Why is There Philosophy of Mathematics at All?

Reviewed by John P. Burgess

Why is There Philosophy of Mathematics At All?
Ian Hacking
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In his latest book, the noted philosopher of science Ian Hacking turns his attention to mathematics, a long-standing interest of his, heretofore seldom indulged. His previous work, though it hasn't always been universally found convincing, has been unfailingly provocative. The present work fits the pattern.

Some mathematicians seeing the title of Hacking's latest book may read it as meaning something like, "Why can't we just get rid of philosophy mathematics?" asked in a tone of voice suggesting that it would be a very good thing if we could. After all, did not Hilbert himself announce, speaking of foundational questions, that what he wanted to do with them was to get rid once and for all [*einfürallemal aus der Welt zu schaffen*] of them? But this is not what is intended. Hacking doesn't think philosophy of mathematics will ever go away for good, be got rid of once and for all, and he genuinely means to ask why this is so: What is it about mathematics that historically has kept drawing philosophers back to it, time and again?

The answer suggested is that there are two factors at work. One is the experience of following a compelling proof. The seeming inevitability of the conclusion, the feeling that it is not something one is free to take or leave as one chooses, Hacking cites as an ultimate motivation behind philosophies that affirm the independent reality of a realm of mathematical facts, from Plato to Hardy [3]. The other factor is the observation that mathematical desk-work again and again proves useful in dealing with the world outside the mathematician's office. How by just sitting and thinking we (or some of us) can arrive at results applicable to the world around us has puzzled thinkers from Kant to Wigner [4].

The two features are separate. That results should prove applicable to the physical universe even though they were obtained by pure desk work, without controlled experimentation on or systematic observation of the material world, can be surprising even if what the desk work produces is not compelling deductive proofs but "only" suggestive heuristic arguments. And with the two factors being separate, the material in the book is divided into two more or less separate parts, though with a lot of back and forth between them, one devoted to proof, the other to applications.

Neither the part about proof nor the part about applications is concerned *only* with their role in perennially drawing the attention of philosophers to mathematics. And beyond the general division into these two broad topics, the book is rather loosely organized and digressive, not to say rambling, in a way that makes it quite impossible for the reviewer to summarize its contents in an even halfway adequate fashion. The analytical table of contents goes on for six pages, and there is nothing I would leave out; but this means that even to list the topics addressed would take up more space than is reasonable for a review.

One thing just leads to another: If a philosophical view is stated, some mathematical example will be wanted to illustrate it, but then at least an informal explanation of the key concepts in the example will be wanted also, and perhaps a capsule bio of the author or authors of the relevant result or results, and even perhaps in cases where they have won prizes something by way of description of the prizes and who established or who awards them, and so on. As a result, in the index one finds Fermat and the Fields Institute, formalism and Foucault, the four-color problem and Frege and Freud all rubbing shoulders.

Not Philosophy but *about* Philosophy

Now it is one thing to write about a field, and another to work in it. Hacking asserts early and emphatically that his is a book *about* philosophy of mathematics, but not a work *of* philosophy of mathematics. He is indeed quite reticent about his own philosophical views, preferring to survey those of others. And the book is wholly free of philosophical polemics: Hacking manages to find something nice to say about almost every writer he discusses or even just mentions in passing, even writers who disagree profoundly with each other, and even writers who disagree profoundly with the views with which Hacking shows himself most sympathetic. (Conflict of interest disclosure: This includes the present reviewer.)

And while Hacking has quite a bit to say about important figures in the history of philosophy, and to a lesser extent of mathematics, he also avoids scholarly controversies over the interpretation of the thought of historical figures. Thus he frequently quotes one of his favorites, Ludwig Wittgenstein, about the exegesis of whose cryptic works there have notoriously been very bitter controversies, but cheerfully says that it doesn't matter to him if he has got Wittgenstein right, though he thinks he has. For what it is worth I am mostly inclined to agree, though I am struck by Hacking's omission of certain of Wittgenstein's dicta that if quoted might make the philosopher seem a less sympathetic figure to mathematicians. Try this one (directed against Hardy): "…what a mathematician is inclined to say about the objectivity and reality of mathematical facts, is not a philosophy of mathematics, but something for philosophical *treatment*. … like the treatment of an illness." ([6], 254-5) Or this one (directed against Hilbert on "Cantor's paradise"): "Philosophical clarity will have the same effect on the growth of mathematics as sunlight has on the growth of potato shoots. (In a dark cellar they grow yards long.)" ([5], 381)

More importantly, while a contrast between what Hacking calls "cartesian" and "leibnizian" proofs or conceptions of proof runs through much of his discussion, he spells the labels with a small *cee* and *el* by way of indicating that he's not especially concerned to defend any claims about the exact content of the thought of the historical Descartes and Leibniz.

One respect, apart from its livelier style, in which Hacking's book differs from conventional philosophical writing about mathematics is that an enormous amount of work goes into gathering food for thought, but much less into boiling it down, chewing it over, and digesting it. For instance, the material on applications goes on at surprising length and depth about etymological issues (when did the expression "mixed mathematics" gave way to "applied mathematics"?) and sociological ones (how the organization of universities into departments in the nineteenth century differ between Britain and Germany?), but it wasn't clear to me what we are supposed to make of all this, fun as it was to read about.

The material on proof also shows a tendency to give priority to information gathering over critical analysis. This is perhaps especially so in connection with the cartesian/leibnizian distinction. It is used as a peg on which to hang the discussion of various issues and episodes and personalities, but the kind of questions that a conventional philosopher of mathematics would feel compelled to address in connection with such a distinction are just not gone into. To cite just one crucial issue, are the various rough characterizations offered of "cartesian" (or "leibnizian") proof equivalent, all pointing towards the same feature? In other words, do we have one distinction here, or are several being run together?

I suspect the latter. Hacking begins by noting that Descartes speaks (though as Hacking significantly admits, not in his *mathematical* writings) of getting an entire proof in the mind all at once. Hacking doesn't, however, mention why Descartes *needs* such a notion in his philosophical system. Descartes, towards the ultimate aim of arguing that all knowledge depends on knowledge of God, suggests that the conclusions of an atheist mathematician, or any conclusion arrived at by a series of steps, can be rendered doubtful by the reflection that maybe a tricky demon was just making it *seem* that one step followed another. Descartes has a argument why we can't really be deceived by such a tricky demon (the main consideration being that there is a God Who would prevent it), but that argument itself consists of a series of steps, and so presumably can be rendered doubtful: Maybe the tricky demon's trickiest trick is to trick Descartes into believing there is no tricky demon. Only if Descartes can see the whole argument in a flash and not as a succession of steps, if he can get the entire proof into his mind all at the same time, can he be freed from the possibility of having his conclusions rendered doubtful.

But Hacking's notion of "cartesian proof" slides from this first characterization, as a proof that one can get the entirety of into one's mind all at once, to a different characterization, as a proof that does not merely convince us that a result is true but explain to us why the result is true. This is a distinction about which in the last decades there has been a great deal of discussion by philosophers of mathematics, especially those who identify themselves as "philosophers of mathematical practice", with rather meager and inconclusive results (a fact that should hardly surprise Hacking, familiar as he must be with how intense philosophical investigation of "scientific explanation" in theoretical physics a couple of decades back led to similarly meager and inconclusive results). And needless to say there are many quotable things mathematicians have said about such a distinction at one time or another, too, not all by any means pointing in the same direction. But what does a proof's being explanatory have to do with our being able to get the entire proof into the mind all at once?

There are lots of proofs of mathematical propositions *p* that one could say explain *why p* and don't just convince *that p*. But I don't find myself able in any interesting case to get the entirety of such a proof into my mind at once, to see the proof as a single step rather than a succession. Hacking cites Littlewood's version of the proof that there can be no decomposition of a cube into cubes all of unequal size, which appears as an epigraph to the book. I myself don't find it easy to take this in as a single step rather than a succession of several — to see in a flash why the result is true, which would presumably include seeing why the same argument doesn't work one dimension down to show that there is no decomposition of a square into squares of unequal size — especially when one fills in the reasoning needed to establish a lemma that Littlewood simply calls obvious. Well, perhaps *my* mind is just too small for this to fit in all at once. One of Hacking's more amusing, and only too true, observations about the experience of compelling proofs is that *most* people don't have it.

Glitter

If Hacking's approach does not stop to carry out the kind of critical analysis that would be needed to establish, say, that in speaking of "cartesian" and "leibnizian" proofs one is looking at a clear, univocally-characterized dichotomy, my saying so is not a matter of *complaining*. It is a matter *explaining* how what Hacking is doing in writing *about* philosophy of mathematics differs from working *in* philosophy of mathematics — how it differs and why it may be more fun. Writing *in* philosophy of mathematics generally must plod along at a slow and deliberate pace. Writing *about* philosophy of mathematics can be breezier and take us to more interesting places in less time.

If I *did* have any complaint, it would not be about the kind of book Hacking has chosen to write, but about his tendency, when he wants to illustrate some phenomenon, to pass over homelier examples and go immediately for the most subtle and sophisticated — and recent. Hacking quotes Wittgenstein as warning against being taken in by "glitter", being distracted in philosophy of mathematics (and not just of mathematics) from what is essential by glamorous results. Hacking himself, as he is not unaware, sometimes runs the risk of being distracted, or distracting his readers, in just this way.

Let me illustrate this rather abstract remark by a concrete case. In speaking of what moves some mathematicians and philosophers to speak of mathematical reality as being "out there" before we discover it, Hacking slides from talking about compelling proofs to talking about compelling results, even when the proofs are long and laborious. He mentions classification theorems, which are indeed excellent examples. There is nothing like learning that there are exactly *N* of something or other to encourage the thought that all *N* of them were "out there" before we found out about them. The five Platonic solids used to be used as an illustrative example in this way. Hacking suggests that, historically important as this example may have been, it has become so familiar that we are now blasé about it, and need a different example.

He then goes at once for the classification of simple groups. This provides opportunities to mention various interesting people and exotic topics, but there is no hope in a work at a semipopular level of explaining what, say, John Conway's "monstrous moonshine" actually amounted to, or even what specifically a sporadic simple group *is*. Surely there must be examples — two-manifolds, perhaps, or non-planar graphs — less hackneyed than the Platonic solids but less ferociously technical than finite simple groups.

Another very interesting phenomenon Hacking gets into, on his way to discussing the extramathematical applicability of mathematics, is *intra*mathematical applicability, beginning with Descartes' application of algebra to geometry. That example by now has something in common with the Platonic solids example, namely, a degree of familiarity that makes it unexciting. Again Hacking goes off in the direction of very sophisticated material indeed — the Langlands program, no less — in search of fresher examples, and gets into material so complicated that he himself is not sure whether one should speak of applying one field to another or merely of seeing analogies between fields.

Again there are near to hand simpler examples of the phenomenon of surprising connections between diverse mathematical specialties that could have been cited instead or as well. There is, to begin with, De Morgan's well-known old story ([1], 284-287) of an encounter with an acquaintance, apparently in the insurance business, in which they were talking of life expectancies and De Morgan cited some actuarial formula, probably related to the normal distribution, involving the symbol π for the famous constant, the ratio of circumference to diameter. The reply was, "Oh, my dear friend! that must be a delusion; what can the circle have to do with the numbers alive at the end of a given time?" This simple example at least shows clearly the first crucial feature that needs to be mentioned in connection with surprising connections: There isn't going to be any *nonmathematical* explanation of them.

Summary

There is much more in the book. I have taken well over two thousand words without really touching on the aspects of the book touted in the publisher's blurb: Discussion of the historical question where proof came from, and where the distinction between applied and pure, and of the question "What is mathematics?" Any answer to this last is likely to look disappointing after the splendid opening to Timothy Gowers' editorial preface to [2], in which he quotes Russell's definition of mathematics and adds that his volume (of 1000+ pages) is about what Russell's definition leaves out. Hacking gives the question only about 35 pages, but he does manage to bring in curious information you won't find in the Gowers volume.

Perhaps I should stop here. But before closing, let me mention one more respect in which Hacking's book *about* philosophy of mathematics differs from books *in* philosophy of mathematics: Since it is as far as possible from being one long, connected argument for a distinctive, unifying thesis, the reader can freely dip into it, open it almost anywhere and just start reading with pleasure and profit.

References

[1] Augustus De Morgan, *A Budget of Paradoxes*, 2nd ed., vol. 1, Open Court, 1915.

[2] Timothy Gowers (ed.), *The Princeton Companion to Mathematics*, Princeton University Press, 2008.

[3] G. H. Hardy, *A Mathematician's Apology*, Cambridge University Press, 1940.

[4] Eugene Wigner, The unreasonable effectiveness of mathematics in the natural sciences, *Communications in Pure and Applied Mathematics* **13** (1960), 1-14.

[5] Ludwig Wittgenstein, *Philosophical Grammar*, ed. Rush Rhees, trans. Anthony Kenney, Blackwell, 1974.

[6] _________, *Philosophical Investigations*, 3rd. ed., trans. G. E. M. Anscombe, Blackwell, 2001.

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