, \tag{5}$$

and f *is convex between* x_{k-1} *and* x_k *for an index* k*, then*

$$x_k - x_{k-1} \le \frac{16}{\sqrt{n}}.$$

It is not clear to me whether or not Erdős and Turán published this result.

An elementary paper of Erdős and Grünwald (Gallai) [39.07] deals with some geometric properties of polynomials with only real zeros. One of their results states that *if* f *is a polynomial of the form (5), then*

$$\int_{x_k}^{x_{k+1}} |f(x)|\, dx \le \frac{2}{3} (x_{k+1} - x_k) \max_{x \in [x_k, x_{k+1}]} |f(x)|.$$

Some extensions of the above are proved in [40.02]. In this paper, Erdős raised a number of questions. For example, he conjectured that if t is a real trigonometric polynomial with only real zeros and with maximum 1 then

$$\int_0^{2\pi} |t(\vartheta)|\, d\vartheta \le 4\,.$$

Concerning polynomials $p \in \mathcal{P}_n$ with all their zeros in $(-1,1)$ and with $\max_{x\in[-1,1]} |p(x)| = 1$, Erdős conjectured that *if $x_k < x_{k+1}$ are two consecutive zeros of p, then*

$$\int_{x_k}^{x_{k+1}} |p(x)|\, dx \le d_n(x_{k+1} - x_k)\,,$$

where

$$d_n := \frac{1}{y_{k+1} - y_k} \int_{y_k}^{y_{k+1}} |T_n(y)|\, dy\,,$$

T_n is the usual Chebyshev polynomial, and $y_k < y_{k+1}$ are two consecutive zeros of T_n. (Note that d_n is independent of k and that $\lim d_n = 2/\pi$.) These conjectures and more have all been proved in 1974, see Saff and Sheil-Small [SaSh], and also Kristiansen [Kr1].

A paper of Erdős [42.05] deals with the uniform distribution of the zeros of certain polynomials. Let

$$1 = x_0 \ge x_1 > x_2 > \cdots > x_n \ge x_{n+1} = -1$$

and let $x_i = \cos\vartheta_i$, where $\vartheta_i \in [0,\pi]$. Let $\omega_n(x) = \prod_{i=1}^n (x - x_i)$. Let $0 \le A < B \le \pi$. Let $N_n(A,B)$ denote the number of ϑ_i in (A,B). Extending the results of an earlier paper [40.08] of his with Turán, Erdős proved that *if there are absolute constants $c_1, c_2 > 0$ such that*

$$\frac{c_1 f(n)}{2^n} \le \max_{x_{k+1}\le x\le x_k} |\omega_n(x)| \le \frac{c_2 f(n)}{2^n}\,, \qquad k = 0,1,\dots,n\,,$$

then

$$N_n(A,B) = \frac{B-A}{\pi} n + O((\log n)(\log f(n)))\,.$$

The gap condition of Fabry states that if $f(z) = \sum a_k z^{n_k}$ is a power series whose radius of convergence is 1, and $\lim n_k/k = \infty$, then the unit circle is the natural boundary of f. Pólya proved the following converse result. *Let (n_k) be an increasing sequence of nonnegative integers for which $\liminf n_k/k < \infty$. Then there exists a power series $\sum a_k z^{n_k}$ with radius of convergence 1 and for which the unit circle is not the natural boundary.* Erdős [45.03] offers a direct and elementary proof of Pólya's result.

Another notable paper of Erdős [47.02], joint with H. Fried, explores some connections between gaps in power series and the zeros of their partial sums. Let $f(z) = 1 + \sum_{n=1}^\infty a_n z^n$ be a power series with radius of convergence 1. The power series is said to have Ostrowsky gaps ϱ if there exists a $\varrho < 1$ and a pair of infinite sequences (m_k) and (n_k), with $m_k < n_k$

and $\lim n_k/m_k > 1$ such that $|a_n| < \varrho^n$ for $m_k \le n \le n_k$. Let $A(n,r)$ denote the number of zeros of $S_n(z) := 1 + \sum_{i=1}^n a_i z^i$ in the open disk centered at 0 with radius r. A theorem of Erdős and Fried states that *a necessary and sufficient condition that a power series have Ostrowsky gaps is that there exists an $r > 1$ such that*

$$\liminf \frac{A(n,r)}{n} < 1 \,.$$

Erdős [67.16] gives an extension of some results of Bernstein and Zygmund. Bernstein had asked the question whether one can deduce boundedness of $|P_n(x)|$ on $[-1,1]$ for polynomials P_n of degree at most n if one knows that $|P_n(x)| \le 1$ for $m > (1+c)n$ values of x with some $c > 0$. His answer was affirmative. He showed that *if $|P_n(x_i^{(m)})| \le 1$ for all zeros $x_i^{(m)}$ of the mth Chebyshev polynomial T_m with $m > (1+c)n$, then $|P_n(x)| \le A(c)$ for all $x \in [-1,1]$, with $A(c)$ depending only on c.* Zygmund had shown that the same conclusion is valid if T_m is replaced by the mth Legendre polynomial L_m. Erdős established a necessary and sufficient condition to characterize the system of nodes

$$-1 \le x_1^{(m)} < x_2^{(m)} < \cdots < x_n^{(m)} \le 1$$

for which

$$|P_n(x_i^{(m)})| \le 1 \,, \qquad i = 1, 2, \dots, m \,, \qquad m > (1+c)n$$

imply $|P_n(x)| \le A(c)$ for all polynomials P_n of degree at most n and for all $x \in [-1,1]$, with $A(c)$ depending only on c. His result contains both that of Bernstein and of Zygmund as special cases. Note that such an implication is impossible if $m \le n+1$, by a well-known result of Faber.

Erdős wrote a paper [46.05] on the coefficients of the cyclotomic polynomials. The cyclotomic polynomial F_n is defined as the monic polynomial whose zeros are the primitive nth roots of unity. It is well known that

$$F_n(x) = \prod_{d|n} (x^{n/d} - 1)^{\mu(d)} \,.$$

For $n < 105$, all coefficients of F_n are ± 1 or 0. For $n = 105$, the coefficient 2 occurs for the first time. Denote by A_n the maximum over the absolute values of the coefficients of F_n. Schur proved that $\limsup A_n = \infty$. Emma Lehmer proved that $A_n > cn^{1/3}$ for infinitely many n. In his paper [46.05], Erdős proved that *for every k, $A_n > n^k$ for infinitely many n.* This is implied by his even sharper theorem to the effect that

$$A_n > \exp\left[c\,(\log n)^{4/3}\right]$$

for $n = 2 \cdot 3 \cdot 5 \cdots p_k$ with k sufficiently large. Recent improvements and generalizations of this can be explored in [Ma1-3].

Erdős has a note [49.08] on the number of terms in the square of a polynomial. Let

$$f_k(x) = a_0 + a_1 x^{n_1} + \cdots + a_{k-1} x^{n_{k-1}}, \qquad 0 \neq a_i \in \mathbb{R}\,,$$

be a polynomial with k terms. Denote by $Q(f_k)$ the number of terms of f_k^2. Let $Q_k := \min Q(f_k)$, where the minimum is taken over all f_k of the above form. Rédei posed the problem whether $Q_k < k$ is possible. Rényi, Kalmár, and Rédei proved that, in fact, $\liminf Q_k/k = 0$, and also that $Q(29) \leq 28$. Rényi further proved that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \frac{Q_k}{k} = 0\,.$$

He also conjectured that $\lim Q_k/k = 0$. In his short note [49.08], Erdős proves this conjecture. In fact, he shows that *there are absolute constants* $c_1 > 0$ *and* $0 < c_2 < 1$ *such that* $Q_k < c_2 k^{1-c_1}$. Rényi conjectured that $\lim Q_k = \infty$. He also asked whether or not Q_k remains the same if the coefficients are complex. These questions remained open (at least in this paper).

Erdős has a number of papers on rational approximation. In [76.20], he proves that *if* f *is a non-vanishing continuous function defined on* $[0,\infty)$ *for which* $\lim_{x\to\infty} f(x) = 0$, *then for every sequence of integers* $0 := n_0 < n_1 < \cdots$ *satisfying* $\sum_{i=1}^{\infty} 1/n_i = \infty$, *there is a sequence of Müntz polynomials* $(p_k) \subset \mathrm{span}\{x^{n_0}, x^{n_1}, \dots\}$ *for which*

$$\lim_{k\to\infty} \left\| \frac{1}{f} - \frac{1}{p_n} \right\|_{L^\infty[0,\infty)} = 0\,. \tag{6}$$

Using a result from the Clarkson–Erdős paper [43.02], he also observes, in [76.20], that *if* f *is a non-vanishing continuous function defined on* $[0,\infty)$ *for which there exists a sequence* $(p_n) \subset \mathrm{span}\{x^{n_0}, x^{n_1}, \dots\}$ *with* $0 := n_0 < n_1 < \cdots$ *and* $\sum_{i=1}^{\infty} 1/n_i < \infty$ *such that (6) holds, then* f *is the restriction to* $[0,\infty)$ *of an entire function.*

A typical result of Erdős, Newman, and Reddy [77.04] deals with rational approximations to e^{-x} on $[0,\infty)$. They prove, among many other results, that *if* p *and* q *are real polynomials of degree at most* $n-1$ *with* $n \geq 2$, *then*

$$\left\| e^{-x} - \frac{p(x)}{q(x)} \right\|_{L^\infty(\mathbb{N})} \geq \frac{(e-1)^n e^{-4n} 2^{-7n}}{n\left(3+2\sqrt{2}\right)^{n-1}}\,.$$

This should be compared with the approximation rate

$$\left\| e^{-x} - \frac{1}{q(x)} \right\|_{L^\infty[0,\infty)} \leq 2^{-n}$$

with $q(x) := \sum_{k=0}^{n} x^k/(k!)$. A substantial collection of various results concerning various kinds of rational approximation can be found in another paper of Erdős written jointly with Reddy [76.46].

Erdős [62.01] proved a significant result related to his conjecture about polynomials with ± 1 coefficients. He showed that if

$$f_n(\vartheta) := \sum_{k=1}^{n} (a_k \cos k\vartheta + b_k \sin k\vartheta)$$

is a trigonometric polynomial with real coefficients,

$$\max_{1 \le k \le n} \{\max\{|a_k|, |b_k|\}\} = 1 \qquad \text{and} \qquad \sum_{k=1}^{n} (a_k^2 + b_k^2) = An\,,$$

then there exists a $c = c(A) > 0$ *depending only on* A *for which* $\lim_{A \to 0} c(A) = 0$ *and*

$$\max_{0 \le \vartheta \le 2\pi} |f(\vartheta)| \ge \frac{1 + c(A)}{\sqrt{2}} \left(\sum_{k=1}^{n} (a_k^2 + b_k^2) \right)^{1/2} .$$

Closely related to this is a problem for which Erdős offered \$100 and which has become one of my favorite Erdős problems: *Is there an absolute constant* $\varepsilon > 0$ *such that the maximum norm on the unit circle of any polynomial* $p(x) = \sum_{j=0}^{n} a_j x^j$ *with each* $a_j \in \{-1, 1\}$ *is at least* $(1+\varepsilon)\sqrt{n}$*?* Erdős conjectured that there is such an $\varepsilon > 0$. Even the weaker version of the above, with $(1+\varepsilon)\sqrt{n}$ replaced by $\sqrt{n} + \varepsilon$ with an absolute constant $\varepsilon > 0$, looks really difficult. (The lower bound $\sqrt{n+1}$ is obvious by the Parseval formula.) Originally, Erdős and D.J. Newman conjectured that there is an absolute constant $\varepsilon > 0$ such that the maximum norm on the unit circle of any polynomial $p(x) = \sum_{j=0}^{n} a_j x^j$ with each $a_j \in \mathbb{C}\,, |a_j| = 1$ is at least $(1+\varepsilon)\sqrt{n}$. An astonishing result of Kahane [K], [L] disproves this by showing the existence of "ultra flat" unimodular polynomials with modulus always between $(1-\varepsilon)\sqrt{n}$ and $(1+\varepsilon)\sqrt{n}$ on the unit circle for an arbitrary prescribed $\varepsilon > 0$.

In [65.19], dedicated to Littlewood on his 80th birthday, Erdős gave an interesting necessary condition insuring that a sequence of integers $0 \le n_0 < n_1 < \cdots$ is not a Zygmund sequence. More precisely, he showed that *if* $0 \le n_0 < n_1 < \cdots$ *is a sequence that contains two subsequences* $(n_{k_i})_{i=1}^{\infty}$ *and* $(n_{l_i})_{i=1}^{\infty}$ *satisfying*

$$k_i \to \infty, \qquad k_i < l_i < k_{i+1}, \quad l_i - k_i \to \infty, \quad (n_{l_i} - n_{k_i})^{1/(l_i - k_i)} \to 1\,,$$

then there is a power series $\sum_{k=0}^{\infty} a_k z^{n_k}$ *with* $|a_k| \to 0$ *that diverges everywhere on the unit circle.* The proof of this theorem utilizes probabilistic arguments which have been used in several earlier papers.

An interesting paper of Erdős [54.07] with Herzog and Piranian deals with sets of divergence of Taylor series and trigonometric series. A typical result of this paper states that *for every subset* E *of the unit circle with logarithmic capacity* 0*, there is a function*

$f(z) = \sum_{n=1}^{\infty} a_n z^n$ so that f is continuous on the closed unit disk, $\sum_{n=1}^{\infty} a_n z^n$ diverges on E, and the sequence of partial sums s_n is uniformly bounded on the unit circle.

In 1911, Lusin constructed a power series $\sum_{n=0}^{\infty} a_n z^n$ with $a_n \to 0$ that diverges at every point on the unit circle. Dvoretzky and Erdős [55.05] gave an interesting extension of this result. They proved that *if $(b_n) \subset \mathbb{C}$ with $|b_n| \geq |b_{n+1}|$ for each n and $\sum_{n=0}^{\infty} |b_n|^2 = \infty$, then there exists a power series $\sum_{n=0}^{\infty} a_n z^n$ with each a_n equal to either b_n or 0 that diverges everywhere on the unit circle.* Here the monotonicity condition cannot be entirely dispensed with, since every power series $\sum_{n=0}^{\infty} a_n z^{t_n}$ with $a_n \to 0$ and $\sum_{n=0}^{\infty} t_n/t_{n+1} < \infty$ converges on a subset of the unit circle which is everywhere dense on the unit circle. The condition $\sum_{n=0}^{\infty} |b_n|^2 = \infty$ cannot be relaxed either by Carleson's theorem. (Carleson's theorem was a conjecture when Dvoretzky and Erdős wrote their paper, so they commented on this as the above assumption "probably cannot be relaxed at all, since it is conjectured that every power series with $\sum_{n=0}^{\infty} b_n z^n$ with $\sum_{n=0}^{\infty} |b_n|^2 < \infty$ converges almost everywhere in C.")

Several topics from Erdős's problem paper [76.14] have already been discussed before. Here is one more interesting group of problems. Let $(z_k)_{k=1}^{\infty}$ be a sequence of complex numbers of modulus 1. Let

$$A_n := \max_{|z|=1} \prod_{k=1}^{n} |z - z_k|\,.$$

What can one say about the growth of A_n? Erdős conjectured that $\limsup A_n = \infty$. In my copy of [76.14] that Erdős gave me a few years ago, there are some handwritten notes (in Hungarian) saying the following. "Wagner proved that $\limsup A_n = \infty$. It is still open whether or not $A_n > n^c$ or $\sum_{k=1}^{n} A_k > n^{1+c}$ happens for infinitely many n (with an absolute constant $c > 0$). These are probably difficult to answer." [W]

Erdős was famous for anticipating the "right" results. "This is obviously true; only a proof is needed" he used to say quite often. Most of the times, his conjectures turned out to be true. Some of his conjectures failed for the more or less trivial reason that he was not always completely precise with the formulation of the problem. However, it happened only very rarely that he was essentially wrong with his conjectures. If someone proved something that was in contrast with Erdős' anticipation, he or she could really boast to have proved a really surprising result. Erdős was always honest with his conjectures. If he did not have a sense about which way to go, he formulated the problem "prove or disprove". Erdős turned even his "ill fated" conjectures into challenging open problems. The following quotation is a typical example for how Erdős treated the rare cases when a conjecture of his was disproved. It is from his problem paper [76.14] entitled "Extremal problems on polynomials". For this quotation, we need to know the following notation. Associated with a monic polynomial $f(z) = \prod_{j=1}^{n} (z - z_j)$, where z_j are complex numbers, let $E_n(f) := \{z \in \mathbb{C} : |f(z)| \leq 1\}$. In his problem paper Erdős writes (in terms of the notation employed here): "In [7] we made the ill fated conjecture that the number of components of $E_n(f)$ with diameter greater than $1 + c$ $(c > 0)$ is less than δ_c, δ_c bounded. Pommerenke [14] showed that nothing could be farther from the truth, in fact he showed that for every $\epsilon > 0$ and $k \in \mathbb{N}$, there is an $E_n(f)$ which has more than k components of diameter greater than $4 - \varepsilon$. Our conjecture can probably be saved as follows: Denote by $\Phi_n(c)$ the largest number of components of diameter greater than $1 + c$ $(c > 0)$ which $E_n(f)$ can have. Surely, for every $c > 0$, $\Phi_n(c) = o(n)$, and hopefully $\Phi_n(c) = o(n^\varepsilon)$ for every $\varepsilon > 0$. I have no guess about a lower bound for $\Phi_n(c)$, also I am not sure whether the growth of $\Phi_n(c)$, $(1 < c < 4)$ depends on c very much."

The list of Erdős' truly ingenious and diverse results concerning polynomials and related topics could be continued for many more pages. One cannot include even all the highlights in a limited space. The reader may correctly think that there are more important results of Erdős in approximation theory than those mentioned in this article. I was concentrating on those results and problems of Erdős that meant the most to me so far and I am looking forward to discovering the beauty in many of his papers that I have not had the chance to read so far.