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AUTOMATIC CONTINUITY FOR ISOMETRY GROUPS

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ABSTRACT. We present a general framework for automatic continuity results for groups of isometries of metric spaces. In particular, we prove automatic continuity property for the groups of isometries of the Urysohn space and the Urysohn sphere, i.e. that any homomorphism from either of these groups into a separable group is continuous. This answers a question of Melleray. As a consequence, we get that the group of isometries of the Urysohn space has unique Polish group topology and the group of isometries of the Urysohn sphere has unique separable group topology. Moreover, as an application of our framework we obtain new proofs of the automatic continuity property for the group $\mathrm{Aut}([0,1],\lambda)$, due to Ben Yaacov, Berenstein and Melleray and for the unitary group of the infinite-dimensional separable Hilbert space, due to Tsankov. The results and proofs are stated in the language of model theory for metric structures.

1. INTRODUCTION

It is well known that every group is isomorphic to the group of isometries of a metric space (or even of a graph). Moreover, if G is the group of isometries of a metric space X, then G carries the topology of pointwise convergence on X. If the space X is separable and its metric is complete, then G is separable and completely metrizable (i.e. *Polish*). In fact, the coverse is also true and any Polish group is isomorphic to the group of isometries of a separable complete metric space [14, Theorem 3.1]. Note that if a group is isomorphic to the group of isometries of a space X, then the structure of this group is completely determined by the metric properties of X. In this paper, we exploit this observation to study the structure of various groups of isometries.

Automatic continuity is a phenomenon that connects the algebraic and topological structures and typically says that any map which preserves an algebraic structure must automatically be continuous. One of the first instances of this phenomenon appears in C*-algebras and Banach algebras, where it is known that any homomorphism from an abelian Banach algebra

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into $\mathbb{C}$ is continuous. More nontrivial results concern continuity of derivations on C*-algebras. Sakai [46] (proving a conjecture of Kaplansky [26]) showed that any derivation on a C*-algebra is norm-continuous. This was generalized first by Kadison [23] who improved it to the continuity in the ultraweak topology and then by Ringrose [41] for derivations of C*-algebras into Banach modules. Johnson and Sinclair [22] on the other hand, showed automatic continuity for derivations on semi-simple Banach algebras. A detailed account on this subject can be found in [7, 8].

In the context of groups and their homomorphisms, one of the first automatic continuity results has been proved by Dudley [9], who showed that any homomorphism from a complete metric or a locally compact group into a normed (e.g. free) group is continuous (see also [47] for a recent generalization to homomorphisms into free products). A general form of automatic continuity phenomenon for groups has appeared in the work of Kechris and Rosendal [30], with connection to the results of Hodges, Hodkinson, Lascar and Shelah [19].

A topological group G has the *automatic continuity property* if for every separable topological group H (or, equivalently, any Polish group H), any group homomorphism from G to H is continuous. Recall [27, Theorem 9.10] that any measurable homomorphism from a Polish group to a separable group must be continuous and the existence of non-measurable homomorphisms on groups such as $(\mathbb{R}, +)$ can be derived from the axiom of choice. So, similarly as amenability, automatic continuity property for a given group can be interpreted in terms of nonexistence (on this group) of pathological phenomena that can follow from the axiom of choice.

Kechris and Rosendal [30] showed that automatic continuity is a consequence of the existence of comeager orbits in the diagonal conjugacy actions of G on G^n for each $n \in \mathbb{N}$ (i.e. *ample generics*, cf [30, Section 1.6]) and discovered a connection between ample generics for the automorphism groups of homogeneous structures and the Fraïssé theory. In consequence, many automorphism groups of homogeneous structures turned out to have the automatic continuity property. However, automatic continuity can hold also for groups which do not have ample generics (and even have meager conjugacy classes). Rosendal and Solecki [45] proved that automatic continuity property holds for the groups of homeomorphisms of the Cantor space and of the real line, and for the automorphism group of $(\mathbb{Q}, <)$. Rosendal [42] showed automatic continuity for the groups of homeomorphisms of compact 2-manifolds. A survey on recent results in this area can be found in [43].

The Urysohn space $\mathbb{U}$ is the separable complete metric space which is *homogeneous* (i.e. any finite partial isometry of $\mathbb{U}$ extends to an isometry of $\mathbb{U}$) and such that any finite metric space embeds into $\mathbb{U}$ isometrically. It is known that these properties of $\mathbb{U}$ determine it uniquely up to isometry and that any separable metric space embeds into $\mathbb{U}$ [38, Theorem 5.1.29]. The analogue of the Urysohn space of diameter 1 also exists and is called the *Urysohn sphere* (or the *bounded Urysohn space of diameter 1*) and denoted by $\mathbb{U}_1$ (see [38, Remark 5.1.31]). The group of isometries of $\mathbb{U}$ is universal among Polish groups, i.e. any Polish group is its closed subgroup [13, Theorem 2.5.2]. The Urysohn space, as well as its group of isometries, have received a considerable amount of attention recently. Tent and Ziegler [52] showed that the quotient of the group of isometries of $\mathbb{U}$ modulo the normal subgroup of bounded isometries is a simple group and recently [51] proved that the group of isometries of the bounded Urysohn space is simple. For more on the structure of the Urysohn space and its group of isometries, see the recent monographs [38, 14, 13] on this subject.

Kechris and Rosendal showed that the group of automorphisms of the *rational Urysohn space* (which is the Fraïssé limit of the class of finite metric spaces with rational distances) has ample generics and deduced from it that the group $\mathrm{Iso}(\mathbb{U})$ has a dense conjugacy class [31, Theorem 2.2]. The question whether the group of isometries of the Urysohn space has the automatic continuity property has been asked by Melleray (cf. [5, Section 6.1]). One of the main applications of the results of this paper is the following.

Theorem 1.1. *The groups of isometries of the Urysohn space and the Urysohn sphere have the automatic continuity property.*

Theorem 1.1 has some immediate consequences on the topological structure of the above groups, in spirit of the results of Kallman [24, 25] and Atim and Kallman [2]. The first one is an abstract consequence of automatic continuity

Corollary 1.2. *The group* $\mathrm{Iso}(\mathbb{U})$ *has unique Polish group topology.*

Recall that a group is *minimal* if it does not admit any strictly corser (Hausdorff) group topology (in this paper we consider only Hausdorff topologies on groups). The second corollary follows from minimality of the group of isometries of the Urysohn sphere, proved by Uspenskij [54]

Corollary 1.3. *The group* $\mathrm{Iso}(\mathbb{U}_1)$ *has unique separable group topology.*

Theorem 1.1 will follow from the following abstract result, which isolates metric (or model-theoretic) properties of a metric structure that imply that the group of automorphisms (with the pointwise convergence topology) of the structure has the automatic continuity property. The definitions of a metric structure and the notions appearing in the statement of the theorem are given in Sections 2 and 3.

Theorem 1.4. *Suppose M is a homogeneous complete metric structure that has locally finite automorphisms, the extension property and admits weakly isolated sequences. Then the group* $\mathrm{Aut}(M)$ *has the automatic continuity property.*

Theorem 1.4 can be also applied to give a unified treatment of previously known automatic continuity results for automorphism groups of some metric structures. It is worth mentioning that up to now, these results have been proved with different methods, varying from case to case. In this paper, we apply Theorem 1.4 to show the automatic continuity property for the group $\mathrm{Aut}([0,1],\lambda)$ (the group of measure-preserving automorphism of the unit interval) and the group $U(\ell_2)$ (unitary operators of the infinite-dimensional separable Hilbert space). Let us also mention that theorem 1.4 trivially covers the automatic continuity results for groups of automorphisms of countable (discrete) structures considered in [31]. This is because in a discrete structure any sequence is weakly isolated and the conjunction of the extension property and locally finite automorphisms implies the existence of ample generics in the automorphism group of that structure

Automatic continuity for the group $\mathrm{Aut}([0,1],\lambda)$ has been proved in a series of two papers [32, 5]. Kittrell and Tsankov [32] showed first that any homomorphism from $\mathrm{Aut}([0,1],\lambda)$ to a separable group must be continuous in the strong topology of $\mathrm{Aut}([0,1],\lambda)$ (see [28, Section 1]) and later, Ben Yaacov, Berenstein and Melleray [5] proved a general result which implies that any homomorphism which is continuous in the strong topology on $\mathrm{Aut}([0,1],\lambda)$ must be continuous in the weak topology on $\mathrm{Aut}([0,1],\lambda)$ (this approach has been recently simplified by Malicki [34]). The group $\mathrm{Aut}([0,1],\lambda)$ (with the weak topology) is isomorphic to the group of automorphism of the measure algebra and applying Theorem 1.4 to the measure algebra we get a new proof of automatic continuity.

Corollary 1.5 (Ben Yaacov, Berenstein, Melleray)**.** *The group* $\mathrm{Aut}([0,1],\lambda)$ *has the automatic continuity property.*

After the work of Ben Yaacov, Berenstein and Melleray [5], Tsankov [53] further showed the automatic continuity property for the infinite-dimensional unitary group. Given that the group $U(\ell_2)$ is the automorphism group of the Hilbert space ℓ_2 (or the isometry group of the sphere in ℓ_2), we can apply Theorem 1.4 to the Hilbert space and get a new proof of this result.

Corollary 1.6 (Tsankov)**.** *The group* $U(\ell_2)$ *has the automatic continuity property.*

Our proof of Theorem 1.4 builds on the work of Kechris and Rosendal [31] and Rosendal and Solecki [45]. In particular, we use the notion of ample generics introduced in [31] and exploit some ideas of [45, Section 3]. The verification of conditions of Theorem 1.4 in the case of the Urysohn space uses a result of Solecki [48] that is based on earlier results of Herwig and Lascar [17]. These result in turn, are connected to a theorem of Ribes and Zalesskiĭ [40], who showed separability of products of finitely-generated subgroups of the free groups (this was later generalized by Minasyan [36] to hyperbolic groups). In Section 8 we present a new proof of Solecki's theorem [48] in the style of Mackey's construction of induced actions and based on the separability result of Ribes and Zalesskiĭ.

This paper is organized as follows. In Sections 2 and 3 we give an overview of model-theoretic notions that appear in the statement of Theorem 1.4. In

Sections 4 and 5 we prove a weak version of ample generics for the automorphism groups of metric structures and Section 6 includes a proof of Theorem 1.4. Section 7 contains a further result on triviality of homomorphisms to groups admitting complete left-invariant metrics. In Section 8 we verify that the assumptions of Theorem 1.4 are satisfied by the Urysohn space, which proves Theorem 1.1. Sections 9 and 10 contain discussion of the cases of the measure algebra and the Hilbert space and proofs of Corollaries 1.5 and 1.6.

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2. First-order continuous model theory

The techniques used in this paper are motivated by the framework and language of model theory for metric structures, developed recently by Ben Yaacov, Berenstein, Henson, Usvyatsov [4] and others. There are, however, some details, that will vary from the original setting. In this paper, a *metric structure* is a tuple $(X, d_X, f_1, f_2, \ldots)$ where X equipped with d_X is a separable metric space and $f_1, f_2, \ldots$ are either closed subsets of $X^n \times \mathbb{R}^m$ (*relations*), for some $n, m \in \mathbb{N}$, or continuous functions, from $X^n \times \mathbb{R}^m$ to $X^k \times \mathbb{R}^l$ for some $n, m, k, l \in \mathbb{N}$ (here we consider the discrete topology on $\mathbb{R}$, i.e. demand continuity only on the arguments from X). Thus, a metric structure is a two-sorted structure with the second sort being a subset of the real line $\mathbb{R}$. Let us note here that in the examples considered in this paper, the structures will contain no relational symbols (only functions) and we allow them in the definition only for sake of generality.

We do not require our metric structures to be complete (as metric spaces) and we say that a metric structure X is *complete* if it is complete with respect to d_X.

Given a metric structure M we write $\mathrm{Aut}(M)$ for the group of automorphisms of M (i.e. bijections of the first sort of M which preserve both the metric and each f_n). $\mathrm{Aut}(M)$ is always endowed with the topology of pointwise convergence and if M is complete, then $\mathrm{Aut}(M)$ is a Polish group. Say that a metric structure X is *homogeneous* if every partial isomorphism between finitely generated substructures of X extends to an automorphism of X. Note that in case X is a metric space, this coicides with the usual notion of homogeneity (sometimes also referred to as *ultrahomogeneity*).

The main difference between continuous logic and our approach lies the syntax. We will consider only a first-order variant of the language. *Terms* are either variables, elements of a structure (of the first sort or the second sort), or expressions of the form $f(\tau_1, \ldots, \tau_n)$, where $\tau_1, \ldots, \tau_n$ are terms and f is a function symbol (e.g. the symbol for the distance function) of appropriate kind (where the numbers and sorts of the variables are correct). The *first-order formulas* in our language will be of the form

- $\tau = \sigma$, or $R(\tau_1, \ldots, \tau_n)$, where $\tau, \sigma, \tau_1, \ldots, \tau_n$ are terms and R is a relational symbol,
- if φ and ψ are first-order formulas and x is a variable of the first sort, then $\exists x\varphi$, $\forall x\varphi$, $\neg\varphi$, $\varphi \vee \psi$ are first-order formulas as well (quantification is only allowed over the first sort).

As usual, a *first-order sentence* is a first-order formula without free variables. The truth value of a first-order sentence in a metric structure is defined as in the classical setting (it is either 0 or 1). We will use the symbol $\prec$ for an elementary substructure, in the following (classical) sense: given a metric structure X and its substructure $Y \subseteq X$ we write $Y \prec X$ if for every first order sentence σ with parameters in Y, if σ is true in Y, then σ is true in X.

Given a metric structure X and a tuple $\bar{a} = (a_1, \ldots, a_m)$ of elements of X, a *quantifier-free type over* $\bar{a}$ is a set of quantifier-free formulas $\varphi(\bar{x}, \bar{a}, \bar{r})$ with parameters $\bar{r}$ from the second sort, for a fixed sequence of variables $\bar{x} = (x_1, \ldots, x_n)$ of the first sort. A *quantifier-free type* is a quantifier-free type over the empty tuple. Quantifier-free types will be denoted by $p(\bar{x})$ (to indicate the variables), or simply p. If $\bar{x} = (x_1, \ldots, x_n)$, then we also say that $p(\bar{x})$ is a *quantifier-free n-type over $\bar{a}$.*

We say that an n-tuple $\bar{b} = (b_1, \ldots, b_n)$ of elements of a metric structure X *realizes* a given quantifier-free n-type p over $\bar{a}$ (write $\bar{b} \models p$) if $X \models \varphi(\bar{b}, \bar{a})$ for every $\varphi(\bar{x}, \bar{a}) \in p$ (the definition of satisfaction in a model is the natural one). Given $Y \subseteq X$ with $\bar{a} \in Y^n$ and a quantifier-free type n-type p over $\bar{a}$ we say that p *is realized in* Y if there is $\bar{b} \in Y^n$ such that $\bar{b} \models p$. Also, abusing notation, if $Y \subseteq X^k$, then we say that p *is realized in* Y if if there is $\bar{b} \in Y$ such that $\bar{b} \models p$. The *quantifier-free type of a tuple* $\bar{b}$ over $\bar{a}$, denoted by $\mathrm{qftp}(\bar{b}/\bar{a})$ is then the set of all quantifier-free formulas $\varphi(\bar{x}, \bar{a})$ such that $X \models \varphi(\bar{b}, \bar{a})$. If $\bar{a}$ is the empty tuple, then we write $\mathrm{qftp}(\bar{b})$ for $\mathrm{qftp}(\bar{b}/\bar{a})$. A quantifier-free type is *consistent* if there is a model that realizes it, and a consistent quantifier-free type p is *complete* if whenever $p \subseteq q$ and q is a consistent quantifer-free type, then $p = q$.

Definition 2.1. Given $n \in \mathbb{N}$, a quantifier-free n-type p over $\bar{a} = (a_1, \ldots, a_n)$ and $\varepsilon > 0$, say that p is an *ε-quantifier-free n-type over* $\bar{a}$ if $\mathrm{qftp}(\bar{a}) \subseteq p$ and $d(x_i, a_i) = \varepsilon_i$ belongs to p for each $i \leq n$ and for some $0 \leq \varepsilon_i < \varepsilon$.

Given a metric structure X, $k \in \mathbb{N}$ and a complete quantifier-free k-type p write $p(X) = \{\bar{a} \in X^k : \bar{a} \models p\}$ and note that $p(X) \subseteq X^k$ is G_δ (closed if there are no relation symbols) in the topology of X^k, so if X is complete, then $p(X)$ becomes a Polish space.

Given three tuples $\bar{a}, \bar{b}$ and $\bar{c}$ in a metric structure X, write $\bar{b} \equiv_{\bar{a}} \bar{c}$ if $\mathrm{qftp}(\bar{b}/\bar{a}) = \mathrm{qftp}(\bar{c}/\bar{a})$. Also, write $\bar{b} \equiv \bar{c}$ to denote $\bar{b} \equiv_{\emptyset} \bar{c}$. If X is a homogeneous metric space, then the former is equivalent to the fact that there is $g \in \mathrm{Iso}(X)$ with $g \upharpoonright \bar{a} = \mathrm{id}$ and $g(\bar{c}) = \bar{b}$.

Definition 2.2. Given a metric structure X, $k \in \mathbb{N}$, a tuple $\bar{a} \in X^k$ with $p = \mathrm{qftp}(\bar{a})$ and $\varepsilon > 0$, say that a subset $Y \subseteq p(X)$ is *relatively ε-saturated over $\bar{a}$* if every ε-quantifier-free k-type over $\bar{a}$ which is realized in X, is also realized in Y.

Note that if Y is relatively ε-saturated over $\bar{a}$, then in particular, Y contains $\bar{a}$. Given two tuples $\bar{a}, \bar{b} \in X^m$ and $\varepsilon > 0$ write $d_X(\bar{a}, \bar{b}) < \varepsilon$ if $d_X(a_k, b_k) < \varepsilon$ for every $k \leq m$.

Definition 2.3. Suppose X is a homogeneous metric structure and $\bar{a} \in X^k$ for some $k \in \mathbb{N}$. Write p for $\mathrm{qftp}(\bar{a})$. Say that a sequence $(\bar{a}_n : n \in \mathbb{N})$ of elements of X^k is an *isolated sequence in p* if every $\bar{a}_n$ realizes p and there exists a sequence of $\varepsilon_n > 0$ and subsets $Y_n \subseteq p(X)$ such that for every $n \in \mathbb{N}$ the set Y_n is relatively ε_n-saturated over $\bar{a}_n$ and for every sequence $\bar{b}_n \in Y_n$ such that $\mathrm{qftp}(\bar{b}_n) = \mathrm{qftp}(\bar{a}_n)$ and $d_X(\bar{a}_n, \bar{b}_n) < \varepsilon_n$ there is an automorphism φ of X with $\varphi(\bar{a}_n) = \bar{b}_n$ for every $n \in \mathbb{N}$.

The following definition will be generalized in Definition 2.8 below.

Definition 2.4. Say that a metric structure X *admits isolated sequences* if for every $k \in \mathbb{N}$, every complete quantifier-free k-type p realized in X, for every nonmeager set $Z \subseteq p(X)$ there is a sequence $(\bar{a}_n : n \in \mathbb{N})$ which is isolated in p and $\bar{a}_n \in Z$ for every n.

The above definitions are enough for the study of the Urysohn space and the measure algebra but in order to deal with the Hilbert space we need to introduce slightly more general notions.

Definition 2.5. Given a metric structure X, $k \in \mathbb{N}$, $\varepsilon > 0$ and tuple $\bar{a} \in X^k$ write $p = \mathrm{qftp}(\bar{a})$. Suppose $T \subseteq p(X)$. Say that $Y \subseteq X^k$ is *T-relatively ε-saturated over $\bar{a}$* if for every $\bar{b} \in T$ with $d_X(\bar{b}, \bar{a}) < \varepsilon$ there is $\bar{b}' \in Y$ such that $\mathrm{qftp}(\bar{b}/\bar{a}) = \mathrm{qftp}(\bar{b}'/\bar{a})$.

Definition 2.6. Suppose X is a metric structure, $\varepsilon > 0$, $k, m \in \mathbb{N}$, $\bar{a} \in X^k$ and $p = \mathrm{qftp}(\bar{a})$. Say that a subset $T \subseteq p(X)$ *(m, ε)-generates an open set over $\bar{a}$* if there is a nonempty open set $U \subseteq p(X)$ such that for every $\bar{b} \in U$ there is a sequence $g_1, \ldots, g_m \in \mathrm{Aut}(X)$ such that

- $g_m \ldots g_1(\bar{a}) = \bar{b}$,
- $g_i(\bar{a}) \in T$ and $d_X(g_i(\bar{a}), \bar{a}) < \varepsilon$ for each $i \leq m$.

Note that, in particular, if T contains an open ball of radius ε around $\bar{a}$ (i.e. $\{\bar{b} \in p(X) : d_X(\bar{b}, \bar{a}) < \varepsilon\}$), then it $(1, \varepsilon)$-generates an open set over $\bar{a}$. Therefore, the following definition is a generalization of Definition 2.2.

Definition 2.7. Suppose X is a metric structure, $\varepsilon > 0$, $k, m \in \mathbb{N}$, $\bar{a} \in X^k$ and $p = \mathrm{qftp}(\bar{a})$. Say that $Y \subseteq p(X)$ is *m-relatively ε-saturated over $\bar{a}$* if there is $T \subseteq p(X)$ such that

- Y is T-relatively ε-saturated over $\bar{a}$,
- T (m,ε)-generates an open set over $\bar{a}$.

Thus, if Y is relatively ε-saturated over $\bar{a}$, then it is 1-relatively ε-saturated over $\bar{a}$.

Definition 2.8. Suppose X is a homogeneous metric structure and $\bar{a} \in X^k$ for some $k \in \mathbb{N}$. Write p for $\mathrm{qftp}(\bar{a})$. Say that a sequence $(\bar{a}_n : n \in \mathbb{N})$ of elements of X^k is a *weakly isolated sequence in* p if every $\bar{a}_n$ realizes p and there exists $m \in \mathbb{N}$ and a sequence of $\varepsilon_n > 0$ and subsets $Y_n \subseteq p(X)$ such that for every $n \in \mathbb{N}$ the set Y_n is m-relatively ε_n-saturated over $\bar{a}_n$ and for every sequence $\bar{b}_n \in Y_n$ such that $\mathrm{qftp}(\bar{b}_n) = \mathrm{qftp}(\bar{a}_n)$ and $d_X(\bar{a}_n, \bar{b}_n) < \varepsilon_n$ there is an automorphism φ of X with $\varphi(\bar{a}_n) = \bar{b}_n$ for every $n \in \mathbb{N}$. Given m as above we will also say that the sequence is m-*weakly isolated*.

Definition 2.9. Say that a metric structure X *admits weakly isolated sequences* if there is $m \in \mathbb{N}$ such that for every $k \in \mathbb{N}$, every complete quantifier-free k-type p realized in X, for every nonmeager set $Z \subseteq p(X)$ there is a sequence $(\bar{a}_n : n \in \mathbb{N})$ which is m-weakly isolated in p and $\bar{a}_n \in Z$ for every n. Given m as above we will also say that the structure admits m-weakly isolated sequences.

Now, Definition 2.4 is a special case of Definition 2.9 since an isolated sequence is obviously weakly isolated.

3. Metric structures

Automatic continuity for automorphism groups of metric structures will depend on the model-theoretic properties of the structure. The key definitions, which stem from the analysis of the work of Kechris and Rosendal [31] are given below.

Below, and throughout of this paper, we use the convention that a finitely generated substructure of a metric structure is always enumerated (a finitely generated substructure is a tuple if there are no function symbols).

Definition 3.1. Let M be a metric structure, $B, C \subseteq M$ be finitely generated substructures. Given a finitely generated substructure $A \subseteq B \cap C$ say that B and C are *independent over* A and write

$$B \mathop{\perp}_{A} C$$

if for every pair of automorphisms $\varphi : B \to B$, $\psi : C \to C$ such that A is closed under φ and ψ and $\varphi \restriction A = \psi \restriction A$, the function $\varphi \cup \psi$ extends to an automorphism of the substructure generated by B and C.

An abstract notion of stationary independence has been considered by Tent and Ziegler in [52]. In general, the above notion is not a stationary independence relation in the sense of [52, Definition 2.1] and satisfies only the Invariance and Symmetry conditions (see also the remarks [52, Example

2.2]). However, in all concrete cases, the examples of independence relation considered in this paper will be the same as in [52]. The following is motivated by a standard property of the independence relation in stable theories (see [4, Theorem 14.12]).

Definition 3.2. Say that a metric structure M has the *extension property* if for every pair $B, C \subseteq M$ of finitely generated substructures and a finitely generated substructure $A \subseteq B \cap C$ there is a finitely generated substructure $C' \subseteq M$ with $C' \equiv_A C$ such that $B \perp_A C'$.

Another property of the metric structures that we will need for automatic continuity is connected with the extension theorems proved by Hrushovski [20], Herwig and Lascar [17] and Solecki [48].

Definition 3.3. Say that a metric structure M *has locally finite automorphisms* if for every $n \in \mathbb{N}$, for any finitely generated substructure N of M, for any partial automorphisms $\varphi_1, \ldots, \varphi_n$ of N, there is a finitely generated structure N' of M containing N such that every φ_i extends to an automorphism of N', for each $i \leq n$.

Note that if finitely generated substructures of M are finite (e.g. when there are no function symbols for functions from M^m to M), then M has locally finite automorphisms if and only if for any finite substructure N of M there is is a finite structure N' of M containing N such that every isomorphism between finite substructures of N extends to an automorphism of N'.

4. First order logic for metric structures

Similarly as in the classical case, the $\mathcal{L}_{\omega_1\omega}$-formulas in first-order logic for metric structures are formed by allowing countable infinite conjunctions and disjunctions of first order formulas as well as finite quantification. Note that the formula $\mathrm{qftp}(\bar{x}) = \mathrm{qftp}(\bar{y})$ belongs to $\mathcal{L}_{\omega_1\omega}$. A property Φ of a metric structures is called a *first-order property* if there is a set $\tilde{\Phi}$ of $\mathcal{L}_{\omega_1\omega}$-sentences such that a metric structure M satisfies Φ if and only if $M \models \phi$ for every $\phi \in \tilde{\Phi}$. Note that if ϕ is an $\mathcal{L}_{\omega_1\omega}$-sentence and $N \prec M$, then we have that $N \models \phi$ if and only if $M \models \phi$.

Lemma 4.1 (Löwenheim–Skolem)**.** *Suppose M is a metric structure and for each $\bar{a}, \bar{b} \in M^{<\omega}$ let $\varphi_{\bar{a}\bar{b}}$ be an automorphism of M. For every countable $M_0 \subseteq M$ there is a countable metric structure $N \subseteq M$ with $M_0 \subseteq N$, such that $N \prec M$ and N is closed under $\varphi_{\bar{a}\bar{b}}$ for each $\bar{a}, \bar{b} \in N^{<\omega}$.*

Proof. The standard Löwenheim–Skolem argument shows that there is a countable M_1 with $M_0 \subseteq M_1$ and $M_1 \prec M$. Construct a chain of countable first-order elementary substructures $M_n \prec M$ with $M_n \subseteq M_{n+1}$ such that M_{n+1} is closed under $\varphi_{\bar{a}\bar{b}}$ for every $\bar{a}, \bar{b} \in M_n^{<\omega}$. Write $N = \bigcup_n M_n$. Then N is as needed. □

Lemma 4.2. *The property saying that a metric structure has locally finite automorphisms is a first-order property for homogeneous metric structures.*

Proof. For each $n \in \mathbb{N}$ write

$$x \in \langle x_1, \ldots, x_n \rangle$$

for the $\mathcal{L}_{\omega_1\omega}$-formula (in variables $x, x_1, \ldots, x_n$) saying that x belongs to the substructure generated by $x_1, \ldots, x_n$. The formula is of the form $\bigvee_{k \in \mathbb{N}} x = g_k(x_1, \ldots, x_n)$ where g_k enumerate all compositions of function symbols in the language. We also write $y_1, \ldots, y_m \in \langle x_1, \ldots, x_n \rangle$ for $\bigwedge_{i \leq m} y_i \in \langle x_1, \ldots, x_n \rangle$.

Note that if a homogeneous metric structure M has locally finite automorphisms and N is its finitely generated substructure, say by $x_1, \ldots, x_n$ and $k \in \mathbb{N}$, then there is a number $n(p,k) \in \mathbb{N}$, depending only on the quantifier-free type p of $x_1, \ldots, x_n$ and k such that any for any substructure N_1 isomorphic to N in M, and any set of partial automorphisms $\varphi_1, \ldots, \varphi_k$ of N_1 there is a substructure N_1' of M containing N_1 and generated by $m \leq n(p)$ elements such that every φ_i extends to an automorphism of N_1'.

Thus, a homogeneous metric structure M has locally finite automorphisms if and only if it satisfies the following $\mathcal{L}_{\omega_1\omega}$-sentences, for every $n, k \in \mathbb{N}$, quantifier-free n-type p and every $n_1, \ldots, n_k \leq n$.

$$\forall x_1, \ldots, x_n \quad [\mathrm{qftp}(x_1, \ldots, x_n) = p] \Rightarrow$$
$$\forall y^1_1, \ldots, y^1_{n_1}, z^1_1, \ldots, z^1_{n_1} y^2_1, \ldots, y^2_{n_2}, z^2_1, \ldots, z^2_{n_2}, \ldots, y^k_1, \ldots, y^k_{n_k}, z^k_1, \ldots, z^k_{n_k}$$
$$\Big[\bigwedge_{i=1}^{k} \big(\bigwedge_{l=1}^{n_i} y^i_l, z^i_l \in \langle x_1, \ldots, x_n \rangle \big) \wedge \mathrm{qftp}(y^i_1, \ldots, y^i_{n_i}) = \mathrm{qftp}(z^i_1, \ldots, z^i_{n_i}) \Big]$$
$$\Rightarrow \quad \Big[\bigvee_{n \leq m \leq n(p,k)} \exists x'_1, \ldots, x'_m \quad x'_1 = x_1 \wedge \ldots \wedge x'_n = x_n$$
$$\bigwedge_{j \leq k} \exists x^k_1, \ldots, x^k_m \in \langle x'_1, \ldots, x'_m \rangle \quad \Big(x'_1, \ldots, x'_m \in \langle x^k_1, \ldots, x^k_m \rangle$$
$$\wedge \quad \mathrm{qftp}(x'_1, \ldots, x'_m, y^k_1, \ldots, y^k_{n_k}) = \mathrm{qftp}(x^k_1, \ldots, x^k_m, z^k_1, \ldots, z^k_{n_k}) \Big) \Big]$$

□

Lemma 4.3. *The extension property is a first-order property for metric structures.*

Proof. The extension property is the conjunction of the following sentences, for all $n, m \in \mathbb{N}$ and $k \leq \min(n, m)$. Below, for a tuple $\bar{x} = (x_1, \ldots, x_n)$ and $\sigma \in S_n$ (the group of permutations of n) we write $\bar{x}_\sigma$ for the tuple

$(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$.

$$\forall x_1, \ldots, x_n \quad \forall y_1, \ldots, y_m \quad (x_1 = y_1 \wedge \ldots \wedge x_k = y_k)$$
$$\Rightarrow \Big[\exists y'_1 \ldots, y'_m (y'_1 = y_1 \wedge \ldots \wedge y'_k = y_k) \wedge \mathrm{qftp}(y_1 \ldots y_m) = \mathrm{qftp}(y'_1 \ldots y'_m)$$
$$\wedge \bigwedge_{\sigma \in S_n} \bigwedge_{\tau \in S_m} \Big(\sigma \restriction k : k \to k \quad \wedge \quad \sigma \restriction k = \tau \restriction k$$
$$\big(\mathrm{qftp}(\bar{x}) = \mathrm{qftp}(\bar{x}_\sigma) \ \wedge\ \mathrm{qftp}(\bar{y}') = \mathrm{qftp}(\bar{y}'_\tau)\big) \Rightarrow \mathrm{qftp}(\bar{x}, \bar{y}) = \mathrm{qftp}(\bar{x}_\sigma, \bar{y}_\tau)\Big)\Big]$$

□

Corollary 4.4. *Suppose M is a homogeneous metric structure which has locally finite automorphisms and the extension property. Let $\varphi_{\bar{a}}$ be an automorphism of M for each $\bar{a} \in M^{<\omega}$. Then there is a countable homogeneous metric structure $N \subseteq M$ which is dense in M, has locally finite automorphisms, the extension property and is closed under all automorphisms $\varphi_{\bar{a}}$ for $\bar{a} \in N^{<\omega}$.*

Proof. Let M_0 be countable dense in M and for each $\bar{a}, \bar{b} \in M^{<\omega}$ which generate isomorphic substructures, let $\varphi_{\bar{a}\bar{b}}$ be an automorphism of M which maps $\bar{a}$ to $\bar{b}$. If $\bar{a}, \bar{b} \in M^{<\omega}$ are of different cardinality or do not generate isomorphic substructures, then put $\varphi_{\bar{a}\bar{b}} = \varphi_{\bar{a}}$. By Proposition 4.1, there is a countable substructure $N \prec M$ which contains N_0 and is closed under all $\varphi_{\bar{a}\bar{b}}$ for $\bar{a}, \bar{b} \in N^{<\omega}$. The latter clearly implies that N is homogeneous and closed under $\varphi_{\bar{a}}$ for $\bar{a} \in N^{<\omega}$. Lemmas 4.3 and 4.2 imply that N has the extension property and locally finite automorphisms. □

5. Ample generics

If a metric structure is countable and has a discrete metric, then its automorphism group is a subgroup of S_∞ and for such groups Kechris and Rosendal [31] developed machinery for proving automatic continuity. Recall that a topological group G has *ample generics* if for every $n \in \mathbb{N}$ there is a comeager class in the diagonal conjugacy action of G on G^n (i.e. the action $g \cdot (g_1, \ldots, g_n) = (gg_1g^{-1}, \ldots, gg_ng^{-1})$).

Recall [18] that given a continuous action of a Polish group G on a Polish space X, a point $x \in X$ is *turbulent* if for every open neighborhood $U \subseteq G$ of the identity and every open $V \ni x$, the *local orbit* $O(x, U, V) = \{x' \in X : \exists g_1, \ldots, g_n \in U \forall i \leq n\ g_i \ldots g_1 x \in U \wedge x' = g_n \ldots g_1 x\}$ is somewhere dense. If G is a subgroup of S_∞, then a point $x \in X$ is turbulent if and only if for every open subgroup $U \leq G$ the set $Ux = \{gx : g \in U\}$ is somewhere dense [31, Proposition 3.2]. Also, in the case of a continuous action of a Polish group, if a point is turbulent and has a dense orbit, then its orbit is actually comeager. This is because in such case the orbit of a turbulent point cannot be meager and hence has to be meager in its closure [31, Proposition 3.2].

The groups of automorphisms of metric structures can be endowed with many topologies and this is the starting point of the analysis of Ben Yaacov, Berenstein and Melleray [5], who consider two types of topologies: the Polish topologies of pointwise convergence and variants of strong (non separable) topologies. We are, however, primarily interested in separable topologies on these groups. By default, the topology on $\mathrm{Aut}(M)$ is that of pointwise convergence with respect to the metric on M. However, if M is countable (but perhaps not complete with respect to its metric), we will also consider the topology inherited from the group S_∞, which coincides with the pointwise convergence topology with respect to a discrete metric on M. If M is countable, then we refer to this topology by saying that the group $\mathrm{Aut}(M)$ is treated *as a subgroup of* S_∞. It can be also viewed as the topology of pointwise convergence on the automorphism group of the structure M endowed with a discrete metric and the original distance function d_M treated as a part of the structure.

Below, and throughout this paper, we use the following notation. If G acts on M and $\bar{a} = (a_1, \ldots, a_n) \in M^{<\omega}$, then $G_{\bar{a}} = \{g \in G : \forall i \leq n \ g(a_i) = a_i\}$.

Lemma 5.1. *Suppose M is a countable homogeneous metric structure. If M has locally finite automorphisms and the extension property, then* $\mathrm{Aut}(M)$ *has ample generics as a subgroup of S_∞.*

Proof. Write G for $\mathrm{Aut}(M)$. Fix $n \in \mathbb{N}$ to find a generic tuple in G^n. Enumerate as $(a_n : n \in \mathbb{N})$ with infinite repetitions all tuples $a = (A, \vec{\varphi}, B, \vec{\psi})$ with $\vec{\varphi} = (\varphi_1, \ldots, \varphi_n)$, $\vec{\psi} = (\psi_1, \ldots, \psi_n)$ such that $A \subseteq B$ are finitely generated substructures of M (possibly generated by the empty set) and for each $i \leq n$ we have $\varphi_i \subseteq \psi_i$, and $\varphi_i : A \to A$, $\psi_i : B \to B$ are automorphisms.

By induction on k construct a sequence of increasing finitely generated substructures $D_k \subseteq M$ together with tuples of increasing automorphisms $\vec{\gamma}_k = (\gamma_k^1, \ldots, \gamma_k^n)$ with $\gamma_k^i : D_k \to D_k$ for each $i \leq n$. Using back-and-forth and homogeneity make sure that $\bigcup_k D_k = M$ and for each $i \leq n$ the function $\bigcup_k \gamma_k^i$ is an automorphism of M. Additionally, for each k make sure that

$$(*) \quad \begin{array}{l} \text{if } a_k = (A_k, \vec{\varphi}_k, B_k, \vec{\psi}_k) \text{ and } A_k \subseteq D_k \text{ are such that} \\ \vec{\varphi}_k = (\varphi_k^1, \ldots, \varphi_k^n),\ \vec{\psi}_k = (\psi_k^1, \ldots, \psi_k^n), \\ \varphi_k^i \restriction A_k : A_k \to A_k,\ \psi_k^i \restriction A_k = \varphi_k^i \restriction A_k \text{ and } \varphi_k^i \subseteq \gamma_k^i \text{ for each } i \leq n, \\ \text{then there is } g \in G_{A_k} \text{ with } g\gamma_{k+1}^i g^{-1} \supseteq \psi_k^i \text{ for each } i \leq n. \end{array}$$

At the induction step k first use locally finite automorphisms to find $D_k' \supseteq D_k$ such that the first k-many elements of M are in D_k' and each φ_k^i extends to an automorphism ${\gamma_k^i}'$ of D_k'. Next, use the extension property to find a finitely generated substructure B_k' of M with $B_k' \equiv_{A_k} B_k$ such that $B_k' \mathop{\smile\!\!\!\!\!\!|}_{A_k} D_k'$. Let $g \in G_{A_k}$ witness that $B_k' \equiv_{A_k} B_k$, i.e. $g(B_k') = B_k$. Define D_{k+1} to be the substructure generated by D_k' and B_k' and for each $i \leq n$

define γ^i_{k+1} so that $\gamma^i_{k+1} \restriction D'_k = {\gamma^i_k}'$ and $\gamma^i_{k+1} \restriction B'_k = g^{-1}\psi_i g$ and use the assumption $B'_k \mathop{\smile\!\!\!\!\!\!|}_{A_k} D'_k$ to extend it to D_{k+1}. Note that g witnesses that $(*)$ is satisfied at the step k.

After this construction is done, write $g_i = \bigcup_k \gamma^i_k$ and note that g_i is an automorphism of M, for each $i \leq n$. To see that $\vec{g} = (g_1, \ldots, g_n)$ is generic, it is enough to see that $\vec{g}$ is turbulent under the diagonal conjugacy action of $\mathrm{Aut}(M)$ on $\mathrm{Aut}(M)^n$ and has a dense orbit.

To see the turbulence, fix an open neighborhood O of the identity in $\mathrm{Aut}(M)$, and, say, $O = G_{\bar{a}}$ for a finite tuple $\bar{a} \subseteq M$. We need to see that $O \cdot \vec{g}$ is somewhere dense. Find m such that $\bar{a} \subseteq D_m$ and write V_i for $\{f \in \mathrm{Aut}(M) : f \restriction D_m = g_i \restriction D_m\}$. Note that $g_i \restriction D_m : D_m \to D_m$ is an automorphism for each $i \leq n$. We claim that $O \cdot \vec{g}$ is dense in $V_1 \times \ldots \times V_n$. To see that, fix a nonempty open subset $W \subseteq V_1 \times \ldots \times V_n$ and without loss of generality assume $W = W_1 \times \ldots \times W_n$. Since M has locally finite automorphisms, we can assume that $W_i = \{f \in \mathrm{Aut}(M) : f \supseteq \psi_i\}$ for each $i \leq n$, where each $\psi_i : B \to B$ is an automorphism of a finitely generated substructure B of M. Note that $\psi_i \supseteq g_i \restriction D_m$ for each $i \leq n$.

We need to see that $(O \cdot \vec{g}) \cap (W_1 \times \ldots \times W_n) \neq \emptyset$. Let $k \in \mathbb{N}$ be such that $a_k = (A_k, \vec{\varphi}_k, B_k, \vec{\psi}_k)$ with $A_k = D_m$, $B_k = B$, $\vec{\varphi}_k = (g_1 \restriction D_m, \ldots, g_n \restriction D_m)$ and $\vec{\psi} = (\psi_1, \ldots, \psi_n)$. By $(*)$ at the step k, there is $g \in G_{A_k} \subseteq G_{\bar{a}}$ such that $g g_i g^{-1} \supseteq \psi_i$ for each $i \leq n$ and so we are done.

The fact that the orbit of $\vec{g}$ is dense follows by analogous arguments. □

Recall that a subset S of a group G is called *countably syndetic* if G can be written as $\bigcup_n a_n S$ for some sequence $a_n \in G$. A subset S of G is called *symmetric* if $S^{-1} = S$.

Corollary 5.2. *Suppose M is a countable homogeneous metric structure and $G = \mathrm{Aut}(M)$. If M has locally finite automorphisms and the extension property, and $W \subseteq \mathrm{Aut}(M)$ is symmetric and countably syndetic, then there is a finite tuple $\bar{a} \subseteq M$ such that $G_{\bar{a}} \subseteq W^{10}$.*

Proof. This is an abstract consequence of ample generics [31, Lemma 6.15] and thus follows from Lemma 5.1. □

6. Automatic continuity

In [45] Rosendal and Solecki isolated an abstract property of a group that implies automatic continuity. We say that G *is Steinhaus* (cf. [49]) if there is a natural number $k \geq 1$ such that for every symmetric countably syndetic set $S \subseteq G$ the set $S^k = \{s_1 \cdot \ldots \cdot s_k : s_1, \ldots, s_k \in S\}$ contains a nonempty open set in G. In such a case G is also called *k-Steinhaus.* If a group G is Steinhaus, then G has the automatic continuity property [45, Proposition 2].

We will need the following lemma.

Lemma 6.1. *Suppose X is a complete homogeneous metric structure that has locally finite automorphisms and the extension property and let $G = \mathrm{Aut}(X)$. If $W \subseteq G$ is symmetric and countably syndetic, then there is $\bar{a} \in X^{<\omega}$ such that $G_{\bar{a}} \subseteq W^{10}$.*

Proof. Suppose otherwise. This means that for each $k \in \mathbb{N}$ and $a \in X^k$ there is $f_{\bar{a}} \in G$ with $f_{\bar{a}}(\bar{a}) = \bar{a}$ and $f_{\bar{a}} \notin W^{10}$. Let $g_n \in G$ be such that $G = \bigcup_n g_n W$. Use Corollary 4.4 to find a countable, dense $X_0 \subseteq X$ such that $X_0 \prec X$ and X_0 is closed under each g_n as well as under $f_{\bar{a}}$ for each $\bar{a} \in (X_0)^{<\omega}$. Since X_0 is elementary in X, it has locally finite automorphisms and the extension property by Lemmas 4.2 and 4.3.

Write W_0 for the set of those automorphisms of X_0 whose unique extension to X belongs to W.

Claim 6.2. *W_0 is symmetric and countably syndetic in $\mathrm{Aut}(X_0)$.*

Proof. It is clear that W_0 is symmetric. To see that it is countably syndetic, pick $f_0 \in \mathrm{Aut}(X_0)$ and let $f \in \mathrm{Aut}(X)$ be the unique extension of f_0 to X. Now, there is $n \in \mathbb{N}$ and $s \in W$ such that $f = g_n s$. Since $s = g_n^{-1} f$ leaves X_0 invariant, we have that $s_0 = s \restriction X_0 \in \mathrm{Aut}(X_0)$ and $s_0 \in W_0$. □

Now, this gives a contradiction since by Corollary 5.2, $(W_0)^{10}$ contains an open neighborhood of the identity in $\mathrm{Aut}(X_0)$, i.e. there is $\bar{a} \in (X_0)^{<\omega}$ such that every automorphism of X_0 which fixes $\bar{a}$ belongs to $(W_0)^{10}$. Write f_0 for $f_{\bar{a}} \restriction X_0$ and note that $f_0 \notin (W_0)^{10}$, by the density of X_0, contradiction. This proves the lemma. □

Theorem 6.3. *Suppose X is a complete homogeneous metric structure that admits weakly isolated sequences, has locally finite automorphisms and the extension property. Then the group $\mathrm{Aut}(X)$ is Steinhaus.*

Proof. Write G for $\mathrm{Aut}(X)$. Suppose $m \in \mathbb{N}$ is such that X admits m-weakly isolated sequences. We will show that G is $(24m + 10)$-Steinhaus. The same argument also shows that if X admits isolated sequences then G is 24-Steinhaus. Let $W \subseteq G$ be symmetric and countably syndentic. Let $g_n \in G$ be such that $G = \bigcup_n g_n W$.

Let $\bar{a}$ be such as in Lemma 6.1 and $k \in \mathbb{N}$ such that $\bar{a} \in X^k$. Write $\bar{a} = (a_1, \ldots, a_k)$, p for $\mathrm{qftp}(\bar{a})$ and $Z = \{w(\bar{a}) : w \in W\} \subseteq p(X)$. Note that since $p(X) = \bigcup_n g_n Z$, the set Z is nonmeager in $p(X)$. Choose an m-weakly isolated sequence $\bar{a}_n$ such that each $\bar{a}_n$ belongs to Z and for each $n \in \mathbb{N}$ let $v_n \in W$ be such that $\bar{a}_n = v_n(\bar{a})$. Let $\varepsilon_n > 0$, and $T_n, X_n \subseteq p(X)$ witness that the sequence of $\bar{a}_n$ is m-weakly isolated (i.e. X_n is T_n-relatively ε_n-saturated in X and T_n (m, ε_n)-generates an open set over $\bar{a}_n$).

Given two subspaces X', X'' of X and a set C of partial automorphisms from X' to X'', say that a subset G_0 of G is *full for C* if every element of C can be extended to an element of G_0 (cf. [45, Claim 1]).

Claim 6.4. *There is $n \in \mathbb{N}$ such that W^2 is full for*

$$C_n = \{\varphi : \bar{a}_n \to \mathrm{ball}_X(\bar{a}_n, \varepsilon_n) \cap X_n : \varphi \textit{ is an isomorphic embedding}\}.$$

Proof. First note that there is $n \in \mathbb{N}$ such that $g_n W$ is full for C_n. If not, then for each $n \in \mathbb{N}$ there is $\varphi_n \in C_n$ such that φ_n cannot be extended to an element of $g_n W$. Since $\bar{a}_n$ are isolated, there is an automorphism φ of X which extends all the φ_n. Then $\varphi \notin \bigcup_n g_n W$, which is a contradiction. Now, if $g_n W$ is full for C_n, then so is $W^2 = (g_n W)^{-1}(g_n W)$ as C_n contains the identity. □

Fix n as in Claim 6.4 and write $T = v_n^{-1}(T_n)$.

Claim 6.5. *We have*

$$\{g \in G : d_X(\bar{a}, g(\bar{a})) < \varepsilon_n \text{ and } g(\bar{a}) \in T\} \subseteq W^{24}.$$

Proof. Let $g \in G$ be such that $g(\bar{a}) \in T$ and $d_X(a_i, g(a_i)) < \varepsilon_n$ for each $i \leq k$.

Let $Y = v_n^{-1}(X_n)$. Note that since v_n is an automorphism and X_n is T_n-relatively ε_n-saturated over $\bar{a}_n$, we get that Y is T-relatively ε_n-saturated over $\bar{a}$. Thus, there is $\bar{b} \in Y$ with

$$\operatorname{qftp}(\bar{b}/\bar{a}) = \operatorname{qftp}(g(\bar{a})/\bar{a}).$$

By homogeneity of X, there is $w_1 \in G_{\bar{a}}$ such that $w_1(g(\bar{a})) = \bar{b}$. Note that $w_1 \in W^{10}$ as $G_{\bar{a}} \subseteq W^{10}$.

Look at $v_n w_1^{-1} g v_n^{-1} \in G$ and note that it maps $\bar{a}_n$ to $\operatorname{ball}_X(\bar{a}_n, \varepsilon_n) \cap X_n$ because v_n maps Y to X_n. By Claim 6.4 and the choice of n, there is $w_2 \in W^2$ which is equal to $v_n w_1^{-1} g v_n^{-1}$ on $\bar{a}_n$. This means that $w_2^{-1} v_n w_1^{-1} g v_n^{-1} \in G_{\bar{a}_n}$ and thus $v_n^{-1} w_2^{-1} v_n w_1^{-1} g \in G_{\bar{a}}$. Therefore, $v_n^{-1} w_2^{-1} v_n w_1^{-1} g \in W^{10}$ and thus $g \in W^{24}$. □

Now note that T (m, ε_n)-generates an open set over $\bar{a}$, so let $U \subseteq p(X)$ be nonempty and such that for every $\bar{b} \in U$ there is a sequence $g_1, \ldots, g_m \in \operatorname{Aut}(X)$ such that

- $g_m \ldots g_1(\bar{a}) = \bar{b}$,
- $g_i(\bar{a}) \in T$ and $d_X(g_i(\bar{a}), \bar{a}) < \varepsilon_n$ for each $i \leq m$.

We claim that

$$\{g \in G : g(\bar{a}) \in U\} \subseteq W^{24m+10}.$$

To see this, let $g \in G$ be such that $g(\bar{a}) \in U$ and let $\bar{b} = g(\bar{a})$. Find $g_1, \ldots, g_m$ as above and note that by Claim 6.5 we have that $g_i \in W^{24}$ for each $i \leq m$. Now, $g^{-1} g_m \ldots g_1 \in G_{\bar{a}}$, so $g \in W^{24m+10}$, as needed.

The set $\{g \in G : g(\bar{a}) \in U\}$ is open in G, so we have that G is $(24m+10)$-Steinhaus. This ends the proof. □

7. Triviality of homomorphisms

In this section we study the circumstances under which one can exclude nontrivial homomorphisms from the groups of the form $\operatorname{Aut}(M)$ to certain topological groups H. Given two groups G and H say that G is H-*trivial* if any homomorphism from G to H is trivial. Tsankov [53] concluded (from the minimality of the unitary group) that whenever H is a separable group which admits a complete left-invariant metric, then the unitary group is H-trivial. In this section, we isolate an abstract property of a metric structure M which implies that $\mathrm{Aut}(M)$ is H-trivial for H as above and in Section 10 we will see that this property is satisfied by the Hilbert space. The same is true for the group $\mathrm{Aut}(X,\mu)$ but here it follows immediately from the fact that $\mathrm{Aut}(X,\mu)$ is simple [10]. The unitary group is not simple but (similarly as the group of isometries of the Urysohn space [52]) has a maximal proper normal subgroup [11]. We do not know, however, if the methods below apply to $\mathbb{U}$.

Given a subset N of a metric structure M and $\bar{a} \in M^{<\omega}$, say that N is *relatively saturated over* $\bar{a}$ if it is α-relatively saturated over $\bar{a}$ for every $\alpha \in [0,\infty)$.

Definition 7.1. Suppose M is a homogeneous metric structure and $\bar{a} \in M^k$ for some $k \in \mathbb{N}$. Write p for $\mathrm{qftp}(\bar{a})$. Say that a sequence $(\bar{a}_n : n \in \mathbb{N})$ of elements of M^k is an *independent sequence in* p if every $\bar{a}_n$ realizes p and there exists a sequence of subsets N_n such that N_n is relatively saturated over $\bar{a}_n$ such that for every sequence $\bar{b}_n \in N_n$ with $\mathrm{qftp}(\bar{b}_n) = \mathrm{qftp}(\bar{a}_n)$ there is an automorphism φ of M with $\varphi(\bar{a}_n) = \bar{b}_n$ for every $n \in \mathbb{N}$.

Definition 7.2. Say that a metric structure M *admits independent sequences* if for every $k \in \mathbb{N}$ and $\bar{a} \in M^k$, for every sequence $(\bar{s}_n : n \in \mathbb{N})$ of finite tuples of elements of M, there is a sequence $\bar{a}_n$ which is independent in $\mathrm{qftp}(\bar{a})$ and is such that $\bar{a}_{n+1} \equiv_{\bar{s}_n} \bar{a}_n$ holds for each $n \in \mathbb{N}$.

Theorem 7.3. *Suppose X is a complete metric structure that admits independent sequences, has locally finite automorphisms and the extension property. Then the group* $\mathrm{Aut}(X)$ *is H-trivial for every Polish group H which has a complete left-invariant metric.*

Proof. Write G for $\mathrm{Aut}(X)$ and suppose H has a complete left-invariant metric. Fix $\varphi : G \to H$ and we will show that φ is trivial. To do this, it is enough to see that whenever $U \subseteq H$ is an open neighborhood of the identity, then $\varphi(G) \subseteq U$.

Let then $U \subseteq H$ be an open neighborhood of the identity. Find $V \subseteq H$ open neighborhood of the identity such that $V^{22} \subseteq U$ and write $T = \varphi^{-1}(V)$. Note that $T \subseteq G$ is countably syndetic, so by Lemma 6.1, there is $k \in \mathbb{N}$ and $\bar{a} \in X^k$ such that $G_{\bar{a}} \subseteq T^{10}$. Write p for the quantifier-free type of $\bar{a}$.

Let d_H be a complete left-invariant metric on H and pick a decreasing sequence of open neighborhoods V_n of the identity in H with $V_0 = V$ such that $\mathrm{diam}_{d_H}(V_n) \leq 2^{-n}$. Let also $W_n \subseteq H$ be open symmetric neighborhoods of the identity in H with $(W_n)^{10} \subseteq V_n$. Note that each $\varphi^{-1}(W_n)$ is countably syndetic in G. Using Lemma 6.1, by induction on $n > 0$ pick increasing sequence $\bar{s}_n$ of finite tuples of elements of X such that $G_{\bar{s}_n} \subseteq \varphi^{-1}(W_n)^{10} \subseteq \varphi^{-1}(V_n)$. Let $\bar{s}_0$ be the empty tuple.

Write $\bar{a}_0 = \bar{a}$. Using the assumption that X admits independent sequences, pick a sequence $g_n \in G$ for $n > 0$ such that $g_n \in G_{\bar{s}_n}$ and the sequence $(\bar{a}_n : n > 0)$ defined as $a_{n+1} = g_n(\bar{a}_n)$, forms an independent sequence in p. Let $X_n \subseteq X$ witness the that the sequence is independent, so that each X_n is relatively saturated over $\bar{a}_n$.

Let $f_n = g_n \dots g_0$ and write $h_n = \varphi(f_n)$. Note that h_n^{-1} is d_H-Cauchy in H and hence convergent. Thus, h_n is convergent in H too. Let $h = \lim_n h_n$.

Claim 7.4. *There is n such that $f_n T^2 f_n^{-1}$ is full for*

$$C_n = \{\varphi : \bar{a}_n \to X_n : \varphi \textit{ is an isomorphic embedding}\}.$$

Proof. We will first prove that there is a sequence $b_n \in G$ such that $G = \bigcup_n b_n T f_n^{-1}$. This will follow from the fact that there is a sequence $a_n \in H$ such that $H = \bigcup_n a_n V h_n^{-1}$ by taking b_n such that $\varphi(b_n) = a_n$.

To see the latter fact, pick a_n so that they are dense in H and we claim that $H = \bigcup_n a_n V h_n^{-1}$. Indeed, if $x \in H$, then note that the sequence

$$z_n = x h_n = x\varphi(g_n) \dots \varphi(g_0)$$

is convergent in H to $z = xh$. Pick a subsequence a_{k_n} converging to z. Since z_{k_n} converges to z, we get that $d_H(z_{k_n}, a_{k_n}) \to 0$. Thus, $d_H(a_{k_n}^{-1} z_{k_n}, 1_H) \to 0$ as well and there is n such that $a_{k_n}^{-1} x h_{k_n} \in V$ and then $x \in a_{k_n} V h_{k_n}^{-1}$.

Now, note that there is $n \in \mathbb{N}$ such that $b_n T f_n^{-1}$ is full for C_n. If not, then for each $n \in \mathbb{N}$ there is $\varphi_n \in C_n$ such that φ_n cannot be extended to an element of $b_n T f_n^{-1}$. As the sequence $\bar{a}_n$ is independent, there is an automorphism φ of X which extends all the φ_n. But then $\varphi \notin \bigcup_n a_n T f_n^{-1}$, which is a contradiction. Finally, if $a_n T f_n^{-1}$ is full for C_n, then so is $f_n T^2 f_n^{-1} = (a_n T f_n^{-1})^{-1}(a_n T f_n^{-1})$ since C_n contains the identity. This proves the claim. $\square$

We need to prove that $G \subseteq \varphi^{-1}(U)$. Let $g \in G$ be arbitrary. Fix n as in Claim 7.4 and note that $f_n(\bar{a}) = \bar{a}_n$. Let $Y = f_n^{-1}(X_n)$. Note that Y and X realize the same quantifier-free n-types over $\bar{a}$. This follows from the fact that X_n is relatively saturated over $\bar{a}_n$ and f_n is an automorphism.

Thus, there is $\bar{b} \in Y$ such that

$$\mathrm{qftp}(\bar{b}/\bar{a}) = \mathrm{qftp}(g(\bar{a})/\bar{a}).$$

Let $w_1 \in G_{\bar{a}}$ be such that $w_1(g(\bar{a})) = \bar{b}$. Note that $w_1 \in T^{10}$ as $G_{\bar{a}} \subseteq W^{10}$.

Look at $f_n w_1^{-1} g f_n^{-1} \in G$ and note that it maps $\bar{a}_n$ to X_n as f_n maps Y to X_n. By Claim 7.4, there is $y \in f_n T^2 f_n^{-1}$ which is equal to $f_n w_1^{-1} g f_n^{-1}$ on $\bar{a}_n$. Write $y = f_n w_2 f_n^{-1}$ for some $w_2 \in W^2$ and note that

$$y^{-1}(f_n w_1^{-1} g f_n^{-1}) = f_n w_2 w_1^{-1} g f_n^{-1} \in G_{\bar{a}_n}.$$

Therefore, $w_2 w_1^{-1} g = f_n^{-1}(y^{-1}(f_n w_1^{-1} g f_n^{-1})) f_n$ fixes $\bar{a}$, which means that

$$w_2 w_1^{-1} g \in T^{10}$$

and thus $g \in T^{22} \subseteq \varphi^{-1}(U)$.

This shows that $G \subseteq \varphi^{-1}(U)$ and since $U \subseteq H$ was an arbitrary open neighborhood of the identity, we have that φ is trivial. This ends the proof. $\square$

8. The Urysohn space

There is no essential difference in verifying that the Urysohn space and the Urysohn sphere satisfy the assumptions of Theorem 1.4. We will focus only on the Urysohn space.

Locally finite automorphisms for the Urysohn space have been already shown by Solecki [48], who proved that for any finite metric space X and finitely many partial isometries of X, there is a metric space Y containing X such that all these partial isometries extend to isometries of Y. Solecki derived his result from an extension theorem of Herwig and Lascar [17]. The theorem of Herwig and Lascar is connected to the Rhodes' Type II Conjecture proved independently by Ash [1] and by Ribes and Zalesskiĭ [40]. The latter results concern the profinite topology on the free groups (cf. also [39]).

Recall that *the profinite topology* on a free group F_n is the one with the basis at the identity consisting of finite-index subgroups of F_n. In the literature, the fact that a set $A \subseteq F_n$ is closed in the profinite topology is usually referred to as by saying that A is *separable*. A classical result of M. Hall, Jr. [16] says that any finitely generated subgroup of F_n is separable. Note that it also implies that any coset of a finitely generated subgroup of F_n is separable (since the multiplication is continuous in the profinte topology). The main result of Ribes and Zalesskiĭ [40] states that products of finitely many finitely generated subgroups of F_n are also separable. Again, note that it immediately implies that products of finitely many cosets of finitely generated subgroups of F_n are separable as well.

An abstract connection between the theorem of Ribes and Zalesskiĭ and extensions of partial isometries was discovered by Rosendal [44], who expressed it in the language of finitely approximable actions and, in particular, gave a new proof of the result of Solecki [48]. On the other hand, the paper of Solecki [48] contains a very elegant argument on the extensions of one isometry. That argument is done in the style of Mackey's constructions of induced actions [33, Page 190] (cf. [3, 2.3.5]) and a similar argument has been used by Hrushovski [20] in the context of extensions of partial isomorphisms of graphs.

Below, we present a new proof of Solecki's theorem [48], which exploits the ideas used in the case of one isometry in [48, Section 3] and is also done in the style of Mackey's construction of induced actions.

Theorem 8.1 (Solecki)**.** *The Urysohn space has locally finite automorphisms.*

Proof (à la Mackey). By the finite extension property of the Urysohn space, it is enough to show that for every finite metric space X, for every tuple $\varphi_1, \ldots, \varphi_n$ of partial isometries of X there is a finite metric space $Y \supseteq X$ such all $\varphi_1, \ldots, \varphi_n$ extend to isometries of Y.

Let X be a finite metric space. Write

$$\delta = \min\{d_X(x,y) : x,y \in X, x \neq y\}$$

and let $\Delta = \mathrm{diam}(X)$. Suppose $\varphi_1, \ldots, \varphi_n$ are partial isometries of X.

Write $a_1, \ldots, a_n$ for the generators of the free group F_n. Write also A for the set $\{a_1, \ldots, a_n, a_1^{-1}, \ldots, a_n^{-1}\}$ and W_n for A^* (the set of all words over A). For a word $w \in W_n$ with $w = v_1 \ldots v_k$, $v_i \in A$ and $x \in X$ say that $w(x)$ *is defined* if there is a sequence of points $x_j \in X$ $(j \leq k)$ with $x_0 = x$ and $x_{j+1} = \varphi_i(x_j)$ if $v_{k-j} = a_i$ and $x_{j+1} = \varphi_i^{-1}(x_j)$ if $v_{k-j} = a_i^{-1}$. If $w(x)$ is defined, then write $w(x) = y$ for $y = x_k$ as above. We also use the notation $w(x)$ if w belongs to F_n (using the reduced word for w).

For each $x, y \in X$ write T_x^y for the set of $w \in F_n$ such that $w(x) = y$.

Claim 8.2. *For every $x, y \in X$ the set T_x^y is either empty or a coset of a finitely generated subgroup of F_n.*

Proof. If T_x^y is nonempty, then let $w \in F_n$ be such that $w(x) = y$. Note that whenever $w' \in T_x^y$, then $w^{-1}w' \in T_x^x$. Now, T_x^x is a finitely generated subgroup of F_n: it is the fundamental group of the graph whose vertices are the points in X and (labelled) edges connect x, y if $\varphi_i(x) = y$ for some $i \leq n$. Therefore, $T_x^y = wT_x^x$ is a coset of a finitely generated subgroup. □

Claim 8.3. *For every $m \in \mathbb{N}$ and $x_1, y_1, \ldots, x_m, y_m \in X$ the set $T_{x_1}^{y_1} \cdot \ldots \cdot T_{x_m}^{y_m}$ is closed in F_n in the profinite topology.*

Proof. This follows from the Ribes–Zalesskiĭ theorem [40] and Claim 8.2. □

We need to define an extension of X. It will be obtained by dividing $X \times F_n$ by certain equivalence relation $\simeq$ so that $x \mapsto (x, e)/\simeq$ is an embedding. We will have to make sure that the extension is finite and define a metric on it so that the embedding is isometric. Before we make this definition precise, let us comment on how the metric on $(X \times F_n)/\simeq$ will be defined. Note that there is a partial distance function d_0 on $X \times F_n$, namely for $(x, w), (y, w)$ with $x, y \in X$ and $w \in F_n$ we put $d_0((x, w), (y, w)) = d_X(x, y)$. Now, if $\simeq$ is an equivalence relation on $X \times F_n$, then there is a natural distance function on $(X \times F_n)/\simeq$ defined as follows. If $C, D \in (X \times F_n)/\simeq$, then put $d_{(X \times F_n)/\simeq}(C, D)$ to be the minimum of Δ and the sums of the form

$$(**) \qquad \sum_{i=0}^{m-1} d_0(z_i, z'_{i+1})$$

such that $z_0, z_1, z'_1, \ldots, z_{m-1}, z'_{m-1}, z'_m \in X \times F_n$, the value $d_0(z_i, z'_{i+1})$ is defined for each $0 \leq i < m$ and there is a sequence $C_0, \ldots, C_m$ of elements of $X \times F_n/\simeq$ with $C_0 = C, C_m = D$ and $z_0 \in C_1$, $z'_m \in C_m$ and $z_j, z'_j \in C_j$ for $0 < j < m$.

Now we will define the equivalence relation $\simeq$ and check the details of the construction described above. For that, we need a couple of definitions.

Given $x, y \in X$, a *chain from x to y* is a sequence

$$z_0, z_1, z_1', \ldots, z_{m-1}, z_{m-1}', z_m, z_m' \in X$$

such that $z_0 = x, z_m = y$ and for each $1 \leq i \leq m$ there exists $w_i \in W_n$ such that $w_i(z_i) = z_i'$. The *distance of a chain* $z_0, z_1, z_1', \ldots, z_{m-1}, z_{m-1}', z_m, z_m'$ is defined as $\sum_{i=0}^{m-1} d_A(z_i, z_{i+1}')$. A *word realization* of a chain as above is a sequence of words $w_1, \ldots, w_m \in W_n$ such that $w_i(z_i) = z_i'$ for each $1 \leq i \leq m$. Call a chain *trivial* if it has a word realization $w_1, \ldots, w_m \in W_n$ such that $w_1 \cdot \ldots \cdot w_m = e$ holds in F_n.

Let now $M \in \mathbb{N}$ be such that $M\delta > \Delta$. Note that since X is finite, there are only finitely many nontrivial chains $z_0, z_1, z_1', \ldots, z_m, z_m' \in X$ with $m \leq M$. For each nontrivial chain $z_0, z_1, z_1', \ldots, z_m, z_m' \in X$ with $m \leq M$, the set $T_{z_1}^{z_1'} \cdot \ldots \cdot T_{z_m}^{z_m'}$ is a closed subset of F_n which does not contain e. Using Claim 8.3, find a finite index normal subgroup $H \lhd F_n$ which is disjoint from every $T_{z_1}^{z_1'} \cdot \ldots \cdot T_{z_m}^{z_m'}$ as above.

Write Z for $X \times F_n$ and define an equivalence relation $\simeq$ on Z as follows. Given $w_1, w_2 \in F_n$ write

$$(x_1, w_1) \simeq (x_2, w_2)$$

if there is $v \in F_n$ with $w_2^{-1} w_1 H = vH$ and $v(x_1) = x_2$. Given $(x, w) \in Z$ write $[x, w]$ for its $\simeq$-class.

Write Y for $Z/\simeq$ and note that Y is finite. The latter follows from the fact that if $F_n/H = \{d_1 H, \ldots, d_t H\}$, then $Y = \{[x, d_i] : x \in X, i \leq t\}$. Now, define a metric d_Y on Y as follows. Let $d_Y([x, w], [y, v])$ be the minimum of Δ and the set of sums of the form

$$\sum_{i=0}^{m-1} d_X(z_i, z_{i+1}')$$

for sequences $z_0, z_1, z_1', \ldots, z_m, z_m'$ of elements of X such that

- $z_0 = x, z_m = y$,
- and there are $w_i \in F_n$ (for $0 \leq i \leq m$) with $w_0 = w, w_m = v$ and $(z_i', w_{i-1}) \simeq (z_i, w_i)$ for each $1 \leq i \leq m$.

Note that a sum as above is equal to 0 exactly when $z_i = z_{i+1}'$ for every $i < m$ and hence the definition of d_Y does not depend on the representatives of $\simeq$-classes and defines a metric on Y. Note that this definition coincides with the formula given by $(**)$.

Define an embedding of X into Y via $x \mapsto [x, e]$. We claim that this is an isometric embedding and that each φ_i extends to an isometry of Y. The second part is clear given the first one since for each $i \leq n$ the map $[x, w] \mapsto [x, a_i w]$ is well-defined and is easily seen to be an isometry of Y which extends φ_i. Thus, we only need to show that $x \mapsto [x, e]$ is an isometric embedding.

Claim 8.4. *For any $x, y \in X$ and $w \in F_n$ the distance $d_Y([x, e], [y, w])$ is equal to the minimum of Δ and the minimal distance of a chain from x to y which has a word realization $v_1, \dots, v_k$ such that $v_1 \cdot \dots \cdot v_k H = wH$.*

Proof. If $z_0, z_1, z'_1, \dots, z_m, z'_m$ in X are such that

$$d_Y([x, e], [y, w]) = \sum_{i=0}^{m-1} d_X(z_i, z'_{i+1})$$

and $w_0, \dots, w_n$ in F_n are such that $w_0 = e, w_m = w$ and $(z'_i, w_{i-1}) \simeq (z_i, w_i)$ for each $1 \le i \le m$, then find $v_i \in F_n$ (for $1 \le i \le m$) such that $v_i H = w_{i-1}^{-1} w_i H$ and $v_i(z'_i) = z_i$. Then $z_0, z_1, z'_1, \dots, z_m, z'_m$ is a chain and $v_1, \dots, v_m$ is its word realization with $v_1 \dots v_m H = w_0^{-1} w_1 \dots w_{m-1}^{-1} w_m H = w_m H = wH$.

On the other hand, if $z_0, z_1, z'_1, \dots, z_m, z'_m$ in X forms a chain from x to y with a word realization $v_1, \dots, v_k$ such that $v_1 \cdot \dots \cdot v_k H = wH$, then write $w_0 = e$ and $w_i = w_{i-1} v_i$ for $1 \le i < m$ and $w_m = w$. Then the sequence $z_0, z_1, z'_1 \dots, z_m, z'_m$ together with $w_0, \dots, w_m$ satisfies the conditions in the definition of d_Y. □

Consequently, as $d_X(x, y) \le \Delta$, by Claim 8.4, we have that $d_Y([x, e], [y, e])$ is the minimal distance of the chains from x to y which have a word realization $w_1, \dots, w_k$ with $w_1 \cdot \dots \cdot w_k \in H$. Say that a chain c from x to y *realizes the distance* if the distance of c is equal to $d_Y([x, e], [y, e])$. We need to show that if a chain c realizes the distance from x to y, then its distance is equal to $d_X(x, y)$.

Claim 8.5. *Suppose $x, y \in X$ and c is a chain from x to y with a word realization $w_1, \dots, w_m \in W_n$. If $w_i = v_i a$ and $w_{i+1} = a^{-1} v_{i+1}$ for some $1 \le i < m$ with $v_i, v_{i+1} \in W_n$ and $a \in A$, then there is a chain c' from x to y which has the same distance as c and a word realization $w'_1, \dots, w'_m \in W_n$ such that $w'_j = w_j$ for $j \ne i, i+1$, $w_j = v_i$ for $j = i, i+1$.*

Proof. Write $c = (z_0, z_1, z'_1, \dots, z_m, z'_m)$. Let $\varphi = \varphi_k$ if $a = a_k$ and $\varphi = \varphi_k^{-1}$ if $a = a_k^{-1}$. Consider the chain $c' = (y_0, y_1, y'_1, \dots, y_m, y'_m)$ with $y_j = z_j$ for $j \ne i$ and $y'_j = z'_j$ for $j \ne i+1$, and $y_i = \varphi(z_i)$, $y'_{i+1} = v_{i+1}(z_{i+1})$. The distance of c' is the same as that of c since

$$d(y_i, y'_{i+1}) = d(\varphi(z_i), v_{i+1}(z_{i+1})) = d(z_i, \varphi^{-1}(v_{i+1}(z_{i+1})))$$

as φ is an isometry. And we have $\varphi^{-1}(v_{i+1}(z_{i+1})) = w_{i+1}(z_{i+1}) = z'_{i+1}$. □

Given two chains $c = (z_0, z_1, z'_1 \dots, z_m, z'_m)$ and $c' = (z_0, z_1, z'_1 \dots, z_k, z'_k)$, both from x to y, say that c is *shorter than* c' if $m < k$ and the distance of c is not greater than that of c'. Say that a word realization $w_1, \dots, w_m$ of a chain *has a trivial element* if there is $0 < i < m$ with $w_i = e$.

Claim 8.6. *If a chain $c = (z_0, z_1, z'_1 \dots, z_m, z'_m)$ from x to y has a word realization $w_1, \dots, w_m$ with a trivial element, then there is a chain from* *x to y which is shorter than c and has a word realization* $w'_1, \dots, w'_k$ *with* $w_1 \dots w_m = w'_1 \dots w'_k$.

Proof. Suppose $w_i = e$, i.e. $z_i = z'_i$. Consider the chain

$$z_0, \dots, z_{i-1}, z'_{i-1}, z_{i+1}, z'_{i+1}, \dots, z_m, z'_m$$

and note that $w_1, \dots, w_{i-1}, w_{i+1}, \dots, w_m \in W_n$ is its word realization. The fact that this chain is shorter than c follows from the triange inequality. □

Claim 8.7. *If a chain* $c = (z_0, z_1, z'_1 \dots, z_m, z'_m)$ *from* x *to* y *has a word realization* $w_1, \dots, w_m$ *and* $z_i = z'_{i+1}$ *for some* $0 < i < m$, *then there is a chain from* x *to* y *which is shorter than* c *and has a word realization* $w'_1, \dots, w'_k$ *with* $w_1 \dots w_m = w'_1 \dots w'_k$.

Proof. If $z_i = z'_{i+1}$, then consider the chain c' of $y_0, y_1, y'_1, \dots, y_{m-1}, y'_{m-1}$ with $y_j = z_j$ for $j < i$, $y_j = z_{j+1}$ for $j \geq i$, $y'_j = z'_j$ for $j \leq i$ and $y'_j = z'_{j+1}$ for $j > i$. Note that it is still a chain from x to y with a word realization $w'_1, \dots, w'_{m-1}$ with $w'_j = w_j$ for $j < i$, $w'_i = w_i w_{i+1}$ and $w'_j = w_{j+1}$ for $j > i$. □

Claim 8.8. *If chain* $c = (z_0, z_1, z'_1 \dots, z_m, z'_m)$ *from* x *to* y *realizes the distance from* x *to* y *and cannot be made shorter, then* $m = 1$ *and* $z_m = z'_m$.

Proof. Note that by Claim 8.7 and the assumption that $M\delta > \Delta$ we have that $m \leq M$. First note that the chain must be trivial. Indeed, since otherwise, for any word realization $w_1, \dots, w_m$ of c we have $w_1 \dots w_m \in T^{z'_1}_{z_1} \cdot \ldots \cdot T^{z'_m}_{z_m}$ and the latter set is disjoint from H if the chain is nontrivial. Now, since the chain is trivial, it has a word realization $w_1, \dots, w_m$ such that $w_1 \dots w_m = e$. Now, if $m \geq 2$, then Claims 8.5 and 8.6 imply that the chain can be made shorter. Therefore, $m = 1$ and $w_m = e$. □

Note finally that since (x, y, y) is a chain from x to y, Claim 8.8 implies that $d_Y([x, e], [y, e]) = d_X(x, y)$ and we have that $x \mapsto [x, e]$ is an isometric embedding, as needed. This ends the proof.

□

Lemma 8.9. *The Urysohn space has the extension property.*

Proof. This is a standard amalgamation argument. Note that since the language of metric spaces does not have any function symbols, instead of finitely generated structures, we talk about finite tuples. Suppose then that $\bar{b} = (b_1, \dots, b_n), \bar{c} = (c_1, \dots, c_m), \bar{a} = (a_1, \dots, a_k)$ are finite tuples in $\mathbb{U}$. Write $B = \{b_1, \dots, b_n\}, C = \{c_1, \dots, c_m\}, A = \{a_1, \dots, a_k\}$ and suppose $A \subseteq B \cap C$. Let C' be copy of C with $B \cap C' = A$ and let $D = B \cup C'$ be a metric space with the metric d_D such that $d_D \restriction B = d_{\mathbb{U}} \restriction B$, $d_D \restriction C' = d_{\mathbb{U}} \restriction C$ (under the natural identification) and if $b \in B, c \in C'$, then $d_D(b, c) = \min\{d_{\mathbb{U}}(b, a) + d_{\mathbb{U}}(a, c) : a \in A\}$. Assume without loss of generality that D is embedded into $\mathbb{U}$ over B and note that $C' \equiv_A C$ and $C' \perp_A B$. This ends the proof. □

To check that the Urysohn space admits isolated sequences, we need to introduce a couple of definitions. Given a metric structure M and a tuple $\bar{a} \in M^k$ and $\varepsilon > 0$ write $\mathrm{ball}_M(\bar{a}, \varepsilon)$ for $\{x \in M : d_M(x, a_i) < \varepsilon$ for some $i \leq k\}$. Suppose M is a homogeneous metric structure, $\bar{a} \in M^k$ for some $k \in \mathbb{N}$ and $p = \mathrm{qftp}(\bar{a})$. We say that a sequence $(\bar{a}_n : n \in \mathbb{N})$ of elements of $p(M)$ is *isometrically isolated* if there exists a sequence of $\varepsilon_n \in (0, \infty)$ and isometric embeddings

$$\eta_n : \mathrm{ball}_M(\bar{a}, \varepsilon_n) \to M$$

such that $\eta_n(\bar{a}) = \bar{a}_n$ and for every sequence $\bar{b}_n \in \mathrm{rng}(\eta_n)$ such that $\mathrm{qftp}(\bar{b}_n) = \mathrm{qftp}(\bar{a}_n)$ and $d_M(\bar{a}_n, \bar{b}_n) < \varepsilon_n$ there is an automorphism φ of M with $\varphi(\bar{a}_n) = \bar{b}_n$ for every $n \in \mathbb{N}$.

Note that any isometrically isolated sequence is isolated since if $p = \mathrm{qftp}(\bar{a})$ and $\eta : \mathrm{ball}_M(\bar{a}, \varepsilon) \to M$ is an isometric embedding for some $\varepsilon > 0$, then $\{\bar{b} = (b_1, \ldots, b_k) \in p(M) : \forall i \leq k \quad b_i \in \mathrm{rng}(\eta_n)\}$ is relatively ε-saturated over $\eta(\bar{a})$.

Given $k \in \mathbb{N}$, say that a sequence $(\bar{a}_n : n \in \mathbb{N})$ of k-tuples of elements of a metric structure M is *nontrivial convergent* if it is convergent as a sequence in M^k and if $\bar{a}_\infty = (a_1^\infty, \ldots, a_k^\infty)$ is its limit and $\bar{a}_n = (a_1^n, \ldots, a_k^n)$, then $a_i^n \neq a_j^\infty$ and $a_i^n \neq a_j^m$ for any $(n, i) \neq (m, j) \in \mathbb{N}^2$. In particular, $a_i^n \neq a_j^n$ for every $n \in \mathbb{N}$ and $i \neq j$. Note that, in case $k = 1$, a nontrivial convergent sequence is a convergent sequence such that all its elements are distinct and different from its limit.

A basic property of the Urysohn space that we will use in the arguments below, due to Huhunaišvili [21] (cf. [38, Proposition 5.1.20]), says that any partial isometry between compact subspaces of $\mathbb{U}$ can be extended to an isometry of $\mathbb{U}$.

Lemma 8.10. *For every $k \in \mathbb{N}$ and a quantifier-free k-type p, any nontrivial convergent sequence in p is isometrically isolated in p.*

Proof. Let p be the quantifier-free type of $\bar{a} \in \mathbb{U}^k$. Write $\bar{a} = (a^1, \ldots, a^k)$ and let $\delta_{ij} = d_{\mathbb{U}}(a^i, a^j)$ for $i, j \leq k$.

Let $\bar{a}_n$ be a nontrivial convergent sequence in p. Assume that $\bar{a}_n$ converges to $\bar{a}_\infty = (a_1^\infty, \ldots, a_k^\infty)$. Write $\bar{a}_n = (a_1^n, \ldots, a_k^n)$ for each $n \in \mathbb{N}$. For each $n, m \in \mathbb{N}$ and $i, j \leq k$ let $\delta_{ij}^{nm} = d_{\mathbb{U}}(a_i^n, a_j^m)$ and $\delta_{ij}^{n\infty} = d_{\mathbb{U}}(a_i^n, a_j^\infty)$ and note that $\lim_{m,n\to\infty} \delta_{ij}^{nm} = \delta_{ij}$ and $\lim_{n\to\infty} \delta_{ij}^{n\infty} = \delta_{ij}$.

For each $n \in \mathbb{N}$ choose $\varepsilon_n > 0$ such that $\varepsilon_n < \delta_{ij}^{nm}$ for each $m \neq n$ and $i, j \leq k$ as well as $\varepsilon_n < \delta_{ij}^{nn} = \delta_{ij}$ for all $i \neq j$, $i, j \leq k$. Such an $\varepsilon_n > 0$ exists since the sequence $(\bar{a}_n : n \in \mathbb{N})$ is nontrivial and $\lim_{m\to\infty} \delta_{ij}^{nm} = d_{\mathbb{U}}(a_i^n, a_j^\infty) > 0$.

For each $n \in \mathbb{N}$ write A_n for $\mathrm{ball}(\{a_1^n, \ldots, a_k^n\}, \varepsilon_n)$. Note that A_n is a disjoint union of balls around the points a_i^n. Consider the metric space B' which is the disjoint union $\bigcup_{n\in\mathbb{N}} B_n'$ with each B_n' a copy of A_n (say the copy of a_i^n in B_k is $a_i^{n\prime}$) and let the metric on B' be defined so that it is equal to the original metric $d_{\mathbb{U}}$ on each B_n' and if $x, y \in \bigcup_{n\in\mathbb{N}} B_n'$ are such that

$x \in B'_n$ and $y \in B'_m$ with $n \neq m$ and $x \in \mathrm{ball}(a_i^{n\prime}, \varepsilon_n)$, $y \in \mathrm{ball}(a_j^{m\prime}, \varepsilon_m)$, then

$$d_{B'}(x,y) = \delta_{ij}^{nm}.$$

Note that $d_{B'}$ is a metric by the choice of the numbers ε_n. Now let $B_\infty = \{a_1^{\infty\prime}, \ldots, a_k^{\infty\prime}\}$ be a copy of $\{a_1^\infty, \ldots, a_k^\infty\}$ and let $B = B' \cup B_\infty$ with the metric $d_B = d_{B'}$ on B', $d_B = d_{B_\infty}$ on B_∞ and if $x \in B'$ is such that $x \in B_n$ with $x \in \mathrm{ball}(a_j^{n\prime}, \varepsilon_n)$ and $i \leq k$, then

$$d_B(x, a_i^{\infty\prime}) = \delta_{ji}^{n\infty}.$$

Since the subspace of B consisting of the points $a_i^{n\prime}$ and $a_i^{\infty\prime}$ for $n \in \mathbb{N}$ and $i \leq k$ is compact and isometric to the subspace of $\mathbb{U}$ consisting of the points a_i^n and a_i^∞ for $n \in \mathbb{N}$ and $i \leq k$, the Huhunaišvili theorem [21] implies that the map $a_i^{n\prime} \mapsto a_i^n$ and $a_i^{\infty\prime} \mapsto a_i^\infty$ extends to an isometric embedding η of B into $\mathbb{U}$.

For each $n \in \mathbb{N}$ write B_n for the image of B'_n under η. Note that each B_n is an isometric copy of $\mathrm{ball}(\bar{a}, \varepsilon_n)$. We claim that the sets B_n (treated as the embeddings of $\mathrm{ball}(\bar{a}, \varepsilon_n)$), together with the numbers ε_n witness that $\bar{a}_n$ is isometrically isolated. For that, pick a sequence of isometric embeddings $\varphi_n : \{a_1^n, \ldots, a_k^n\} \to B_n$ with $d_{\mathbb{U}}(\varphi_n(a_i^n), a_i^n) < \varepsilon_n$ for each $i \leq k$. Consider a partial isometry φ' of $\mathbb{U}$ with $\mathrm{dom}(\varphi') = \bigcup_{n \in \mathbb{N}} \{a_1^n \ldots, a_k^n\} \cup \{a_1^\infty, \ldots, a_k^\infty\}$ such that $\varphi'(a_i^\infty) = a_i^\infty$ and $\varphi'(a_i^n) = \varphi_n(a_i^n)$. Note that φ' is a partial isometry of the Urysohn space with compact domain, so again by the Huhunaišvili theorem [21], there is an isometry $\varphi \in \mathrm{Iso}(\mathbb{U})$ that extends φ'. Clearly, φ extends each φ_n, which shows that B_n are as needed and the sequence is isometrically isolated. This ends the proof. □

Proposition 8.11. *The Urysohn space admits isolated sequences.*

Proof. Suppose p is the quantifier-free k-type of a tuple $\bar{a} = (a_1, \ldots, a_k)$. First note that we can assume that $\bar{a}$ consists of distinct elements. Indeed, otherwise one can remove repetitions from $\bar{a}$ and work with a quantifier-free m-type q for some $m < k$. Then, for every m-tuple $\bar{b} \in q(M)$ there is a unique tuple $\bar{b}' \in p(M)$, which contains $\bar{b}$ such that

- if $(\bar{b}_n : n \in \mathbb{N})$ is isolated in q, then $(\bar{b}'_n : n \in \mathbb{N})$ is isolated in p,
- the map $\bar{b} \mapsto \bar{b}'$ is a homeomorphism of $q(M)$ and $p(M)$.

Now, suppose $Z \subseteq p(\mathbb{U})$ is nonmeager. Without loss of generality (restricting to an open subset of $p(\mathbb{U})$ if neccessary), assume that Z is nonmeager in every nonempty open set. Pick any $\bar{a}_\infty \in p(\mathbb{U})$ with $\bar{a}_\infty = (a_1^\infty, \ldots, a_k^\infty)$ and note that $a_i^\infty \neq a_j^\infty$ for $i \neq j$. Using the assumption that $Z \cap V$ is nonmeager for every open neighborhood V of $\bar{a}_\infty$, construct a sequence $\bar{a}_n$ of elements of Z convergent to $\bar{a}_\infty$ such that if $\bar{a}_n = (a_1^n, \ldots, a_k^n)$, then $a_i^n \neq a_j^\infty$ and $a_i^n \neq a_j^m$ for any $n, m \in \mathbb{N}$ and $i, j \leq k$ with $(n, i) \neq (m, j)$. This sequence is then nontrivial convergent and hence isolated by Proposition 8.10. □

9. The measure algebra

Recall that given a standard probability space $(X, \mathcal{B}, \mu)$ we define the equivalence $\approx$ on $\mathcal{B}$ by $A \approx B$ if $\mu(A \Delta B) = 0$ and the measure algebra is the family of equivalence classes of sets in $\mathcal{B}$. Given $A \in \mathcal{B}$ write $[A]$ for its $\approx$-equivalence class (although we will often abuse notation and write only A instead of $[A]$). The measure algebra is then the family of $\approx$-classes of the sets in $\mathcal{B}$. It becomes a metric space with the metric $d_{\mathrm{MALG}}([A],[B]) = \mu(A\Delta B)$ and we treat it as a metric structure together with this metric, the operation of symmetric difference Δ and the empty set as a constant. We write MALG for the structure $(\mathcal{B}/\approx, d_{\mathrm{MALG}}, \Delta, \emptyset)$. The Sikorski duality [27, Theorem 15.9] connects automorphisms of MALG and measure-preserving bijections on the space X. In particular, it implies that the group of automorphisms of MALG with the topology of pointwise convergence is isomorphic to the group of measure-preserving bijections $\mathrm{Aut}(X, \mu)$ with the weak topology (see [28, Section 1]). For more details about the measure algebra and the standard measure space we refer the reader to [37, Chapter 22] and to [12, Chapter 32].

Throughout the proofs below we often use the fact (cf [29, Lemma 7.10]) that whenever $A, B \subseteq X$ have the same measure, then there is a measure-preserving bijection $f : X \to X$ such that $f(A) = B$. Below, given a finite subalgebra $\mathcal{A}$ of MALG, we write $\mathrm{atom}(\mathcal{A})$ for the set of atoms of $\mathcal{A}$.

Lemma 9.1. *The measure algebra* MALG *has locally finite automorphisms.*

Proof. Note that finitely generated substructures of MALG are finite subalgebras. Thus, to show locally finite automorphism we need to prove the following. For every finite subalgebra $\mathcal{A} \subseteq$ MALG there exists a finite algebra $\mathcal{B} \subseteq$ MALG with $\mathcal{A} \subseteq \mathcal{B}$ such that every partial automorphism of $\mathcal{A}$ extends to an automorphism of $\mathcal{B}$. To see this, we need a couple of notions. Given finite $\mathcal{A} \subseteq \mathcal{B} \subseteq$ MALG and $A_1, A_2 \in \mathcal{A}$ say that A_1 and $\mathcal{A}_2$ *are identically partitioned by* $\mathcal{B}$ if the sets $\{\mu(A_1 \cap B) : B \in \mathrm{atom}(\mathcal{B}), B \subseteq A_1\}$ and $\{\mu(A_2 \cap B) : B \in \mathrm{atom}(\mathcal{B}), B \subseteq A_2\}$ (both counted with repetitions) are equal (up to a permutation). Note that if $\mathcal{A} \subseteq \mathcal{B} \subseteq$ MALG are finite and such that every $A_1, A_2 \in \mathcal{A}$ of the same measure are identically partitioned by $\mathcal{B}$, then any partial automorphism of $\mathcal{A}$ extends to an automorphism of $\mathcal{B}$. Moreover, it is enough to guarantee this for A_1 and A_2 disjoint.

Say that a finite extension $\mathcal{A} \subseteq \mathcal{B}$ is *good* if every two atoms of $\mathcal{A}$ of the same measure are identically partitioned by $\mathcal{B}$. Note that this is a transitive relation and if $\mathcal{A} \subseteq \mathcal{B}$ is good and $A_1, A_2 \in \mathcal{A}$ are identically partitioned by $\mathcal{A}$, then A_1 and A_2 are identically partitioned by $\mathcal{B}$.

Enumerate as $((A_i, B_i) : 1 \leq i < N)$ the set of all pairs A, B of disjoint sets in $\mathcal{A}$ of the same measure. By induction on $i \leq N$ construct a sequence of finite subalgebras $\mathcal{A}_i \subseteq$ MALG with $\mathcal{A}_0 = \mathcal{A}$ such that for every $1 \leq i \leq N$ we have

- $A_{i-1} \subseteq \mathcal{A}_i$ is good

- A_i and B_i are identically partitioned by $\mathcal{A}_{i+1}$.

After this is done, the algebra $\mathcal{B} = \mathcal{A}_N$ will be as needed.

It is enough to describe the induction step construction of $\mathcal{A}_{i+1}$ from $\mathcal{A}_i$. Note first that (by shrinking A_i and B_i if neccessary) we can assume that for every $A, B \in \mathrm{atom}(\mathcal{A}_i)$ with $A \subseteq A_i$ and $B \subseteq B_i$ we have $\mu(A) \neq \mu(B)$. Write $R = \{\mu(A) : A \in \mathrm{atom}(\mathcal{A}_i), A \subseteq A_i\}$ and $S = \{\mu(B) : B \in \mathrm{atom}(\mathcal{A}_i), B \subseteq B_i\}$, so that $R \cap S = \emptyset$. Write $a = \mu(A_i) = \mu(B_i)$ and let $U \subseteq X$ be a measurable set of measure a. Let $\mathcal{C}_1$ be the algebra of subsets of A_i equal to $\mathcal{A}_i \restriction A_i$ and $\mathcal{C}_2$ be the algebra of subsets of B_i equal to $\mathcal{A}_i \restriction B_i$. Find two algebras $\mathcal{C}'_1$ and $\mathcal{C}'_2$ of subsets of U such that $\mathcal{C}'_1$ is isomorphic to $\mathcal{C}_1$, $\mathcal{C}'_2$ is isomorphic to $\mathcal{C}_2$ and $\mathcal{C}'_1, \mathcal{C}'_2$ are (stochastically) independent. Fix measure-preserving bijections $\varphi : A_i \to U$, $\psi : B_i \to U$ such that φ maps $\mathcal{C}_1$ to $\mathcal{C}'_1$ and ψ maps $\mathcal{C}_2$ to $\mathcal{C}'_2$. Write $\mathcal{C}'$ for the algebra of subsets of U generated by $\mathcal{C}'_1$ and $\mathcal{C}'_2$ and let $\mathcal{C}$ be the algebra of subsets of $A_i \cup B_i$ generated by the preimages $\varphi^{-1}(C)$ and $\psi^{-1}(C)$ for $C \in \mathcal{C}'$. Note that the sets A_i and B_i are identically partitioned by the algebra generated by $\mathcal{A}_i$ and $\mathcal{C}$. Note also that if $C \subseteq A_i$ is an atom of $\mathcal{A}_i$ of measure r, then C is partitioned by $\mathcal{C}$ into sets of measures $r \cdot \mu(D)$ for $D \in \mathrm{atom}(\mathcal{C}_2)$ and analogously, if $E \subseteq B_i$ is an atom of $\mathcal{A}_i$ of measure s, then D is partitioned by $\mathcal{C}$ into sets of measures $s \cdot \mu(F)$ for $F \in \mathrm{atom}(\mathcal{C}_2)$. In order to construct a good extension of $\mathcal{A}_i$, partition every atom $A \in \mathrm{atom}(\mathcal{A}_i)$ that is disjoint from $A_i \cup B_i$ as follows:

(i) if $\mu(A) \in R$, then partition A into sets of measures $\mu(A) \cdot \mu(D)$, for $D \in \mathrm{atom}(\mathcal{C}_2)$,
(ii) if $\mu(A) \in S$, then partition A into sets of measures $\mu(A) \cdot \mu(F)$, for $F \in \mathrm{atom}(\mathcal{C}_1)$.

Let $\mathcal{A}_{i+1}$ be an extension of $\mathcal{A}_i$ generated by all the partitions as in (i) and (ii) above and by $\mathcal{C}$. Now $\mathcal{A}_{i+1}$ is a good extension of $\mathcal{A}_i$ and the sets A_i and B_i are identically partitioned by $\mathcal{A}_{i+1}$. This ends the construction and the proof. □

Lemma 9.2. *The measure algebra* MALG *has the extension property.*

Proof. Suppose A, B, C are finitely generated subalgebras of MALG with $A \subseteq B \cap C$. Write $A_1, \ldots, A_n$ for the set of atoms of A. Find an automorphism φ of the measure space which fixes $A_1, \ldots, A_n$ and within each A_i sends the atoms of C contained in A_i to sets which are (stochastically) independent from the atoms of B contained in A_i. It is easy to see that $\varphi(C) \perp_A B$. □

To see that MALG admits isolated sequences, we need to understand which quantifier-free ε-types are realized over finite tuples in MALG.

Definition 9.3. Suppose $k \in \mathbb{N}$ and $\mathcal{P} = (A_1, \ldots, A_k)$ is a partition of X into positive measure sets. Let $E = (e_{ij} : 1 \leq i, j \leq k)$ be a matrix of reals. Say that E is *$\mathcal{P}$-additive* if the following conditions hold:

- $e_{ii} \geq 0$ and $0 \leq e_{ij} \leq \mu(A_i) + \mu(A_j)$ for every $i, j \leq k$,

- the following equations are satisfied:

$$e_{ii} = \sum_{j\neq i} \mu(A_i) + \mu(A_j) - e_{ij}$$

$$e_{ii} = \sum_{j\neq i} \mu(A_i) + \mu(A_j) - e_{ji}$$

Claim 9.4. *Suppose $\mathcal{P} = (A_1, \ldots, A_k)$ is a partition of X into positive measure sets and $\varphi \in \mathrm{Aut}(\mathrm{MALG})$. Let $e_{ij} = d_{\mathrm{MALG}}(A_i, \varphi(A_j))$. Then the matrix $E = (e_{ij} : 1 \leq i, j \leq k)$ is $\mathcal{P}$-additive.*

Proof. Let $f : X \to X$ be a measure-preserving bijection that induces φ. For $i \neq j$ write $\varepsilon_{ij} = \mu(f(A_i) \cap A_j)$. Note that

$$e_{ij} = \mu(f(A_i)\Delta A_j) = \mu(A_j) + \mu(A_j) - 2\varepsilon_{ij}.$$

This implies that

$$e_{ii} = \mu(f(A_i)\Delta A_i) = 2\sum_{j\neq i} \varepsilon_{ij} = \sum_{j\neq i} \mu(A_i) + \mu(A_j) - e_{ij}$$

On the other hand,

$$e_{ii} = \mu(f(X \setminus A_i)\Delta(X \setminus A_i)) = 2\sum_{j\neq i} \varepsilon_{ji} = \sum_{j\neq i} \mu(A_i) + \mu(A_j) - e_{ji}.$$

□

Lemma 9.5. *Let $\bar{a} = (A_1, \ldots, A_k)$ be a partition of X into positive measure sets and let $p = \mathrm{qftp}(\bar{a})$. Suppose $C_1, \ldots, C_k$ are such that $C_i \subseteq A_i$ for each $i \leq k$ and $\mu(C_1) = \ldots = \mu(C_k) > 0$. Let*

$$M = \{(B_1, \ldots, B_k) \in p(\mathrm{MALG}) : \forall i \neq j \leq k \;\; B_i \cap A_j \subseteq C_j \;\wedge\; A_i \setminus B_i \subseteq C_i\}.$$

Then M is relatively $2\mu(C_1)$-saturated over $\bar{a}$.

Proof. Write $\varepsilon = 2\mu(C_1)$. Let $E_1, \ldots, E_k \in \mathrm{MALG}$ be such that $\mu(E_i\Delta A_i) < \varepsilon$ and $\mathrm{qftp}(E_1, \ldots, E_k) = p$. Write $e_{ij} = \mu(E_i\Delta A_j)$ and note that $E = (e_{ij} : i, j \leq k)$ is $\bar{a}$-additive by Claim 9.4. We need to find $(B_1, \ldots, B_k) \in M$ such that $\mu(B_i\Delta A_j) = e_{ij}$ for each $i, j \leq k$. For each $i \neq j$ write

$$\varepsilon_{ij} = \frac{1}{2}(\mu(A_i) + \mu(A_j) - e_{ij})$$

and note that by $\bar{a}$-additivity we have

$$\sum_{j\neq i} \varepsilon_{ji} = \frac{1}{2} e_{ii} < \frac{1}{2}\varepsilon = \mu(C_i)$$

Thus, we can find disjoint measurable sets $D_{ji} \subseteq C_i$ such that $\mu(D_{ji}) = \varepsilon_{ji}$. Write $D_i = \bigcup_{j\neq i} D_{ji}$ and note that $\mu(D_i) = \frac{1}{2}e_{ii}$. Put $B_i = A_i \setminus D_i \cup \bigcup_{j\neq i} D_{ij}$ and note that since (by $\bar{a}$-additivity)

$$\sum_{j\neq i} \varepsilon_{ji} = \sum_{j\neq i} \varepsilon_{ij}$$

we have that $\mu(B_i) = \mu(A_i)$. The sets B_i are pairwise disjoint since the sets D_{ij} are disjoint and so $\mathrm{qftp}(B_1, \ldots, B_k) = \mathrm{qftp}(A_1, \ldots, A_k)$. Also, we have

$$\mu(B_i \Delta A_j) = \mu(B_i) + \mu(A_j) - 2\varepsilon_{ij} = e_{ij}$$

if $j \neq i$ and

$$\mu(B_i \Delta A_i) = 2 \sum_{j \neq i} \varepsilon_{ij} = e_{ii}.$$

This shows that $(B_1, \ldots, B_k)$ is as needed and M is relatively ε-saturated over $\bar{a}$. □

Definition 9.6. Let $\bar{a} = (A_1, \ldots, A_k)$ be a tuple in MALG such that $\bar{a}$ is a partition of X into positive measure sets. Write p for $\mathrm{qftp}(\bar{a})$. Given a sequence $\bar{a}_n = (A^n_1, \ldots, A^n_k)$ in p say that it is *weakly independent* if there is a sequence $((C^n_1, \ldots, C^n_k) : n \in \mathbb{N})$ such that

- $C^n_i \subseteq A^n_i$ for each $i \leq k$ and $n \in \mathbb{N}$,
- $\mu(C^n_1) = \ldots = \mu(C^n_k) > 0$ for each $n \in \mathbb{N}$,
- all sets $\{C^n_i : i \leq k, n \in \mathbb{N}\}$ are pairwise disjoint,
- if $m \neq n$, then $C^n_i \subseteq A^m_1$ for every $i \leq k$.

Lemma 9.7. *If a sequence $\bar{a}_n$ in* MALG *is weakly independent, then it is isolated.*

Proof. Let $k \in \mathbb{N}$ be such that each $\bar{a}_n = (A^n_1, \ldots, A^n_k)$ is an k-element partition of X. Suppose $((C^n_1, \ldots, C^n_k) : n \in \mathbb{N})$ witnesses that the sequence is weakly independent and let $\varepsilon_n = 2\mu(C^n_1)$. Write p for the quantifier-free type of $\bar{a}_n$ and let

$$M_n = \{(B_1, \ldots, B_k) \in p(\mathrm{MALG}) : \forall i \neq j \leq k \ \ B_i \cap A^n_j \subseteq C^n_j \wedge A^n_i \setminus B_i \subseteq C^n_i\}.$$

We claim that the sequence of sets M_n together with ε_n witness that the sequence $\bar{a}_n$ is isolated. The fact that M_n is relatively ε_n-saturated over $\bar{a}_n$ follows directly from Lemma 9.5.

Suppose now that $\bar{b}_n = (B^n_1, \ldots, B^n_k)$ are such that each $\bar{b}_n$ belongs to M_n and $\mathrm{qftp}(\bar{b}_n) = \mathrm{qftp}(\bar{a}_n)$. We need to find $\varphi \in \mathrm{Aut}(\mathrm{MALG})$ such that $\varphi(\bar{a}_n) = \bar{b}_n$ for each $n \in \mathbb{N}$. For every $n \in \mathbb{N}$ and $i \neq j$ let $D^n_{ij} = B^n_i \cap A^n_j \subseteq C^n_j$ and let $\varepsilon^n_{ij} = \mu(D^n_{ij})$. For each $i \leq k$ write $D^n_i = \bigcup_{j \neq i} D^n_{ji}$. Let $E^n_i = A^n_i \setminus B^n_i \subseteq C^n_i$ and note that

$$\mu(E^n_i) = \sum_{j \neq i} \varepsilon^n_{ij} = \sum_{j \neq i} \varepsilon^n_{ji}$$

since $\mu(B^n_i) = \mu(A^n_i)$. For every $j \neq i$ find measurable sets $E^n_{ji} \subseteq E^n_i$ such that $E^n_i = \bigcup_{j \neq i} E^n_{ji}$ and $\mu(E^n_{ji}) = \varepsilon^n_{ji}$. Now, for each $n \in \mathbb{N}$ find a measure-preserving bijection

$$f_n : \bigcup_{i \leq k} D^n_i \cup E^n_i \to \bigcup_{i \leq k} D^n_i \cup E^n_i$$

such that

- $f_n(E_i^n) = D_i^n$ for each $i \leq k$,
- $f_n(D_{ij}^n) = E_{ij}^n$ for every $i \neq j \leq k$

Let $f : X \to X$ be a measure-preserving bijection such that $f \supseteq f_n$ for each $n \in \mathbb{N}$ and f is equal to the identity on the complement of the set $\bigcup_{n\in\mathbb{N}} \bigcup_{i\leq k} D_i^n \cup E_i^n$. Note that for $m \neq n$ and $i \leq k$, the set $D_i^m \cup E_i^m$ is contained in C_i^m, so the function f_m maps A_1^n into itself and the domain of f_m is disoint from A_i^m for $i > 1$. This implies that $f(A_i^n) = B_i^n$ and hence the autmomorphism of MALG induced by f is as needed. This ends the proof. □

Proposition 9.8. *The measure algebra* MALG *admits isolated sequences.*

Proof. Suppose p is a quantifier-free k-type of a tuple $\bar{a} = (A_1, \ldots, A_k)$ in MALG. First note that we can assume that the elements of $\bar{a}$ form a partition of X into positive measure sets. Otherwise, one can consider the atoms of the algebra generated by $\bar{a}$ and work with a quantifier-free m-type q for m equal to the number of these atoms. Then, for every m-tuple $\bar{b} \in q(M)$ there is a unique tuple $\bar{b}' \in p(M)$, such that the algebras generated by $\bar{b}$ and $\bar{b}'$ are the same and such that

- if $(\bar{b}_n : n \in \mathbb{N})$ is isolated in q, then $(\bar{b}'_n : n \in \mathbb{N})$ is isolated in p,
- the map $\bar{b} \mapsto \bar{b}'$ is a homeomorphism of $q(M)$ and $p(M)$.

Now, suppose $Z \subseteq p(\mathrm{MALG})$ is nonmeager and assume (restricting to an open subset if neccessary) that Z is nonmeager in every nonempty open set. Construct a sequence of $(A_1^n, \ldots, A_k^n) \in Z$ and positive measure pairwise disjoint sets $D_i^n \subseteq A_i^n$ (for $i \leq k$) together with positive reals δ_i^n such that for every $n \in \mathbb{N}$ we have

(i) $D_1^n, \ldots, D_k^n \subseteq A_1^m$ if $m < n$,
(ii) $\mu(D_i^n \cap A_1^{n+1}) = \delta_i^n$ for every $i \leq k$,
(iii) $d_{\mathrm{MALG}}(A_k^{n+l+1}, A_k^{n+l}) < \delta_k^n/2^{l+1}$ for every $l \geq 1$,
(iv) $\mu(\bigcap_{m\leq n} A_1^m \setminus D_1^m) > 0$.

After this is done, for every $i \leq k$ and $n \in \mathbb{N}$ put $C_i^n = D_i^n \cap \bigcap_{m>n} A_1^m$ and note that by (ii) and (iii) above we have that $\mu(C_i^n) > 0$ for each $i \leq k$ and $n \in \mathbb{N}$. By shrinking the sets C_i^n if neccessary, we can assume that $\mu(C_1^n) = \ldots = \mu(C_k^n)$. Then the definition of C_i^n and condition (i) above imply that if $m \neq n$, then $C_i^n \subseteq A_1^m$. Given that D_i^n are pariwise disjoint, so are the sets C_i^n and so the sequence $(A_1^n, \ldots, A_k^n)$ is weakly independent, as witnessed by C_i^m and hence isolated by Lemma 9.7.

To perform the induction step, suppose we have constructed $(A_1^j, \ldots, A_k^j) \in Z$ for $j \leq n$, the sets D_i^j for $i \leq n$ and $i \leq k$ as well as δ_i^j for $j \leq n-1$ and $i \leq k$. Consider the open set

$$U = \{(A_1, \ldots, A_k) \in p(\mathrm{MALG}) : \forall i \leq k \ \ d_{\mathrm{MALG}}(A_i, A_i^n) \leq \min_{m<n} \frac{\delta_i^m}{2^{n-m+1}}\}.$$

Write also $F = \bigcap_{m\leq n} A_1^m \setminus D_1^m$ and note that by the inductive assumption (iv), we have $\mu(F) > 0$. Using the fact that for any positive measure set

E and $i \leq k$ the set $\{(A_1, \ldots, A_n) \in p(\mathrm{MALG}) : \mu(A_i \cap E) = 0\}$ is closed nowhere dense, find $(A_1^{n+1}, \ldots, A_k^{n+1}) \in U \cap Z$ such that

(a) $\mu(F \cap A_i^{n+1}) > 0$ for every $i \leq k$,
(b) $\mu(A_1^{n+1} \cap D_i^n) > 0$ for every $i \leq k$.

Now, using (a) above, find $D_i^{n+1} \subseteq A_i^{n+1} \cap F$ such that

$$0 < \mu(D_i^{n+1}) < \frac{1}{2}\mu(A_i^{n+1} \cap F).$$

This implies that the inductive condition (iv) will be satisfied at the next step. Put $\delta_i^n = \mu(A_1^{n+1} \cap D_i^n)$ and note that $\delta_i^n > 0$ by (b) above. Condition (iii) holds because $(A_1^{n+1}, \ldots, A_k^{n+1}) \in U$ and thus, this concludes the induction step. This ends the proof. □

10. The Hilbert space

The orthogonal group $O(\ell_2)$ is the group of automorphism of the (real) Hilbert space. The Hilbert space here is treated as the metric structure with the first sort being $(\ell_2, 0, +)$ and the second sort being the real line with the field structure (including the inverse function defined on non-zero elements by $x \mapsto x^{-1}$ and mapping 0 to 0, as well as the function $x \mapsto -x$) and constants for the rationals. We also add to the language the multiplication by scalars function $\cdot : \mathbb{R} \times \ell_2 \to \ell_2$ (i.e. $(a, v) \mapsto a \cdot v$) as well as the inner product function $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$.

Recall that by the Mazur–Ulam theorem [35] any isometry of a normed vector space which preserves zero, is a linear isomorphism (in case of the Hilbert space this is even simpler than the general case of the Mazur–Ulam theorem), so we could also consider the structure only with the constant 0 and the inner product function. Still another way would be to look at the unit sphere in the Hilbert space equipped only with the metric (as a metric space with no additional structure) and then the orthogonal group would be the group of isometries of the sphere. We will however, use the above language, as it seems the most natural, and we will make use of it in order to talk about substructures of the Hilbert space.

The unitary group $U(\ell_2)$ is the automorphism group of the complex Hilbert space and the arguments below apply in the same way to the complex Hilbert space, so we will focus only of the real Hilbert space.

Claim 10.1. *If A is a finitely generated substructure of the Hilbert space, then there exists a countable field $\mathbb{K} \subseteq \mathbb{R}$ such that $\mathbb{Q} \subseteq \mathbb{K}$ and A is a finite-dimensional $\mathbb{K}$-vector space*

Proof. Let $\mathbb{K}$ consist of the elements of A which are of the second sort. Since the language contains constants for the rationals, we have $\mathbb{Q} \subseteq \mathbb{K}$ and since the language contains the language of fields, $\mathbb{K}$ is a field. Clearly then A is a $\mathbb{K}$-vector space and the dimension is bounded by the number of generators of A. □

Lemma 10.2. *The Hilbert space ℓ_2 has locally finite automorphisms.*

Proof. In fact, ℓ_2 has the following stronger property. For any finitely generated substructure $A \subseteq \ell_2$, any isomoprhism between finitely generated substructures of A extends to an automorphism of A. To see this, let $A_1, A_2 \subseteq A$ be finitely generated substructures and $\varphi : A_1 \to A_2$ be an isomorphism. Let $\mathbb{K}$ and $\mathbb{K}_1, \mathbb{K}_2 \subseteq \mathbb{K}$ be such that A is a $\mathbb{K}$-vector space, A_1 is $\mathbb{K}_1$-vector space and A_2 is a $\mathbb{K}_2$-vector space. Write A_1' for the $\mathbb{K}$-vector space generated by A_1 and A_2' for the $\mathbb{K}$-vector space generated by A_2. Note that since φ preserves the inner product, it is an isometry and since both $\mathbb{K}_1$ and $\mathbb{K}_2$ contain $\mathbb{Q}$, the map φ can be extended to an isomorphism $\varphi' : A_1' \to A_2'$. Now, since $\mathbb{K}$ is a field, the usual Gram–Schmidt orthogonalization process gives orthogonal bases $\{b_1^1, \ldots, b_k^1\}$ and $\{b_1^2, \ldots, b_k^2\}$ for the orthogonal complements of A_1' in A and A_2 in A (respectively). The map which extends φ and maps b_i^1 to b_i^2 extends to an automorphism of A. $\square$

Note that the above proof also shows that given a finitely generated substructure A of the Hilbert space and its finitely generated substructure $C \subseteq A$, we can form the orthogonal complement $A \ominus C$ inside A using the standard Gram–Schmidt process. The extension property for the Hilbert space is then straightforward and based on the following claim.

Claim 10.3. *Given finitely generated substructures $A, B, C \subseteq \ell_2$ with $C \subseteq A \cap B$, if $A \ominus C \perp B \ominus C$, then $A \mathop{\smile\!\!\!\!\!\!|}_C B$.*

Proof. This is elementary linear algebra and the proof is analogous to that of Lemma 10.2. $\square$

Corollary 10.4. *The Hilbert space ℓ_2 has the extension property.*

Proof. Given finite-dimensional subspaces $A, B, C \subseteq \ell_2$ with $C \subseteq A \cap B$ find a copy $D \subseteq \ell_2$ of $B \ominus C$ which is orthogonal to A. Then $C \oplus D$ witnesses the extension property by Claim 10.3. $\square$

Before we show that the Hilbert space admits isolated sequences, we need a couple of lemmas. Below, given a closed subspace $V \subseteq \ell_2$ and a vector $v \in \ell_2$ write $\pi_V(v)$ for the projection of v onto V. Also, $\mathrm{ball}_{\ell_2}(v, \varepsilon)$ stands for the open ball $\{w \in \ell_2 : ||w - v|| < \varepsilon\}$ and $\mathrm{S}_{\ell_2}(v, \varepsilon)$ stands for the sphere $\{w \in \ell_2 : ||w - v|| = \varepsilon\}$. Recall also that $\bar{v} = (v_1, \ldots, v_k) \in \ell_2$ is an *orthonormal tuple* if $||v_i|| = 1$ and $v_i \perp v_j$ for $i \neq j$

Lemma 10.5. *Suppose $\bar{v} = (v_1, \ldots, v_k)$ is an orthonormal tuple in ℓ_2 and let $H \subseteq \ell_2$ be an infinite-dimensional closed subspace. Suppose $V_1, \ldots, V_k \subseteq \ell_2$ are closed infinite-dimensional subspaces such that $v_i \in V_i$ and $V_i \perp V_j$ for $i \neq j$. Write $H_i = V_i \cap H$ and suppose H_i is infinite-dimensional and that $\pi_{H_i}(v_i) \neq 0$ for each $i \leq k$. Then there exists $\varepsilon > 0$ such that for every $\bar{v}' = (v_1', \ldots, v_k')$ such that*

$$\bar{v}' \equiv \bar{v}, \quad v_i' \in V_i \quad \textit{and} \quad ||v_i' - v_i|| < \varepsilon$$

for each $i \leq k$, there exists $\bar{v}'' = (v''_1, \dots, v''_k)$ such that

$$\bar{v}'' \equiv_{\bar{v}} \bar{v}' \quad \text{and} \quad v''_i - v_i \in H_i$$

for each $i \leq k$.

Proof. Note that since the subspaces V_i are mutually orthogonal, it is enough to prove the lemma for $k = 1$. Assume then $V_1 = \ell_2$ and write $v = v_1$ so that $\pi_H(v) \neq 0$ (i.e. $v \not\perp H$). We need to show that there exists $\varepsilon > 0$ such that for every $v' \in \mathrm{S}_{\ell_2}(0,1)$ with $||v' - v|| < \varepsilon$ there exists $v'' \in \mathrm{S}_{\ell_2}(0,1)$ with $v'' - v \in H$ and $v'' \equiv_v v'$. The latter is equivalent to $||v'' - v|| = ||v' - v||$ (since v', v'' have the same norm). Since $v \not\perp H$, there exists $w \in \mathrm{S}_{\ell_2}(0,1)$ such that $w \neq v$ and $w - v \in H$. Let $\varepsilon = ||w - v||$. Write $S = \mathrm{S}_{\ell_2}(0,1) \cap (H + v)$ and note that $S = \mathrm{S}_{\ell_2}(0,1) \cap A$ for some infinite-dimensional closed affine subspace A of ℓ_2. Hence, S is homeomorphic to the sphere $\mathrm{S}_{\ell_2}(0,1)$ and thus is connected. By the intermediate-value theorem, the function $f : S \to \mathbb{R}$ given by $f(s) = ||s - v||$ assumes all values between 0 and ε on S, and so for every $v' \in \mathrm{S}_{\ell_2}(0,1)$ with $||v' - v|| < \varepsilon$ there exists $v'' \in S$ with $||v'' - v|| = ||v' - v||$. This ends the proof. □

Lemma 10.6. *Suppose $\bar{v} = (v_1, \dots, v_k)$ is an orthonormal tuple in ℓ_2 and $V_1, \dots, V_k \subseteq \ell_2$ are closed infinite-dimensional subspaces such that $v_i \in V_i$ and $V_i \perp V_j$ for $i \neq j$. Write*

$$T = \{\bar{w} = (w_1, \dots, w_k) : \bar{w} \equiv \bar{v} \ \wedge \ \forall i \leq k \ w_i \in V_i\}.$$

Then T $(2, \varepsilon)$-generates an open set, for every $\varepsilon > 0$.

Proof. Fix $\varepsilon > 0$. Find $\bar{v}' = (v'_1, \dots, v'_k)$ in ℓ_2 such that

- $\bar{v}' \equiv \bar{v}$
- $v'_i \perp v_j$ for every $i \neq j$
- for every $i \leq k$ we have $\pi_{V_j}(v_i) \neq 0$ for every $j \neq i$.

For each $i, j \leq k$ write v'_{ij} for $\pi_{V_j}(v'_i)$ and note that if $i \neq j$, then $v'_{ij} \perp v_j$.

Find $\delta > 0$ such that for every $i \leq k$ the following holds: for every sequence $(v''_j : j \neq i, j \leq k)$ of vectors in V_i such that $||v''_j - v'_{ji}|| < \delta$ there exists $\tilde{v} \in V_i$ with $||\tilde{v}|| = 1$, $\tilde{v} \perp v''_j$ for every $j \neq i$ and $||\tilde{v} - v_i|| < \varepsilon/2$. Assume without loss of generality that $\delta < \varepsilon/2$. Write

$$U = \{\bar{v}'' = (v''_1, \dots, v''_k) : \bar{v}'' \equiv \bar{v} \ \wedge \ d_{\ell_2}(\bar{v}'', \bar{v}') < \delta\}.$$

Claim 10.7. *For every $\bar{v}'' \in U$ there are $\varphi_1, \varphi_2 \in O(\ell_2)$ such that*

$$\varphi_2 \varphi_1(\bar{v}) = \bar{v}''$$

and $\varphi_1(\bar{v}), \varphi_2(\bar{v}) \in T$, as well as $d_{\ell_2}(\varphi_1(\bar{v}), \bar{v}) < \varepsilon$ and $d_{\ell_2}(\varphi_2(\bar{v}), \bar{v}) < \varepsilon$.

Proof. Fix $\bar{v}''$ in U. Note that, by the choice of δ, for each $i \leq k$ there exists $\tilde{v}_i \in V_i$ such that $||\tilde{v}_i|| = 1$, $||\tilde{v}_i - v_i|| < \varepsilon/2$ and $\tilde{v}_i \perp \pi_{V_i}(v''_j)$ for every $j \neq i$. Now, for every $i \leq k$ find $w_i \in V_i$ such that

$$(\dagger) \qquad \mathrm{qftp}(w_i / v_i) = \mathrm{qftp}(v''_i / \tilde{v}_i).$$

Such vectors w_i exist since each V_i is isomorphic to ℓ_2. Now, (†) implies that

$$\mathrm{qftp}(w_i v_i) = \mathrm{qftp}(v_i'' \tilde{v}_i)$$

for each $i \leq k$ and hence the map ψ_i such that

$$\psi_i : w_i \mapsto v_i'', \quad \psi_i : v_i \mapsto \tilde{v}_i$$

is a partial automorphism of ℓ_2 for each $i \leq k$. Now, since for $i \neq j$ both the domains and ranges of ψ_i and ψ_j are pairwise orthogonal, the map $\bigcup_{i \leq k} \psi_i$ is a partial automorphism of ℓ_2. Extend $\bigcup_{i \leq k} \psi_i$ to $\varphi_2 \in O(\ell_2)$. Find also $\varphi_1 \in O(\ell_2)$ such that

$$\varphi_1 : v_i \mapsto w_i$$

for each $i \leq k$.

Note that since $||\tilde{v}_i - v_i|| < \varepsilon/2$ and $||v_i'' - v_i|| < \varepsilon/2$, we have $||v_i'' - \tilde{v}_i|| < \varepsilon$ and hence (†) implies that $||w_i - v_i|| < \varepsilon$ for each $i \leq k$. Therefore, $d_{\ell_2}(\varphi_1(\bar{v}), \bar{v}) < \varepsilon$. Also $d_{\ell_2}(\varphi_2(\bar{v}), \bar{v}) < \varepsilon$, as well as $\varphi_1(\bar{v}) \in T$ and $\varphi_2(\bar{v}) \in T$. As we clearly have $\varphi_2\varphi_1(\bar{v}) = \bar{v}''$, this proves the claim. □

Claim 10.7 clearly means that T $(2, \varepsilon)$-generates an open set, so this ends the proof. □

Lemma 10.8. *Suppose $\bar{v} = (v_1, \ldots, v_k)$ is an orthonormal tuple in ℓ_2 and let $H \subseteq \ell_2$ be an infinite-dimensional closed subspace such that the vectors $\pi_H(v_1), \ldots, \pi_H(v_k)$ are linearly independent. Write*

$$N = \{\bar{w} = (w_1, \ldots, w_k) : \bar{w} \equiv \bar{v} \ \wedge \ \forall i \leq k \ w_i - v_i \in H\}.$$

Then there exists $\varepsilon > 0$ such that N is 2-relatively ε-saturated over $\bar{v}$.

Proof. Write $w_j = \pi_H(v_i)$ for each $i \leq k$.

Claim 10.9. *There exist $w_1', \ldots, w_k' \in H$ such that $w_i' \perp w_j'$ and $w_i' \perp w_j$ for $i \neq j \leq k$ and $w_i' \not\perp w_i$ for every $i \leq k$.*

Proof. Inductively on $i \leq k$ construct $w_i' \in H$ such that $w_i' \not\perp w_i$ and $w_i' \perp w_j$ for $j \neq i$ and $w_i' \perp w_j'$ for $j < i$ as well as

$$w_1, w_1', \ldots, w_i, w_i', w_{i+1}, \ldots, w_k$$

are linearly independent. Suppose $w_1', \ldots, w_{i-1}'$ have been constructed. Let

$$W_i = \{w_1, w_1', \ldots, w_{i-1}, w_{i-1}', w_{i+1}, \ldots, w_k\}^{\perp} \cap H$$

and note that since $w_i \notin \mathrm{span}(w_1, w_1', \ldots, w_{i-1}, w_{i-1}', w_{i+1}, \ldots, w_k)$, we have that $W_i' = W_i \cap \{w_i\}^{\perp}$ is a proper subspace of W_i. Also, $W_i'' = W_i \cap \mathrm{span}\{w_1, w_1', \ldots, w_i, w_i', w_{i+1}, \ldots, w_k\}$ is a proper subspace of W_i since W_i is infinite-dimensional. Now, $W_i' \cup W_i''$ do not cover W_i, so find $w_i' \in W_i \setminus (W_i' \cup W_i'')$ and note that it is as needed. □

Using Claim 10.9, find closed infinite-dimensional subspaces V_i for $i \leq k$ such that for each $i \neq j \leq k$ we have

- $v_i \in V_i$ and $V_i \perp V_j$,

- $H \cap V_i$ is infinite-dimensional,
- $\pi_{H\cap V_i}(v_i) \neq 0$.

Find $\varepsilon > 0$ as in Lemma 10.5 and let

$$T = \{\bar{w} = (w_1, \ldots, w_k) : \bar{w} \equiv \bar{v} \ \wedge \ \forall i \leq k \ w_i \in V_i\}.$$

Then N is T-relatively ε-saturated by Lemma 10.5 and T $(2, \varepsilon)$-generates an open set, by Lemma 10.6. This ends the proof. □

Definition 10.10. Say that a sequence of k-tuples $\bar{a}_n$ in ℓ_2 is *strongly linearly independent* if there is a sequence of infinite-dimensional closed subspaces $V_n \subseteq \ell_2$ such that

- $V_n \perp V_m$ for $n \neq m$,
- $\bar{a}_m \perp V_n$ for $n \neq m$,
- the projections of the elements of $\bar{a}_n$ to V_n are linearly independent.

Lemma 10.11. *If p is a quantifier-free type of an orthonormal tuple in ℓ_2, then any strongly linearly independent sequence in $p(\ell_2)$ is 2-weakly isolated.*

Proof. Suppose $\bar{a}_n = (a_1^n, \ldots, a_k^n)$ is strongly linearly independent in p. Note that $a_1^n, \ldots, a_k^n$ form an orthonormal tuple. Let

$$N_n = \{\bar{v} = (v_1, \ldots, v_k) \in p(\ell_2) : \forall i \leq k \ v_i - a_i^n \in V_n\}.$$

We claim that there are $\varepsilon_n > 0$ such that the sequence of N_n and ε_n witnesses that $\bar{a}_n$ is 2-weakly isolated. For each n find $\varepsilon_n > 0$ as in Lemma 10.8 for $\bar{v} = \bar{a}_n$. Then N_n is 2-relatively ε_n-saturated over $\bar{a}_n$.

Suppose now that $\bar{b}_n = (b_1^n, \ldots, b_k^n) \in N_n$ are such that $\mathrm{qftp}(\bar{b}_n) = \mathrm{qftp}(\bar{a}_n)$ for each $n \in \mathbb{N}$ and $d_{\ell_2}(\bar{b}_n, \bar{a}_n) < \varepsilon_n$. Then $b_i^n - a_i^n \in V_n$ for each $i \leq k$. Find $\varphi_n \in O(V_n)$ such that $\varphi_n(\pi_{V_n}(\bar{a}_n)) = \pi_{V_n}(\bar{b}_n)$ and let $\varphi \in O(\ell_2)$ be such that φ extends all the φ_n and is equal to the identity on the orthogonal complement of the union of V_n's. Then $\varphi(\bar{a}_n) = \bar{b}_n$ for each $n \in \mathbb{N}$. This ends the proof. □

Proposition 10.12. *The Hilbert space ℓ_2 admits 2-weakly isolated sequences.*

Proof. Suppose p is a quantifier-free k-type of a tuple $\bar{a} = (a_1, \ldots, a_k)$ in ℓ_2. First note that we can assume that the elements of $\bar{a}$ form an orthonormal set. Otherwise, one can consider a tuple which is an orthonormal basis for the space spanned by $\bar{a}$ and work with a quantifier-free m-type q for some $m \leq n$. Then, for every m-tuple $\bar{b} \in q(M)$ there is a unique tuple $\bar{b}' \in p(M)$ such that the linear spans of $\bar{b}$ and $\bar{b}'$ are the same and

- if $(\bar{b}_n : n \in \mathbb{N})$ is isolated in q, then $(\bar{b}'_n : n \in \mathbb{N})$ is isolated in p,
- the map $\bar{b} \mapsto \bar{b}'$ is a homeomorphism of $q(M)$ and $p(M)$.

In fact, for simplicity of notation, assume that $k = 1$ (the argument for arbitrary k is analogous).

Suppose now that $Z \subseteq p(\ell_2)$ is nonmeager. Restricting to an open subset of $p(\ell_2)$ if neccessary, we can assume that Z is nonmeager in every nonempty open subset of ℓ_2.

Write $\mathrm{Gr}(\ell_2)$ for the space of all closed subspaces of ℓ_2 and $\mathrm{Gr}(\ell_2,\infty)$ for the space of infinite-dimensional closed subspaces of ℓ_2. The topology on $\mathrm{Gr}(\ell_2)$ is induced from the strong operator topology via the map $V \mapsto \pi_V$. Write d_{Gr} for a compatible metric on $\mathrm{Gr}(\ell_2)$. Note that there is a sequence of functions $\rho_n : \mathrm{Gr}(\ell_2,\infty) \to (0,\infty)$ such that whenever $W_n \in \mathrm{Gr}(\ell_2,\infty)$ is a decreasing sequence of infinite-dimensional closed subspaces of ℓ_2 and $d_{\mathrm{Gr}}(W_n, W_{n+1}) < \rho_{n+1}(W_n)$, then $\bigcap_n W_n$ is also infinite-dimensional.

By induction on $n \in \mathbb{N}$ find vectors $a_n \in Z$, positive reals δ_n and pairwise orthogonal infinite-dimensional closed subspaces $W_n \subseteq \ell_2$ such that:

(i) $W_1 \oplus \ldots \oplus W_n$ is co-infinite dimensional,
(ii) $\pi_{W_n}(a_n) \neq 0$.
(iii) $||\pi_{W_n \cap \mathrm{span}(a_{n+1})^\perp}(a_n)|| = \delta_n$,
(iv) if $m < n$, then $a_m \perp W_n$
(v) if $m < n$, then we have

$$||\pi_{W_m \cap (\bigcup_{i=m+1}^{n} \mathrm{span}(a_i))^\perp}(a_m)|| > \frac{1}{2}\delta_m,$$

(vi) if $m < n$ and $\varepsilon = \rho_n(W_m \cap (\bigcup_{i=m+1}^{n-1} \mathrm{span}(a_i))^\perp)$, then

$$d_{\mathrm{Gr}}(W_m \cap (\bigcup_{i=m+1}^{n} \mathrm{span}(a_i))^\perp, W_m \cap (\bigcup_{i=m+1}^{n-1} \mathrm{span}(a_i))^\perp) < \varepsilon.$$

After this is done, put $V_n = W_n \cap (\bigcup_{m>n} \mathrm{span}(a_m))^\perp$. Note that V_n are infinite-dimensional by (vi) and mutually orthogonal given that W_n are mutually orthogonal. Also, (iv) and the definition of V_n imply that if $n \neq m$, then $a_m \perp V_n$. The projection of a_n onto V_n is nonzero by the condition (v) and hence $\bar{a}_n$ is strongly linearly independent, as witnessed by V_n and hence 2-weakly isolated by Lemma 10.11.

To perform the induction step, suppose $a_1, \ldots, a_n$ and $W_1, \ldots, W_n$ as well as $\delta_1, \ldots, \delta_{n-1}$ are chosen. Using the fact that a proper subspace of ℓ_2 is meager as well as the assumption that Z is nonmeager in any nonempty open set, find $a_{n+1} \in Z$ which does not belong to $\mathrm{span}(\bigcup_{i=1}^n W_i \cup \{a_i\})$ and

$$a_{n+1} \not\perp W_n \cap \mathrm{span}(\pi_{W_n}(a_n))^\perp \qquad (\dagger\dagger)$$

and a_{n+1} is so close to a_n that for $m < n$ we have

$$||\pi_{W_m \cap (\bigcup_{i=m+1}^{n} \mathrm{span}(a_i))^\perp}(a_m)|| > \frac{1}{2}\delta_m$$

and for every $m < n$, writing $\varepsilon_m^n = \rho_{n+1}(W_m \cap (\bigcup_{i=m+1}^{n} \mathrm{span}(a_i))^\perp)$ we have

$$d_{\mathrm{Gr}}(W_m \cap (\bigcup_{i=m+1}^{n+1} \mathrm{span}(a_i))^\perp, W_m \cap (\bigcup_{i=m+1}^{n} \mathrm{span}(a_i))^\perp) < \varepsilon_m^n.$$

This implies that (v) and (vi) are satisfied at the induction step.

Note that the projection of a_{n+1} to $(\mathrm{span}(\bigcup_{i=1}^n W_i \cup \{a_i\}))^\perp$ is nonzero. Find an infinite-dimensional closed space W_{n+1} such that

- W_{n+1} is orthogonal to $\mathrm{span}(\bigcup_{i=1}^n W_i \cup \{a_i\})$,
- the projection of a_{n+1} onto W_{n+1} is nonzero,
- $W_1 \oplus \ldots \oplus W_{n+1}$ is co-infinite dimensional.

This gives (i), (ii) and (iv). Finally, we claim that $\pi_{W_n \cap \mathrm{span}(a_{n+1})^\perp}(a_n)$ is nonzero. Indeed, otherwise

$$a_n \perp W_n \cap \mathrm{span}(a_{n+1})^\perp$$

and so

$$\pi_{W_n}(a_n) \perp W_n \cap \mathrm{span}(a_{n+1})^\perp.$$

But then, since $a_{n+1} \notin W_n^\perp$ (by (††)), we have that

$$W_n \cap (\mathrm{span}(\pi_{W_n}(a_n)))^\perp = W_n \cap (\mathrm{span}(a_{n+1}))^\perp$$

and so $a_{n+1} \perp W_n \cap (\mathrm{span}(\pi_{W_n}(a_n)))^\perp$, which contradicts (††). Let then $\delta_n = ||\pi_{W_n \cap \mathrm{span}(a_{n+1})^\perp}(a_n)|| > 0$. This ends the construction and the proof. □

Finally, we verify that the stronger property discussed in Section 7 holds for the Hilbert space. Say that a sequence of tuples $\bar{a}_n \in \ell_2$ is a *proper orthogonal sequence* if the subspaces spanned by different $\bar{a}_n$ are pairwise ortogonal and the orthogonal complement of their union is infinite-dimensional.

Claim 10.13. *Any proper orthogonal sequence in ℓ_2 is independent.*

Proof. Let $\bar{a}_n$ be a proper orthogonal sequence in the quantifier-free type of a given $\bar{a}$ and let H_n be a sequence of orthogonal infinite-dimensional subspaces of the orthogonal complement of the space spanned by the vectors in all $\bar{a}_n$'s. Write H'_n for the space spanned by H_n and $\bar{a}_n$ and note that the subspaces H'_n witness that the sequence $\bar{a}_n$ is independent. □

Lemma 10.14. *The Hilbert space ℓ_2 admits independent sequences.*

Proof. Fix $k \in \mathbb{N}$ and $\bar{a} = (a_1, \ldots, a_k) \in \mathbb{U}^k$. Let $(\bar{s}_n : n \in \mathbb{N})$ be a sequence of finite tuples and without loss of generality assume that $\bar{s}_n$ is a subtuple of $\bar{s}_{n+1}$. We need to find an independent sequence $\bar{a}_n$ in the quantifier-free type of $\bar{a}$ such that $\bar{a}_n \equiv_{\bar{s}_n} \bar{a}_{n+1}$. Find the sequence $\bar{a}_n$ of tuples as well as additional vectors v_n so that

- The elements of $\bar{a}_n$ are orthogonal to all elements of $\bar{s}_n$, to all elements of $\bar{a}_i$'s for $i < n$ as well as to v_n
- v_{n+1} is orthogonal to all elements of $\bar{a}_i$'s for $i \leq n$
- $\bar{a}_n \equiv_{\bar{s}_n} \bar{a}_{n+1}$

The sequence is easy to construct using the fact that if $\bar{b}$ and $\bar{s}$ are two tuples whose elements are pairwise orthogonal, then the orbit of $\bar{b}$ with respect to the stabilizer of $\bar{s}$ contains vectors orthogonal to any finite tuple. The sequence is then proper orthogonal, and hence independent by Claim 10.13. □

11. Questions

There are still many natural examples of automorphism groups for which the automatic continuity (and even the uniqueness of Polish group topology) is open. Here we list some of them.

Question 11.1. Does the group of automorphisms of the Cuntz algebra $\mathcal{O}_2$ have the automatic continuity property?

Question 11.2. Does the group of automorphisms of the hyperfinite II_1 factor have the automatic continuity property?

Question 11.3. Does the group of linear isometries of the Gurariĭ space have the automatic continuity property?

Finally, the problem of uniqueness of separable topology for the group $\mathrm{Iso}(\mathbb{U})$ remains open. For other groups considered in this paper, uniqueness of separable topology follows from the combination of automatic continuity property and minimality (or even *total minimality* which says that any Hausdorff quotient of the group is minimal). For the unitary group this has been proved by Stojanov [50] and for the group $\mathrm{Aut}([0,1],\lambda)$ by Glasner [15] (see also [6] for a recent general framework for these kind of results).

Question 11.4. Is the group $\mathrm{Iso}(\mathbb{U})$ minimal?

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