BIG IMAGE OF GALOIS REPRESENTATIONS
AND CONGRUENCE IDEALS

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CONTENTS

1. INTRODUCTION

Let $n \geq 1$; we consider the $n \times n$ antidiagonal unit matrix $s = (\delta_{i,n+1-j})_{1 \leq i,j \leq n}$ and the $2n \times 2n$ antisymmetric matrix $J = \begin{pmatrix} 0 & s \\ -s & 0 \end{pmatrix}$; we denote by $G = \mathrm{GSp}_{2n}$ the Chevalley group of symplectic similitude matrices for J and by $B = TN$ its standard Borel consisting of upper triangular matrices in G ; let ρ be the half-sum of positive roots for (G, B, T) and $\lambda \in X^*(T)$ a dominant weight. Let π be a cuspidal automorphic representation on $G(\mathbf{A}_{\mathbb{Q}})$ of level $M \geq 1$, whose infinity type is in the discrete series, with infinitesimal character $\lambda + \rho$. It occurs in the cohomology of the genus n Siegel variety of level M, with coefficients in the local system of highest weight λ. It implies that there exists a number field K_0 whose ring of integers $\mathcal{O}_0$ contains all Hecke eigenvalues of π (for prime-to-M Hecke operators). Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and $\widehat{G}$ be the dual Chevalley group for (G, B, T) ; one has $\widehat{G} = \mathrm{GSpin}_{2n+1}$ (see below) ; we assume that the compatible system of Galois representations $\rho_{\pi,\iota_\ell} \colon G_{\mathbb{Q}} \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ associated to π and to embeddings $\iota_\ell \colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$ is constructed (for the moment, it is known for $n = 1, 2$). Let $p \nmid M$ be a rational prime. Assume that π is ordinary for ι_p ; there exists an n-variable Hida family passing by π ; more precisely, let Λ_n be the n-variable Iwasawa algebra and $\mathbf{T}^M$ be the ordinary prime-to-M Hida-Hecke algebra [MT02], [H02], [Pi12] ; it

is finite torsion-free over Λ_n. A Hida family is a Λ_n-algebra homomorphism $\theta\colon \mathbf{T}^M \to \mathbb{I}$ where $\mathbb{I}$ is an integrally closed finite torsion-free extension of Λ_n.

By interpolating the pseudo-representations associated to the arithmetic specializations of θ (see Sect.4.2), one constructs a continuous homomorphism $\rho_\theta\colon G_{\mathbb{Q}} \to \widehat{G}(\mathbb{I})$ associated to θ (see Lemma 4.3 in the text). We assume throughout that the residual representation is irreducible. Following the ideas of [H13b], we prove first, under some assumptions (mostly $\mathbb{I} = \Lambda_n$ and $\mathbb{Z}_p$-regularity), that if the Hida family θ is "generic", the image of ρ_θ is "Λ_n-full" (see 4.8) ; more precisely, there is a non-zero ideal $\mathfrak{l}$ of Λ_n such that the image of Galois contains the principal congruence subgroup $\Gamma_{\widehat{G}'}(\mathfrak{l})$ in $\widehat{G}'(\Lambda_n)$, where $\widehat{G}'$ denotes the derived group of $\widehat{G}$. An important point of the proof is to create a Λ_n-structure on the submodule of a Lie algebra associated to $\operatorname{Im} \rho_\theta$; in the case $n = 1$, Hida [H13b] used conjugation by certain elements of the inertia ; a suitable generalization using repeated use of Poisson brackets provides the desired structure of Λ_n-module. Moreover, we ask the relation between the maximal such ideal $\mathfrak{l}_\theta$, called the Galois level of θ, and the schematic intersection of the irreducible component $\operatorname{Spec}(\mathbb{I})$ with other components, more precisely, at least if $\mathbb{I} = \Lambda_n$, we ask whether the set of primes containing $\mathfrak{l}_\theta$ coincides with the union of the zero loci in Λ_n of the greatest common divisor of the congruence ideal of θ and those of other non-generic families congruent to θ. The equality of these has been established for $n = 1$ by Hida [H13b]. We prove this equality for $n = 2$ in the case of twisted Yoshida lift congruence. In a subsequent paper, we plan to formulate more precisely this relation (including exponents of height one prime ideals) and generalize it to more general groups. The question of replacing Λ_n-fullness by $\mathbb{I}$-fullness, or rather $\mathbb{I}'$-fullness for a suitable subring of $\mathbb{I}$ is currently investigated, by a student of one of the authors[1].

2. Galois representations associated to Siegel modular forms

Let $G = \operatorname{GSp}_{2n}$ be the Chevalley group of symplectic similitude matrices for J. Let $\nu\colon G \to \mathbb{G}_m$ be the similitude factor character. Its kernel Sp_{2n} coincides with the derived group G' of G. We denote by $B = TU$ its standard Borel (consisting of upper triangular matrices in G).

Let $S_k^{(n)}(M)$ be the space of genus n holomorphic Siegel cusp forms of weight $k = (k_1, \ldots, k_n)$ with $k_1 \geq k_2 \geq \ldots \geq k_n \geq n+1$, and level M (say, for the principal congruence subgroup of level M). Let $\mathbf{T}_k^M$ be the M-spherical $\mathbb{Z}[\frac{1}{M}]$-algebra generated by the Hecke operators $T_{\ell,i}$ for all rational primes ℓ prime to M and all $i = 1, \ldots, n$, acting on $S_k^{(n)}(M)$. Recall that $T_{\ell,i}$ denotes the action of the double class $\Gamma(M) \cdot d_{\ell,i} \cdot \Gamma(M)$ where $d_{\ell,1} = \operatorname{diag}(1_n, \ell \cdot 1_n)$ and $d_{\ell,i} = \operatorname{diag}(1_{n+1-i}, \ell \cdot 1_{2i}, \ell^2 \cdot 1_{n+1-i})$ for $i = 2, \ldots n$. Let $f \in S_k^{(n)}(M)$ be an eigenform for $\mathbf{T}_k^M$; let K_0 be a number field containing its eigenvalues ; let $\mathcal{O}_0$ be its ring of integers. It gives rise to a character $\theta_f\colon \mathbf{T}_k^M \to \mathcal{O}_0$. For any prime ℓ not dividing M, we denote by $P_{f,\ell}(X)$ the Hecke polynomial (see [FC90] Chapt.VII, Sect.1); it is monic of degree 2^n.

Fix an odd prime p relatively prime to M, an embedding $\iota_p\colon \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and assume that f is ordinary for ι_p. It means that the multiset of p-adic valuations of the (images by ι_p of the) roots of the Hecke polynomial $P_{f,p}(X)$ coincides with the multiset $\{\sum_{i \in I}(k_i - i); I \subset [1, n]\}$ of Hodge weights.

To the data (G, B, T), one can associate a dual reductive group over $\mathbb{Z}[\frac{1}{2}]$ with a standard Borel and a standard torus $(\widehat{G}, \widehat{B}, \widehat{T})$, with an identification $X^*(T) = X_*(\widehat{T})$. By comparing the root data (with épinglage, see [MT02] Sect.3.2.2), one has a canonical identification $\widehat{G} = \operatorname{GSpin}_{2n+1}$ over $\mathbb{Z}[\frac{1}{2}]$. Therefore $\widehat{G}$ is endowed with a linear representation $\operatorname{spin}\colon \widehat{G} \to \operatorname{GL}_{2^n}$ and an orthogonal representation $\pi\colon \widehat{G} \to \operatorname{SO}_{2n+1}$ whose kernel is the center $\mathbb{G}_m$ of $\widehat{G}$ (the central inclusion $\mathbb{G}_m \to \widehat{G}$ is dual to the similitude factor $G \to \mathbb{G}_m$). It also carries a similitude factor $\widehat{G} \to \mathbb{G}_m$, dual to the central inclusion $\mathbb{G}_m \to G$; its kernel coincides with the derived group $\widehat{G}' = \operatorname{Spin}_{2n+1}$.

For any number field F, we put $G_F = \operatorname{Gal}(\overline{F}/F)$ and for any finite place v of F, we denote by D_v a decomposition subgroup at v in G_F ; let $\epsilon\colon G_F \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character. One still denotes by ϵ its restriction to D_v. We assume that there exists a Galois representation $\rho_f\colon G_{\mathbb{Q}} \to \widehat{G}(\overline{\mathbb{Q}}_p)$ unramified outside Mp and such that for any rational prime ℓ not dividing Mp, the

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characteristic polynomial $\mathrm{Char}(\mathrm{spin} \circ \rho_f(\mathrm{Frob}_\ell))$ coincides with $\iota_p(P_{f,\ell}(X))$. It is called the Galois representation associated to f. The integer $w = k_1 + \cdots + k_n - \frac{n(n+1)}{2}$ is called the motivic weight of ρ_f.

The existence of ρ_f is known for $n = 1$ (Deligne) and for $n = 2$ (due to R. Taylor, Laumon and Weissauer, in [Ta93], [We05] and [Lau05]). By compactness, there exists a p-adic field $K \subset \overline{\mathbb{Q}}_p$ such that ρ_f is defined over K. We may assume that $\iota_p(K_0) \subset K$, hence if $\mathcal{O}$ denotes its valuation ring, we have $\iota_p(\mathcal{O}_0) \subset \mathcal{O}$. One can choose an $\mathcal{O}$-lattice stable in the orthogonal representation $\pi \circ \rho_f$. We assume it is unimodular. It implies that ρ_f takes values in $\widehat{G}(\mathcal{O})$. Let ϖ be a uniformizing parameter of $\mathcal{O}$ and $\mathbb{F} = \mathcal{O}/\varpi\mathcal{O}$ its residue field. Let $\overline{\rho}_f \colon G_{\mathbb{Q}} \to \widehat{G}(\mathbb{F})$ be the residual representation.

Under the assumption of "automorphic ordinarity", it is conjectured that the following "Galois ordinarity" holds. Let $\widetilde{\rho} = \sum_{i=1}^{n} \varpi_i$ be the character of T sum of all the fundamental weights of G ; it is given by ρ on $T' = T \cap G'$ and by $z \mapsto z^{\frac{n(n+1)}{2}}$ on the center. Let us view $\lambda + \widetilde{\rho}$ as a cocharacter of $\widehat{T}$ via $X^*(T) = X_*(\widehat{T})$. For $\widehat{g}, \widehat{h} \in \widehat{G}$, let us put $\widehat{h}^{\widehat{g}} = \widehat{g}^{-1}\widehat{h}\widehat{g}$. Write D_p for a (chosen) decomposition group at p inside $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ with the inertia subgroup I_p.

(GO) There exists $\widehat{g} \in \widehat{G}(\mathcal{O})$ such that $\rho_f(D_p) \subset \widehat{g}\cdot\widehat{B}(\mathcal{O})\cdot\widehat{g}^{-1}$ and for any $\sigma \in I_p$, $\rho_f^{\widehat{g}}(\sigma) \pmod{\widehat{N}(\mathcal{O})}$ is given by $(\lambda + \widetilde{\rho}) \circ \epsilon(\sigma)$.

"Automorphic ordinarity" implies "Galois ordinarity" is known for $n = 1$ (due to Wiles) and for $n = 2$ (due to Urban [Ur05]). We assume in the sequel it holds for any n. It would follow from Katz-Messing theorem if one knew that $\mathrm{spin} \circ \rho_f$ is motivic.

In order to motivate the assumptions we shall make in genus n, let us prove some results in genus 2, where the results have less conditions.

3. Fullness of the image for Galois representations in $\mathrm{GSp}(4)$

3.1. **Irreducibility and open image.** Let $n = 2$ and f be a genus 2 cusp eigenform of weight $k = (k_1, k_2)$, $k_1 \geq k_2 \geq 3$, level M, as above. Fix an odd prime p prime to M at which f is ordinary; let $P_{f,p}(X) = (X - \alpha)(X - \beta)(X - \gamma)(X - \delta)$ be the Hecke polynomial of f at p, with roots ordered with increasing p-adic valuation (so that $\mathrm{ord}_p(\alpha) = 0$, $\mathrm{ord}_p(\beta) = k_2 - 2$, $\mathrm{ord}_p(\gamma) = k_1 - 1$ and $\mathrm{ord}_p(\alpha) = k_1 + k_2 - 3$) ; we assume that all the Hecke eigenvalues, together with α, β, γ and δ the are contained in $\mathcal{O}$ and that the Galois representation associated to f ([Ta93], [We05] and [Lau05]) is defined over $\mathcal{O}$: $\rho_f \colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathcal{O})$ where the group of symplectic similitudes is

relative to the matrix $J = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$. Recall that the spin representation provides an

isomorphism $\mathrm{GSpin}_5 \cong \mathrm{GSp}_4 \subset \mathrm{GL}_4$. Let (L, ψ) be the $\mathcal{O}$-module of the representation endowed with a unimodular symplectic pairing ψ ; we fix a symplectic $\mathcal{O}$-basis (e_1, e_2, e_3, e_4) of L with $\psi(e_1, e_4) = \psi(e_2, e_3) = 1$ so that we have an identification $\mathrm{GSp}(L, \psi) = \mathrm{GSp}_4(\mathcal{O})$; let $V = L[\frac{1}{p}]$ be the K-vector space spanned by L. We have

$$\rho_f|_{D_p} \sim \begin{pmatrix} \epsilon^{k_1+k_2-3}ur(\frac{\delta}{p^{k_1+k_2-3}}) & * & * & * \\ 0 & \epsilon^{k_1-1}ur(\frac{\gamma}{p^{k_1-1}}) & * & * \\ 0 & 0 & \epsilon^{k_2-2}ur(\frac{\beta}{p^{k_2-2}}) & * \\ 0 & 0 & 0 & ur(\alpha) \end{pmatrix}.$$

Here $ur(x) \colon D_p \to \mathcal{O}^\times$ is the unramified character sending Frob_p to $x \in \mathcal{O}^\times$. We shall assume in the sequel that the conjugation needed for this description is the identity. We also put $\overline{\rho} \colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\mathbb{F})$ its reduction modulo ϖ.

We first show

Proposition 3.1. *If f is neither CAP nor endoscopic, and if either*
  (i) Langlands transfer holds from $\mathrm{GSp}(4)$ to $\mathrm{GL}(4)$ (no need of the ordinarity assumption then),
  (ii) $p > 7$, the semisimplification $(\overline{\rho}|_{G_{\mathbb{Q}(\zeta_p)}})^{ss}$ contains a copy of $\mathrm{SL}_2(\mathbb{F}_p)$, and the p-adic units $\frac{\delta}{p^{k_1+k_2-3}}$, $\frac{\gamma}{p^{k_1-1}}$, $\frac{\beta}{p^{k_2-2}}$ and α are distinct modulo ϖ,
  then ρ_f is absolutely irreducible.

In the first case, one can use [CaGe13]. The transfer has been established by Arthur for $\mathrm{Sp}(4)$, but not yet for $\mathrm{GSp}(4)$, unless f admits a generic form with the same eigenvalues.

In the second case, we proceed by case inspection. If there is a stable line $D = \langle e_2' \rangle$ in L, let $P_1 = \langle e_2', e_3' \rangle$ be a hyperbolic plane of L containing D and $P_2 = \langle e_1', e_4' \rangle$ its orthogonal, so that $P_1 \perp P_2 = L$ be a decomposition of L into two orthogonal hyperbolic planes (it is possible by unimodularity of the symplectic pairing on L). By taking (e_1', e_2', e_3', e_4') as symplectic basis of L adapted to this decomposition, we see in this basis that

$$
\rho_f = \begin{pmatrix} * & 0 & \bullet & * \\ \bullet & \chi & \bullet & \bullet \\ 0 & 0 & \chi' & 0 \\ * & 0 & \bullet & * \end{pmatrix}.
$$

hence,

- ρ_f admits a degree 1 subrepresentation χ which is locally algebraic at p and finitely ramified, hence it is given by an (A_0)-type Hecke character of $\mathbb{Q}$, which cannot be of finite order because $\chi(p)$ cannot be of archimedean absolute value one by Deligne purity for α, β, γ and δ, so that χ cannot have Hodge-Tate weight 0 : it must be $k_2 - 2$ or $k_1 - 1$
- ρ_f has a degree 2 subquotient $\sigma = \begin{pmatrix} * & * \\ * & * \end{pmatrix}$, which is odd because its determinant is the similitude factor of ρ, which is odd.

Moreover, σ is p-ordinary with two distinct Hodge-Tate weights (0 and $k_1 + k_2 - 3$) and its residual representation is irreducible over $G_{\mathbb{Q}(\zeta_p)}$ by assumption. Therefore, the representation σ is modular : $\sigma = \sigma_g$ for an ordinary form g of weight $\ell \geq 2$ and level prime to p by Emerton's theorem on Fontaine-Mazur Conjecture ([Em14, Cor.1.2.2]) ; this implies by the classification of Vogan-Zuckerman (see [Ta93] Section 1 or [Ti09] Section 6) that, $k_1 = k_2 = k$, $\ell = 2k - 2$, $\chi = \epsilon^{k-1} \cdot (finite)$, and that the L-function of f is of the form $L(\chi, s)L(g, s)L(\chi, s-1)$, hence f is CAP of Siegel type (*i.e.* is a Saito-Kurokawa lift), contrary to the assumption.

Similarly, if there is a stable isotropic plane, ρ_f contains a two-dimensional subrepresentation; it is p-ordinary with distinct Hodge-Tate weights, $p > 7$, and the image of its reduction modulo ϖ restricted to $G_{\mathbb{Q}(\zeta_p)}$ contains $\mathrm{SL}_2(\mathbb{F})$, it is odd by Cor.1.3 of Calegari [Ca13] ; so again by [Em14, Cor.1.2.2], Fontaine-Mazur Conjecture holds and shows that f is CAP of Klingen type, contradiction. Finally, if there is a stable hyperbolic plane, we see by a similar argument (but without using Calegari's argument, hence without assuming $p > 7$ and the $\mathrm{SL}_2(\mathbb{F}_p)$ condition, because the oddness is obvious in this case), that f is a Yoshida lift, so it is endoscopic. QED.

Proposition 3.2. *Assume that the adjoint representation of ρ_f on the Lie algebra $\mathfrak{sp}_4$ is irreducible. Then (without assuming ordinarity), the image of Galois is full, i.e. contains a conjugate in $G(\overline{\mathbb{Q}}_p)$ of a congruence subgroup of $G(\mathbb{Z}_p)$.*

We first show that the image of Galois is open in its Zariski closure. Say that $\mathrm{Im}\,\rho_f \subset \mathrm{GSp}_4(E)$ for a finite extension $E/\mathbb{Q}_p$, and let $G = \mathrm{Res}_{E/\mathbb{Q}_p}\mathrm{GSp}_4$. Let $\mathfrak{g}$, resp. $\mathfrak{G}$, be the Lie algebra over $\mathbb{Q}_p$ of image of Galois (with its natural structure of p-adic Lie group), resp. of its $\mathbb{Q}_p$-Zariski closure in G. For any embedding $\sigma \colon E \to \overline{\mathbb{Q}}_p$, let $\mathfrak{G}_\sigma$ resp. $\mathfrak{g}_\sigma$ be the σ-projection of $\mathfrak{G} \otimes \overline{\mathbb{Q}}_p$ resp. $\mathfrak{g} \otimes \overline{\mathbb{Q}}_p$; By a theorem of Chevalley [Che51, p.177, Th.15], we see that, for any σ, the derived Lie algebras $\mathfrak{g}_\sigma'$ and $\mathfrak{G}_\sigma'$ coincide. By absolute irreducibility of ρ_f, the center of $\mathfrak{g}_\sigma$ has rank at most one. Moreover, since the similitude factor $\nu \circ \rho$ is a non-trivial power of the p-adic cyclotomic character, the rank of the center is exactly one for any σ, so that $\mathfrak{g} \otimes \overline{\mathbb{Q}}_p = \mathfrak{G} \otimes \overline{\mathbb{Q}}_p$.

The possibilities for $\mathfrak{G}' \otimes \overline{\mathbb{Q}}_p$ are listed below

- (F) $\mathfrak{G}' \otimes \overline{\mathbb{Q}}_p \sim \mathfrak{sp}_4$.
- (Y) $\mathfrak{G}' \otimes \overline{\mathbb{Q}}_p \sim \mathfrak{sl}(2) \times \mathfrak{sl}(2)$
- (K) $\mathfrak{G}' \otimes \overline{\mathbb{Q}}_p \sim \mathfrak{sl}(2)$ with $\mathfrak{sl}(2)$ embedded into a Klingen parabolic subalgebra.
- (C) $\mathfrak{G}' \otimes \overline{\mathbb{Q}}_p \sim \mathfrak{sl}(2)$ via the symmetric cube representation of $\mathrm{SL}(2)$,
- (S) $\mathfrak{G}' \otimes \overline{\mathbb{Q}}_p \sim \mathfrak{sl}(2)$ with $\mathfrak{sl}(2)$ embedded into a Siegel parabolic subalgebra.
- (T) $\mathfrak{G}' \otimes \overline{\mathbb{Q}}_p \sim \{1\}$

Let us show that in all cases except (F), the adjoint representation of ρ_f on $\mathfrak{gsp}_4$ (denoted by adding the superscript "ad") would be reducible. Indeed, in all cases except (F), we see that $\mathfrak{G}^{ad}$ is reducible (note that $\left(\mathrm{Symm}^3\mathfrak{sl}(2)\right)^{ad}$ is also reducible). This shows that the $\mathbb{Q}_p$-Lie algebra $\mathfrak{g}$ of $\mathrm{Im}\,\rho_f$ is a $\mathbb{Q}_p$-form of $\mathfrak{gsp}_4(\mathbb{Q}_p)$ contained in $\mathfrak{gsp}_4(\overline{\mathbb{Q}}_p)$. Since $\mathrm{Aut}(\mathfrak{gsp}_4) = \mathrm{Int}(\mathfrak{gsp}_4)$, it implies that there exists $\alpha \in G(\overline{\mathbb{Q}}_p)$ such that

$$\mathfrak{g} = \alpha \cdot \mathfrak{gsp}_4(\mathbb{Q}_p) \cdot \alpha^{-1}.$$

In order to get a more complete result without assuming adjoint irreducibility, one needs to analyze the connected components of the Galois image.

3.2. **Connected components of the image of Galois.** Let $\rho = \{\rho_l\}_l$ be an n-dimensional compatible system of l-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/k)$ satisfying the assertions of [P98a, Theorems 3.2–3]. Write $V_l \cong \mathbb{Q}_l^n$ for the space of ρ_l. Replacing ρ_l by its semi-simplification, we may assume that ρ_l is semi-simple for all l. Define an algebraic group $G_{p/\mathbb{Q}_p}$ over $\mathbb{Q}_p$ by the $\mathbb{Q}_p$-Zariski closure in $\mathrm{GL}(V_{p/\mathbb{Q}_p})$ of $\mathrm{Im}(\rho_p)$ for a prime p, and let G_p° be its connected component. Since ρ_p is semi-simple, G_p is reductive. Then by a result of Serre, G_p/G_p° and $\mathrm{rank}\,G_p^\circ$ are independent of p (see [P98a, Theorem 3.6]). Let k^{conn} for the fixed field of $\rho_p^{-1}(G_p^\circ(\mathbb{Q}_p))$. We fix a prime p and write $\mathfrak{l}$ for a prime ideal of k^{conn} prime to p. Let $\rho_p(\mathrm{Frob}_\mathfrak{l})^{ss}$ be the semi-simplification of $\rho_p(\mathrm{Frob}_\mathfrak{l})$ inside $G_p(\mathbb{Q}_p)$. Write $T_\mathfrak{l}$ for the smallest algebraic subgroup over $\mathbb{Q}_p$ containing $\rho(\mathrm{Frob}_\mathfrak{l})^{ss}$ inside $G_p^\circ(\mathbb{Q}_p)$. Then the connected component $T_\mathfrak{l}^\circ$ is a torus, and it is isomorphic to a base-change to $\mathbb{Q}_p$ of a $\mathbb{Q}$-torus in $\mathrm{GL}(n)$. If $T_\mathfrak{l}$ is a maximal torus of G_p°, then $\rho_p(\mathrm{Frob}_\mathfrak{l})$ is already semi-simple.

Lemma 3.3 (Serre). *Let p be a prime. The algebraic group G_p° has a dense Zariski open subset U stable under conjugation such that T_l is connected and maximal if $\rho_p(\mathrm{Frob}_l)^{ss}$ is in $U(\mathbb{Q}_p)$. In particular, for any prime $\mathfrak{l}$ in a density one subset of primes of k^{conn}, $T_\mathfrak{l}$ is maximal.*

This follows from the argument in [Ch92, Theorem 3.7]. Though Chi assumes that ρ comes from an abelian variety, his argument works in general.

3.3. **Connected semi-simple subgroups of** $\mathrm{GSp}(4)$. Let $\rho = \{\rho_\mathfrak{l}\}_\mathfrak{l}$ be a strict compatible system of symplectic semi-simple representations of $\mathrm{Gal}(\overline{k}/k)$ into $\mathrm{GSp}_4(T_\mathfrak{l})$ for a number field T (associated to a motive). Assume that T is the smallest coefficient field (i.e., it is generated by trace of Frobenii). Here $\mathfrak{l}$ runs over primes of T. Let l be the residual characteristic of $\mathfrak{l}$. Let $\mathrm{Res}\,\rho = \{\rho_l\}_l$ for $\rho_l = \prod_{\mathfrak{l}|l}\rho_\mathfrak{l}$ regarded to have values in $\mathrm{GSp}_4(T \otimes_\mathbb{Q} \mathbb{Q}_l)$. Then $\mathrm{Res}\,\rho$ is a compatible system with coefficients in $\mathbb{Q}$ acting on the $\mathbb{Q}_l$-vector spaces $V_l = (T \otimes_\mathbb{Q} \mathbb{Q}_l)^4$. Consider the $\mathbb{Q}_l$-Zariski closure G_l of $\mathrm{Im}(\rho_l)$ in $\mathrm{GL}(V_l)$.

As mentioned earlier, G_l is a reductive subgroup of $\mathrm{GL}(V_l)$ defined over $\mathbb{Q}_l$, $\mathrm{rank}\,G_l^\circ$ is independent of l and G_l/G_l° is independent of l and is isomorphic to $\mathrm{Gal}(k^{conn}/k)$ for the fixed field k^{conn} of $\rho_l^{-1}(G_l^\circ)$

Since G_l commutes with the action of T on V_l, for each prime factor $\mathfrak{l}$, we have its projection $G_\mathfrak{l}$ in $\mathrm{GSp}_4(T_\mathfrak{l})$. Let G_l° be the connected component of G_l (and $G_\mathfrak{l}^\circ$ for the $\mathfrak{l}$-component of G_l°; so, $G_\mathfrak{l}^\circ$ is the connected component of $G_\mathfrak{l}$).

Let $G_\mathfrak{l}'$ be the derived group of $G_\mathfrak{l}^\circ$. We may regard $G_\mathfrak{l}' \subset \mathrm{Res}_{T_\mathfrak{l}/\mathbb{Q}_l}\mathrm{Sp}(4)$. Thus the projections of $G_\mathfrak{l}'(\overline{\mathbb{Q}}_l)$ to each simple component $\mathrm{Sp}_4(\overline{\mathbb{Q}}_l)$ of $(\mathrm{Res}_{T_\mathfrak{l}/\mathbb{Q}_l}\mathrm{Sp}(4))(\overline{\mathbb{Q}}_l)$ are mutually isomorphic. We write $G_\mathfrak{l}' \sim G'$ for a semi-simple connected subgroup G' of $\mathrm{Sp}(4)_{/\overline{\mathbb{Q}}_l}$ if its projections to any simple factor $\mathrm{Sp}_4(\overline{\mathbb{Q}}_l)$ of $\mathrm{Res}_{T_\mathfrak{l}/\mathbb{Q}_l}\mathrm{Sp}(4)_{/\overline{\mathbb{Q}}_l}$ are isomorphic to G' over $\overline{\mathbb{Q}}_l$. Assume that ρ has weight w, i.e., $\det\rho$ is, up to finite character, given by $\{\epsilon_l^w\}_\mathfrak{l}$ for the l-adic cyclotomic character ϵ_l.

Let $\mathcal{G}$ be a connected semi-simple subgroup of $\mathrm{Sp}(4)_{/k}$ for a field k, and write $\mathfrak{g}$ for its Lie algebra. Write V_n for the symplectic space of dimension $2n$ on which $\mathrm{Sp}(2n)$ acts. Then by [LS98, Theorem 1], the largest proper semi-simple connected subgroup $\mathcal{G}$ of $\mathrm{Sp}(4)$ is isomorphic to $\mathrm{SL}(2) \times \mathrm{SL}(2)$ over an algebraic closure $\overline{k}$. Making an identification $V_2 \cong V_1 \oplus V_1$ as symplectic spaces, this group $\mathcal{G}$ acts diagonally on $V_1 \oplus V_1$. As for smaller semi-simple connected subgroups, we have two possibilities $\mathcal{G} = 1$ and $\mathcal{G} \cong \mathrm{SL}(2)$. The second possibility includes the case where the isomorphism $\mathrm{SL}(2) \cong \mathcal{G}$ is given by the symmetric cube representation of $\mathrm{SL}(2)$. Thus irreducibility of $\mathcal{G}$-module $\mathfrak{g}$ (under the

adjoint action) is satisfied if the derived group $\mathcal{G}$ is either $\mathrm{SL}(2)$ (the image of symmetric cube) or $\mathrm{Sp}(4)$.

For a prime $\mathfrak{P}$ over p of k, we suppose to have

(Reg) $$\rho|_{I_\mathfrak{P}} \sim \begin{pmatrix} \epsilon^{w_1} & * & * & * \\ 0 & \epsilon^{w_2} & * & * \\ 0 & 0 & \epsilon^{w_3} & * \\ 0 & 0 & 0 & \epsilon^{w_4} \end{pmatrix} \quad (w_1 > w_2 > w_3 > w_4)$$

up to finite error for the p-adic cyclotomic character $\epsilon = \epsilon_p$ of the inertia group $I_\mathfrak{P}$ of $\mathfrak{P}$. Thus we have six possibilities of the derived group $G'_\mathfrak{p}$ (the group $G^\circ_\mathfrak{p}$ is split over $T_\mathfrak{p}$ by (Reg)):

(F) $G'_\mathfrak{p} \sim \mathrm{Sp}(4)$.

(Y) $G'_\mathfrak{p} \sim \mathrm{SL}(2) \times \mathrm{SL}(2)$ with $\rho^{conn}_\mathfrak{p} = \rho_{1,\mathfrak{p}} \oplus \rho_{2,\mathfrak{p}}$, where $\rho_{1,\mathfrak{p}}$ is not twist equivalent to $\rho_{2,\mathfrak{p}}$.

(K) $G'_\mathfrak{p} \sim \mathrm{SL}(2)$ with $\rho^{conn}_\mathfrak{p} = \chi \oplus \chi' \oplus \rho_{1,\mathfrak{p}}$ with irreducible two dimensional $\rho_{1,\mathfrak{p}}$.

(C) $G'_\mathfrak{p} \sim \mathrm{SL}(2)$ via the symmetric cube representation of $\mathrm{SL}(2)$.

(S) $G'_\mathfrak{p} \sim \mathrm{SL}(2)$ with $\rho^{conn}_\mathfrak{p} = \rho_{1,\mathfrak{p}} \oplus (\rho_{1,\mathfrak{p}} \otimes \chi)$ with absolutely irreducible two dimensional $\rho_{1,\mathfrak{p}}$.

(T) $G'_\mathfrak{p} \sim \{1\}$ with $\rho_\mathfrak{p} = \mathrm{Ind}^\mathbb{Q}_F \chi$ for a degree 4 CM field $F/\mathbb{Q}$.

Here we write $\rho^{conn}_\mathfrak{p}$ for the restriction of $\rho_\mathfrak{p}$ to $\mathrm{Gal}(\overline{\mathbb{Q}}/k^{conn})$.

3.4. $\mathbb{Z}_p$-Fullness for $\mathrm{GSp}(4)$ and its subgroups.
Note that $\mathrm{Sp}_4 \cong \mathrm{Spin}_5$ over $\mathbb{Z}_p$ (for $p > 2$). Suppose that ρ is the (semi-simple) system associated to a Siegel cusp form (so $k = \mathbb{Q}$) with regular weight and that $\rho_\mathfrak{p}$ (for $\mathfrak{p}|p$) is ordinary as in Proposition 3.1. Recall that $G_\mathfrak{p}$ is the Zariski closure of $\mathrm{Im}(\rho_\mathfrak{p})$, that $G^\circ_\mathfrak{p}$ is the identity connected component of $G_\mathfrak{p}$ and that $G'_\mathfrak{p}$ is the derived group of $G^\circ_\mathfrak{p}$ for a prime $\mathfrak{p}|p$ of T. For a semi-simple $\mathbb{Q}$-split subgroup G of $\mathrm{Sp}(4)$ and a semi-simple $T_\mathfrak{p}$ subgroup H, we write $H \sim G$ if $H \cong G \times_\mathbb{Q} T_\mathfrak{p}$. Then as seen in Section 3.3, we have the following six possibilities of $G'_\mathfrak{p}$:

(F) $G'_\mathfrak{p} \sim G$ for $G = \mathrm{Sp}(4)$.

(Y) $G'_\mathfrak{p} \sim G$ for $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$ with $\rho^{conn}_\mathfrak{p} = \rho_{1,\mathfrak{p}} \oplus \rho_{2,\mathfrak{p}}$, where $\rho_{1,\mathfrak{p}}$ is not twist equivalent to $\rho_{2,\mathfrak{p}}$.

(K) $G'_\mathfrak{p} \sim G$ for $G = \mathrm{SL}(2)$ with $\rho^{conn}_\mathfrak{p} = \chi \oplus \chi' \oplus \rho_{1,\mathfrak{p}}$ with irreducible two dimensional $\rho_{1,\mathfrak{p}}$.

(C) $G'_\mathfrak{p} \sim G$ for $G = \mathrm{SL}(2)$ via the symmetric cube representation of $\mathrm{SL}(2)$.

(S) $G'_\mathfrak{p} \sim G$ for $G = \mathrm{SL}(2)$ with $\rho^{conn}_\mathfrak{p} = \rho_{1,\mathfrak{p}} \oplus (\rho_{1,\mathfrak{p}} \otimes \chi)$ with absolutely irreducible two dimensional $\rho_{1,\mathfrak{p}}$.

(T) $G'_\mathfrak{p} \sim \{1\}$ with $\rho_\mathfrak{p} = \mathrm{Ind}^\mathbb{Q}_F \chi$ for a degree 4 CM field $F/\mathbb{Q}$.

Here $\mathrm{Im}(\rho_{j,\mathfrak{p}})$ $(j = 1, 2)$ contains an open subgroup of $\mathrm{SL}_2(\mathbb{Z}_p)$ by the definition of $G'_\mathfrak{p}$, and χ, χ' are characters. In the automorphic side, Case (Y) corresponds to the Yoshida lift from an automorphic form $\mathrm{GL}(2) \times \mathrm{GL}(2)_{/\mathbb{Q}}$ or $\mathrm{Res}_{F/\mathbb{Q}}\mathrm{GL}(2)$ for a real quadratic field F, under the hypothesis that the starting automorphic form is not endo-scopic (i.e., non-CM). Cases (K) and (S) are either CAP or Eisenstein associated to the Siegel parabolic (S) and Klingen parabolic (K), assuming the corresponding $\mathrm{GL}(2)$ automorphic form is not endoscopic. Case (C) is the symmetric cube lift from $\mathrm{GL}(2)_{/\mathbb{Q}}$.

If G is absolutely simple, let $\mathfrak{g}$ be the Lie algebra of G and put $\mathfrak{g}_\mathfrak{p} = \mathfrak{g} \otimes_\mathbb{Q} T_\mathfrak{p}$ and regard it as the Lie algebra of $G'_\mathfrak{p}$. Since $\mathrm{Im}(\rho_\mathfrak{p})$ is Zariski dense in $G'_\mathfrak{p}$, the adjoint action of $\rho_\mathfrak{p}$ is absolutely irreducible. Write $Ad(\rho_\mathfrak{p})$ for the adjoint representation acting on $\mathfrak{g}_\mathfrak{p}$, and let $E_\mathfrak{p}$ be the subfield of $T_\mathfrak{p}$ generated over $\mathbb{Q}_p$ by $\mathrm{Tr}(Ad(\rho_\mathfrak{p}))$ on the Galois group. Let W be the $\mathfrak{p}$-adic integer ring of $E_\mathfrak{p}$. If $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$ (i.e., we are in Case (Y)), we consider $Ad(\rho_{j,\mathfrak{p}})$ acting on $\mathfrak{sl}_2(T_\mathfrak{p})$. We then define E_j by the subfield of $T_\mathfrak{p}$ generated over $\mathbb{Q}_p$ by the values of $\mathrm{Tr}(Ad(\rho_{j,\mathfrak{p}}))$ over $\mathrm{Gal}(\overline{\mathbb{Q}}/k^{conn})$. Then we write W_j for the $\mathfrak{p}$-adic integer ring of E_j.

Proposition 3.4. *Let the notation be as above. Suppose $p > 3$. If G is absolutely simple, $\mathrm{Im}(\rho_\mathfrak{p})$ contains an open subgroup of $G(W)$. If $G = \mathrm{SL}(2) \times \mathrm{SL}(2)$ (i.e., in Case (Y)), then $\mathrm{Im}(\rho_\mathfrak{p})$ contains an open subgroup of $\mathrm{SL}_2(W_1) \times \mathrm{SL}_2(W_2)$.*

Proof. Suppose first that G is absolutely simple (i.e., we are not in Case (Y)); so, G is either $\mathrm{SL}(2)$ or $\mathrm{Sp}(4)$. Thus the adjoint representation of G on its Lie algebra $\mathfrak{g}$ is absolutely irreducible. We

regard $Ad(\rho_{\mathfrak{p}})$ restricted to $\mathrm{Gal}(\overline{\mathbb{Q}}/k^{conn})$ as having values in G^{ad} (the adjoint group of G). Since $\mathrm{Im}(\rho_{\mathfrak{p}}^{conn})$ is Zariski dense in G, $Ad(\rho_{\mathfrak{p}}^{conn})$ is absolutely irreducible. Then by a theorem of Weisfeiler (see [P98b, Theorem 0.7]), there exists a linear algebraic group H defined over $E_{\mathfrak{p}}$ in $T_{\mathfrak{p}}$ such that $H \times_{E_{\mathfrak{p}}} T_{\mathfrak{p}} = G^{ad} \times_{\mathbb{Q}} T_{\mathfrak{p}}$ with $\mathrm{Im}(Ad(\rho_{\mathfrak{p}}^{conn})$ being an open subgroup of $H(W)$. Since the Siegel cusp form has regular weight, $\mathrm{Im}(\rho_{\mathfrak{p}}^{conn})$ contains p-inertia regular element g such that any of its power g^n ($0 < n \in \mathbb{Z}$) has a split torus of maximal rank in G as its centralizer. Thus H is split, and the result follows as $G \to G^{ad}$ is a central isogeny.

Now we assume that we are in Case (Y). Applying the above argument to $\rho_{j,\mathfrak{p}}$, $\mathrm{Im}(\rho_{j,\mathfrak{p}})$ contains an open subgroup of $\mathrm{SL}_2(W_j)$. Thus we need to show that $Ad(\rho_1) \times Ad(\rho_2)$ contains an open subgroup of $\mathrm{PGL}_2(W_1) \times \mathrm{PGL}_2(W_2)$. Let PG be $\mathrm{PGL}(2)$ defined over F; so, $PG(F) = \mathrm{PGL}_2(T_{\mathfrak{p}}) \times \mathrm{PGL}_2(T_{\mathfrak{p}})$. By [P98b, Main Theorem 0.2 (a)], there exist a semi-simple $\mathbb{Q}_p$-subalgebra E of $F := T_{\mathfrak{p}} \oplus T_{\mathfrak{p}}$ and a connected adjoint group $H_{/E}$ and an isogeny $\varphi : H \times_E F \to PG$ such that $\mathrm{Im}(\rho_{\mathfrak{p}}^{conn}))$ is an open subgroup of $\varphi(H(E))$. If E is a field, we have $E = E_1 = E_2$, and $\rho_{1,\mathfrak{P}} \cong \rho_{2,\mathfrak{p}}$, and hence we are in Case (S), a contradiction. Thus $E = E_1 \oplus E_2$, and $H(E) = \mathrm{PGL}_2(E_1) \times \mathrm{PGL}_2(E_2)$ as desired. $\square$

3.5. **Pink's Lie algebra theory.** Let G be a split (smooth) connected reductive $\mathbb{Z}_p$-group with maximal split torus T. Let G' be the derived group of G, and put $\Gamma_A^G(\mathfrak{c})$ to be the kernel of the reduction map $G'(A) \twoheadrightarrow G'(A/\mathfrak{c})$ for a non-zero ideal $\mathfrak{c}$ of A for a $\mathbb{Z}_p$-algebra A. For a $\mathbb{Z}_p$-algebra $B \subset A$, a subgroup H of $G(A)$ is B-full if $H \supset \Gamma_B^G(\mathfrak{c})$ for a non-zero B-ideal $\mathfrak{c}$. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/K) \to G(A)$ be a continuous representation for a finite extension $K/\mathbb{Q}$. Suppose A is an integral domain with quotient field $Q(A)$. We call ρ B-*full with respect to G* if after replacing ρ by a conjugate of ρ under an element of $G(Q(A))$, $\mathrm{Im}(\rho)$ is B-full. Suppose that B is a local ring with maximal ideal $\mathfrak{m}_B$. We call ρ B-*regular* if $\mathrm{Im}(\rho)$ contains a regular element b in the torus $T(B)$ such that $\alpha(b) \not\equiv \beta(b) \mod \mathfrak{m}_B$ for all distinct roots α and β.

Via Pink's theory of tight correspondence between p-profinite Lie subalgebras of $\mathfrak{sl}(2)$ and p-profinite subgroups, we want to show that if a p-profinite subgroup of Spin_{2n+1} contains sufficiently large unipotent subgroups, it is full. Let us recall Pink's theory briefly. To study a general p-profinite subgroup $\mathcal{G}$ of $\mathrm{SL}_2(A)$ for a general p-profinite ring A, we want to have an explicit relation between p-profinite subgroups $\mathcal{G}$ of the form $\mathrm{SL}_2(A) \cap (1 + X)$ for a Lie $\mathbb{Z}_p$-subalgebra $X \subset \mathfrak{gl}_2(A)$. Assuming $p > 2$, Pink [P93] found a functorial explicit relation between closed subgroups in $\mathrm{SL}_2(A)$ and Lie subalgebras of $\mathfrak{gl}_2(A)$ (valid even for A of characteristic p). We call subgroups of the form $\mathrm{SL}_2(A) \cap (1 + X)$ (for a p-profinite Lie $\mathbb{Z}_p$-subalgebra X of $\mathfrak{gl}_2(A)$) *basic subgroups* following Pink.

We prepare some notation to quote here the results in [P93]. Let A be a semi-local p-profinite ring (not necessarily of characteristic p). Since Pink's result allows semi-local p-profinite algebra, we do not assume A to be local in the exposition of his result. We assume $p > 2$. Define maps $\Theta : \mathrm{SL}_2(A) \to \mathfrak{sl}_2(A)$ and $\zeta : \mathrm{SL}_2(A) \to Z(A)$ for the center $Z(A)$ of the algebra $M_2(A)$ by

$$\Theta(x) = x - \frac{1}{2}\mathrm{Tr}(x)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \quad \text{and} \quad \zeta(x) = \frac{1}{2}(\mathrm{Tr}(x) - 2)\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

For each p-profinite subgroup $\mathcal{G}$ of $\mathrm{SL}_2(A)$, define L by the closed additive subgroup of $\mathfrak{sl}_2(A)$ topologically generated by $\Theta(x)$ for all $x \in \mathcal{G}$. Then we put $C = \mathrm{Tr}(L \cdot L)$. Here $L \cdot L$ is the closed additive subgroup of $M_2(A)$ generated by $\{xy | x, y \in L\}$ for the matrix product xy, similarly L^j is the closed additive subgroup generated by iterated products (j times) of elements in L. We then define $L_1 = L$ and inductively $L_{j+1} = [L, L_j]$; so, $L_2 = [L, L]$, where $[L, L_j]$ is the closed additive subgroup generated by Lie bracket $[x, y] = xy - yx$ for $x \in L$ and $y \in L_n$. Then by [P93, Proposition 3.1], we have

(3.1) $\quad [L, L] \subset L, \ C \cdot L \subset L, \ L = L_1 \supset \cdots \supset L_j \supset L_{j+1} \supset \cdots \ \text{and} \ \bigcap_{j \geq 1} L_n = \bigcap_{j \geq 1} L^j = 0.$

In particular, L is a Lie $\mathbb{Z}_p$-subalgebra of $\mathfrak{sl}_2(A)$. Put $\mathcal{M}_j(\mathcal{G}) = C\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \oplus L_j \subset M_2(A) = \mathfrak{gl}_2(A)$, which is a closed Lie $\mathbb{Z}_p$-subalgebra by (3.1). Define

$$\mathcal{H}_j = \{x \in \mathrm{SL}_2(A) | \Theta(x) \in L_j, \ \mathrm{Tr}(x) - 2 \in C\} \ \text{for } j \geq 1.$$

If $x \in \mathcal{H}_n$, then $x = \Theta(x) + \zeta(x) + \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$; thus, $\mathcal{H}_j \subset \mathrm{SL}_2(A) \cap (1 + \mathcal{M}_j(\mathcal{G}))$. If we pick $x \in \mathrm{SL}_2(A) \cap (1 + \mathcal{M}_j(\mathcal{G}))$, then $x = 1 + c \cdot 1 + y$ with $y \in L_n$ and $c \in C$. Thus $\mathrm{Tr}(x) - 2 = 2c \in C$ and

$\Theta(x) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + c \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) + y - \frac{1}{2}(2 + 2c) \cdot \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = y$. This shows

$$\mathcal{H}_j = \mathrm{SL}_2(A) \cap (1 + \mathcal{M}_j(\mathcal{G})).$$

Here is a result of Pink (Theorem 3.3 combined with Theorem 2.7 both in [P93]):

Theorem 3.5 (Pink). *Let the notation be as above. Suppose $p > 2$, and let A be a semi-local p-profinite commutative ring with identity. Take a p-profinite subgroup $\mathcal{G} \subset \mathrm{SL}_2(A)$. Then we have*

(1) *$\mathcal{G}$ is a normal closed subgroup of $\mathcal{H}_1$ (defined as above for $\mathcal{G}$),*

(2) *$\mathcal{H}_n$ is a p-profinite subgroup of $\mathrm{SL}_2(A)$ inductively given by $\mathcal{H}_{j+1} = (\mathcal{H}_1, \mathcal{H}_j)$ which is the closed subgroup topologically generated by commutators (x, y) with $x \in \mathcal{H}_1$ and $y \in \mathcal{H}_j$,*

(3) *$\{\mathcal{H}_j\}_{j \geq 2}$ coincides with the descending central series of $\{\mathcal{G}_j\}_{j \geq 2}$ of $\mathcal{G}$, where $\mathcal{G}_{j+1} = (\mathcal{G}, \mathcal{G}_j)$ starting with $\mathcal{G}_1 = \mathcal{G}$.*

In particular, we have

(P) *The topological commutator subgroup $\mathcal{G}'$ of $\mathcal{G}$ is the subgroup given by $\mathrm{SL}_2(A) \cap (1 + \mathcal{M}_2(\mathcal{G}))$ for the closed Lie subalgebra $\mathcal{M}_2(\mathcal{G}) \subset M_2(A)$ as above.*

Recall that G is a split (smooth) connected reductive $\mathbb{Z}_p$-group with maximal split torus T. Let G' be the derived group of G. Now let $\mathcal{G}$ be a p-profinite subgroup of $G'(A)$. Then on $\mathcal{G}_1 = \mathcal{G} \cap \Gamma_A^G(p)$ the logarithm $\sum_{m=1}^{\infty} (-1)^{m+1} \frac{(x-1)^m}{m}$ converges to an element $\mathrm{Log}(x) \in Lie(G)(A)$. Note that in general $\mathrm{Log}(\mathcal{G}) = \{\mathrm{Log}(x) | x \in \mathcal{G}\}$ is neither an abelian group nor a Lie algebra. Fix a Borel subgroup $B_{/\mathbb{Z}_p}$ of G containing T with $B = TU^+$ for the unipotent radical U^+. Write U^- for the unipotent subgroup opposite to U^+; so, TU^- is the opposite Borel subgroup of B.

Lemma 3.6. *Suppose that A is a p-profinite complete local integral domain over $\mathbb{Z}_p$ for $p > 2$. If $\mathcal{G}_1$ contains $\Gamma_A^{U^+}(\mathfrak{a})$ and $\Gamma_A^{U^-}(\mathfrak{b})$ for non-zero ideals $\mathfrak{a}$ and $\mathfrak{b}$ of A. Then $\mathcal{G}_1$ (and hence $\mathcal{G}$) contains $\Gamma_A^G(\mathfrak{a}^2\mathfrak{b}^2)$.*

Proof. We write $\mathfrak{u}^+$ (resp. $\mathfrak{u}^-$, $\mathfrak{t}$) for the Lie subalgebra of $Lie(G)$ corresponding to U^+ (resp. U^-, T). We start with the case where $G = \mathrm{SL}(2)$. Note that $\mathrm{GSpin}_3 \cong \mathrm{SL}(2)$.

In our setting, we assume that A is local. Then by the definition as above, under the notation in Theorem 3.5, we have $\mathrm{Log}(\mathcal{G}_1 \cap U^+(A)) = \mathcal{M}_1(\mathcal{G}_1) \cap U^+(A)$ and $\mathrm{Log}(\mathcal{G}_1 \cap U^-(A)) = \mathcal{M}_1(\mathcal{G}_1) \cap U^-(A)$. Thus $\mathcal{M}_1(\mathcal{G}_1)$ contains $\mathfrak{u}^+(\mathfrak{a}) = \left\{ \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \big| a \in \mathfrak{a} \right\}$ and $\mathfrak{u}^-(\mathfrak{a}') = \left\{ \left(\begin{smallmatrix} 0 & 0 \\ a & 0 \end{smallmatrix}\right) \big| a \in \mathfrak{a}' \right\}$. Thus $[\mathfrak{u}^+(\mathfrak{a}), \mathfrak{u}^-(\mathfrak{a}')] \supset \mathfrak{t}(\mathfrak{a}\mathfrak{a}') := \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \big| a \in \mathfrak{a}\mathfrak{a}' \right\}$ is contained in $\mathcal{M}_1(\mathcal{G}')$. In other words, $\mathcal{M}_1(\mathcal{G}_1) \supset \mathcal{M}_1(\Gamma_A^G(\mathfrak{a}\mathfrak{a}'))$. This implies $\mathcal{G}_1'$ contains the derived group of $\Gamma_A^G(\mathfrak{a}\mathfrak{a}')$ which contains $\Gamma_A^G(\mathfrak{a}^2\mathfrak{a}'^2)$. This finishes the proof for $n = 1$.

For general G, we pick a positive root α of T with root subgroup $U_\alpha \subset U^+$. Then we have an embedding $i_\alpha : \mathrm{SL}(2) \hookrightarrow G$ sending the upper (resp. lower) triangular unipotent subgroup of $\mathrm{SL}(2)$ to U_α (resp. $U_{-\alpha} \subset U^-$). Applying the argument in the case of $\mathrm{SL}(2)$ to $\mathcal{G}_\alpha := i_\alpha^{-1}(\mathcal{G})$, we have $\mathcal{G}_\alpha \supset \Gamma_A^{\mathrm{SL}(2)}(\mathfrak{a}^2\mathfrak{b}^2)$. It is easy to see that $\{i_\alpha(\Gamma_A^{\mathrm{SL}(2)}(\mathfrak{a}^2\mathfrak{b}^2))\}_\alpha$ generates $\Gamma_A^G(\mathfrak{a}^2\mathfrak{b}^2)$, and hence the desired assertion holds. $\qquad\square$

3.6. **Λ-fullness for the endoscopic cases.** We return to the setting of Section 3.4. We now study Λ-fullness in the one variable case of $\Lambda = \Lambda_1 = \mathbb{Z}_p[[T]]$ in the endoscopic cases (Y), (K), (C) and (S) below. In each endoscopic case, we have a factor $\mathrm{GL}(2)$ of G. Even if the component of the Hecke algebra has two variables, the Galois image of the projected factor falls in $\mathrm{GL}_2(\mathbb{I})$ for $\mathbb{I}$ finite over Λ (for a suitable quotient Λ of Λ_2). Thus we do not lose any generality assuming that $\mathbb{I}$ is finite over one variable Λ. To describe the quotient map $\Lambda_2 \to \Lambda$ explicitly, we normalize first $\Lambda_2 = \mathbb{Z}_p[[T_1, T_2]]$ so that the Galois image to contain

$$\mathcal{T} = \{\mathrm{diag}[(t_1 t_2)^s, t_1^s, t_2^s, 1] \in T(\Lambda_2) | s \in \mathbb{Z}_p, t_j = 1 + T_j\}$$

as the wild p-inertia image. Then we normalize the Galois representation attached an $\mathbb{I}$-adic ordinary Hecke eigenform form as follows depending on cases for $\rho = \rho_{\mathbb{I}}$.

- In Case (Y), we write $\rho \cong \phi \times \varphi$ with $\phi(\sigma) = \left(\begin{smallmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{smallmatrix}\right)$, $\varphi(\sigma) = \left(\begin{smallmatrix} \alpha(\sigma) & \beta(\sigma) \\ \gamma(\sigma) & \delta(\sigma) \end{smallmatrix}\right)$ and

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & 0 & 0 & b(\sigma) \\ 0 & \alpha(\sigma) & \beta(\sigma) & 0 \\ 0 & \gamma(\sigma) & \delta(\sigma) & 0 \\ c(\sigma) & 0 & 0 & d(\sigma) \end{pmatrix} \quad \text{and} \quad \Lambda = \Lambda_2/(t_1 t_2 - t_1) \text{ with } t_1 \mapsto t = 1 + T.$$

- In Case (K), $\rho \cong \chi \oplus \chi' \oplus \phi$ with $\phi(\sigma) = \begin{pmatrix} a(\sigma) & b(\sigma) \\ c(\sigma) & d(\sigma) \end{pmatrix}$ and

$$\rho(\sigma) = \begin{pmatrix} a(\sigma) & 0 & 0 & b(\sigma) \\ 0 & \chi(\sigma) & 0 & 0 \\ 0 & 0 & \chi'(\sigma) & 0 \\ c(\sigma) & 0 & 0 & d(\sigma) \end{pmatrix} \quad \text{and } \Lambda = \Lambda_2/(t_1 - t_2) \text{ with } t_1 t_2 \mapsto t = 1 + T.$$

- In Case (S), $\rho \cong (\phi^t \otimes \chi) \oplus \phi$; i.e.,

$$\rho(\sigma) = \begin{pmatrix} \chi(\sigma) s^t \phi(\sigma)^t s^{-1} & 0 \\ 0 & \phi(\sigma) \end{pmatrix} \quad \text{and } \Lambda = \Lambda_2/(t_1 - 1) \text{ with } t_2 \mapsto t = 1 + T.$$

- In Case (C), and $\Lambda = \Lambda_2/(t_1 - t_2^2)$ with $t_2 \mapsto t = 1 + T$.

Suppose

(y) In Case (Y), $n = 2$, $\det \varphi = \det \rho = \nu \rho$ and for $\sigma \in I_p$,

$$\phi(\sigma) = \begin{pmatrix} (t_1 t_2)^{\log_p(\epsilon(\sigma)/\log_p(1+p)} & * \\ 0 & 1 \end{pmatrix}, \quad \varphi(\sigma) = \begin{pmatrix} t_1^{\log_p(\epsilon(\sigma)/\log_p(1+p)} & * \\ 0 & t_2^{\log_p(\epsilon(\sigma)/\log_p(1+p)} \end{pmatrix},$$

and ϕ has values in a finite extension $\mathbb{I}_\phi \subset \mathbb{I}$ of $\Lambda_\phi = \mathbb{Z}_p[[T_\phi]]$ for $T_\phi = t_1 t_2 - 1$ and $\varphi \otimes \kappa^{-1}$ has values in a finite extension $\mathbb{I}_\varphi \subset \mathbb{I}$ of $\Lambda_\varphi = \mathbb{Z}_p[[T_\varphi]]$ for $T_\varphi = t_1 t_2^{-1} - 1$, where $\kappa : G_{\mathbb{Q}} \to \Lambda_\varphi^\times$ is a character such that $\kappa(\sigma) = t_2^{\log_p(\epsilon(\sigma)/\log_p(1+p)}$.

(k) In Case (K), $n = 2$ and for $\sigma \in I_p$, $\chi(\sigma) = t_1^{\log_p(\epsilon(\sigma)/\log_p(1+p)}$, $\chi'(\sigma) = t_2^{\log_p(\epsilon(\sigma)/\log_p(1+p)}$ and

$$\phi(\sigma) = \begin{pmatrix} (t_1 t_2)^{\log_p(\epsilon(\sigma)/\log_p(1+p)} & * \\ 0 & 1 \end{pmatrix},$$

and ϕ has values in a finite extension $\mathbb{I}_\phi \subset \mathbb{I}$ of $\Lambda_\phi = \mathbb{Z}_p[[T_\phi]]$ for $T_\phi = t_1 t_2 - 1$.

(s) In Case (S), $n = 2$ and $\chi(\sigma) = t_1^{\log_p(\epsilon(\sigma)/\log_p(1+p)}$ and $\phi(\sigma) = \begin{pmatrix} t_2^{\log_p(\epsilon(\sigma)/\log_p(1+p)} & * \\ 0 & 1 \end{pmatrix}$ for $\sigma \in I_p$, and ϕ has values in a finite extension $\mathbb{I}_\phi \subset \mathbb{I}$ of $\Lambda_\phi = \mathbb{Z}_p[[T_\phi]]$ for $T_\phi = t_2 - 1$.

(c) In Case (C), $n = 1$ and for $\sigma \in I_p$,

$$\rho(\sigma) = \begin{pmatrix} t_1^{3\log_p(\epsilon(\sigma)/\log_p(1+p)} & * & * & * \\ 0 & t_1^{2\log_p(\epsilon(\sigma)/\log_p(1+p)} & * & * \\ 0 & 0 & t_1^{\log_p(\epsilon(\sigma)/\log_p(1+p)} & * \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note here $\mathrm{GSpin}_3 \cong \mathrm{SL}(2)$ and $\mathrm{GSpin}_5 = \mathrm{GSp}(4)$ over $\mathbb{Z}_p$ (for p odd). Let $\mathbb{I}$ be a finite flat normal extension of Λ as before, and $\rho_{\mathbb{I}} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSpin}_{2n+1}(\mathbb{I})$ be a Galois representation. Suppose

(G0) $G \supset \mathcal{T}$; so, $\mathrm{rank}\, G = \mathrm{rank}\, \mathrm{GSpin}_{2n+1}$,

(G1) there exists a point $P \in \mathrm{Spec}(\Lambda)(\overline{\mathbb{Q}}_p)$ such that $\rho_P = (\rho_{\mathbb{I}} \mod P)$ is $\mathbb{Z}_p$-full with respect to G, regarding $\mathbb{I}/P\mathbb{I}$ as a $\mathbb{Z}_p$-algebra.

(G2) the semi-simplification of $\rho_{\mathbb{I}}$ restricted to the wild p-inertia group I_p^w is isomorphic to the diagonal representation $\mathrm{diag}[\boldsymbol{\kappa}^{2n-1}, \boldsymbol{\kappa}^{2n-2}, \cdots, 1]$ for the character $\boldsymbol{\kappa} : I_p^w \to \Lambda^\times$ such that $\boldsymbol{\kappa}([1+p, \mathbb{Q}_p]) = t = 1 + T$.

(G3) $(\nu \circ \rho_{\mathbb{I}})/\boldsymbol{\kappa}^{n(2n-1)}$ has finite order.

(G4) We have $\mathrm{diag}[\boldsymbol{\kappa}^{2n-1}, \boldsymbol{\kappa}^{2n-2}, \cdots, 1]$ in $\rho_{\mathbb{I}}(I_p^w)$.

Conjecture 3.7. *Assume $p > 2n - 1 > 2$ in addition to (G0–4). Then $\rho_{\mathbb{I}}$ is Λ-full with respect to G.*

Theorem 3.8. *Suppose that $n = 1, 2$ and that we are not in Case (F) nor in Case (T) (so we are in the endoscopic cases). Let $\rho_{\mathbb{I}}$ be associated to an $\mathbb{I}$-adic ordinary Hecke eigen cusp form. Assume $\mathbb{Z}_p$-regularity if $\mathbb{I} \neq \Lambda$ or in Case (Y). Then the above conjecture holds.*

We prove below the Λ-fullness including $p = 3$.

Proof. When $n = 1$, this follows from [H13b, Theorem I]. If $\mathbb{I} = \Lambda$, by [Z14], we have a non-trivial upper unipotent element in $\rho_{\mathbb{I}}(I_p)$ and also a lower one; so, (G4) is actually not necessary if $\mathbb{I} = \Lambda$. If $\mathbb{I} \neq \Lambda$, $\mathbb{Z}_p$-regularity is used in [H13b] to show non-trivial unipotents can be found in $\mathrm{SL}_2(\Lambda)$ (not just in $\mathrm{SL}_2(\mathbb{I})$ shown in [Z14]). Since we are in the cases different from (F) and (T), G' is either split $\mathrm{SL}(2)$ or $\mathrm{SL}(2) \times \mathrm{SL}(2)$. This implies that, by $\mathrm{GL}(2)$-theory, we can find an arithmetic prime $P \in \mathrm{Spec}(\Lambda)$ (sufficiently regular) such that $\rho := \rho_{\mathfrak{P}}$ for every prime $\mathfrak{P} \in \mathrm{Spec}(\mathbb{I})$ above P belongs

to the same case except for Case (F). Then by Proposition 3.4 applied to ρ, $\rho_{j,\mathfrak{p}}$ is $\mathbb{Z}_p$-full, and hence the argument in [H13b] works.

Since we have one simple component SL(2) except for Case (Y), if we are not in Cases (Y) and (F), there is no difference in the argument proving the theorem from the one given in [H13b]. Thus we give more details in Case (Y). In this case, $\rho_{\mathbb{I}}$ has values in $\mathrm{GL}_2(\mathbb{I}) \times \mathrm{GL}_2(\mathbb{I})$. We write $\rho_L, \rho_R : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{I})$ for the projection of $\rho_{\mathbb{I}}$ to the left and right factor $\mathrm{GL}_2(\mathbb{I})$. Each projection $\rho_{\mathbb{I},j}$ $(j = 1, 2)$ to each simple component GL(2) is Λ-full by the argument in [H13b]. Pink's theory can be also applied to semi-local p-profinite rings, like $A = \mathbb{I} \oplus \mathbb{I}$. Put $\mathcal{G} = \mathrm{Im}(\rho_{\mathbb{I}}) \cap \Gamma^{\mathrm{SL}(2)}_{\mathbb{I} \oplus \mathbb{I}}(\mathfrak{m}_{\mathbb{I}})$ for the maximal ideal $\mathfrak{m}_{\mathbb{I}}$. and $\mathcal{G}_P = \mathrm{Im}(\rho_P) \cap \Gamma^{\mathrm{SL}(2)}_{\mathbb{I}/P \oplus \mathbb{I}/P}(\mathfrak{m}_{\mathbb{I}/P})$. Since the reduction map modulo P is onto for Pink's Lie algebra, we have a surjective morphism $\mathcal{M}_j(\mathcal{G}) \twoheadrightarrow \mathcal{M}_j(\mathcal{G}_P)$. Adding the superscript "+" (resp. "−", "0") to Pink's Lie algebra, we indicates the upper nilpotent subalgebra (resp. the lower nilpotent subalgebra, the diagonal subalgebra). Then by $\mathbb{Z}_p$-regularity, $\mathcal{M}_j(\mathcal{G}) = \mathcal{M}_j^+(\mathcal{G}) \oplus \mathcal{M}_j^0(\mathcal{G}) \oplus \mathcal{M}_j^-(\mathcal{G})$ and the same for $\mathcal{M}_j(\mathcal{G}_P)$. Thus the reduction map modulo P induces a sujection $\mathcal{M}_j^*(\mathcal{G}) \to \mathcal{M}_j^*(\mathcal{G}_P)$ for $* = \pm, 0$.

Let 1_L (resp. 1_R) be the idempotent of the left (resp. the right) factor $\mathbb{I}$ of A. Since $\mathcal{G}_P$ is $\mathbb{Z}_p$-full, for $\mathcal{M}_j(\mathcal{G}_P)$, $1_L \mathcal{M}_j^*(\mathcal{G}_P) \neq 0$ and $1_R \mathcal{M}_j^*(\mathcal{G}_P) \neq 0$ for $* = \pm, 0$. Take $g \in \mathrm{Im}(\rho_{\mathbb{I}})$ giving the $\mathbb{Z}_p$-regularity, $Ad(g)$ acts on $1_L \mathcal{M}_j^{\pm}(\mathcal{G}_P) \cap \mathcal{M}_j^{\pm}(\mathcal{G}_P)$ and $1_R \mathcal{M}_j^{\pm}(\mathcal{G}_P) \cap \mathcal{M}_j^{\pm}(\mathcal{G}_P)$ by the different set of eigenvalues. Thus $1_? \mathcal{M}_j^{\pm}(\mathcal{G}_P) \cap \mathcal{M}_j^{\pm}(\mathcal{G}_P) \neq 0$ for $? = L, R$. We have eigenspace decompositions under the action of $Ad(g)$: $\mathcal{M}_j^{\pm}(\mathcal{G}_P) = (1_L \mathcal{M}_j^{\pm}(\mathcal{G}_P) \cap \mathcal{M}_j^{\pm}(\mathcal{G}_P)) \oplus (1_L \mathcal{M}_j^{\pm}(\mathcal{G}_P) \cap \mathcal{M}_j^{\pm}(\mathcal{G}_P))$ and $\mathcal{M}_j^{\pm}(\mathcal{G}) = (1_L \mathcal{M}_j^{\pm}(\mathcal{G}) \cap \mathcal{M}_j^{\pm}(\mathcal{G})) \oplus (1_L \mathcal{M}_j^{\pm}(\mathcal{G}) \cap \mathcal{M}_j^{\pm}(\mathcal{G}))$. This implies $1_L \mathcal{M}_j^{\pm}(\mathcal{G}) \cap \mathcal{M}_j^{\pm}(\mathcal{G}) \neq 0$ and $1_R \mathcal{M}_j^{\pm}(\mathcal{G}) \cap \mathcal{M}_j^{\pm}(\mathcal{G}) \neq 0$ as the modulo P reduction map of $1_? \mathcal{M}_j^{\pm}(\mathcal{G}) \cap \mathcal{M}_j^{\pm}(\mathcal{G})$ to $1_? \mathcal{M}_j^{\pm}(\mathcal{G}_P) \cap \mathcal{M}_j^{\pm}(\mathcal{G}_P)$ is surjective for $? = L, R$. This implies that $(1_? \mathcal{M}_j^{\pm}(\mathcal{G}) \cap \mathcal{M}_j^{\pm}(\mathcal{G})) = 1_? \mathcal{M}_j^{\pm}(\mathcal{G})$ for $? = L, R$ and for all $j > 0$. After replacing $\rho_{\mathbb{I}}$ by a conjugate under $G(Q(\mathbb{I}))$, by [H13b, Lemma 2.9], that $\mathrm{Im}(\rho_L) \supset \Gamma^{\mathrm{SL}(2)}_{\Lambda}(\mathfrak{c}_L)$ and $\mathrm{Im}(\rho_R) \supset \Gamma^{\mathrm{SL}(2)}_{\Lambda}(\mathfrak{c}_R)$ for non-zero Λ-ideals $\mathfrak{c}_L$ and $\mathfrak{c}_R$. In particular, by Lemma 3.6, we get $\mathrm{Im}(\rho_{\mathbb{I}}) \supset \Gamma^{\mathrm{SL}(2)}_A(\mathfrak{c})$ for $\mathfrak{c} = \mathfrak{c}_L^2 \cap \mathfrak{c}_R^2$ as desired. $\qquad\square$

3.7. $\mathbb{Z}_p[[S_1]] \oplus \mathbb{Z}_p[[S_2]]$-**fullness in Case (Y).** We can think of fullness property in two variable setting. Case (F) will be treated in the following section in the general setting of GSpin representations as $\mathrm{GSpin}_5 \cong \mathrm{GSp}(4)$. In the cases other than (F) and (Y), the abelian part consumes one variable out of the two, we do not have a properly two variable Galois representation $\rho_{\mathbb{I}}$. Thus we assume that there exists a finite extension $k/\mathbb{Q}$ such that $\rho := \rho_{\mathbb{I}}|_{\mathrm{Gal}(\overline{\mathbb{Q}}/k)}$ has values in

$$G_2 = \{(x, y) \in \mathrm{GL}(2) \times \mathrm{GL}(2) \mid \det(x) = \det(y)\}.$$

We suppose that G_2 is embedded into $\mathrm{GSp}(4)$ in the following way:

$$G_2 \ni \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) \mapsto \begin{pmatrix} a & 0 & 0 & b \\ 0 & \alpha & \beta & 0 \\ 0 & \gamma & \delta & 0 \\ c & 0 & 0 & d \end{pmatrix} \in \mathrm{GSp}(4).$$

As before, we suppose that $\rho(I_p^w)$ for the wild p-inertia group I_p^w contains the following split torus

$$\mathcal{T} = \{\mathrm{diag}[t_1^s t_2^s, t_1^s, t_2^s, 1] \in T \mid s \in \mathbb{Z}_p\}$$

for $t_j = 1 + T_j$ and $\Lambda_2 = \mathbb{Z}_p[[T_1, T_s]]$. We make the following variable change $s_1 = t_1 t_2$ and $s_2 = t_1/t_2$; so, $\mathbb{Z}_p[[S_1, S_2]] = \Lambda_2$ for $S_j = s_j - 1$. We put $A_j = \mathbb{Z}_p[[S_j]]$; so, $\Lambda_2 = \mathbb{Z}_p[[S_1, S_2]] = A_1 \widehat{\otimes}_{\mathbb{Z}_p} A_2$.

Proposition 3.9. *Let $\rho_{\mathbb{I}}$ is associated to $\mathbb{I}$-adic ordinary Hecke eigenform on $\mathrm{GSp}(4)$ for an algebra $\mathbb{I}$ finite torsion-free over Λ_2. Suppose $\mathbb{Z}_p$-regularity for $\rho_{\mathbb{I}}$ and that for an arithmetic point $P \in \mathrm{Spec}(\Lambda_2)(\overline{\mathbb{Q}}_p)$ and for all primes $\mathfrak{P} \in \mathrm{Spec}(\mathbb{I})(\overline{\mathbb{Q}}_p)$ above P, $\rho_{\mathfrak{P}} = \rho \mod \mathfrak{P}$ belongs to Case (Y) in Section 3.4. Then, if $p \geq 3$, replacing ρ by a conjugate of ρ by an element in $\mathrm{GL}_2(Q(A_1)) \times \mathrm{GL}_2(Q(A_2))$ if necessary, there exist non-zero A_j-ideals $\mathfrak{c}_j$ such that $\mathrm{Im}(\rho)$ contains*

$$\Gamma^{\mathrm{SL}(2)}_{A_1}(\mathfrak{c}_1) \times \Gamma^{\mathrm{SL}(2)}_{A_2}(\mathfrak{c}_2) \subset \mathrm{SL}_2(A_1) \times \mathrm{SL}_2(A_2) \subset G_2(\Lambda_2).$$

Proof. Let $\{i, j\} = \{1, 2\}$. We can write $P = P_i \otimes P_j$ for $P_? \in \mathrm{Spec}(A_?)(\overline{\mathbb{Q}})$ for $? = i, j$. Consider the projection $P_i \times \mathrm{id} : \Lambda_2 := A_i \widehat{\otimes}_{\mathbb{Z}_p} A_j \to W_i \otimes_{\mathbb{Z}_p} A_j = W_i[[S_j]]$ for $W_i = A_i/P_i$. Take a prime $\widetilde{\mathfrak{P}}_i \in \mathrm{Spec}(\mathbb{I})$ above $\mathrm{Ker}(P_i \times \mathrm{id})$. Let $\rho_j = \rho \mod \widetilde{\mathfrak{P}}_i$. Then by Theorem 3.8 (or more precisely,

by the same argument proving the theorem), replacing ρ_j by its conjugate under an element of $\mathrm{GL}_2(Q(A_j))$, $\mathrm{Im}(\rho_j)$ contains $\Gamma_{A_j}^{\mathrm{SL}(2)}(\mathfrak{a}_j)$ for a non-zero A_j-ideal $\mathfrak{a}_j$.

Put $\mathcal{G} = \mathrm{Im}(\rho) \cap \Gamma_{\Lambda_2}^{\mathrm{Sp}(4)}(\Lambda_2)$, and consider Pink's Lie algebra $\mathcal{M}_2(\mathcal{G})$ with respect to $A = A_1 \oplus A_2$ regarding $\mathrm{SL}_2(A_1) \times \mathrm{SL}_2(A_2) \subset G_2(\Lambda_2)$ as $\mathrm{SL}_2(A_1 \oplus A_2)$. Write $\mathfrak{u}_j^+$ for the upper nilpotent Lie algebra of the j-th component of $\mathrm{SL}(2) \times \mathrm{SL}(2)$, and let $\mathfrak{u}_j^-$ be the opposite algebra of $\mathfrak{u}_j^+$. By $\mathbb{Z}_p$-regularity of ρ, we have $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/K)$ normalizing $\mathcal{G}$ whose adjoint action $Ad(\rho(\sigma))$ has distinct $\mathbb{Z}_p$-eigenvalues on $\mathfrak{u}_j^{\pm}$ for $j = 1, 2$.

Since $\mathrm{Im}(\rho_j)$ contains $\Gamma_{A_j}^{\mathrm{SL}(2)}(\mathfrak{a}_j)$, $\mathfrak{n}_j^{\pm} := \mathcal{M}_2(\mathcal{G}) \cap \mathfrak{u}_j^{\pm}$ is non-trivial. By the adjoint action of $\mathcal{T}$, $\mathfrak{n}_j^{\pm}$ is a A_j-module. Then by [H13b, Lemma 2.9] applied to each projection of ρ to $\mathrm{GL}(2)$, we conclude a suitable conjugate of ρ has image containing the product of congruence subgroups as in the proposition. $\qquad\square$

4. Fullness of the Galois image in general Spin groups over Iwasawa algebras

4.1. **Spin groups.** Let us first recall the definition and the basic properties of the $\mathbb{Z}[\frac{1}{2}]$-group scheme GSpin_{2n+1} of spinorial similitudes. Consider the quadratic form $Q(\underline{x}) = 2x_1 x_{2n+1} + 2x_2 x_{2n} + \ldots + 2x_{n-1}x_{n+1} + x_n^2$ on $L = \bigoplus_{i=1}^{2n+1} \mathbb{Z}[\frac{1}{2}] \cdot e_i$. Let $C = C(L, Q)$ be the Clifford algebra associated to (L, Q) and $C = C^+ \oplus C^-$ be its decomposition as $\mathbb{Z}/2\mathbb{Z}$-graded algebra into even and odd Clifford elements ; C^+ is a subalgebra. The main (anti-)involution $*$ sends a pure element $v_1 \cdot \ldots \cdot v_r$ to $v_r \cdot \ldots \cdot v_1$. It leaves C^+ stable ; the module C, resp. C^+, is locally free of rank 2^{2n+1}, resp. 2^{2n} ; L embeds in C^- by $v \mapsto v$. Then $\mathrm{GSpin}_{2n+1} = \{x \in (C^+)^{\times}; x \cdot L \cdot x^* = L\}$ and $\mathrm{Spin}_{2n+1} = \{x \in (C^+)^{\times}; x \cdot L \cdot x^* = L, x^*x = 1\}$. We have two exact sequences of group schemes

$$1 \to \mathrm{Spin}_{2n+1} \to \mathrm{GSpin}_{2n+1} \xrightarrow{\mu} \mathbb{G}_m \to 1$$

by $\mu\colon x \mapsto x^*x$, and

$$1 \to \mathbb{G}_m \to \mathrm{GSpin}_{2n+1} \to \mathrm{SO}(L, Q) \to 1$$

by $x \mapsto (v \mapsto xvx^{-1})$. Since 2 is invertible, we can write $L = W \oplus W^{\vee} \oplus U$ where W, resp. $W^{\vee}$, is the standard lagrangian generated by $e_1, \ldots, e_n$, resp. its dual, identified to the direct factor generated by $e_{n+2}, \ldots, e_{2n+1}$, and $U = \mathbb{Z}e_{n+1}$. There is a natural isomorphism $C^+ \cong \mathrm{End}(\bigwedge^{\bullet} W)$ (Fulton-Harris, Lemma 20.16, the proof works with $\mathbb{C}$ replaced by $\mathbb{Z}[\frac{1}{2}]$) ; by restriction to GSpin, it induces the spin representation $\mathrm{spin}\colon \mathrm{GSpin} \to \mathrm{GL}(\bigwedge^{\bullet} W) = \mathrm{GL}_{2^n}$. Let T_{spin} be the diagonal maximal torus of GSpin_{2n+1}, and put $T'_{\mathrm{spin}} = T_{\mathrm{spin}} \cap \mathrm{Spin}_{2n+1}$. We denote by $(t_1, \ldots, t_n; \mu) \mapsto [t_1, \ldots, t_n; \mu]$ the standard isomorphism $\mathbb{G}_m^n \times \mathbb{G}_m \cong T_{\mathrm{spin}}$, with $\mu([t_1, \ldots, t_n; \mu]) = \mu$. In particular $(t_1, \ldots, t_n) \mapsto [t_1, \ldots, t_n; 1]$ is an isomorphism $\mathbb{G}_m^n \cong T'_{\mathrm{spin}}$. We define the upper, resp. lower, triangular Borel subgroup $B_{\mathrm{spin}} = T'_{\mathrm{spin}} \cdot U_{\mathrm{spin}}$ resp. ${}^t B_{\mathrm{spin}} = T'_{\mathrm{spin}} \cdot {}^t U_{\mathrm{spin}}$, of Spin_{2n+1} as $\mathrm{spin}^{-1}(B_{2^n})$, resp. $\mathrm{spin}^{-1}({}^t B_{2^n})$, where $B_{2^n} = T_{2^n} \cdot U_{2^n}$ denotes the upper triangular Borel subgroup of GL_{2^n}.

The triple $(\mathrm{GSpin}_{2n+1}, B_{\mathrm{spin}}, T_{\mathrm{spin}})$ identifies to the dual $(\widehat{G}, \widehat{B}, \widehat{T})$ of the triple (G, B, T) (see [MT02, Sect.3.2]). In particular, the parametrization of (the semisimple part of) the standard torus $T'_{\mathrm{spin}} = \widehat{T}$ of $\widehat{G} = \mathrm{GSpin}_{2n+1}$ by $[t_1, \ldots, t_n; 1]$ can be written as $\prod_i [e_i - \frac{1}{2}f](t_i)$ via $\widehat{T} = \mathbb{G}_m \otimes X^*(T)$, where $e_i\colon (x_1, \ldots, x_n, \nu x_n^{-1}, \ldots, \nu x_1^{-1}) \mapsto x_i$ and $f\colon (x_1, \ldots, x_n, \nu x_n^{-1}, \ldots, \nu x_1^{-1}) \mapsto \nu$.

4.2. **Genus n Hida families.** Let $n \geq 1$ and $d = \frac{n(n+1)}{2}$. Let $\mathfrak{H}$ be the genus n Siegel half space ; it is a d-dimensional hermitian symmetric domain. Let $G = \mathrm{GSp}_{2n}$ with its standard Borel subgroup B and its diagonal torus T as before. For any torsion-free compact open subgroup $K \subset G(\mathbb{A}^{\infty})$, we form the d-dimensional Shimura Siegel variety $X_K = G(\mathbb{Q}) \backslash (\mathfrak{H} \times G(\mathbb{A}^{\infty}))/K$. These are non-connected d-dimensional complex varieties; the inclusions $K' \subset K$ induce a projective system of finite coverings $\pi_{K',K}\colon X_{K'} \to X_K$. These varieties and morphisms are algebraic and admit quasi-projective canonical models over $\mathbb{Q}$ which we still denote by X_K and $\pi_{K',K}$. Let $\lambda \in X^*(T)$ be a dominant weight for (G, B, T) and V_{λ} the irreducible representation of G over $\mathbb{Q}$ associated to λ. It defines a projective system of étale local systems $V_{\lambda,K}(\overline{\mathbb{Q}}_p)$ over the X_K's : $\pi_{K',K}^* V_{\lambda,K}(\overline{\mathbb{Q}}_p) = V_{\lambda,K'}(\overline{\mathbb{Q}}_p)$. Let $M \geq 3$ and $K(M)$ be the level M principal congruence subgroup of $G(\widehat{\mathbb{Z}})$. Let us fix an odd prime p not dividing M and consider the subgroups $I_m^+ \subset I_m$ of $K_p = G(\mathbb{Z}_p)$ where

$I_m = \{g \in K_p; g \pmod{p^m} \in B(\mathbb{Z}/p^m\mathbb{Z})\}$ and $I_m^+ = \{g \in K_p; g \pmod{p^m} \in N(\mathbb{Z}/p^m\mathbb{Z})\}$ denotes its (normal) pro-p-Sylow. Let $K_m = K(M)^p \times I_m^+$. Consider the middle degree cohomology

$$\mathbb{H} = \lim_m H^d(X_{K_m} \times \overline{\mathbb{Q}}, V_{\lambda,K_m}(\overline{\mathbb{Q}}_p))$$

Note that any $\mathbb{Z}_p$-model V_λ of the representation of G defines a compatible system of integral étale sheaves $V_{\lambda,K_m}(\overline{\mathbb{Z}}_p))$, hence an integral structure $\mathbb{H}_{\mathbb{Z}_p}$ of $\mathbb{H}$: $\mathbb{H} = \mathbb{H}_{\mathbb{Z}_p} \otimes \mathbb{Q}_p$; note that $\mathbb{H}_{\mathbb{Z}_p}$ may have torsion. Let $\epsilon = (1, \ldots, 1; 0) \in X_*(T)$; we put $w = \langle \lambda + \rho, \epsilon \rangle$; it is the motivic weight of these cohomology groups.

For any prime ℓ not dividing M, let $d_{\ell,1} = \mathrm{diag}(1_n, \ell \cdot 1_n)$ and for any $i = 2 \ldots, n$, let $d_{\ell,i} = \mathrm{diag}(1_{n+1-i}, \ell \cdot 1_{2n-i}, \ell^2 \cdot 1_{n+1-i})$. Consider the groups $K_m' = d_{\ell,i} K_m d_{\ell,i}^{-1}$ and $K_m'' = d_{\ell,i}^{-1} K_m d_{\ell,i}$, we have an isomorphism $\pi(d_{\ell,i}): X_{K_m'} \to X_{K_m''}$. We define an action of the double class $K_m d_{\ell,i} K_m$ on the pair $(X_{K_m}, V_{\lambda,K_m}(\overline{\mathbb{Q}}_p))$ by using the two finite morphisms π_{K_m', K_m}, π_{K_m'', K_m} and the morphism of sheaves

$$(\pi(d_{\ell,i}), d_{\ell,i}^*): V_{\lambda,K_m''}(\overline{\mathbb{Q}}_p) \to V_{\lambda,K_m'}(\overline{\mathbb{Q}}_p)$$

where for $t \in T(\mathbb{Q}_p)$, we write $t^* = \nu(t) \cdot t^{-1} \in T(\mathbb{Q}_p)$; and we consider its action on $V_\lambda(\mathbb{Q}_p)$. Note that for any $\ell \neq p$, $d_{\ell,i}^* = d_{\ell,i}^* = 1$ on $V_\lambda(\mathbb{Q}_p)$, in particular, it preserves any integral structure of V_λ. For $\ell = p$, We see that $d_{\ell,i}^*$ defines an isomorphism of $V_\lambda(\mathbb{Q}_p)$, that it preserves $V_\lambda^{\max}(\mathbb{Z}_p) = \mathrm{Ind}_{B^-}^G \lambda$, but has non trivial cokernel ; we won't make use of this integral structure in the sequel. We define for any prime ℓ not dividing M, and any $i = 1, \ldots, n$:

$$[K_m d_{\ell,i} K_m] = \pi_{K_m', K_m, *} \circ (\pi d_{\ell,i}, d_{\ell,i}^*))^* \circ \pi_{K_m'', K_m}^*.$$

These actions are compatible when m varies and define $\mathbb{Q}_p$-linear endomorphisms on $\mathbb{H}$, denoted by $T_{\ell,i}$ for ℓ prime to Mp, resp. by $U_{p,i}$ for $\ell = p$. These operators are the Hecke correspondences acting on $\mathbb{H}$. The space $\mathbb{H}$ carries a Galois action which commutes to the $T_{\ell,i}$'s and $U_{p,i}$'s. These actions preserve the integral structures.

It carries also a normal action of the torus $T(\mathbb{Z}_p)$ viewed as the quotient $B(\mathbb{Z}_p)/N(\mathbb{Z}_p) = \varprojlim I_m/I_m^+$. This action is continuous. One can decompose $T(\mathbb{Z}_p)$ as $T_f \times T_1$ where T_f is the torsion subgroup and T_1 is the pro-p-Sylow, kernel of the reduction modulo p. We fix a topological basis $(u_1, \ldots, u_n)$ of T_1 by fixing $u = 1 + p$ as topological basis of $1 + p\mathbb{Z}_p$ and defining u_i as the image of u by the cocharacter $t \mapsto \mathrm{diag}(E_i(t), E_{n+1-i}(t^{-1}))$ where $E_i(t) = \mathrm{diag}(1, \ldots, t, \ldots, 1)$ (t is the i-th component). By this choice, we identify the completed group algebra $\mathbb{Z}_p[[T_1]]$ to the n-variable Iwasawa algebra $\Lambda_n = \mathbb{Z}_p[[X_1, \ldots, X_n]]$ via $u_i \mapsto X_i + 1$. $\mathbb{H}$ is therefore a $\Lambda_n[T_f]$-module. This action commutes to the $T_{\ell,i}$ and the $U_{p,i}$'s ; moreover it preserves any integral structure given by a $\mathbb{Z}_p$-model V_λ of the representation of G. We consider the largest subspace $H_0^d(X_{K_m} \times \overline{\mathbb{Q}}, V_{\lambda,K_m}(\overline{\mathbb{Q}}_p))$, resp. $\mathbb{H}_0$, of $H^d(X_{K_m} \times \overline{\mathbb{Q}}, V_{\lambda,K_m}(\overline{\mathbb{Q}}_p))$, resp. of $\mathbb{H}$, on which the eigenvalues of the operators $U_{p,i}$ are p-adic units ; it is the sum of all the generalized eigenspaces for such eigenvalues. We denote by $\mathbf{T}_\lambda^M(K_m)$ the $\mathbb{Z}_p$-algebra generated by the $T_{\ell,i}$'s, the $U_{p,i}$'s and the group action of $T(\mathbb{Z}/p^m\mathbb{Z})$ on $H_0^d(X_{K_m} \times \overline{\mathbb{Q}}, V_{\lambda,K_m}(\overline{\mathbb{Q}}_p))$; similarly, let $\mathbf{T}^M$ be the Λ_n-subalgebra of $\mathrm{End}_{\Lambda_n} \mathbb{H}_0$ generated by the $T_{\ell,i}$'s, the $U_{p,i}$'s and the action of T_f. Recall ([M04] Prop.6.4.1) that $\mathbb{H}_0$ is a finite Λ_n-module. Therefore, $\mathbf{T}^M$ is a finite Λ_n-algebra. Note however that it is not cuspidal, in general (this would mean that all specializations at cohomological weights would be cuspidal). It is conjectured that cuspidality holds for the localization of $\mathbf{T}^M$ at a "weakly non-Eisenstein" maximal ideal (assuming the existence of the $\widehat{G}$-valued p-adic Galois representation and its ordinarity at p, "weak non-Eisensteinness" means that the corresponding residual Galois representation is absolutely irreducible).

However, another obstacle in this definition (due to the presence of torsion in the cohomology groups) is that the exact control theorem has not been established for all cohomological weights, unless one makes some strong extra assumptions. This seems to prevent to show in full generality that $\mathbf{T}^M$ is torsion-free as a Λ_n-algebra. Let $\lambda' \in X^*(T)$ be a dominant weight such that $\lambda'|_{T_f} = \lambda|_{T_f}$, its restriction to T_1 defines by linearity and continuity an algebra homomorphism $\lambda': \Lambda_n \to \mathbb{Z}_p$; we denote by $P_{\lambda'}$ the prime ideal of Λ_n kernel of λ'.

Let $C > 0$ be a real number. The set of C-regular weights is the set of λ''s such that $\langle \lambda', \alpha \rangle > C$ for all simple roots α of G.

Definition 4.1. *We say that control theorem, resp. $\mathfrak{m}$-control theorem, holds for the weight λ' (congruent to λ mod. $p-1$), if the surjective morphism $\mathbf{T}^M \otimes_{\Lambda_n, \lambda'} \mathbb{Z}_p \to \mathbf{T}^M_{\lambda'}(K_1)$, resp. $\mathbf{T}^M_{\mathfrak{m}} \otimes_{\Lambda_n, \lambda'} \mathbb{Z}_p \to \mathbf{T}^M_{\lambda'}(K_1)_{\mathfrak{m}}$, has finite kernel.*

If one assumes that $p - 1 > w$ (for the motivic weight w) and if the maximal ideal $\mathfrak{m}$ is "strongly non-Eisenstein" (in the sense of [MT02] Sect.9 : it means that the image of the residual representation contains the normalizer of the $\mathbb{F}_p$-points of the standard maximal torus $\widehat{T}$) then cuspidality and Λ_n-torsion-freeness hold for the $\mathfrak{m}$-localized Hecke algebra and $\mathfrak{m}$-control theorem do hold for all cohomological weights λ' congruent to λ mod. $p - 1$ (see Th.10 of [MT02]), otherwise, without assuming the existence of the Galois representations and without localization, there exists a constant C depending on the data (λ, p) such that control theorem holds for all C-regular weights (but it does not imply the torsion-freeness of the Hecke algebra, except for Siegel-Hilbert varieties of genus 2, see [TU99], or for unitary groups in three variables, see [M04]).

In order to obtain a control theorem for all weights without localization, one needs to follow another approach via coherent cohomology, started by Hida [H02]. We identify the character group $X^*(T)$ to the index two sublattice of $\mathbb{Z}^{n+1}$ by sending the character

$$\lambda \colon \operatorname{diag}(t_1, \ldots, t_n, \nu t_n^{-1}, \ldots, \nu t_1^{-1}) \mapsto t_1^{k_1} \ldots t_n^{k_n} \nu^{(c - k_1 - \ldots - k_n)/2}$$

to $(k_1, \ldots, k_n; c)$. Let $t = (1, \ldots, 1; 1)$; we fix the cohomological weight $k = \lambda + (n+1)t = (k_1, \ldots, k_n; k_1 + \cdots + k_n)$ and the sheaf ω^k_{cusp} over X_{K_1} (see [H02, Sect.2.3] and [Pi12, Sect.4]). These objects are now viewed on the rigid analytic site over $\mathbb{Q}_p$. One starts from a holomorphic cusp form $f \in H^0(X_{K_1}, \omega^k_{\text{cusp}})$; instead of the whole tower of finite coverings $X_{K_m} \to X_{K_1}$, one forms the Igusa tower $T_m = \operatorname{Isom}_{X^0_{K_1}} (\mu^n_{p^m}, A[p^m]^\circ)$ over the ordinary locus $X^0_{K_1}$ of X_{K_1} (here, $A[p^m]^\circ$ denotes the connected component of the p^m-torsion of the universal abelian variety over $X^0_{K_1}$); T_m can be viewed as the rigid tube of the "multiplicative type" connected component of the ordinary locus in the reduction modulo p of X_{K_m}. Consider the $\mathbb{Q}_p$-vector space $\mathbb{HC} = \lim_m H^0(T_m, \omega^k_{\text{cusp}})$; one defines similarly action of the Hecke operators $T_{\ell, i}$'s and $U_{p, i}$'s ($i = 1, \ldots, n$); these are compatible with the transition morphisms of the Igusa tower ; the Galois group of the Igusa tower is $T(\mathbb{Z}_p)$ which therefore acts on $H^0(T_m, \omega^k_{\text{cusp}})$ and $\mathbb{HC}$. One defines the $\mathbb{Z}_p$-Hecke algebras $\mathbf{T}^M(T_m)$, resp. $\mathbf{T}^M(T_\infty)$ as generated by $T_{\ell, i}$'s, $U_{p, i}$'s and the image of $T(\mathbb{Z}_p)$ acting on $H^0(T_m, \omega^k)$, resp. $\mathbb{HC}$. Similarly, one could define $\mathbf{T}^{M, \text{holcusp}}$ as the $\mathbb{Z}_p$-algebra acting on the ordinary part $\mathbb{H}^{\text{holcusp}}_0$ of $\mathbb{H}^{\text{holcusp}} = \lim_m H^0(X_{K_m}, \omega^k_{\text{cusp}})$; by density of classical Siegel modular cusp forms in p-adic modular cusp forms, the natural morphism $\mathbf{T}^M(T_\infty) \to \mathbf{T}^{M, \text{holcusp}}$ is an isomorphism of Λ_n-algebras (note though that one has only surjections $\mathbf{T}^M(T_m) \to \mathbf{T}^{M, \text{holcusp}}(K_m)$). By Hodge–Tate decomposition (see Faltings–Chai Chapter VI, using BGG theory), $\mathbf{T}^{M, \text{holcusp}}$ is a quotient of $\mathbf{T}^M$ (it cannot be injective since the left-hand side contains non cuspidal families, but after localization at any weakly non-Eisenstein maximal ideal should be an isomorphism).

By [Pi12, Th.6.7 and Th.1.2], and [PiSt, Th.6.3.1], $\mathbf{T}^{M, \text{holcusp}}$ is finite torsion-free over Λ_n and (cuspidal) Control Theorem holds without localization (and without assuming the existence of the Galois representations) ; in other words, for any dominant weight λ' such that $\lambda'|_{T_f} = \lambda|_{T_f}$, the surjective morphism $\mathbf{T}^{M, \text{holcusp}} \otimes_{\Lambda_n, \lambda'} \mathbb{Z}_p \to \mathbf{T}^{M, \text{holcusp}}_{\lambda'}(K_1)$ has finite kernel.

Let $\mathbb{I}$ be the normal closure of an irreducible component of $\operatorname{Spec}(\mathbf{T}^{M, \text{holcusp}})$. Let $\theta \colon \mathbf{T}^{M, \text{holcusp}} \to \mathbb{I}$ be the corresponding Hida family. Assuming the existence of the Galois representations $\rho_f \colon G_{\mathbb{Q}} \to \widehat{G}(\overline{\mathbb{Z}}_p)$ associated to the holomorphic cusp forms of level K_1 occurring in the family, let us construct the Galois representation $G_{\mathbb{Q}} \to \widehat{G}(\mathbb{I})$ associated to θ. We say that a cusp eigenform f' of level K_1, weight $k' = \lambda' + (n+1)t$ and eigensystem $\theta_{f'}$ occurs in the family if there exists a prime P' of $\mathbb{I}$ above $P_{\lambda'}$ such that, via the control theorem, one has $\theta_{f'} = \theta \pmod{P'}$.

Definition 4.2. *We say that the irreducible component $\mathbb{I}$ is weakly non-Eisenstein if for some (hence all) cusp eigenform f' occurring in the family, the residual representation composed with* spin *is absolutely irreducible.*

Lemma 4.3. *For any weakly non-Eisenstein irreducible component $\mathbb{I}$, there exists a continuous homomorphism $\rho_\theta \colon G_{\mathbb{Q}} \to \widehat{G}(\mathbb{I})$ such that for any dominant weight λ' such that $\lambda'|_{T_f} = \lambda|_{T_f}$, and*

for any cusp eigenform f' of weight $k' = \lambda' + (n+1)t$ in the family, the representation $\rho_{f'}$ is conjugate in $\widehat{G}(\overline{\mathbb{Q}}_p)$ to $\rho_\theta \pmod{P'}$.

Proof. We give the proof only for $\mathbb{I} = \Lambda_n$. Let us consider the $\mathbb{Z}_p$-morphism $\mathrm{Ad} \colon \widehat{G} \to \mathrm{SO}_{2n+1}$; note that $\mathrm{Ker}(\mathrm{Ad})$ is the center $\widehat{Z} \cong \mathbb{G}_m$ of $\widehat{G}$. We first construct a representation $R \colon G_\mathbb{Q} \to \mathrm{SO}_{2n+1}(\mathbb{I})$ and then lift it to $\widehat{G}(\mathbb{I})$. For any arithmetic prime P' of $\mathbb{I}$ above a dominant weight λ' such that $\lambda'|_{T_f} = \lambda|_{T_f}$, let $f_{P'}$ be a cusp eigenform occurring in the family with this weight ; by residual irreducibility of $\mathrm{spin} \circ \rho_{f_{P'}}$ and Carayol's theorem [Car94], we can assume that this representation is defined over $\mathbb{I}/P'$; by injectivity of spin, it implies that $\rho_{P'} = \rho_{f_{P'}}$ takes values in $\widehat{G}(\mathbb{I}/P')$: $\rho_{P'} \colon G_\mathbb{Q} \to \widehat{G}(\mathbb{I}/P')$.

On the other hand, by residual irreducibility, it follows from Carayol's theorem [Car94] that $\mathrm{Ad} \circ \overline{\rho}$ lifts to a continuous representation $R \colon G_\mathbb{Q} \to \mathrm{GL}_{2n+1}(\mathbb{I})$. Since $\mathrm{Ad} \circ \overline{\rho}$ is orthogonal, and each specialization modulo an arithmetic prime is orthogonal, we see that R is orthogonal. Since $\mathbb{I}$ is profinite and by formal smoothness of the group $\widehat{G}$, there exists a continuous lift $\widetilde{R} \colon G_\mathbb{Q} \to \widehat{G}(\mathbb{I})$, which gives rise to a 2-cocycle $c_{\sigma,\tau} \in Z^2(G_\mathbb{Q}, \mathbb{I}^\times)$.

For any prime P', the existence of the representation $\rho_{P'}$ implies that the image of $c_{\sigma,\tau}$ is a coboundary : $c_{\sigma,\tau} = b_{\sigma\tau} b_\sigma^{-1} b_\tau^{-1}$ over $(\mathbb{I}/P')^\times$. Take $\mathbb{I} = \Lambda_n$ and consider the Iwasawa ideal $\Omega_{n,\underline{k}}$ generated by the $(1+T_i)^{p^n} - u^{k_i p^n}$, $= 1, \ldots, n$. The quotient ring $\Lambda_{n,\underline{k}} = \Lambda/\Omega_{n,\underline{k}}$ embeds in the product of cyclotomic rings $\Lambda(\zeta_1, \ldots, \zeta_n) = \Lambda_n/I(\zeta_1, \ldots, \zeta_n)$ where

$$I(\zeta_1, \ldots, \zeta_n) = (1 + T_i - u^{k_i}\zeta_i, \quad i = 1 \ldots, n)$$

where ζ_i is a p^n-th root of unity. The condition for a family $(b(\zeta_1, \ldots, \zeta_n))_{\zeta_1, \ldots, \zeta_n}$ to be in the image of the embedding is that for any pair $(\zeta_1, \ldots, \zeta_n)$ and $(\zeta_1', \ldots, \zeta_n')$ the components $b(\zeta_1, \ldots, \zeta_n)$ and $b(\zeta_1', \ldots, \zeta_n')$ are congruent modulo $I(\zeta_1, \ldots, \zeta_n) + I(\zeta_1', \ldots, \zeta_n')$. Let us check it is the case for the family of 1-cochains $(b(\zeta_1, \ldots, \zeta_n)_\sigma)$ associated to the specialization $c_{\sigma,\tau}^{n,\underline{k}}$ of $c_{\sigma,\tau}$ over $(\Lambda_{n,\underline{k}})^\times$. Indeed, let us compare the reductions modulo $I(\zeta_1, \ldots, \zeta_n) + I(\zeta_1', \ldots, \zeta_n')$ of the representations $\mathrm{spin} \circ \widetilde{R}(\sigma) b(\zeta_1, \ldots, \zeta_n)_\sigma$ and $\mathrm{spin} \circ \widetilde{R}(\sigma) b(\zeta_1', \ldots, \zeta_n')_\sigma$; they have the same characteristic polynomial and they both lift the irreducible representation $\mathrm{spin} \circ \overline{\rho}$. By Carayol's theorem [Car94], they are conjugate, so that $b(\zeta_1, \ldots, \zeta_n)_\sigma = b(\zeta_1', \ldots, \zeta_n')_\sigma$ in $\Lambda_n/I(\zeta_1, \ldots, \zeta_n) + I(\zeta_1', \ldots, \zeta_n')$.

From this, we conclude that there exists a 1-cochain $b_{\sigma,\tau}^{n,\underline{k}}$ with values in $(\Lambda_{n,\underline{k}})^\times$ such that $c_{\sigma,\tau}^{n,\underline{k}} = b_{\sigma\tau}^{n,\underline{k}} b_\sigma^{n,\underline{k}-1} b_\tau^{n,\underline{k}-1}$. This construction is compatible when n grows and gives rise to a 1-cochain b_σ with values in $\Lambda_n^\times$ as desired. The resulting map $\sigma \mapsto \widetilde{R}(\sigma) b_\sigma$ is the desired representation $G_\mathbb{Q} \to \widehat{G}(\Lambda_n)$. $\qquad\square$

Moreover ρ_θ is ordinary at p. More precisely, recall that ϵ denotes the p-adic cyclotomic character ; we define for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the cyclotomic logarithm $\ell(\sigma) = \frac{\log_p(\epsilon(\sigma))}{\log_p(1+p)}$ (for the p-adic cyclotomic character ϵ). Let $t_i = 1 + X_i$ $(i = 1, \ldots, n)$, then

Lemma 4.4. *Assume $p - 1 > w$. The representation ρ_θ is ordinary : there exists $\widehat{g} \in \widehat{G}(\mathbb{I})$ such that $\rho_\theta(D_p) \subset \widehat{g}\widehat{B}(\mathbb{I})\widehat{g}^{-1}$ and for any $\sigma \in I_p$,*

$$\widehat{g}^{-1}\rho_\theta(\sigma)\widehat{g} \equiv [\epsilon(\sigma)^{-1}t_1^{\ell(\sigma)}, \epsilon(\sigma)^{-2}t_2^{\ell(\sigma)} \ldots, \epsilon(\sigma)^{-n}t_n^{\ell(\sigma)}; \epsilon(\sigma)^{-d}] \pmod{U(\mathbb{I})}$$

where $d = \frac{n(n+1)}{2}$.

Proof. Since all Borel subgroups are conjugated in $\widehat{G}$, it is enough to apply the spin representation and to find a D_p-stable flag in $\mathbb{I}^{2^n}$ with action of I_p on the graded pieces given by the mutually distinct characters $\prod_{i \in I} \epsilon(\sigma)^{-i} t_i^{\ell(\sigma)}$ for all subset I's of $\{1, \ldots, n\}$. Over $\mathbb{J} = \prod_{P'} \mathcal{O}_{P'}$ as in the previous proof, we have such a flag (in $\mathbb{J}^{2^n}$). If we assume $p - 1 > w$ so that all the characters above take values in $\mathbb{I}$ and are mutually distinct modulo p, we see that the intersection of this flag with $\mathbb{I}^{2^n}$ induces a flag as desired. $\qquad\square$

4.3. Λ_n-**fullness.** In the sequel, we consider an irreducible component $\mathbb{I}$ of $\mathbf{T}^M$ such that $\mathbb{I} = \Lambda_n = \mathbb{Z}_p[[X_1, \ldots X_n]]$ (with $p > 2$). Let $u = 1 + p$. Let $\mathbb{Q}_p^{\mathrm{cyc}}$ be the p-power-cyclotomic extension of $\mathbb{Q}_p$. For any weight $\underline{k} = (k_1, \ldots, k_n)$ dominant for (G, B, T), and cohomological, we denote by P_k the point of $\mathrm{Spec}\, \Lambda_n$ given by $P_k = (1 + X_1 - u^{k_1}, \ldots, 1 + X_n - u^{k_n})$. the point

We define for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ the cyclotomic logarithm $\ell(\sigma) = \frac{\log_p(\epsilon(\sigma))}{\log_p(1+p)}$ (for the p-adic cyclotomic character ϵ). We put, as above, $t_i = 1 + X_i$ $(i = 1, \ldots, n)$, and we assume that the semi-simplification of ρ restricted to the p-inertia group I_p is

$$[\epsilon(\sigma)^{-1} t_1^{\ell(\sigma)}, \epsilon(\sigma)^{-2} t_2^{\ell(\sigma)} \ldots, \epsilon(\sigma)^{-n} t_n^{\ell(\sigma)}; \epsilon(\sigma)^{-d} (t_1 \cdot \ldots \cdot t_n)^{\ell(\sigma)}],$$

where $d = \frac{n(n+1)}{2}$. This is conjecturally the case if ρ is associated to an n-variable p-adic Hida family of Siegel forms of genus n, passing by a form f of level prime to p and weight $\underline{k}^0 = (k_1^0, \ldots, k_n^0)$, with $k_1^0 \geq \ldots \geq k_n^0 \geq n + 1$. It is actually proven if $n = 2$.

We consider the Galois representation $\rho = \rho_\theta : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSpin}_{2n+1}(\Lambda_n)$ associated to the family $\theta \colon \mathbf{T}^{M,\mathrm{holcusp}} \to \mathbb{I} = \Lambda_n$; the representation is ρ ordinary at p ; After conjugation, we can and will assume in the sequel that $\rho(D_p) \subset \widehat{B}(\Lambda_n)$. For any root α of $(\widehat{G}, \widehat{T})$, we denote by U_α the one-dimensional Lie subgroup of Spin_{2n+1}, by $u_\alpha \colon \mathbf{G}_a \to U_\alpha$ the corresponding one-parameter subgroup and by $\mathfrak{u}_\alpha = \mathrm{Lie}(U_\alpha) \subset \mathfrak{spin}_{2n+1}$ the corresponding Lie subalgebra (say over $\mathbb{Z}_p$ or any base change thereof). Because of these notations, it will be convenient to write U instead of $\widehat{N}$ for the unipotent radical of the standard Borel subgroup $\widehat{B} = \widehat{T}\widehat{N}$. We have the following key lemma

Lemma 4.5. Let $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSpin}_{2n+1}(\Lambda_n)$ be the Galois representation associated to a family $\theta \colon \mathbf{T}^{M,\mathrm{holcusp}} \to \mathbb{I} = \Lambda_n$. Suppose

(1) there is a cohomological weight k^0 for which the specialization $\rho_{P_{k^0}} = \rho_f$ has $\mathbb{Z}_p$-full image; in other words, $\mathrm{Im}(\rho_{P_{k^0}})$ contains an open subgroup of $\mathrm{Spin}_{2n+1}(\mathbb{Z}_p)$,

(2) strong regularity, (compare to 3.5) $\mathrm{Im}(\rho)$ contains an element $\delta \in \widehat{T}(\Lambda_n)$ such that for any two roots $\alpha \neq \alpha'$ of SO_{2n+1}, $\alpha(\mathrm{Ad}(\delta)) \not\equiv \alpha'(\mathrm{Ad}(\delta))$ $(\mathrm{mod}\ \mathfrak{m}_{\Lambda_n})$,

Then for any root α of $(\widehat{G}, \widehat{T})$, $\mathrm{Im}\, \rho \cap U_\alpha(\Lambda_n) \neq 0$

Remarks :

1) Note that the roots of $(\widehat{G} = \mathrm{GSpin}_{2n+1}, \widehat{T})$ are the same as those of $(\mathrm{SO}_{2n+1}, T_{\mathrm{SO}_{2n+1}})$.

2) If instead of Assumption 2 we only assume that there exists an element $\delta \in \widehat{B}(\Lambda_n)$ such that its projection $\delta_1 \in \widehat{T}(\Lambda_n)$ is strongly regular mod. $\mathfrak{m}_{\Lambda_n}$, by which we mean that for any two roots $\alpha \neq \alpha'$ of SO_{2n+1}, $\alpha(\mathrm{Ad}(\delta_1)) \not\equiv \alpha'(\mathrm{Ad}(\delta_1))$ $(\mathrm{mod}\ \mathfrak{m}_{\Lambda_n})$, then , we can modify it by conjugations by elements $u_\alpha(x)$ for positive roots α roots and suitable elements x so that it belongs to $\widehat{T}(\Lambda_n)$ and it satisfies the exact statement of Assumption 2.

3) If we assume furthermore that $p - 1 > 2w$ and $\underline{k}^0 = (k_1^0, \ldots, k_n^0)$ is such that for any $i \neq j$, $k_i^0 - i \not\equiv \pm(k_j^0 - j)$ $(\mathrm{mod}\ p - 1)$, we see by Lemma 4.4 above that if we choose $\sigma \in I_p$ such that $\kappa(\sigma) = \omega(\sigma)$ is of exact order $p - 1$, we can take $\delta = \rho(\sigma)$.

Proof. Since the argument is the same for positive and negative roots, we restrict ourselves to the "upper triangular" unipotent radical U (corresponding to positive roots α) of $\mathrm{Spin}(2n + 1)$. Let $K = \mathrm{Im}\, \rho \subset \widehat{G}(\Lambda_n)$. We write $P = P_{k^0}$ for the arithmetic point such that the specialization ρ_P of the representation ρ is $\mathbb{Z}_p$-full. Fix a positive root α and define

(4.1)
$$\Gamma_{U_\alpha}(P^j) = \{x \in \mathrm{Spin}_{2n+1}(\Lambda_n) | x \mod P^j \in U_\alpha(\Lambda_n/P^j)\}$$
$$\Gamma_{B_\alpha}(P^j) = \{x \in \mathrm{Spin}_{2n+1}(\Lambda_n) | x \mod P^j \in \widehat{T}(\Lambda_n/P^j) U_\alpha(\Lambda_n/P^j)\}$$

and

(4.2)
$$K_{U_\alpha}(P^j) = K \cap \Gamma_{U_\alpha}(P^j)$$
$$K_{B_\alpha}(P^j) = K \cap \Gamma_{B_\alpha}(P^j)$$

Since $U(\Lambda_n)$ and $\Gamma_{\Lambda_n}(P)$ are p-profinite, the groups $\Gamma_{U_\alpha}(P^j)$ and $K_{U_\alpha}(P^j)$ for all $j \geq 1$ are also p-profinite. Recall

$$\Gamma_{\Lambda_n}(P^i) = \{x \in \mathrm{Spin}_{2n+1}(\Lambda_n) | (x \mod P^i) = 1\}$$

is the kernel of the reduction morphism $\pi_j \colon \mathrm{Spin}_{2n+1}(\Lambda_n) \to \mathrm{Spin}_{2n+1}(\Lambda_n/P^j)$. Recall the embedding $\iota_\alpha \colon \mathrm{SL}(2) \to \mathrm{Spin}_{2n+1}$ associated to the root α (i.e., the upper (resp. lower) unipotent subgroup $U^+_{\mathrm{SL}(2)}$ (resp. $U^-_{\mathrm{SL}(2)}$) of $\mathrm{SL}(2)$ is sent to U_α (resp. $U_{-\alpha}$) by ι_α). Thus, we see

$$\Gamma_{U_\alpha}(P^j) = \Gamma_{\Lambda_n}(P^j)\iota_\alpha(U_{\mathrm{SL}(2)}(\Lambda_n)).$$

Note that

$$\left[\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right), \left(\begin{smallmatrix} e & f \\ g & -e \end{smallmatrix}\right)\right] = \left(\begin{smallmatrix} bg-cf & 2(af-be) \\ 2(ce-ag) & cf-bg \end{smallmatrix}\right).$$

From this, we have

Lemma 4.6. *If $X, Y \in \mathfrak{sl}_2(\Lambda_n) \cap \left(\begin{smallmatrix} P^j & P^k \\ P^i & P^j \end{smallmatrix}\right)$ with $i \geq j \geq k$, $[X, Y] \in \left(\begin{smallmatrix} P^{i+k} & P^{j+k} \\ P^{i+j} & P^{i+k} \end{smallmatrix}\right)$.*

This tells us, for the topological commutator subgroup $D\Gamma_{U_\alpha}(P^j) := (\Gamma_{U_\alpha}(P^j), \Gamma_{U_\alpha}(P^j))$ that

(4.3) $$D\Gamma_{U_\alpha}(P^j) \subset \Gamma_{B_\alpha}(P^{2j}) \cap \Gamma_{U_\alpha}(P^j).$$

Using now the $\mathbb{Z}_p$-regularity assumption (in Section 3.5), we consider an element an element $\delta \in \widehat{T}(\Lambda_n) \in K$, with distinct root values modulo $\mathfrak{m}_{\Lambda_n}$. Replacing δ by $\lim_{m \to \infty} \delta^{p^m}$, we may assume that $\delta \in \widehat{T}(\mathbb{Z}_p)$ has finite order a. The order a is prime to p (indeed it is a factor of $p-1$). Note that δ normalizes $K_{U_\alpha}(P^j)$ and $\Gamma_{B_\alpha}(P^j)$. Since H is p-profinite, $x \in H$ has unique a-th root inside H. We define

$$\Delta_\alpha(x) = \sqrt[a]{x \cdot (\delta x \delta^{-1})^{\alpha(\delta)^{-1}} \cdot (\delta^2 x \delta^{-2})^{\alpha(\delta)^{-2}} \cdots (\delta^{a-1} x \delta^{1-a})^{\alpha(\delta)^{1-a}}} \in H.$$

Lemma 4.7. *If $u \in \Gamma_{U_\alpha}(P^j)$ $(j \geq 1)$, then $\Delta_\alpha^2(u) \in \Gamma_{U_\alpha}(P^{2j})$ and $\pi_j(\Delta_\alpha(u)) = \pi_j(u)$.*

Proof. If $u \in \Gamma_{U_\alpha}(P^j)$, we have $\pi_j(\Delta_\alpha(u)) = \pi_j(u)$ as Δ_α is the identity map on $U_\alpha(\Lambda_n/P^j)$. Let $D\Gamma_{U_\alpha}(P^j)$ be the topological commutator subgroup of $\Gamma_{U_\alpha}(P^j)$. Since Δ_α induces the projection of the $\mathbb{Z}_p$-module $\Gamma_{U_\alpha}(P^j)/D\Gamma_{U_\alpha}(P^j)$ onto its $\alpha(\delta)$-eigenspace for $\mathrm{Ad}(\delta)$, it is a projection onto $U_\alpha(\Lambda_n)D\Gamma_{U_\alpha}(P^j)/D\Gamma_{U_\alpha}(P^j)$. The fact that this is exactly the $\alpha(\delta)$-eigenspace comes from the Iwahori decomposition of $\Gamma_{U_\alpha}(P^j)$ which shows it is generated by the $U_\beta(P^j\Lambda_n)$ for all roots $\beta \neq \alpha$ and by $U_\alpha(\Lambda_n)$, hence a similar direct sum decomposition holds in the abelianization $\Gamma_{U_\alpha}(P^j)/D\Gamma_{U_\alpha}(P^j)$.

By (4.3) $D\Gamma_{U_\alpha}(P^j) \subset \Gamma_{B_\alpha}(P^{2j}) \cap \Gamma_{U_\alpha}(P^j)$. Since the $\alpha(\delta)$-eigenspace of $\Gamma_{U_\alpha}(P^j)/D\Gamma_{U_\alpha}(P^j)$ is inside $\Gamma_{B_\alpha}(P^{2j})$, Δ_α projects $u\Gamma_{U_\alpha}(P^j)$ to

$$\overline{\Delta}_\alpha(u) \in (\Gamma_{B_\alpha}(P^{2j}) \cap \Gamma_{U_\alpha}(P^j))/D\Gamma_{U_\alpha}(P^j).$$

In particular, $\Delta_\alpha(u) \in \Gamma_{B_\alpha}(P^{2j}) \cap \Gamma_{U_\alpha}(P^j)$. Again apply Δ_α. Since $\Gamma_{B_\alpha}(P^{2j})/\Gamma_{\Lambda_n}(P^{2j})$ is sent to to $\Gamma_{U_\alpha}(P^{2j})/\Gamma_{\Lambda_n}(P^{2j})$ by Δ_α, we get $\Delta_\alpha^2(u) \in \Gamma_{U_\alpha}(P^{2j})$ as desired. $\square$

We can now prove the key Lemma 4.5. Pick $0 \neq \overline{u}_\alpha \in U_\alpha(A/P) \cap \mathrm{Im}(\rho_P)$. By $\mathbb{Z}_p$-fullness of ρ_P, $\overline{u}_\alpha$ exists. Since the reduction map $\mathrm{Im}(\rho) \to \mathrm{Im}(\rho_1)$ is surjective, we have $u_\alpha \in \mathrm{Im}(\rho)$ such that $u_\alpha \mod P = \overline{u}_\alpha$. Take $v_1 \in U_\alpha(\Lambda_n)$ such that $v_1 \mod P = \overline{u}_\alpha$. Such a v_1 exists as the reduction map $U_\alpha(\Lambda_n) \to U_\alpha(\Lambda_n/P)$ is onto. Then $u_\alpha v_1^{-1} \in \Gamma_{\Lambda_n}(P)$, we find that $u_\alpha \in K_{U_\alpha}(P)$.

By the compactness of $K_{U_\alpha}(P)$, by Lemma 4.7, starting with u_α as above, $\lim_{m \to \infty} \Delta_\alpha^m(u_\alpha)$ converges P-adically to $\Delta_\alpha^\infty(u_\alpha) \in U_\alpha(\Lambda_n) \cap \mathrm{Im}(\rho)$ with $\Delta_\alpha^\infty(u_\alpha)\Gamma_{\Lambda_n}^{\mathrm{Spin}(2n+1)}(P) = u_\alpha \Gamma_{\Lambda_n}^{\mathrm{Spin}(2n+1)}(P)$ (i.e., $\overline{u}_\alpha = (\Delta_\alpha^\infty(u_\alpha) \mod P)$). Thus $\Delta_\alpha^\infty(u_\alpha)$ is a non-trivial unipotent element in $U_\alpha(\Lambda_n) \cap \mathrm{Im}(\rho)$. $\square$

From this lemma we can deduce the following main theorem (we still assume that $\rho(D_p) \subset \widehat{B}(\Lambda_n)$).

Theorem 4.8. *Let $\rho \colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GSpin}_{2n+1}(\Lambda_n)$ be a continuous representation. Suppose*

(1) *there is a cohomological weight k^0 for which the specialization $\rho_{P_{k^0}} = \rho_f$ has $\mathbb{Z}_p$-full image ; in other words, $\mathrm{Im}(\rho_{P_{k^0}})$ contains an open subgroup of $\mathrm{Spin}_{2n+1}(\mathbb{Z}_p)$,*

(2) *$\mathrm{Im}(\rho)$ contains an element $\delta \in \widehat{B}(\Lambda_n)$, whose projection $\delta_1 \in \widehat{T}(\Lambda_n)$ is $\mathfrak{spin}_{2n+1}$-regular mod. $\mathfrak{m}_{\Lambda_n}$, that means that for any two roots $\alpha \neq \alpha'$ of SO_{2n+1}, $\alpha(\mathrm{Ad}(\delta_1)) \not\equiv \alpha'(\mathrm{Ad}(\delta_1))$ $(\mod \mathfrak{m}_{\Lambda_n})$,*

Then $\mathrm{Im}\,\rho$ contains a principal congruence subgroup of $\mathrm{Spin}_{2n+1}(\Lambda_n)$

Definition 4.9. *The largest ideal of Λ_n such that $\Gamma_{\Lambda_n}(\mathfrak{l}_\theta) \subset \operatorname{Im} \rho_\theta$ is called the Galois level of θ, and is denoted by $\mathfrak{l}_\theta$.*

There are two proofs. It can either be deduced from Pink's theory using Lemma 3.6, as the only missing point was the non-triviality of $U_\alpha(\Lambda_n) \cap \operatorname{Im}(\rho)$ for all α's (which follows from Lemma 4.5), or it can be proven using Lie algebras as follows.

Let $H = \operatorname{Im} \rho \cap \Gamma_{\Lambda_n}(p)$. We have seen that for any root α, $\operatorname{Im} \rho \cap U_\alpha(\Lambda_n) \neq \{1\}$; therefore, by taking possibly a p-th power of th non trivial element in this intersection, we can assume that for any root α, $H \cap U_\alpha(\Lambda_n) \neq \{1\}$. The series defining Log still provides a bijection between H and the set $\operatorname{Log} H$ which is contained in $p \cdot \mathfrak{spin}_{2n+1}(\Lambda_n)$, but may not be stable by addition ; however, the Log map induces group isomorphisms from $H \cap U_\alpha(\Lambda_n)$ to the (possibly infinite dimensional) $\mathbb{Z}_p$-Lie algebras $\operatorname{Log}(H) \cap \mathfrak{u}_\alpha(\Lambda)$, which are therefore non-trivial $\mathbb{Z}_p$-modules for all roots α. Recall that the $\mathbb{Q}_p \cdot \operatorname{Log}(H)$ is a $\mathbb{Q}_p$-Lie algebra, denoted $\operatorname{Log}_{\mathbb{Q}_p} H$ below.

Using the adjoint action of δ, we have a decomposition into root spaces

$$(*) \quad \operatorname{Log}_{\mathbb{Q}_p} H = (\operatorname{Log}_{\mathbb{Q}_p} H)_0 \oplus \bigoplus_{\alpha \in \Phi} (\operatorname{Log}_{\mathbb{Q}_p} H)_\alpha$$

compatible with the root decomposition

$$\mathfrak{spin}_{2n+1} = \operatorname{Lie}(\widehat{T}) \oplus \bigoplus_\alpha \mathfrak{u}_\alpha$$

which is valid over Λ_n, hence also over $\Lambda_n[\frac{1}{p}]$

Let $[t_1, \ldots, t_n] = \prod_i [e_i - \frac{1}{2} f](t_i)$ denote the parametrization of (the semisimple part of) the standard torus $\mathcal{T}' = \widehat{T}$ of $\widehat{G} = \operatorname{GSpin}_{2n+1}$ via $\widehat{T} = \mathbb{G}_m \otimes X^*(T)$, where $e_i \colon (x_1, \ldots, x_n, \nu x_n^{-1}, \ldots, \nu x_1^{-1}) \mapsto x_i$ and $f \colon (x_1, \ldots, x_n, \nu x_n^{-1}, \ldots, \nu x_1^{-1}) \mapsto \nu$. For any $\sigma \in I_p$, we put

$$t(\sigma) = [(1 + T_1)^{\ell(\sigma)}, \ldots, (1 + T_n)^{\ell(\sigma)}]$$

We know that for any $\sigma \in I_p$,

$$\rho'(\sigma) \equiv t(\sigma) \pmod{U(\Lambda_n)}$$

The simple roots of $\widehat{G}$ (for the standard "upper triangular" Borel) are $\alpha_i \colon [t_1, \ldots, t_n] \mapsto t_i/t_{i+1}$ ($i = 1, \ldots, n-1$) and $\alpha_n \colon [t_1, \ldots, t_n] \mapsto t_n$. The positive roots are then t_i/t_j for $1 \leq i < j \leq n$, t_i, $i = 1, \ldots, n$, and $t_i t_j$ for $1 \leq i < j \leq n$; they can be written respectively as $\alpha_i + \ldots + \alpha_{j-1}$, $\alpha_i + \ldots + \alpha_n$, and $\alpha_i + \ldots + \alpha_{j-1} + 2(\alpha_j + \ldots + \alpha_n)$. Conjugation by $\rho(\sigma)$ acts on $\operatorname{Log}_{\mathbb{Q}_p} H_\alpha = \operatorname{Log}_{\mathbb{Q}_p} H \cap \mathfrak{u}_\alpha(\Lambda_n)$ by $\alpha(t(\sigma))$; one can choose $\sigma \in I_p$ such that $\ell(\sigma) = 1$. We find that we can let $\frac{1 + X_i}{1 + X_j}$, resp. $1 + X_i$, resp. $(1 + X_i)(1 + X_j)$ act on $(\operatorname{Log}_{\mathbb{Q}_p} H)_{t_i/t_j}$ resp. $\operatorname{Log}_{\mathbb{Q}_p} H_{t_i}$, resp. $(\operatorname{Log}_{\mathbb{Q}_p} H)_{t_i t_j}$. We have an action of $1 + X_n$ on the non-zero subgroup $(\operatorname{Log}_{\mathbb{Q}_p} H)_{\alpha_n}$ and of $\frac{1 + X_{n-1}}{1 + X_n}$ on the non-zero subgroup $(\operatorname{Log}_{\mathbb{Q}_p} H)_{\alpha_{n-1}}$; by forming the Poisson bracket of these two submodules, we find a non-zero modules of $(\operatorname{Log}_{\mathbb{Q}_p} H)_{\alpha_{n-1} + \alpha_n}$ on which both $(1 + X_n)$ and $\frac{1 + X_{n-1}}{1 + X_n}$ act ; therefore $(1 + X_n)$ and $1 + X_{n-1}$ both act on this non zero subgroup of $(\operatorname{Log}_{\mathbb{Q}_p} H)_{\alpha_{n-1} + \alpha_n}$. By repeating this procedure, we find, for $\alpha = \alpha_1 + \ldots + \alpha_n$, a non-zero submodule of $(\operatorname{Log}_{\mathbb{Q}_p} H)_\alpha$ on which all $1 + X_i$'s act. This is a non zero Λ_n-submodule of $U_\alpha(\Lambda_n)$, contained in $(\operatorname{Log} H)_\alpha$. We then use successive Poisson brackets with $(\operatorname{Log}_{\mathbb{Q}_p} H)_{-\alpha_1}, \ldots, (\operatorname{Log}_{\mathbb{Q}_p} H)_{-\alpha_j}$ to construct inside all $(\operatorname{Log}_{\mathbb{Q}_p} H)_{\alpha_i}$'s a non-trivial Λ_n-submodule of $\mathfrak{u}_{\alpha_i}(\Lambda_n)$. Once we have non-zero Λ_n-submodules in $(\operatorname{Log}_{\mathbb{Q}_p} H)_{\alpha_i}$ for all the simple roots, one can construct non-trivial Λ_n-submodules in $(\operatorname{Log}_{\mathbb{Q}_p} H)_\alpha$ for all the roots α. After multiplying by a power of p, we obtain non-trivial Λ_n-submodules in $(\operatorname{Log}_{\mathbb{Q}_p} H)_\alpha$ for all roots. Taking the exponential, we see that there exists a non zero ideal $\mathfrak{l}'$ of Λ_n such that $U_\alpha(\mathfrak{l}') \subset \operatorname{Im} \rho$. Since these subgroups generate a group containing a principal congruence subgroup of $\operatorname{Spin}_{2n+1}(\Lambda_n)$, we see that there exists a non zero ideal $\mathfrak{l}$ of Λ_n such that $\Gamma_{\Lambda_n}(\mathfrak{l}) \subset \operatorname{Im} \rho$ as desired.

$\square$

We give a lemma which follows from the proof above. For any simple root α_i of $\mathfrak{spin}_{2n+1}$, let $\overline{\mathfrak{u}}_{\alpha_i} = \mathfrak{u}_{\alpha_i}(\Lambda_n[\frac{1}{p}]) \cap \operatorname{Lie}_{\mathbb{Q}_p} \operatorname{Im} \rho_\theta$. For any characteristic zero quotient A of Λ_n, let ρ_A be the

specialization of ρ_θ to A and let $\overline{\mathfrak{u}}_{\alpha_i}^A$ be the image of $\overline{\mathfrak{u}}_{\alpha_i}$ in $\mathfrak{spin}_{2n+1}(A[\frac{1}{p}])$. Let $\overline{\mathfrak{u}}_{\alpha_\bullet}^{A,j}$ the sequence defined by $\overline{\mathfrak{u}}_{\alpha_\bullet}^{A,1} = \overline{\mathfrak{u}}_{\alpha_1}^A$, $\overline{\mathfrak{u}}_{\alpha_\bullet}^{A,2} = [\overline{\mathfrak{u}}_{\alpha_1}^A, \overline{\mathfrak{u}}_{\alpha_2}^A]$, and for $j \leq n-1$:

$$\overline{\mathfrak{u}}_{\alpha_\bullet}^{A,j+1} = [\overline{\mathfrak{u}}_{\alpha_\bullet}^{A,j}, \overline{\mathfrak{u}}_{\alpha_{j+1}}^A]$$

Note that by calculations in the proof of Theorem 4.8, $\overline{\mathfrak{u}}_{\alpha_\bullet}^{A,n}$ is an $A[\frac{1}{p}]$-module.

Lemma 4.10. *For any $n \geq 1$, assume ρ_θ is full over Λ_n. For any characteristic zero quotient A of Λ_n, there exists a unique Lie subalgebra Lie_A of the $\mathbb{Q}_p$-Lie algebra $\mathrm{Lie}_{\mathbb{Q}_p} \mathrm{Im}\, \rho_A$ which is an $A \otimes \mathbb{Q}_p$-Lie algebra, such that $\mathrm{Lie}_A \cap \overline{\mathfrak{u}}_{\sum_i \alpha_i}^A = \overline{\mathfrak{u}}_{\alpha_\bullet}^{A,n}$ and which is stable by the adjoint action of $\mathrm{Im}\, \rho_A$. If $\Lambda_n \to A \to B$ are two characteristic zero quotients of Λ_n, then $\mathrm{Lie}_A \to \mathrm{Lie}_B$ is surjective.*

Note however that for some quotients A, this algebra may be zero because $\overline{\mathfrak{u}}_{\alpha_\bullet}^{A,n}$ may be zero, although it is not the case over Λ_n. This is why we'll need a modified construction later (see Lemma 5.4).

Proof. To simplify notations, we prove it for $\mathfrak{sp}_4$. Let $\alpha_1 = t_1 t_2^{-1}$ and $\alpha_2 = t_2^2$ be the two simple roots. We form $\overline{\mathfrak{u}}_{\alpha_\bullet}^{A,2} = [\overline{\mathfrak{u}}_{\alpha_1}^A, \overline{\mathfrak{u}}_{\alpha_2}^A]$ which is an A-module by calculations at the end of the proof of Theorem 4.8. If it is non zero, by Poisson bracket of $\overline{\mathfrak{u}}_{\alpha_\bullet}^{A,2}$ with all the $\overline{\mathfrak{u}}_\gamma^A$'s for all the root (positive or negative) of $\mathfrak{sp}_4$, we construct $A[\frac{1}{p}]$-submodules $\overline{\mathfrak{u}}_{\alpha_1+\alpha_2+\gamma}^{A,\prime}$ of $\overline{\mathfrak{u}}_{\alpha_1+\alpha_2+\gamma}^A$. Note that in particular, $\overline{\mathfrak{u}}_{\alpha_1+\alpha_2}^{A,\prime} = \overline{\mathfrak{u}}_{\alpha_1+\alpha_2}^A$. We also define a Cartan subalgebra $\overline{\mathfrak{t}}^{A,\prime}$ by $[\overline{\mathfrak{u}}_{\alpha_1}^{A,\prime}, \overline{\mathfrak{u}}_{-\alpha_1}^{A,\prime}] \oplus [\overline{\mathfrak{u}}_{\alpha_2}^{A,\prime}, \overline{\mathfrak{u}}_{-\alpha_2}^{A,\prime}]$, and we then define an $A \otimes \mathbb{Q}_p$-module by

$$\mathrm{Lie}_A = \overline{\mathfrak{t}}^{A,\prime} \oplus \bigoplus_\gamma \overline{\mathfrak{u}}_\gamma^{A,\prime}$$

Using the Jacobi identity for the Poisson bracket, one sees easily that it is an $A \otimes \mathbb{Q}_p$-Lie algebra which has all the desired properties. $\square$

It follows from the construction above that if ρ_θ is full over Λ_n; then for all roots γ, $\overline{\mathfrak{u}}_\gamma'$ which is the $\gamma(\delta)$-eigenspace for the action of δ as in Theorem 4.8) on Lie_{Λ_n}, is non zero. More precisely,

Corollary 4.11. *If ρ_θ is full over Λ_n, there exists a non zero largest $\Lambda_n[\frac{1}{p}]$-ideal $\mathfrak{l}_\theta'$ such that*

$$\mathfrak{l}_\theta' \cdot \mathfrak{sp}_4(\Lambda_n \otimes \mathbb{Q}_p) \subset \mathrm{Lie}_{\Lambda_n}$$

We have $\mathfrak{l}_\theta' \subset \mathfrak{l}_\theta \otimes \mathbb{Q}_p$ and these two ideals have the same radical : $\sqrt{\mathfrak{l}_\theta'} = \sqrt{\mathfrak{l}_\theta \otimes \mathbb{Q}_p}$ in $\Lambda_n[\frac{1}{p}]$.

5. Galois level and Congruence Ideals

5.1. Congruence ideals.
Let $\theta \colon \mathbf{T}^{M,\mathrm{holcusp}} \to \mathbb{I}_0$ be an irreducible component of $\mathrm{Spec}\, \mathbf{T}^{M,\mathrm{holcusp}}$; we denote by $\mathbb{I}$ the normalization of $\mathbb{I}_0$ and by $\mathcal{K}$ its field of fractions; let $\mathbf{T}_{\mathbb{I}} = \mathbf{T}^{M,\mathrm{holcusp}} \otimes_{\Lambda_n} \mathbb{I}$ and $\theta_{\mathbb{I}} = m_{\mathbb{I}} \circ \theta \otimes \mathrm{Id}_{\mathbb{I}}$ the composition of the extension of scalars with the multiplication on $\mathbb{I}$; by the diagonalisability of $\mathbf{T}^{M,\mathrm{holcusp}}$ over $\mathcal{K}$, the character $\theta_{\mathbb{I}} \colon \mathbf{T}_{\mathbb{I}} \to \mathbb{I}$ splits over $\mathcal{K}$, and there is an isomorphism of $\mathcal{K}$-algebras :

$$\mathbf{T}_{\mathbb{I}} \otimes_{\mathbb{I}} \mathcal{K} \cong \mathcal{K} \times \mathbf{T}_{\mathcal{K}}'$$

with the first projection pr_1 given by $\theta_{\mathcal{K}} = \theta_{\mathbb{I}} \otimes \mathcal{K}$.

(Mult 1) We assume that $\theta_{\mathbb{I}}$ does not factor through pr_2. In other words, the family $\theta_{\mathbb{I}}$ occurs with multiplicity 1 in $\mathbf{T}_{\mathbb{I}}$.

If $n = 1$, this assumption amounts to saying that the family $\theta_{\mathbb{I}}$ is M-new (*i.e.* all the forms occurring in the family are M-new); if $n = 2$, using the Roberts-Schmidt theory of newforms, and assuming a folklore conjecture which says that automorphic forms on GSp_4 which are neither CAP nor endoscopic satisfy multiplicity one, one can prove that it holds for M-new families of squarefree level M. For $n \geq 3$, we simply assume (Mult 1).

Let $\mathbf{T}'$ be the image of $\mathbf{T}_{\mathbb{I}}$ by the second projection pr_2. We define then $\mathfrak{c}_1 = \mathbf{T}_{\mathbb{I}} \cap (\mathbb{I} \times \{0\})$, which is an ideal of $\mathbf{T}_{\mathbb{I}}$ and of $\mathbb{I} \times \{0\}$, and $\mathfrak{c}_2 = \mathbf{T}_{\mathbb{I}} \cap (\{0\} \times \mathbf{T}')$ which is an ideal of $\mathbf{T}_{\mathbb{I}}$ and of $\times \mathbf{T}' \times \{0\}$. Note that $\mathfrak{c}_1 = \mathrm{Ker}\, \mathrm{pr}_2$ and $\mathfrak{c}_2 = \mathrm{Ker}\, \mathrm{pr}_1$, and that for $i = 1, 2$, one can identify $\mathfrak{c}_i$ with its image $\mathrm{pr}_i(\mathfrak{c}_i)$. We also define $\mathfrak{c} = \mathfrak{c}_1 + \mathfrak{c}_2$. The projections pr_i induce isomorphisms of $\mathbb{I}$-modules

$$\overline{\mathrm{pr}}_1 \colon \mathbf{T}_{\mathbb{I}}/\mathfrak{c} \to \mathbb{I}/\mathfrak{c}_1 \text{ and } \quad \overline{\mathrm{pr}}_2 \colon \mathbf{T}_{\mathbb{I}}/\mathfrak{c} \to \mathbf{T}'/\mathfrak{c}_2$$

We thus obtain the congruence isomorphism $\overline{\mathrm{pr}}_2 \circ \overline{\mathrm{pr}}_1^{-1} : \mathbb{I}/\mathfrak{c}_1 \cong \mathbf{T}'/\mathfrak{c}_2$.

Definition 5.1. *The ideal $\mathfrak{c}_\theta = \mathfrak{c}_1$ is called the congruence ideal associated to the family θ.*

For any prime P of $\mathbb{I}$ containing $\mathfrak{c}_\theta$, there exists another family $\theta' : \mathbf{T}_\mathbb{I} \to \mathbb{J}$ (for some finite normal extension $\mathbb{J}$ of $\mathbb{I}$) and a prime Q of $\mathbb{J}$ above P such that $\theta_\mathbb{I} \cong \theta' \pmod{Q}$.

Let $\theta_2 : \mathbf{T}^{M,\mathrm{holcusp}} \to \mathbb{J}$ be a second family, different from θ, hence it factors through pr_2. Let $\mathfrak{c}_{\theta_2} \subset \mathbb{J}$ be its congruence ideal ; if Q is a prime of $\mathbb{J}$ containing both the ideals $\mathfrak{c}_\theta$ and $\mathfrak{c}_{\theta_2}$, and if $P = \mathbb{I} \cap Q$, the congruence isomorphism induces a congruence injection $\mathbb{I}/P \cong \mathbb{J}/Q$ which expresses a congruence between the two specific families θ and θ_2 modulo Q. Therefore the intersection locus $V(\mathfrak{c}_\theta + \mathfrak{c}_{\theta_2})$ in $\mathrm{Spec}\, \mathbf{T}^{M,\mathrm{holcusp}}$ of the (normalizations of the) two irreducible components $\mathrm{Spec}\, \mathbb{I}$ and $\mathrm{Spec}\, \mathbb{J}$ is also the congruence locus of the two families θ and θ_2. Note that this locus is empty unless both families factor through the same local factor of the semilocal algebra $\mathbf{T}^{M,\mathrm{holcusp}}$; in other words a necessary condition for this locus to be non empty is that the residual representations $\overline{\rho}_\theta$ and $\overline{\rho}_{\theta_2}$ coincide.

5.2. **Comparison between Galois level and congruence ideals.** In this section we assume again that the family $\theta : \mathbf{T}^{M,\mathrm{holcusp}} \to \mathbb{I}$ is Λ_n-valued, *i.e.* $\mathbb{I} = \Lambda_n$. We also assume that its residual Galois representation $\overline{\rho}_\theta$ is irreducible and that $\rho = \rho_\theta$ satisfies the assumptions of Theorem 4.8 : there is an arithmetic specialization for which $\mathrm{Im}\, \rho_{P_{k^0}}$ is full and that there is a $\mathbb{Z}_p$-regular element δ in $\mathrm{Im}\, \rho$. By Theorem 4.8, the representation ρ_θ is full; we denote by $\mathfrak{l}_\theta$ the Galois level of θ, that is, the largest ideal of Λ_n such that $\Gamma_{\Lambda_n}(\mathfrak{l}_\theta) \subset \mathrm{Im}\, \rho_\theta$.

Let us assume that there exists another family θ_2 in the same local component of $\mathbf{T}^{M,\mathrm{holcusp}}$ as θ (so that its residual representation coincides with that of θ, in particular it is irreducible) but such that ρ_{θ_2} is not full. Let $\mathfrak{c}(\theta, \theta_2) = \mathfrak{c}_\theta \cdot \mathbb{J} + \mathfrak{c}_{\theta_2}$ be the ideal of $\mathbb{J}$ defining the congruence locus between θ and θ_2 and let $c(\theta) = \bigcap_{\theta_2} \mathfrak{c}(\theta, \theta_2) \cap \Lambda_n$, where the intersection of the $\mathfrak{c}(\theta, \theta_2)$'s is taken over all the families θ_2 congruent to θ but not full. Let Q be a prime ideal of $\mathbb{J}$ containing $\mathfrak{c}(\theta, \theta_2)$, and let $P = Q \cap \Lambda_n$. Then, we have a congruence $\rho_\theta \equiv \rho_{\theta_2} \pmod{Q}$; then $\rho_\theta \pmod{P}$ is not full ; hence we must have $\mathfrak{l}_\theta \subset Q$. Since for any prime P of Λ_n containing $c(\theta)$ there exists a prime Q above P in a finite extension which contains $\bigcap_{\theta_2} \mathfrak{c}(\theta, \theta_2)$, we see that any prime ideal P containing $c(\theta)$ contains $\mathfrak{l}_\theta$. For any ideal $\mathfrak{a}$ of Λ_n, let $V(\mathfrak{a})$ be the set of prime ideals containing $\mathfrak{a}$. Recall that in the case $n = 1$, Hida has proved in [H13b] that if the family θ is not CM but there is another family θ_2 which is CM for a given imaginary quadratic field, then the sets $V(\mathfrak{l}_\theta)$ and $V(c(\theta))$ coincide; actually under slightly stronger assumptions (see [H13b, Th.7.2]), he could use Pink's Lie algebra theory for SL_2 to compare the exponents of the height one primes occurring in (the reflexive envelopes of) the two ideals $\mathfrak{l}_\theta$ and $c(\theta)$, see [H13b, Th.8.5.].

For general n, a natural question is to ask whether the converse is true, namely that if a prime P divides $\mathfrak{l}_\theta$, there exists a non full family θ_2 congruent to θ at a prime Q above P. Assume again $n = 2$, recall that given a real quadratic field F of discriminant D, and a Hilbert modular cusp form for F of level $\mathfrak{n}$ and weight (m_1, m_σ) with $m_1 = m_\sigma + 2r$ with $r \geq 1$ and $m_\sigma \geq 2$, eigen for the Hecke operators and p-ordinary, one can define its theta lift to GSp_4 (see [Y79] and [R01, Th.8.6]), which we call its twisted Yoshida lift. It gives rise to an eigensystem of $\mathbf{T}^{M,\mathrm{holcusp}}$ with $M = N(\mathfrak{n})D^2$ of weight (k_1, k_2) with $k_1 = m_\sigma + r$ and $k_2 = r + 2$, (so, $k_1 \geq k_2 \geq 3$), see for instance [MT02, Section 7.3]. Assuming moreover that $\mathfrak{n}$ is prime to p and that g is p-ordinary, there exists a unique 2-variable p-adic family of p-ordinary Hilbert modular forms $\mathbf{g}$ of auxiliary level $\mathfrak{n}$ passing through g (after p-stabilization). It defines a homomorphism of Λ_2-algebras $\theta_\mathbf{g} : \mathbf{T}_F^\mathfrak{n} \to \mathbb{J}$ of the ordinary Hida Hecke algebra $\mathbf{T}_F^\mathfrak{n}$ for Hilbert cusp forms over F, of auxiliary level $\mathfrak{n}$. For any place w of F prime to $\mathfrak{n}p$, this homomorphism sends the Hecke operator T_w to the eigenvalue $\theta_\mathbf{g}(T_w)$ on $\mathbf{g}$. The map $(m_1, m_\sigma) \mapsto (k_1, k_2)$ defines an automorphism of Λ_2, denoted by τ.

Lemma 5.2. *Given $(F, \mathbf{g})$ of a real quadratic field and a 2-variable family of p-ordinary cusp forms $\theta_\mathbf{g} : \mathbf{T}_F^\mathfrak{n} \to \mathbb{J}$, then, for $M = N(\mathfrak{n})D^2$, there exists a 2-variable p-adic family $\theta_2 : \mathbf{T}^{M,\mathrm{holcusp}} \to \mathbb{J}^\tau$, which is a homomorphism of Λ_2-algebras via $\tau : \theta_2(\alpha \cdot T) = \tau(\alpha) \cdot \theta_2(T)$, such that for every prime q not dividing Mp,*

$$\theta_2(T_{q,1}) = \lambda_q(\mathbf{g}) \,\mathrm{and}\quad \theta_2(T_{q,2}) = \mu_q(\mathbf{g})$$

where the Hecke eigenvalues $\lambda_q(\mathbf{g})$ and $\mu_q(\mathbf{g})$ are defined as follows

- If q splits in F, let w_1 and w_2 the places above q, then $\lambda_q(\mathbf{g}) = \theta_{\mathbf{g}}(T_{w_1}) + \theta_{\mathbf{g}}(T_{w_2})$ and $q \cdot \mu_q(\mathbf{g}) = q^2 + q\theta_{\mathbf{g}}(T_{w_1})\theta_{\mathbf{g}}(T_{w_2}) - 1$
- If q is inert, $\lambda_q(\mathbf{g}) = 0$ and $q \cdot \mu_q(\mathbf{g}) = -(q^2 + q\theta_{\mathbf{g}}(T_w) + 1)$

We call $\theta_2 = \theta_2(F, \mathbf{g})$ the twisted Yoshida lift of $(F, \mathbf{g})$

The proof is by comparing for each prime q not dividing Mp the characteristic polynomial of $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \rho_h$ for all classical Hilbert forms h in the family $\mathbf{g}$ and the GSp_4-Hecke polynomial at q of its Yoshida lift.

From now on, we fix a real quadratic field F of discriminant D, an auxiliary level $\mathfrak{n}$ in F prime to p and we pose $M = N(\mathfrak{n})D^2$; we give ourselves a Λ_2-valued family $\theta \colon \mathbf{T}^{M,\mathrm{holcusp}} \to \Lambda_2$ which satisfies the assumptions of §4.3, so that its Galois representation $\rho_\theta \colon G_{\mathbb{Q}} \to \mathrm{GSp}_4(\Lambda_2)$ is full.

In the following theorem, we assume there exists a Hilbert modular form g of weight $m_1 = m_\sigma + 2r$ with $r \geq 1$ (hence the form g does not descend to $\mathbb{Q}$) and $m_\sigma \geq 2$, such that

- $\overline{\rho}_g$ is "full" : that is, $\mathrm{SL}_2(\mathbb{F}_p) \subset \mathrm{Im}\,\overline{\rho}_g$,
- there exists no character $\xi \colon G_{\mathbb{Q}} \to \overline{\mathbb{F}}_p^{\times}$ such that $\overline{\rho}_{g^\sigma} \cong \overline{\rho}_g \otimes \xi$

We finally assume that $\overline{\rho}_\theta = \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \overline{\rho}_g$. By the previous lemma, there exists a "twisted Yoshida lift" family $\theta_2 = \theta_2(F, \mathbf{g})$ in the same local component of $\mathbf{T}^{M,\mathrm{holcusp}}$ as θ. Note that if there is another quadratic field F' and a Hilbert form g' on F' such that $\overline{\rho}_\theta = \mathrm{Ind}_{G_{F'}}^{G_{\mathbb{Q}}} \overline{\rho}_{g'}$ then $F' = F$. Indeed, if this equality holds and $F' \neq F$, we have an isomorphism of $G_{\mathbb{Q}}$-modules $\overline{\rho}_\theta \otimes \xi \cong \overline{\rho}_\theta$ where ξ is the (non trivial) quadratic character associated to F'; hence, by irreducibility of the G_F-modules $\overline{\rho}_g$ and $\overline{\rho}_{g^\sigma}$, we have an isomorphism of G_F-modules $\overline{\rho}_g \otimes \xi \cong \overline{\rho}_g$ (which implies that $\overline{\rho}_g \cong \mathrm{Ind}_{FF'}^{F} \varphi$ for a character φ and is impossible by fullness of $\overline{\rho}_g$) or $\overline{\rho}_g \otimes \xi \cong \overline{\rho}_{g^\sigma}$ which is impossible by assumption.

Theorem 5.3. For $n = 2$, assume that ρ_θ satisfies the assumptions of Theorem 4.8, hence is full, and that there is a Hilbert cusp form g for F as above such that $\overline{\rho}_\theta = \mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \overline{\rho}_g$, then we have $V(c(\theta)[\frac{1}{p}]) = V(\mathfrak{l}_\theta[\frac{1}{p}])$.

Note that we are not yet able to study the prime (p) ; more precisely, we can't prove that if $(p) \in V(\mathfrak{l}_\theta)$, then $(p) \in V(c(\theta))$. We need first a variant of 4.10 and 4.11, because in our case 4.10 applied to the quotient A of interest for us yields a trivial $A \otimes \mathbb{Q}_p$-Lie algebra. Since ρ_θ is congruent to $\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \rho_g$ and since its image contains the diagonal element δ with distinct eigenvalues, we see that the diedral group generated by δ and by the element w of the Weyl group which exchanges t_1 and t_2 in $\mathrm{diag}(t_1, t_2, t_2^{-1}, t_1^{-1})$ lifts to characteristic zero as a subgroup of $\mathrm{Im}\,\rho_\theta$ (by a theorem of P. Hall, as the kernel of $\mathrm{Im}\,\rho_\theta \to \mathrm{Im}\,\mathrm{Ind}_{G_F}^{G_{\mathbb{Q}}} \rho_g$ is pro-p); hence $\mathrm{Im}\,\rho_\theta$ contains the element w of the Weyl group which exchanges t_1 and t_2. Let $\alpha_1 = t_1 t_2^{-1}$ and $\alpha_2 = t_2^2$ be the two simple roots. We recall the notations $\overline{\mathfrak{u}}_\gamma^A = \mathfrak{u}_\gamma(A) \cap \mathrm{Lie}_{\mathbb{Q}_p} \mathrm{Im}\,\rho_A$ for all roots γ's.

Lemma 5.4. For any characteristic zero quotient A of Λ_2, there exists a Lie subalgebra Lie_A of the $\mathbb{Q}_p$-Lie algebra $\mathrm{Lie}_{\mathbb{Q}_p} \mathrm{Im}\,\rho_A$ which is an $A \otimes \mathbb{Q}_p$-Lie algebra, which contains $\overline{\mathfrak{u}}_{\alpha_2}^A$ and which is stable by the adjoint action of $\mathrm{Im}\,\rho_A$. If $\Lambda_2 \to A \to B$ are two characteristic zero quotients of Λ_2, then $\mathrm{Lie}_A \to \mathrm{Lie}_B$ is surjective.

Proof. By action of the inertia as in the proof of Theorem 4.8, we see that $\overline{\mathfrak{u}}_{\alpha_2}$, resp. $\overline{\mathfrak{u}}_{2\alpha_1+\alpha_2}$, carries an action of $(1+T_1)(1+T_2)^{-1}$, resp. of $(1+T_1)(1+T_2)$; moreover $\overline{\mathfrak{u}}_{\alpha_2} = w(\overline{\mathfrak{u}}_{2\alpha_1+\alpha_2})$ hence $\overline{\mathfrak{u}}_{\alpha_2}$ carries a structure of A-module by transport of structure. We construct Lie_A as the Lie algebra generated by $\overline{\mathfrak{u}}_{\alpha_2}^A$, $\overline{\mathfrak{u}}_{2\alpha_1+\alpha_2}^A$ and their Poisson brackets by the $\overline{\mathfrak{u}}_{\alpha_2}^A$'s for all the root (positive or negative) of $\mathfrak{sp}_4$. This defines the desired $A \otimes \mathbb{Q}_p$-Lie algebra. $\square$

Again, it follows from the construction above that if ρ_θ is full over Λ_2; then for all roots γ, the $\gamma(\delta)$-eigenspace for the action of δ as in Theorem 4.8) on Lie_{Λ_2}, is non zero. More precisely,

Corollary 5.5. If ρ_θ is full over Λ_2, there exists a non zero largest $\Lambda_n[\frac{1}{p}]$-ideal $\mathfrak{l}'_\theta$ such that

$$\mathfrak{l}'_\theta \cdot \mathfrak{sp}_4(\Lambda_n[\frac{1}{p}]) \subset \mathrm{Lie}_{\Lambda_2}$$

We have $\mathfrak{l}'_\theta \subset \mathfrak{l}_\theta \otimes \mathbb{Q}_p$ and these two ideals have the same radical : $\sqrt{\mathfrak{l}'_\theta} = \sqrt{\mathfrak{l}_\theta \otimes \mathbb{Q}_p}$.

Proof. Simply review the construction of Lie_{Λ_2} starting from $\mathfrak{u}_\gamma(\mathfrak{l}_\theta) \subset \overline{\mathfrak{u}}_\gamma$ for all γ's, and take Poisson brackets. $\qquad\square$

Proof of the theorem. Let us prove that if P is a characteristic zero prime containing $\mathfrak{l}_\theta$, then it contains $c(\theta)$. We can assume that P is an isolated component in the primary decomposition of the ideal $\mathfrak{l}'_\theta$ defined in the previous corollary. Let $\mathfrak{q}$ be the primary component of $\mathfrak{l}'_\theta$ whose radical is P. Let Λ_P be the localization of Λ_2 at P, $\kappa_P = \mathrm{Frac}(\Lambda_2/P)$ its residue field. Let $\widetilde{\Lambda} = \Lambda_2/\mathfrak{q}$ and $\widetilde{\Lambda}_P$ its localization at P; it is an artinian Λ_P-algebra with residue field κ_P.

Let $\widetilde{\rho}$ be the reduction of ρ_θ modulo $\mathfrak{q}$; it takes values in $G(\widetilde{\Lambda})$. We consider an ideal $\mathfrak{a}$ of Λ_2 containing $\mathfrak{q}$ and $\widetilde{\mathfrak{a}} = \mathfrak{a}/\mathfrak{q}$, such that $\widetilde{\mathfrak{a}}_P$ is one-dimensional over κ_p. Consider the $\widetilde{\Lambda} \otimes \mathbb{Q}_p$-Lie algebra $\mathrm{Lie}_{\widetilde{\Lambda}}$ of Lemma 5.4 in $\mathfrak{gsp}_4(\widetilde{\Lambda})$ and the Lie subalgebra $\mathfrak{s} = \widetilde{\mathfrak{a}} \cdot \mathfrak{sp}_4(\widetilde{\Lambda}) \cap \mathrm{Lie}_{\widetilde{\Lambda}}$ inside the Lie algebra $\widetilde{\mathfrak{a}} \cdot \mathfrak{sp}_4(\widetilde{\Lambda})$, after localization at P, the latter is isomorphic to $\mathfrak{sp}_4(\kappa_P)$. So we can identify $\mathfrak{s}_P$ to a κ_P-Lie subalgebra of $\mathfrak{sp}_4(\kappa_P)$. Let us prove that $\mathfrak{s}_P \neq \mathfrak{sp}_4(\kappa_P)$. Indeed if equality holds, we have $\widetilde{\mathfrak{a}} \cdot \mathfrak{sp}_4(\widetilde{\Lambda}) \subset \mathrm{Lie}_{\widetilde{\Lambda}}$. hence $\mathfrak{a} \cdot \mathfrak{sp}_4(\Lambda_2) \subset \mathrm{Lie}_{\Lambda_2} + \mathfrak{q} \cdot \mathfrak{sp}_4(\Lambda_2)$ so after localizing at P, noting that $\mathfrak{l}'_{\theta,P} = \mathfrak{q}_P$ and that $\mathfrak{l}'_\theta \cdot \mathfrak{sp}_4(\Lambda_2) \subset \mathrm{Lie}_{\Lambda_2}$, we have $\mathfrak{a}_P \cdot \mathfrak{sp}_4(\Lambda_P) \subset \mathrm{Lie}_{\Lambda_P}$. Consider the ideal $\mathfrak{b} = \mathfrak{l}'_\theta + \mathfrak{a}$, it is strictly larger than $\mathfrak{l}'_\theta$ and we have $\mathfrak{b} \cdot \mathfrak{sp}_4(\Lambda_2) \subset \mathrm{Lie}_{\Lambda_2}$; this contradicts the definition of $\mathfrak{l}'_\theta$.

In conclusion, $\mathfrak{s}_P$ is a strict κ_P-Lie subalgebra of $\mathfrak{sp}_4(\kappa_P)$. It is semisimple because $\widetilde{\rho}$ is irreducible. Hence it is either 0, or an $\mathfrak{sl}_2$ in the Levi of a maximal parabolic, or $\mathrm{Sym}^3 \mathfrak{sl}_2$, or the endoscopic $\mathfrak{sl}_2 \times \mathfrak{sl}_2$. Since it is normalized by $\mathrm{Im}\,\rho_P$ (recall ρ_P is the representation over Λ_2/P), $\mathrm{Im}\,\rho_P$ is contained in the corresponding normalizer. However the second and third cases are impossible. Indeed, for any root γ occurring in $\mathfrak{sl}_2 \times \mathfrak{sl}_2$, one can show using the projection Δ_γ as Lemma 4.5 and Lemma 4.7, that $U_\gamma(\widetilde{\Lambda}) \cap \mathrm{Im}\,\widetilde{\rho}$ maps surjectively to $U_\gamma(\mathbb{F}_p) \cap \mathrm{Im}\,\overline{\rho}$. Hence by the assumption of fullness of $\overline{\rho}_g$ and $\overline{\rho}_{g^\sigma}$, these groups are non trivial. For instance for $\gamma = \alpha_2$, by taking a power of p and taking the Log of a non trivial element, we find a non trivial element of $\overline{\mathfrak{u}}_{\alpha_2}^{\widetilde{\Lambda}} \subset \mathrm{Lie}_{\widetilde{\Lambda}}$, since both sides are $\widetilde{\Lambda}$-modules, one can assume that this non zero element is in $\overline{\mathfrak{u}}_{\alpha_2}^{\widetilde{\Lambda}}(\widetilde{\mathfrak{a}}) \subset \mathrm{Lie}_{\widetilde{\Lambda}}$, hence $\mathfrak{s}_P \neq 0$. The only remaining case is the endoscopic $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ and $\mathrm{Im}\,\rho_P \subset N(\mathrm{GL}_2 \times \mathrm{GL}_2)^\circ)$. Comparing with the reduction modulo the maximal ideal, we find a two-dimensional Galois representation of G_F over Λ_2/P congruent to $\overline{g}$ modulo the maximal ideal of Λ_2/P.

From this, one can deduce, using an $R_F = \mathbf{T}_F$ of Skinner-Wiles [SW01], that there exists a Hilbert family $\mathbf{g}$ such that the family $\theta_2 \colon \mathbf{T}^{M,\mathrm{holcusp}} \to \mathbb{J}$ associated to $(F, \mathbf{g})$ which is congruent to θ modulo a prime Q of $\mathbb{J}$ above P. $\qquad\square$

Questions : 1) By this theorem, we can ask more precisely about the primary ideals (for minimal primes) occurring in $c(\theta)[\frac{1}{p}]$ and $\mathfrak{l}_\theta[\frac{1}{p}]$. Are they equal?

2) Note that for a Hilbert form g on a real quadratic field F, the L function of $\mathrm{Ad}\,\mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \rho_g$ can be decomposed as

$$L(\mathrm{Ad}\,\mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \rho_g, s) = L(\mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \mathrm{Ad}\rho_g, s)L(R_g, s)$$

where R_g is the extension to $G_\mathbb{Q}$ of the representation $\rho_g \otimes \rho_{g^\sigma}^\vee$ of G_F appearing as a factor of $\mathrm{Ad}\,\mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \rho_g$.

By a theorem of [Di05], the normalized special value at 1 of the first factor control congruences between the Hilbert form g and other Hilbert forms, hence a similar normalized special value at 1 of the second factor should control the congruences between the Yoshida lift of g and other Siegel forms which are not theta lifts of Hilbert forms. The p-adic L function interpolating the values of the first factor for g varying in a Hida family $\mathbf{g}$ has been defined by Hida. It might be possible to define the p-adic L function interpolating the second factor for g varying in a Hida family. Then one could ask how it is related to the determination of $\mathfrak{l}_\theta$ for a family θ congruent to $\Theta(\mathbf{g})$.

3) More generally, we hope to treat other cases of congruences between a "full" family and other families of automorphic forms with "small but irreducible Galois image", for $\mathrm{GSp}(4)$ and other groups.

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