Recovery of a Manifold with Boundary and its Continuity as a Function of its Metric Tensor

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Abstract

A basic theorem from differential geometry asserts that, if the Riemann curvature tensor associated with a field $\mathbf{C}$ of class $\mathcal{C}^2$ of positive-definite symmetric matrices of order n vanishes in a connected and simply-connected open subset Ω of $\mathbb{R}^n$, then there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega;\mathbb{R}^n)$, uniquely determined up to isometries in $\mathbb{R}^n$, such that $\mathbf{C}$ is the metric tensor field of the manifold $\boldsymbol{\Theta}(\Omega)$, then isometrically immersed in $\mathbb{R}^n$. Let $\dot{\boldsymbol{\Theta}}$ denote the equivalence class of $\boldsymbol{\Theta}$ modulo isometries in $\mathbb{R}^n$ and let $\mathcal{F} : \mathbf{C} \to \dot{\boldsymbol{\Theta}}$ denote the mapping determined in this fashion.

The first objective of this paper is to show that, if Ω satisfies a certain "geodesic property" (in effect a mild regularity assumption on the boundary $\partial\Omega$ of Ω) and if the field $\mathbf{C}$ and its partial derivatives of order ≤ 2 have continuous extensions to $\overline{\Omega}$, the extension of the field $\mathbf{C}$ remaining positive-definite on $\overline{\Omega}$, then the immersion $\boldsymbol{\Theta}$ and its partial derivatives of order ≤ 3 also have continuous extensions to $\overline{\Omega}$.

The second objective is to show that, under a slightly stronger regularity assumption on $\partial\Omega$, the above extension result combined with a fundamental theorem of Whitney leads to a stronger extension result: There exist a connected open subset $\widetilde{\Omega}$ of $\mathbb{R}^n$ containing $\overline{\Omega}$ and a field $\widetilde{\mathbf{C}}$ of positive-definite symmetric matrices of class $\mathcal{C}^2$ on $\widetilde{\Omega}$ such that $\widetilde{\mathbf{C}}$ is an extension of $\mathbf{C}$ and the Riemann curvature tensor associated with $\widetilde{\mathbf{C}}$ still vanishes in $\widetilde{\Omega}$.

The third objective is to show that, if Ω satisfies the geodesic property and is bounded, the mapping $\mathcal{F}$ can be extended to a mapping that is locally Lipschitz-continuous with respect to the topologies of the Banach spaces $\mathcal{C}^2(\overline{\Omega})$ for the continuous extensions of the symmetric matrix fields $\mathbf{C}$, and $\mathcal{C}^3(\overline{\Omega})$ for the continuous extensions of the immersions $\boldsymbol{\Theta}$.

Résumé

Un théorème de base de la géométrie différentielle affirme que, si le tenseur de courbure de Riemann associé à un champ $\mathbf{C}$ de classe $\mathcal{C}^2$ de matrices symétriques définies positives d'ordre n s'annule sur un ouvert Ω de $\mathbb{R}^n$ connexe et simplement connexe, alors il existe une immersion $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$, définie de façon unique aux isométries de $\mathbb{R}^n$ près, telle que $\mathbf{C}$ soit le champ de tenseurs métriques de la variété $\mathbf{\Theta}(\Omega)$, celle-ci étant plongée isométriquement dans $\mathbb{R}^n$. Soit $\dot{\mathbf{\Theta}}$ la classe d'équivalence de $\mathbf{\Theta}$ modulo les isométries de $\mathbb{R}^n$ et soit $\mathcal{F} : \mathbf{C} \to \dot{\mathbf{\Theta}}$ l'application ainsi définie.

Le premier objectif de cet article est d'établir que, si Ω satisfait une certaine "propriété géodésique" (en fait une hypothèse peu restrictive sur la régularité de la frontière $\partial\Omega$ de Ω) et si le champ $\mathbf{C}$ et ses dérivées partielles d'ordre ≤ 2 ont des prolongements continus à $\overline{\Omega}$, le prolongement du champ $\mathbf{C}$ restant défini positif sur $\overline{\Omega}$, alors l'immersion $\mathbf{\Theta}$ et ses dérivées partielles d'ordre ≤ 3 ont également des prolongements continus à $\overline{\Omega}$.

Le second objectif est d'établir que, moyennant une hypothèse de régularité légèrement plus forte sur $\partial\Omega$, le résultat de prolongement ci-dessus combiné avec un théorème fondamental de Whitney conduit à un résultat plus fort de prolongement: Il existe un ouvert $\widetilde{\Omega}$ connexe de $\mathbb{R}^n$ contenant $\overline{\Omega}$ et un champ $\widetilde{\mathbf{C}}$ de matrices symétriques définies positives de classe $\mathcal{C}^2$ sur $\widetilde{\Omega}$ tels que $\widetilde{\mathbf{C}}$ soit un prolongement de $\mathbf{C}$ et le tenseur de courbure de Riemann associé à $\widetilde{\mathbf{C}}$ reste nul sur $\widetilde{\Omega}$.

Le troisième objectif est d'établir que, si Ω satisfait la propriété géodésique et est borné, l'application $\mathcal{F}$ peut être prolongée en une application qui est localement Lipschitz-continue pour les topologies usuelles des espaces de Banach $\mathcal{C}^2(\overline{\Omega})$ pour les prolongements continus des champs de matrices symétriques $\mathbf{C}$, et $\mathcal{C}^3(\overline{\Omega})$ pour les prolongements continus des immersions $\mathbf{\Theta}$.

Key words: Differential geometry, nonlinear elasticity

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1 Introduction

All the notations used, but not defined, here are defined in the next sections. Let Ω be a connected and simply-connected open subset of $\mathbb{R}^n$, let $\mathbb{S}^n$, resp. $\mathbb{S}^n_>$, denote the set of symmetric, resp. positive-definite symmetric, matrices of order n, and let a *Riemannian metric* $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ be given that satisfies

$$R^p_{\cdot ijk} := \partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik}\Gamma^p_{j\ell} - \Gamma^\ell_{ij}\Gamma^p_{k\ell} = 0 \text{ in } \Omega,$$

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where

$$\Gamma_{ij}^k := \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}) \text{ and } (g^{k\ell}) := (g_{ij})^{-1},$$

i.e., the *Riemann curvature tensor* associated with the metric (g_{ij}) vanishes in Ω.

Then a basic theorem of differential geometry (recalled in Theorem 3.1 for convenience) asserts that there exists an immersion $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$, uniquely determined up to isometries in $\mathbb{R}^n$, such that

$$\partial_i \boldsymbol{\Theta}(x) \cdot \partial_j \boldsymbol{\Theta}(x) = g_{ij}(x) \text{ for all } x \in \Omega,$$

i.e., such that the manifold $\boldsymbol{\Theta}(\Omega)$ is isometrically immersed in $\mathbb{R}^n$.

Hence there exists a mapping $\mathcal{F}$ that associates with any matrix field $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ satisfying $R^p_{\cdot ijk} = 0$ in Ω a well-defined element in the quotient set $\mathcal{C}^3(\Omega; \mathbb{R}^n)/\mathcal{R}$, where $(\boldsymbol{\Phi}; \boldsymbol{\Theta}) \in \mathcal{R}$ means that there exist a vector $\mathbf{a} \in \mathbb{R}^n$ and an orthogonal matrix $\mathbf{Q}$ of order n such that $\boldsymbol{\Phi}(x) = \mathbf{a} + \mathbf{Q}\boldsymbol{\Theta}(x)$ for all $x \in \Omega$.

Our *first objective* is to extend this classical existence and uniqueness result "*up to the boundary*" of the set Ω. More specifically, we assume that the set Ω satisfies what we call the "*geodesic property*" (in effect, a mild smoothness assumption on the boundary $\partial\Omega$; cf. Definition 2.2) and that the functions g_{ij} and their partial derivatives of order ≤ 2 can be extended by continuity to the closure $\overline{\Omega}$, the symmetric matrix field extended in this fashion remaining positive-definite over the set $\overline{\Omega}$. Then we show that the immersion $\boldsymbol{\Theta}$ and its partial derivatives of order ≤ 3 can be also extended by continuity to $\overline{\Omega}$ (cf. Theorem 3.3).

Let $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$, resp. $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$, denote the set formed by the positive-definite symmetric matrix fields, resp. the space formed by the vector fields, that, together with their partial derivatives of order ≤ 2, resp. ≤ 3, admit such continuous extensions, the extensions of the matrices remaining positive-definite on $\overline{\Omega}$. Then the above result shows that there exists a mapping $\overline{\mathcal{F}}$ that associates with any matrix field $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ satisfying $R^p_{\cdot ijk} = 0$ in Ω a well-defined element in the quotient set $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)/\mathcal{R}$. The mapping $\overline{\mathcal{F}}$ thus maps matrix fields defined "*up to the boundary*" into equivalence classes of vector fields also defined "*up to the boundary*".

Our *second objective* is to show that, if in addition the geodesic distance is equivalent to the Euclidean distance on Ω (a property stronger than the "geodesic property", but that is in particular satisfied if the boundary $\partial\Omega$ is Lipschitz-continuous), then a Riemannian metric $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ with a Riemann curvature tensor vanishing in Ω can be extended to a Riemannian metric $(\widetilde{g}_{ij}) \in \mathcal{C}^2(\widetilde{\Omega}; \mathbb{S}^n_>)$ defined on a connected open set $\widetilde{\Omega}$ containing $\overline{\Omega}$ and whose Riemann curvature tensor still vanishes in $\widetilde{\Omega}$.

As shown in Theorem 4.3, this result relies on the existence (established in Theorem 3.3) of continuous extensions to $\overline{\Omega}$ of the immersion $\mathbf{\Theta}$ and its partial derivatives of order ≤ 3 and on a deep extension theorem of Whitney [27].

Our *third objective* is to study the continuity of the mapping $\overline{\mathcal{F}}$. In this direction, we show that, if the set Ω is *bounded* and again satisfies the "*geodesic property*", the mapping $\overline{\mathcal{F}}$ is *locally Lipschitz-continuous* when the vector spaces $\mathcal{C}^2(\overline{\Omega};\mathbb{S}^n)$ and $\mathcal{C}^3(\overline{\Omega};\mathbb{R}^n)$ are equipped with their natural norms of Banach space (cf. Theorem 5.2 and Corollary 5.3).

Note that the issue of continuity of the mapping $\mathcal{F}$ described earlier, i.e., "when the boundary of the open set Ω is ignored", was recently addressed by Ciarlet & Laurent [9] who showed, *albeit* by means of a completely different approach, that the mapping $\mathcal{F}$ is continuous when both spaces $\mathcal{C}^2(\Omega;\mathbb{S}^n)$ and $\mathcal{C}^3(\Omega;\mathbb{R}^n)$ are equipped with their natural Fréchet topologies.

The main feature of the results of the present paper is thus that they hold "up to, or beyond, the boundary". This theoretical aspect does not seem to have been previously considered in the existing literature on differential geometry (at least to the best of our knowledge).

Another, more "applied", motivation behind the present work stems from *nonlinear three-dimensional elasticity* (an extensive account of which may be found in Ciarlet [5]). As already noted by Antman [3], one possible approach to this theory consists in considering the matrix field $\mathbf{C}$ as the "primary unknown", instead of the vector field $\mathbf{\Theta}$ itself as is customary. In this context, where $n = 3$, the matrix field $\mathbf{C}$ is called the *Cauchy-Green tensor field* and the immersion $\mathbf{\Theta} : \Omega \to \mathbb{R}^3$, which is called a *deformation*, should be in addition injective in Ω so as to avoid interpenetrability of matter, an issue which is not addressed here.

Indeed, the *stored energy function* of hyperelastic materials is naturally expressed in terms of the Cauchy-Green tensor (the particular form of this dependence played a decisive rôle in the landmark existence theory of Ball [4]). However, the part of the energy that takes into account the *applied forces* is naturally expressed in terms of the deformation; hence the need to study the dependence of a deformation in terms of its Cauchy-Green tensor field. In the same vein, the *boundary conditions* that are found in the traditional boundary value problems of nonlinear elasticity are aptly expressed in terms of boundary values of the deformation or of its gradient; hence the need to study the same dependence, this time "up to the boundary".

In this spirit, the local Lipschitz-continuity of the mapping $\overline{\mathcal{F}}$ established in Theorem 5.2 and Corollary 5.3 is to be compared with the earlier landmark estimates of John [17,18] and Kohn [19] and the recent and far-reaching "theorem on geometric rigidity" of Friesecke, James and Müller [15]. Such estimates are more powerful than those found here, in the sense that they are established for Sobolev norms. However, they only imply continuity at *rigid body deformations*, i.e., corresponding to $\mathbf{C} = \mathbf{I}$, whereas our estimates hold "at any Cauchy-Green tensor $\mathbf{C}$".

In all fairness, the present study only constitutes a preliminary stage of the above programme, the completion of which should also include the consideration of Sobolev-type norms, more likely to arise in, e.g., an existence theory undertaken from this perspective. In this direction, the recent contribution of Reshetnyak [24] is particularly noteworthy.

Similar questions, this time motivated by *nonlinear shell theory* and accordingly relative to *surfaces in* $\mathbb{R}^3$, are considered in Ciarlet [7] and Ciarlet & Mardare [13].

The results of this paper have been announced in [11] and [12].

2 Preliminaries

This section gathers the main conventions, notations, and definitions used in this article, as well as various preliminary results that will be subsequently needed.

An integer $n \geq 2$ is chosen *once and for all* throughout this article. It is then understood that Latin indices and exponents vary in the set $\{1, 2, \ldots, n\}$, save when they are used for indexing sequences. The summation convention with respect to repeated indices and exponents is systematically used in conjunction with this rule.

The Euclidean inner product of $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$ and the Euclidean norm of $\mathbf{a} \in \mathbb{R}^n$ are denoted by $\mathbf{a} \cdot \mathbf{b}$ and $|\mathbf{a}|$. The notations $\mathbb{M}^n, \mathbb{S}^n, \mathbb{S}^n_>$, and $\mathbb{O}^n$, respectively designate the sets of all square matrices, of all symmetric matrices, of all positive-definite symmetric matrices, and of all orthogonal matrices, of order n. The notation (a_{ij}) designates the matrix of $\mathbb{M}^n$ with a_{ij} as its elements, the first index being the row index. The spectral norm of a matrix $\mathbf{A} \in \mathbb{M}^n$ is

$$|\mathbf{A}| := \sup\{|\mathbf{A}\mathbf{v}|;\ \mathbf{v} \in \mathbb{R}^n,\ |\mathbf{v}| \leq 1\}.$$

In any metric space, the open ball with center x and radius $\delta > 0$ is denoted $B(x; \delta)$. The notation $f|_U$ designates the restriction to a set U of a function f.

The coordinates of a point $x \in \mathbb{R}^n$ are denoted x_i. Partial derivative operators of order $\ell \geq 1$ are denoted $\partial^{\boldsymbol{\alpha}}$, where $\boldsymbol{\alpha} = (\alpha_i) \in \mathbb{N}^n$ is a multi-index satisfying $|\boldsymbol{\alpha}| := \sum_i \alpha_i = \ell$. Partial derivative operators of the first, second, and third order are also denoted $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$.

The space of all continuous functions from a normed space X into a normed space Y is denoted $\mathcal{C}^0(X; Y)$, or simply $\mathcal{C}^0(X)$ if $Y = \mathbb{R}$.

Let Ω be an open subset of $\mathbb{R}^n$. For any integer $\ell \geq 1$, the space of all real-valued functions that are ℓ times continuously differentiable in Ω is denoted $\mathcal{C}^\ell(\Omega)$. Similar definitions hold for the spaces $\mathcal{C}^\ell(\Omega; \mathbb{R}^n)$, $\mathcal{C}^\ell(\Omega; \mathbb{M}^n)$, and $\mathcal{C}^\ell(\Omega; \mathbb{S}^n)$. If $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ and $x \in \Omega$, the notation $\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$ denotes the matrix in $\mathbb{M}^n$ whose i-th column is the vector $\partial_i \boldsymbol{\Theta}(x) \in \mathbb{R}^n$. We recall that a mapping $\boldsymbol{\Theta} \in \mathcal{C}^1(\Omega; \mathbb{R}^n)$ is an *immersion* if the matrix $\boldsymbol{\nabla}\boldsymbol{\Theta}(x)$ is invertible at all points $x \in \Omega$. We also define the set

$$\mathcal{C}^2(\Omega; \mathbb{S}^n_>) := \{\mathbf{C} \in \mathcal{C}^2(\Omega; \mathbb{S}^n);\ \mathbf{C}(x) \in \mathbb{S}^n_> \text{ for all } x \in \Omega\}.$$

Central to this paper is the following notion of spaces of functions, vector fields, or matrix fields, "of class $\mathcal{C}^\ell$ up to the boundary of Ω".

Definition 2.1 Let Ω be an open subset of $\mathbb{R}^n$. For any integer $\ell \geq 1$, we define the space $\mathcal{C}^\ell(\overline{\Omega})$ as the subspace of the space $\mathcal{C}^\ell(\Omega)$ that consists of all functions $f \in \mathcal{C}^\ell(\Omega)$ that, together with all their partial derivatives $\partial^{\boldsymbol{\alpha}} f$, $1 \leq |\boldsymbol{\alpha}| \leq \ell$, possess continuous extensions to the closure $\overline{\Omega}$ of Ω. Equivalently, a function $f : \Omega \to \mathbb{R}$ belongs to $\mathcal{C}^\ell(\overline{\Omega})$ if $f \in \mathcal{C}^\ell(\Omega)$ and, at each point x_0 of the boundary $\partial\Omega$ of Ω, $\lim_{x \in \Omega \to x_0} f(x)$ and $\lim_{x \in \Omega \to x_0} \partial^{\boldsymbol{\alpha}} f(x)$ for all $1 \leq |\boldsymbol{\alpha}| \leq \ell$ exist. Analogous definitions hold for the spaces $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{R}^n), \mathcal{C}^\ell(\overline{\Omega}; \mathbb{M}^n)$, and $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{S}^n)$.

All the continuous extensions appearing in such spaces will be identified by a bar. Thus for instance, we shall denote by $\overline{f} \in \mathcal{C}^0(\overline{\Omega})$ and $\overline{\partial^{\boldsymbol{\alpha}} f} \in \mathcal{C}^0(\overline{\Omega})$, $1 \leq |\boldsymbol{\alpha}| \leq \ell$, the continuous extensions to $\overline{\Omega}$ of the functions f and $\partial^{\boldsymbol{\alpha}} f$ if $f \in \mathcal{C}^\ell(\overline{\Omega})$; similarly, we shall denote by $\overline{\partial_i \boldsymbol{\Theta}} \in \mathcal{C}^0(\overline{\Omega}; \mathbb{R}^n)$ and $\overline{\boldsymbol{\nabla}\boldsymbol{\Theta}} \in \mathcal{C}^0(\overline{\Omega}; \mathbb{M}^n)$ the continuous extensions to $\overline{\Omega}$ of the fields $\partial_i \boldsymbol{\Theta} \in \mathcal{C}^0(\Omega; \mathbb{R}^n)$ and $\boldsymbol{\nabla}\boldsymbol{\Theta} \in \mathcal{C}^0(\Omega; \mathbb{M}^n)$ if $\boldsymbol{\Theta} \in \mathcal{C}^1(\overline{\Omega}; \mathbb{R}^n)$; etc.

Finally, we also define the set

$$\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>) := \{\mathbf{C} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n);\ \overline{\mathbf{C}}(x) \in \mathbb{S}^n_> \text{ for all } x \in \overline{\Omega}\}. \qquad \square$$

Remark. The above definition of the space $\mathcal{C}^\ell(\overline{\Omega})$ coincides with that given in Adams [1, Definition 1.26] when the set Ω is bounded. $\square$

Given a differentiable real-valued, vector-valued, or matrix-valued, function of a single variable, its first-order derivative is indicated by a prime. Thus for instance

$$\gamma_i'(t) := \frac{\mathrm{d}\gamma_i}{\mathrm{d}t}(t) \text{ and } \boldsymbol{\gamma}'(t) := \frac{\mathrm{d}\boldsymbol{\gamma}}{\mathrm{d}t}(t),\ 0 \le t \le 1,\ \text{ if } \boldsymbol{\gamma} = (\gamma_i) \in \mathcal{C}^1([0,1];\mathbb{R}^n),$$

$$\mathbf{Z}'(t) := \frac{\mathrm{d}\mathbf{Z}}{\mathrm{d}t}(t),\ 0 \le t \le 1,\ \text{ if } \mathbf{Z} \in \mathcal{C}^1([0,1];\mathbb{M}^n),\ \text{ etc.}$$

Remark. Since the definition of $\mathcal{C}^\ell$-differentiability on a *closed interval of* $\mathbb{R}$ is straightforward, the definition of vector-valued functions in a space such as $\mathcal{C}^1([0,1];\mathbb{R}^n)$ does not require the consideration of continuous extensions from $]0,1[$ to $[0,1]$. There is thus no inconsistency with the above definition of the space $\mathcal{C}^1(\overline{\Omega})$ (see Definition 2.1), which is given for an open set Ω in $\mathbb{R}^n$ with $n \ge 2$. $\square$

Let Ω be a connected open subset of $\mathbb{R}^n$. Given two points $x, y \in \Omega$, a *path joining x to y in* Ω is any mapping $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\mathbb{R}^n)$ that satisfies $\boldsymbol{\gamma}(t) \in \Omega$ for all $t \in [0,1]$ and $\boldsymbol{\gamma}(0) = x$ and $\boldsymbol{\gamma}(1) = y$. Note that there always exist such paths. Given a path $\boldsymbol{\gamma}$ joining x to y in Ω, its *length* is defined by

$$L(\boldsymbol{\gamma}) := \int_0^1 |\boldsymbol{\gamma}'(t)|\,\mathrm{d}t.$$

Let Ω be a connected open subset of $\mathbb{R}^n$. The *geodesic distance in* Ω between two points $x, y \in \Omega$ is defined by

$$d_\Omega(x,y) = \inf\{L(\boldsymbol{\gamma}); \boldsymbol{\gamma} \text{ is a path joining } x \text{ to } y \text{ in } \Omega\}.$$

Most results of this paper will be established under a specific, but mild, regularity assumption on the boundary of an open subset of $\mathbb{R}^n$, according to the following definition:

Definition 2.2 An open subset Ω of $\mathbb{R}^n$ satisfies the *geodesic property* if it is connected and, given any point $x_0 \in \partial\Omega$ and any $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon) > 0$ such that

$$d_\Omega(x,y) < \varepsilon \text{ for all } x, y \in \Omega \cap B(x_0;\delta). \qquad \square$$

Remarks. (1) Replacing "given any point $x_0 \in \partial\Omega$" by "given any point $x_0 \in \overline{\Omega}$" does not alter this definition.

(2) Any connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary, in the sense of Adams [1, Definition 4.5] or Nečas [23, pp. 14–15] satisfies the geodesic property.

(3) Let $I = \{(x_1, x_2) \in \mathbb{R}^2;\ 0 \le x_1 \le 1,\ x_2 = 0\}$. Then $\mathbb{R}^2 - I$ is an instance of a connected open subset of $\mathbb{R}^2$ that does not satisfy the geodesic property. $\square$

Let Ω be a connected open subset of $\mathbb{R}^n$. The *geodesic diameter* of Ω is defined by

$$D_\Omega := \sup_{x,y\in\Omega} d_\Omega(x, y).$$

Note that $D_\Omega = +\infty$ is not excluded. The following lemma gives a useful characterization of boundedness in terms of the geodesic diameter.

Lemma 2.3 *An open subset Ω of $\mathbb{R}^n$ that satisfies the geodesic property is bounded if and only if $D_\Omega < +\infty$.*

PROOF. Clearly, Ω is bounded if $D_\Omega < +\infty$. Before proving in (ii) that $D_\Omega < +\infty$ if Ω is bounded and satisfies the geodesic property, we establish in (i) a useful property of the geodesic distance, which holds for *any* connected open subset of $\mathbb{R}^n$.

(i) *Let Ω be a connected open subset of $\mathbb{R}^n$. Then*

$$d_\Omega(x, z) \le d_\Omega(x, y) + d_\Omega(y, z) \text{ for all } x, y, z \in \Omega.$$

Let $x, y, z \in \Omega$ and $\varepsilon > 0$ be given. By definition of the geodesic distance, there exist a path $\boldsymbol{\gamma}^1$ joining x to y in Ω and a path $\boldsymbol{\gamma}^2$ joining y to z in Ω such that

$$L(\boldsymbol{\gamma}^1) \le d_\Omega(x, y) + \varepsilon \text{ and } L(\boldsymbol{\gamma}^2) \le d_\Omega(y, z) + \varepsilon.$$

Define a mapping $\widetilde{\boldsymbol{\gamma}} \in [0, 1] \to \mathbb{R}^n$ by letting

$$\widetilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}^1(2t) \text{ for } 0 \le t \le \frac{1}{2} \text{ and } \widetilde{\boldsymbol{\gamma}}(t) = \boldsymbol{\gamma}^2(2t - 1) \text{ for } \frac{1}{2} < t \le 1.$$

Since $y \in \Omega$ and Ω is open, there exists an open ball with center y and contained in Ω. By smoothing the mapping $\widetilde{\boldsymbol{\gamma}}$ around $t = \frac{1}{2}$, one can construct a path $\boldsymbol{\gamma}$ joining x to z in Ω that satisfies

$$L(\boldsymbol{\gamma}) \le L(\boldsymbol{\gamma}^1) + L(\boldsymbol{\gamma}^2) + \varepsilon \le d_\Omega(x, y) + d_\Omega(y, z) + 3\varepsilon.$$

Since $d_\Omega(x, z) \le L(\boldsymbol{\gamma})$ and $\varepsilon > 0$ is arbitrary, the announced inequality holds.

(ii) Let Ω be a bounded open subset of $\mathbb{R}^n$ that satisfies the geodesic property. Since the set $\overline{\Omega}$ is compact, there exist finitely many open balls B^j, $1 \le j \le J$, with centers in the set $\overline{\Omega}$ such that

$$d_\Omega(x, y) < 1 \text{ for all } x, y \in \Omega \cap B^j,\ 1 \le j \le J, \text{ and } \overline{\Omega} \subset \bigcup_{j=1}^{J} B^j.$$

For each $j \in \{1, \ldots, J\}$, pick a point $y^j \in \Omega \cap B^j$.

Given any points $x, y \in \Omega$, there exist $p, q \in \{1, \ldots J\}$ such that $x \in B^p$ and $y \in B^q$. The inequality established in (i) thus yields

$$\begin{aligned} d_\Omega(x, y) &\leq d_\Omega(x, y^p) + d_\Omega(y^p, y^q) + d_\Omega(y^q, y) \\ &\leq 2 + \max_{p,q \in \{1,\ldots,J\}} d_\Omega(y^p, y^q). \end{aligned}$$

Hence $D_\Omega = \sup_{x,y\in\Omega} d_\Omega(x, y) < +\infty$. $\square$

Remark. By (i), the function d_Ω defines a *distance* on any connected open subset Ω of $\mathbb{R}^n$ (the other properties of a distance clearly hold). $\square$

The next lemma records a well-known property of the mapping that associates with any symmetric positive-definite matrix $\mathbf{C}$ its *square root* $\mathbf{C}^{1/2}$. Its proof, recalled here for convenience, is found in, e.g., Gurtin [16, Sect. 3].

Lemma 2.4 *Given any matrix $\mathbf{C} \in \mathbb{S}^n_>$, there exists a unique matrix $\mathbf{C}^{1/2} \in \mathbb{S}^n_>$ such that $(\mathbf{C}^{1/2})^2 = \mathbf{C}$, and the mapping*

$$\mathbf{\Phi} : \mathbf{C} \in \mathbb{S}^n_> \to \mathbf{\Phi}(\mathbf{C}) = \mathbf{C}^{1/2} \in \mathbb{S}^n_>$$

defined in this fashion is of class $\mathcal{C}^\infty$.

PROOF. The existence and uniqueness of $\mathbf{C}^{1/2}$ for any $\mathbf{C} \in \mathbb{S}^n_>$ is well-known; see, e.g., Ciarlet [6, Theorem 3.2-1].

Let $\boldsymbol{\psi} : \mathbb{S}^n_> \to \mathbb{S}^n_>$ denote the inverse mapping of $\mathbf{\Phi}$, thus defined by $\boldsymbol{\psi}(\mathbf{B}) = \mathbf{B}^2$ for all $\mathbf{B} \in \mathbb{S}^n_>$. Then the Fréchet derivative $\boldsymbol{\psi}'(\mathbf{B}) \in \mathcal{L}(\mathbb{S}^n)$ of the mapping $\boldsymbol{\psi}$ at each $\mathbf{B} \in \mathbb{S}^n_>$, which is defined by

$$\boldsymbol{\psi}'(\mathbf{B})\mathbf{H} = \mathbf{B}\mathbf{H} + \mathbf{H}\mathbf{B} \text{ for any } \mathbf{H} \in \mathbb{S}^n,$$

is an *isomorphism*. To see this, let $\mathbf{H} \in \mathbb{S}^n$ be such that $\boldsymbol{\psi}'(\mathbf{B})\mathbf{H} = \mathbf{0}$, let $\mathbf{p}_i$, $1 \leq i \leq n$, be a basis of $\mathbb{R}^n$ consisting of eigenvectors of $\mathbf{B}$ and let $\lambda_i > 0$ be the eigenvalue of $\mathbf{B}$ corresponding to $\mathbf{p}_i$. Then, for $i = 1, 2, \ldots, n$,

$$\boldsymbol{\psi}'(\mathbf{B})\mathbf{H}\mathbf{p}_i = \mathbf{B}\mathbf{H}\mathbf{p}_i + \lambda_i \mathbf{H}\mathbf{p}_i = \mathbf{0},$$

so that $\mathbf{H}\mathbf{p}_i = \mathbf{0}$, for otherwise $\mathbf{H}\mathbf{p}_i$ would be an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue $-\lambda_i < 0$. Hence $\mathbf{H} = \mathbf{0}$, which shows that $\boldsymbol{\psi}'(\mathbf{B}) \in \mathcal{L}(\mathbb{S}^n)$ is an isomorphism (the space $\mathbb{S}^n$ is finite-dimensional).

Since the mapping $\boldsymbol{\psi} : \mathbb{S}^n_> \to \mathbb{S}^n_>$ is of class $\mathcal{C}^\infty$, its inverse mapping $\boldsymbol{\Phi} : \mathbb{S}^n_> \to \mathbb{S}^n_>$ is thus also of class $\mathcal{C}^\infty$ by the *inverse function theorem* (see, e.g., Dieudonné [14, Theorem 10.2.5]). $\square$

We conclude this section by a useful estimate.

Lemma 2.5 *Let there be given matrix fields* $\mathbf{A}, \mathbf{B} \in \mathcal{C}^0([0,1], \mathbb{M}^n)$ *and* $\mathbf{Z} \in \mathcal{C}^1([0,1]; \mathbb{M}^n)$ *that satisfy*

$$\mathbf{Z}'(t) = \mathbf{Z}(t)\mathbf{A}(t) + \mathbf{B}(t),\ 0 \le t \le 1.$$

Then

$$|\mathbf{Z}(t)| \le |\mathbf{Z}(0)| \exp\Big(\int_0^t |\mathbf{A}(\tau)|\,\mathrm{d}\tau\Big) + \int_0^t |\mathbf{B}(s)| \exp\Big(\int_s^t |\mathbf{A}(\tau)|\,\mathrm{d}\tau\Big)\,\mathrm{d}s,\ 0 \le t \le 1.$$

PROOF. Since

$$|\mathbf{Z}'(t)| \le |\mathbf{Z}(t)||\mathbf{A}(t)| + |\mathbf{B}(t)|,\ 0 \le t \le 1,$$

it suffices to apply Gronwall's lemma for vector fields (see, e.g., Schatzman [25, Lemma 15.2.6]). $\square$

3 Recovery of a manifold with boundary from a prescribed metric tensor

Let a Riemannian metric $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ be given over an open subset Ω of $\mathbb{R}^n$. The Christoffel symbols of the second kind associated with this metric are then defined by

$$\Gamma^k_{ij} := \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}),\ \text{where } (g^{k\ell}) := (g_{ij})^{-1},$$

and the mixed components $R^p_{\cdot ijk} \in \mathcal{C}^0(\Omega)$ of its Riemann curvature tensor are defined by

$$R^p_{\cdot ijk} := \partial_j \Gamma^p_{ik} - \partial_k \Gamma^p_{ij} + \Gamma^\ell_{ik}\Gamma^p_{j\ell} - \Gamma^\ell_{ij}\Gamma^p_{k\ell}.$$

If this tensor vanishes in Ω and Ω is simply-connected, a classical result in differential geometry asserts that (g_{ij}) is the metric tensor field of a *manifold* $\boldsymbol{\Theta}(\Omega)$ that is isometrically immersed in $\mathbb{R}^n$ and, if Ω is connected, such a manifold is unique up to isometries in $\mathbb{R}^n$. More precisely, we have (see, e.g., Malliavin [20], or Ciarlet & Larsonneur [8, Theorem 2] for an elementary and self-contained proof):

Theorem 3.1 *Let Ω be a connected and simply-connected open subset of $\mathbb{R}^n$. Let a matrix field $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ be given that satisfies*

$$R^p_{\cdot ijk} = 0 \text{ in } \Omega.$$

Then there exists an immersion $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ that satisfies

$$\boldsymbol{\nabla}\mathbf{\Theta}(x)^T \boldsymbol{\nabla}\mathbf{\Theta}(x) = \mathbf{C}(x) \text{ for all } x \in \Omega.$$

If an immersion $\mathbf{\Phi} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ satisfies

$$\boldsymbol{\nabla}\mathbf{\Phi}(x)^T \boldsymbol{\nabla}\mathbf{\Phi}(x) = \mathbf{C}(x) \text{ for all } x \in \Omega,$$

then there exist a vector $\mathbf{a} \in \mathbb{R}^n$ and a matrix $\mathbf{Q} \in \mathbb{O}^n$ such that

$$\mathbf{\Phi}(x) = \mathbf{a} + \mathbf{Q}\mathbf{\Theta}(x) \text{ for all } x \in \Omega. \qquad \square$$

Remark. The existence of immersions $\mathbf{\Theta}$ satisfying $\boldsymbol{\nabla}\mathbf{\Theta}(x)^T \boldsymbol{\nabla}\mathbf{\Theta}(x) = \mathbf{C}(x)$ for all $x \in \Omega$ holds under weaker regularity assumptions on the matrix field $\mathbf{C}$; see C. Mardare [21] and S. Mardare [22]. Likewise, their uniqueness up to isometries in $\mathbb{R}^n$ still holds under weaker regularity assumptions on the mappings $\mathbf{\Theta}$ and $\mathbf{\Phi}$; see Ciarlet & Larsonneur [8, Theorem 3] and Ciarlet & Mardare [10, Theorem 1]. $\square$

While the immersions $\mathbf{\Theta}$ found in Theorem 3.1 are only defined up to isometries in $\mathbb{R}^n$, they become uniquely determined if they are required to satisfy *ad hoc* additional conditions, according to the following corollary to Theorem 3.1.

Corollary 3.2 *Let the assumptions on the set Ω and on the matrix field $\mathbf{C}$ be as in Theorem 3.1 and let a point $x_0 \in \Omega$ be given. Then there exists one and only one immersion $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ that satisfies*

$$\boldsymbol{\nabla}\mathbf{\Theta}(x)^T \boldsymbol{\nabla}\mathbf{\Theta}(x) = \mathbf{C}(x) \text{ for all } x \in \Omega,$$
$$\mathbf{\Theta}(x_0) = \mathbf{0} \text{ and } \boldsymbol{\nabla}\mathbf{\Theta}(x_0) = \mathbf{C}(x_0)^{1/2}.$$

PROOF. Given any immersion $\mathbf{\Phi} \in \mathcal{C}^3(\Omega; \mathbb{R}^3)$ that satisfies $\boldsymbol{\nabla}\mathbf{\Phi}(x)^T \boldsymbol{\nabla}\mathbf{\Phi}(x) = \mathbf{C}(x)$ for all $x \in \Omega$ (such immersions exist by Theorem 3.1), let the mapping $\mathbf{\Theta} : \Omega \to \mathbb{R}^n$ be defined by

$$\mathbf{\Theta}(x) := \mathbf{C}(x_0)^{1/2} \boldsymbol{\nabla}\mathbf{\Phi}(x_0)^{-1}(\mathbf{\Phi}(x) - \mathbf{\Phi}(x_0)) \text{ for all } x \in \Omega.$$

Then it is immediately verified that this mapping $\mathbf{\Theta}$ satisfies the announced properties.

Besides, it is uniquely determined. To see this, let $\mathbf{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ and $\mathbf{\Phi} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ be two immersions that satisfy

$$\nabla\mathbf{\Theta}(x)^T \nabla\mathbf{\Theta}(x) = \nabla\mathbf{\Phi}(x)^T \nabla\mathbf{\Phi}(x) \text{ for all } x \in \Omega.$$

Hence there exist (again by Theorem 3.1) $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{Q} \in \mathbb{O}^n$ such that $\mathbf{\Phi}(x) = \mathbf{a} + \mathbf{Q}\mathbf{\Theta}(x)$ for all $x \in \Omega$; hence $\nabla\mathbf{\Phi}(x) = \mathbf{Q}\nabla\mathbf{\Theta}(x)$ for all $x \in \Omega$. The relation $\nabla\mathbf{\Theta}(x_0) = \nabla\mathbf{\Phi}(x_0)$ then implies that $\mathbf{Q} = \mathbf{I}$ and the relation $\mathbf{\Theta}(x_0) = \mathbf{\Phi}(x_0)$ in turn implies that $\mathbf{a} = \mathbf{0}$. $\square$

Remark. In fact, any additional conditions of the form $\mathbf{\Theta}(x_0) = \mathbf{a}_0$ and $\nabla\mathbf{\Theta}(x_0) = \mathbf{F}_0$, where $\mathbf{a}_0$ is any vector in $\mathbb{R}^n$ and $\mathbf{F}_0$ is any matrix in $\mathbb{M}^n$ that satisfies $\mathbf{F}_0^T\mathbf{F}_0 = \mathbf{C}(x_0)$, likewise imply the uniqueness of the mapping $\mathbf{\Theta}$. The particular choice $\mathbf{F}_0 = \mathbf{C}(x_0)^{1/2}$ made here insures that the associated mapping $\mathbf{C}(x_0) \in \mathbb{S}^n_> \to \mathbf{F}_0 \in \mathbb{M}^n$ is smooth (cf. Lemma 2.4), a property that will be used later on. Another choice for the matrix $\mathbf{F}_0$ that fulfills the same criterion is the upper triangular matrix that arises in the Cholesky factorization of the matrix $\mathbf{C}(x_0)$. $\square$

The *first objective* of this paper is to establish (cf. the next theorem) that a *manifold with boundary*, i.e., a subset of $\mathbb{R}^n$ of the form $\mathbf{\Theta}(\overline{\Omega})$, can be likewise recovered from a metric tensor field that, together with some partial derivatives, can be continuously extended to the *closure* $\overline{\Omega}$. In other words, we now extend the above existence and uniqueness results "up to the boundary".

In what follows, sets such as $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ or $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ and extensions such as $\overline{\nabla\mathbf{\Theta}}$ or $\overline{\mathbf{C}}$ are meant according to Definition 2.1 and the "geodesic property" is that of Definition 2.2.

Theorem 3.3 *Let Ω be a simply-connected open subset of $\mathbb{R}^n$ that satisfies the geodesic property. Let there be given a matrix field $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ that satisfies*

$$R^p_{\cdot ijk} = 0 \text{ in } \Omega.$$

Then there exists a mapping $\mathbf{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ that satisfies

$$\overline{\nabla\mathbf{\Theta}}(x)^T \overline{\nabla\mathbf{\Theta}}(x) = \overline{\mathbf{C}}(x) \text{ for all } x \in \overline{\Omega}.$$

If $\mathbf{\Phi} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ satisfies

$$\overline{\nabla\mathbf{\Phi}}(x)^T \overline{\nabla\mathbf{\Phi}}(x) = \overline{\mathbf{C}}(x) \text{ for all } x \in \overline{\Omega},$$

then there exist a vector $\mathbf{a} \in \mathbb{R}^n$ and a matrix $\mathbf{Q} \in \mathbb{O}^n$ such that

$$\overline{\mathbf{\Phi}}(x) = \mathbf{a} + \mathbf{Q}\overline{\mathbf{\Theta}}(x) \text{ for all } x \in \overline{\Omega}.$$

PROOF. The proof is broken into five steps, numbered (i) to (v).

(i) *Preliminaries.* Given any mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega; \mathbb{R}^n)$ that satisfies

$$\boldsymbol{\nabla}\boldsymbol{\Theta}(x)^T \boldsymbol{\nabla}\boldsymbol{\Theta}(x) = \mathbf{C}(x) \text{ for all } x \in \Omega$$

(such mappings exist by Theorem 3.1), we must show that $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$, i.e., that the vector fields $\boldsymbol{\Theta} \in \mathcal{C}^0(\Omega; \mathbb{R}^n)$ and $\partial^{\boldsymbol{\alpha}}\boldsymbol{\Theta} \in \mathcal{C}^0(\Omega; \mathbb{R}^n)$, $1 \leq |\boldsymbol{\alpha}| \leq 3$, admit continuous extensions on $\overline{\Omega}$. To begin with, let

$$\mathbf{F}(x) := \boldsymbol{\nabla}\boldsymbol{\Theta}(x) \in \mathbb{M}^n \text{ and } \boldsymbol{\Gamma}_i(x) := (\Gamma_{ij}^k(x)) \in \mathbb{M}^n \text{ for all } x \in \Omega,$$

with k as the row index and j as the column index. Then an immediate computation shows that the matrix fields $\mathbf{F} \in \mathcal{C}^2(\Omega; \mathbb{M}^n)$ and $\boldsymbol{\Gamma}_i \in \mathcal{C}^1(\Omega; \mathbb{M}^n)$ defined in this fashion satisfy

$$\partial_i \mathbf{F}(x) = \mathbf{F}(x)\boldsymbol{\Gamma}_i(x) \text{ for all } x \in \Omega.$$

The assumption $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ implies that $\det(\overline{g_{ij}}(x)) > 0$ for all $x \in \overline{\Omega}$. Hence the expressions of the functions $g^{k\ell} \in \mathcal{C}^0(\Omega)$ and $\partial^{\boldsymbol{\alpha}} g^{k\ell} \in \mathcal{C}^0(\Omega)$, $1 \leq |\boldsymbol{\alpha}| \leq 2$, as rational fractions of the functions g_{ij} and their derivatives, show that $(g^{k\ell}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$. Consequently, the definition of the Christoffel symbols Γ_{ij}^k implies that they belong to the space $\mathcal{C}^1(\overline{\Omega})$ or equivalently, that *the matrix fields* $\boldsymbol{\Gamma}_i$ *belong to the space* $\mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$.

(ii) *Let K be any compact subset of* $\mathbb{R}^n$*. Then* $\sup_{x \in K \cap \Omega} |\mathbf{F}(x)| < \infty$.

For any $x \in \Omega$,

$$|\mathbf{F}(x)|^2 = |\mathbf{F}(x)^T \mathbf{F}(x)|^{1/2} = |\mathbf{C}(x)|^{1/2}.$$

Hence

$$\sup_{x \in K \cap \Omega} |\mathbf{F}(x)|^2 = \sup_{x \in K \cap \overline{\Omega}} |\mathbf{C}(x)| < +\infty,$$

since the function $\overline{g_{ii}}$ belongs to the space $\mathcal{C}^0(\overline{\Omega})$ by assumption.

(iii) *The matrix field* $\mathbf{F} \in \mathcal{C}^2(\Omega; \mathbb{M}^n)$ *belongs to the space* $\mathcal{C}^2(\overline{\Omega}; \mathbb{M}^n)$.

Fix a point $x_0 \in \partial\Omega$ and let $K_0 = \overline{B(x_0; 1)}$. Then the properties established in (i) and (ii) together imply that

$$c_0 := \Big(\sup_{x \in K_0 \cap \Omega} |\mathbf{F}(x)|\Big)\Big(\sup_{x \in K_0 \cap \Omega} \Big(\sum_i |\boldsymbol{\Gamma}_i(x)|^2\Big)^{1/2}\Big) < +\infty.$$

Let $\varepsilon > 0$ be given. Because Ω satisfies the geodesic property, there exists $\delta(\varepsilon) > 0$ such that, given any two points $x, y \in B(x_0; \delta(\varepsilon)) \cap \Omega$, there exists a path $\boldsymbol{\gamma} = (\gamma_i)$ joining x to y in Ω whose length satisfies $L(\boldsymbol{\gamma}) \leq \dfrac{\varepsilon}{\max\{c_0, 2\}}$.

To ensure that the set $\boldsymbol{\gamma}([0,1])$ is contained in the set K_0, we assume, without loss of generality, that $\varepsilon \leq 1$ and $\delta(\varepsilon) \leq \dfrac{1}{2}$.

Since $\partial_i \mathbf{F}(x) = \mathbf{F}(x)\boldsymbol{\Gamma}_i(x)$ for all $x \in \Omega$ by part (i), the matrix field $\mathbf{Y} := \mathbf{F} \circ \boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\mathbb{M}^n)$ associated with any such path $\boldsymbol{\gamma}$ satisfies

$$\mathbf{Y}'(t) = \gamma_i'(t)\mathbf{Y}(t)\boldsymbol{\Gamma}_i(\boldsymbol{\gamma}(t)) \text{ for all } 0 \leq t \leq 1.$$

Expressing that $\mathbf{Y}(1) = \mathbf{Y}(0) + \int_0^1 \mathbf{Y}'(t)\,\mathrm{d}t$, we thus have, for any two points $x, y \in B(x_0;\delta(\varepsilon)) \cap \Omega$,

$$|\mathbf{F}(y) - \mathbf{F}(x)| = |\mathbf{Y}(1) - \mathbf{Y}(0)| \leq \Big(\sup_{0\leq t\leq 1} |\mathbf{F}(\boldsymbol{\gamma}(t))|\Big) \int_0^1 |\gamma_i'(t)||\boldsymbol{\Gamma}_i(\boldsymbol{\gamma}(t))|\,\mathrm{d}t$$
$$\leq \Big(\sup_{x\in K_0\cap\Omega} |\mathbf{F}(x)|\Big)\Big(\sup_{x\in K_0\cap\Omega}\Big(\sum_i |\boldsymbol{\Gamma}_i(x)|^2\Big)^{1/2}\Big)L(\boldsymbol{\gamma}) \leq \varepsilon.$$

Let $(x_m)_{m\geq 1}$ be any sequence of points $x_m \in \Omega$ such that $\lim_{m\to\infty} x_m = x_0$. Since, for any $\varepsilon > 0$, there exists $m_0(\varepsilon)$ such that $x_m \in B(x_0;\delta(\varepsilon))$ for all $m \geq m_0(\varepsilon)$, the last inequality shows that the sequence $(\mathbf{F}(x_m))_{m\geq 1}$ is a Cauchy sequence. Hence $\lim_{m\to\infty} \mathbf{F}(x_m)$ exists and this limit is clearly independent of the sequence $(x_m)_{m\geq 1}$. This shows that *the field* $\mathbf{F} \in \mathcal{C}^2(\Omega;\mathbb{M}^n)$ *can be extended to a field that is continuous on* $\overline{\Omega}$.

Since $\partial_i \mathbf{F} = \mathbf{F}\boldsymbol{\Gamma}_i$ in Ω and the fields $\boldsymbol{\Gamma}_i$ belong to the space $\mathcal{C}^1(\overline{\Omega};\mathbb{M}^n)$ by part (i), each field $\partial_i \mathbf{F} \in \mathcal{C}^1(\Omega;\mathbb{M}^n)$ can be extended to a field that is continuous on $\overline{\Omega}$; hence $\mathbf{F} \in \mathcal{C}^1(\overline{\Omega};\mathbb{M}^n)$. Differentiating the relations $\partial_i \mathbf{F} = \mathbf{F}\boldsymbol{\Gamma}_i$ in Ω further shows that $\mathbf{F} \in \mathcal{C}^2(\overline{\Omega};\mathbb{M}^n)$.

(iv) *The vector field* $\boldsymbol{\Theta} \in \mathcal{C}^3(\Omega;\mathbb{R}^n)$ *belongs to the space* $\mathcal{C}^3(\overline{\Omega};\mathbb{R}^n)$.

Given $x_0 \in \partial\Omega$, we proceed as in (iii), the number $\delta(\varepsilon) > 0$ being now chosen in such a way that $L(\boldsymbol{\gamma}) \leq \dfrac{\varepsilon}{\max\{c_1, 2\}}$, where

$$c_1 := \frac{1}{\sqrt{n}}\Big(\sup_{x\in K_0\cap\Omega} |\mathbf{F}(x)|\Big)^{-1} < \infty.$$

Again without loss of generality, we assume that $\varepsilon \leq 1$ and $\delta(\varepsilon) \leq \dfrac{1}{2}$.

For each $x \in \Omega$, let $\mathbf{g}_i(x)$ denote the i-th column vector of the matrix $\mathbf{F}(x)$. The relations $\partial_i \boldsymbol{\Theta}(x) = \mathbf{g}_i(x)$ for all $x \in \Omega$ then imply that the vector field $\mathbf{y} := \boldsymbol{\Theta} \circ \boldsymbol{\gamma} \in \mathcal{C}^1([0,1];\mathbb{R}^n)$ associated with each such path $\boldsymbol{\gamma}$ joining x to y in Ω satisfies

$$\mathbf{y}'(t) = \gamma_i'(t)\mathbf{g}_i(\boldsymbol{\gamma}(t)) \text{ for all } 0 \leq t \leq 1,$$

so that, for any two points $x, y \in B(x_0; \delta(\varepsilon)) \cap \Omega$,

$$|\boldsymbol{\Theta}(y) - \boldsymbol{\Theta}(x)| = |\mathbf{y}(1) - \mathbf{y}(0)| \leq \int_0^1 |\gamma_i'(t)\mathbf{g}_i(\boldsymbol{\gamma}(t))|\,\mathrm{d}t$$
$$\leq L(\boldsymbol{\gamma}) \sup_{x \in K_0 \cap \Omega} \Big(\sum_i |\mathbf{g}_i(x)|^2\Big)^{1/2} \leq \sqrt{n}\, L(\boldsymbol{\gamma}) \sup_{x \in K_0 \cap \Omega} |\mathbf{F}(x)| \leq \varepsilon.$$

Arguing as in (iii), we thus conclude that *the field* $\boldsymbol{\Theta} \in \mathcal{C}^0(\Omega; \mathbb{R}^n)$ *can be extended to a field that is continuous on* $\overline{\Omega}$.

Noting that $\mathbf{g}_i \in \mathcal{C}^2(\overline{\Omega}; \mathbb{R}^n)$ by (iii) and differentiating the relations $\partial_i \boldsymbol{\Theta} = \mathbf{g}_i$ in Ω, we finally conclude that the fields $\partial^{\boldsymbol{\alpha}} \boldsymbol{\Theta} \in \mathcal{C}^0(\Omega)$, $1 \leq |\boldsymbol{\alpha}| \leq 3$, can be extended to fields that are continuous on $\overline{\Omega}$. Hence $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$.

(v) *Uniqueness up to isometries in* $\mathbb{R}^n$. If $\boldsymbol{\Phi} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ satisfies

$$\overline{\boldsymbol{\nabla}\boldsymbol{\Phi}}(x)^T \overline{\boldsymbol{\nabla}\boldsymbol{\Phi}}(x) = \overline{\mathbf{C}}(x) \text{ for all } x \in \overline{\Omega},$$

then by Theorem 3.1, there exist $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{Q} \in \mathbb{O}^n$ such that

$$\boldsymbol{\Phi}(x) = \mathbf{a} + \mathbf{Q}\boldsymbol{\Theta}(x) \text{ for all } x \in \Omega.$$

Consequently, the continuous extensions $\overline{\boldsymbol{\Phi}}$ and $\overline{\boldsymbol{\Theta}}$ satisfy

$$\overline{\boldsymbol{\Phi}}(x) = \mathbf{a} + \mathbf{Q}\overline{\boldsymbol{\Theta}}(x) \text{ for all } x \in \overline{\Omega}. \qquad \square$$

The existence and uniqueness result of Corollary 3.2 can be also extended to the mappings $\overline{\boldsymbol{\Theta}}$ found in Theorem 3.3:

Corollary 3.4 *Let the assumptions on the set* Ω *and on the matrix field* $\mathbf{C}$ *be as in Theorem 3.3 and let a point* $x_0 \in \Omega$ *be given. Then there exists one and only one mapping* $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ *that satisfies*

$$\overline{\boldsymbol{\nabla}\boldsymbol{\Theta}}(x)^T \overline{\boldsymbol{\nabla}\boldsymbol{\Theta}}(x) = \overline{\mathbf{C}}(x) \text{ for all } x \in \overline{\Omega},$$
$$\boldsymbol{\Theta}(x_0) = \mathbf{0} \text{ and } \boldsymbol{\nabla}\boldsymbol{\Theta}(x_0) = \mathbf{C}(x_0)^{1/2}.$$

PROOF. The proof is a straightforward adaptation of that of Corollary 3.2 and, for this reason, is omitted. $\square$

As illustrated in the proof of Theorem 3.3 (and later in that of Theorem 5.2), the interest of the *geodesic property* introduced in Definition 2.2 is to provide estimates on the solutions of ordinary differential equations along a path joining two points in Ω that eventually depend only on the length of the path, but not on the path itself.

4 Extension of a Riemannian metric with vanishing curvature

The *second objective* of this paper is to provide sufficient conditions guaranteeing that a Riemannian metric $(g_{ij}) \in \mathcal{C}^2(\Omega; \mathbb{S}^n_>)$ with a Riemann curvature tensor vanishing in an open subset Ω of $\mathbb{R}^n$ can be extended to a Riemannian metric $(\widetilde{g}_{ij}) \in \mathcal{C}^2(\widetilde{\Omega}; \mathbb{S}^n_>)$ on a connected open set $\widetilde{\Omega}$ containing $\overline{\Omega}$, in such a way that the Riemann curvature tensor associated with this extension still vanishes in $\widetilde{\Omega}$.

To this end, we begin by introducing another definition based on the geodesic distance, which is stronger than that of Definition 2.2.

Definition 4.1 An open subset Ω of $\mathbb{R}^n$ satisfies the *strong geodesic property* if it is connected and there exists a constant C_Ω such that

$$d_\Omega(x, y) \leq C_\Omega |x - y| \text{ for all } x, y \in \Omega,$$

where d_Ω designates the geodesic distance in Ω (cf. Section 2). □

Remarks. (1) Since $|x - y| \leq d_\Omega(x, y)$ for all $x, y \in \Omega$, the geodesic distance is thus equivalent to the Euclidean distance on an open set that satisfies the strong geodesic property.

(2) The strong geodesic property clearly implies the geodesic property, but not conversely; consider, e.g., a bounded open subset of $\mathbb{R}^2$ whose boundary is a cardioid.

(3) Any connected open subset of $\mathbb{R}^n$ with a Lipschitz-continuous boundary satisfies the strong geodesic property; for a proof, see, e.g., Proposition 5.1 in Anicic, Le Dret & Raoult [2]. □

The following theorem, which hinges in particular on a profound result of Whitney [27] shows that, when an open set Ω satisfies the strong geodesic property, the spaces $\mathcal{C}^\ell(\overline{\Omega})$ introduced in Definition 2.1 admit a remarkably simple characterization.

Theorem 4.2 *Let Ω be an open subset of $\mathbb{R}^n$ that satisfies the strong geodesic property. Then for any integer $\ell \geq 1$, the space $\mathcal{C}^\ell(\overline{\Omega})$ of Definition 2.1 can be also defined as*

$$\mathcal{C}^\ell(\overline{\Omega}) = \{f|_\Omega \in \mathcal{C}^\ell(\Omega); f \in \mathcal{C}^\ell(\mathbb{R}^n)\}.$$

PROOF. For convenience, the proof is broken into four parts. Note that the assumption that Ω satisfies the strong geodesic property is not needed until part (iii).

(i) To begin with, we list some *notations* used throughout this proof. Given a multi-index $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbb{N}^n$, we let

$$|\boldsymbol{\alpha}| := \sum_i \alpha_i \text{ and } \partial^{\boldsymbol{\alpha}} := \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}} \text{ if } |\boldsymbol{\alpha}| \geq 1,$$

as before; in addition, we also let

$$\mathbf{0} := (0, 0, \dots, 0) \text{ and } \partial^{\mathbf{0}} f := f,$$
$$\mathbf{0}! := 1 \text{ and } \boldsymbol{\alpha}! := (\alpha_1!)(\alpha_2!) \cdots (\alpha_n!).$$

If $x = (x_i)$ and $y = (y_i)$ are two points in $\mathbb{R}^n$, we let

$$(y - x)^{\mathbf{0}} := 1 \text{ and } (y - x)^{\boldsymbol{\alpha}} := (y_1 - x_1)^{\alpha_1} (y_2 - x_2)^{\alpha_2} \cdots (y_n - x_n)^{\alpha_n}.$$

Concurrently with the multi-index notation $\partial^{\boldsymbol{\alpha}} f$ for partial derivatives and depending on the context, we shall also use the notations

$$\partial_{i_1} f := \frac{\partial f}{\partial x_{i_1}}, \partial_{i_1 i_2} f := \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}, \dots, \partial_{i_1 i_2 \dots i_m} f := \frac{\partial^m f}{\partial x_{i_1} \partial x_{i_2} \cdots \partial x_{i_m}},$$

with the understanding that, whenever a summation involves such indices $i_1, i_2, \dots, i_m$, then they range in the set $\{1, 2, \dots, n\}$ independently of each other; thus for instance,

$$\partial_{i_1 i_2} f(x) h_{i_1} h_{i_2} = \sum_{i_1=1}^n \sum_{i_2=1}^n \frac{\partial^2 f}{\partial x_{i_1} \partial x_{i_2}}(x) h_{i_1} h_{i_2}.$$

(ii) *Let Ω be a connected open subset of $\mathbb{R}^n$, let x and y be two points in Ω, let $\boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^n)$ be a path joining x to y in Ω, and let a function $f \in \mathcal{C}^m(\Omega)$, $m \geq 1$, be given. Then*

$$\Big| f(y) - \sum_{|\boldsymbol{\beta}| \leq m} \frac{1}{\boldsymbol{\beta}!} \partial^{\boldsymbol{\beta}} f(x) (y - x)^{\boldsymbol{\beta}} \Big| \leq L(\boldsymbol{\gamma})^m \Big\{ \sum_{|\boldsymbol{\alpha}| = m} \frac{1}{\boldsymbol{\alpha}!} \sup_{z \in \boldsymbol{\gamma}([0,1])} |\partial^{\boldsymbol{\alpha}} f(z) - \partial^{\boldsymbol{\alpha}} f(x)|^2 \Big\}^{1/2},$$

where $L(\boldsymbol{\gamma})$ denotes the length of the path $\boldsymbol{\gamma}$.

To give a flavor of the kind of computations involved, assume for instance that $m = 2$, in which case we may write

$$
\begin{aligned}
&f(y) - f(x) = f(\boldsymbol{\gamma}(1)) - f(\boldsymbol{\gamma}(0)) = \int_0^1 \partial_{i_1} f(\boldsymbol{\gamma}(t_1))\gamma'_{i_1}(t_1)\,\mathrm{d}t_1 \\
&= \partial_{i_1} f(x) \int_0^1 \gamma'_{i_1}(t_1)\,\mathrm{d}t_1 + \int_0^1 \{\partial_{i_1} f(\boldsymbol{\gamma}(t_1)) - \partial_{i_1} f(x)\}\gamma'_{i_1}(t_1)\,\mathrm{d}t_1 \\
&= \partial_{i_1} f(x) \int_0^1 \gamma'_{i_1}(t_1)\,\mathrm{d}t_1 + \int_0^1 \left(\int_0^{t_1} \partial_{i_1 i_2} f(\boldsymbol{\gamma}(t_2))\gamma'_{i_2}(t_2)\,\mathrm{d}t_2\right)\gamma'_{i_1}(t_1)\,\mathrm{d}t_1 \\
&= \partial_{i_1} f(x) \int_0^1 \gamma'_{i_1}(t_1)\,\mathrm{d}t_1 + \partial_{i_1 i_2} f(x) \int_0^1 \left(\int_0^{t_1} \gamma'_{i_2}(t_2)\,\mathrm{d}t_2\right)\gamma'_{i_1}(t_1)\,\mathrm{d}t_1 \\
&\quad + \int_0^1 \left(\int_0^{t_1} \{\partial_{i_1 i_2} f(\boldsymbol{\gamma}(t_2)) - \partial_{i_1 i_2} f(x)\}\gamma'_{i_2}(t_2)\,\mathrm{d}t_2\right)\gamma'_{i_1}(t_1)\,\mathrm{d}t_1.
\end{aligned}
$$

Denoting by x_i and y_i the coordinates of the points x and y, we also have

$$
\begin{aligned}
&\partial_{i_1} f(x) \int_0^{t_1} \gamma'_{i_1}(t_1)\,\mathrm{d}t_1 = \frac{1}{1!}\partial_{i_1} f(x)(y_{i_1} - x_{i_1}), \\
&\partial_{i_1 i_2} f(x) \int_0^1 \left(\int_0^{t_1} \gamma'_{i_2}(t_2)\,\mathrm{d}t_2\right)\gamma'_{i_1}(t_1)\,\mathrm{d}t_1 \\
&\qquad = \partial_{i_1 i_2} f(x) \int_0^1 (\gamma_{i_2}(t_1) - \gamma_{i_2}(0))[\gamma_{i_1}(t_1) - \gamma_{i_1}(0)'\,\mathrm{d}t_1 \\
&\qquad = \frac{1}{2}\partial_{i_1 i_2} f(x) \int_0^1 \left[(\gamma_{i_1}(t_1) - \gamma_{i_1}(0))(\gamma_{i_2}(t_1) - \gamma_{i_2}(0))\right]'\mathrm{d}t_1 \\
&\qquad = \frac{1}{2!}\partial_{i_1 i_2} f(x)(y_{i_1} - x_{i_1})(y_{i_2} - x_{i_2}),
\end{aligned}
$$

so that the above relations together imply that, when $m = 2$,

$$
\begin{aligned}
&f(y) - \left\{f(x) + \frac{1}{1!}\partial_{i_1} f(x)(y_{i_1} - x_{i_1}) + \frac{1}{2!}\partial_{i_1 i_2} f(x)(y_{i_1} - x_{i_1})(y_{i_2} - x_{i_2})\right\} \\
&= \int_0^1 \left(\int_0^{t_1} \left\{\partial_{i_1 i_2} f(\boldsymbol{\gamma}(t_2)) - \partial_{i_1 i_2} f(x)\right\}\gamma'_{i_2}(t_2)\,\mathrm{d}t_2\right)\gamma'_{i_1}(t_1)\,\mathrm{d}t_1.
\end{aligned}
$$

When $m \geq 2$, similar computations likewise lead to the identity:

$$
\begin{aligned}
&f(y) - \left\{f(x) + \frac{1}{1!}\partial_{i_1} f(x)(y_{i_1} - x_{i_1}) + \cdots \right. \\
&\qquad\qquad \left. + \frac{1}{m!}\partial_{i_1 \ldots i_m} f(x)(y_{i_1} - x_{i_1}) \cdots (y_{i_m} - x_{i_m})\right\} \\
&\quad = \int_0^1 \left(\cdots\left(\int_0^{t_{m-2}}\left(\int_0^{t_{m-1}} \left\{\partial_{i_1 \cdots i_m} f(\boldsymbol{\gamma}(t_m)) - \partial_{i_1 \cdots i_m} f(\boldsymbol{\gamma}(0))\right\}\right.\right.\right. \\
&\qquad\qquad\qquad \left.\left.\left. \times\, \gamma'_{i_m}(t_m)\,\mathrm{d}t_m\right)\gamma'_{i_{m-1}}(t_{m-1})\,\mathrm{d}t_{m-1}\right)\cdots\right)\gamma'_{i_1}(t_1)\,\mathrm{d}t_1,
\end{aligned}
$$

which implies that

$$\Big|f(y)-\sum_{|\boldsymbol{\beta}|\le m}\frac{1}{\beta!}\partial^{\boldsymbol{\beta}}f(x)(y-x)^{\boldsymbol{\beta}}\Big|\le$$

$$\le\int_0^1\cdots\Big(\int_0^{t_{m-2}}\Big(\int_0^{t_{m-1}}C_{i_1\ldots i_m}|\gamma'_{i_m}(t_m)|\,\mathrm{d}t_m\Big)\,|\gamma'_{i_{m-1}}(t_{m-1})|\,\mathrm{d}t_{m-1}\Big)\cdots\,\mathrm{d}t_1,$$

where

$$C_{i_1\ldots i_m}:=\sup_{z\in\boldsymbol{\gamma}([0,1])}|\partial_{i_1\ldots i_m}f(z)-\partial_{i_1\ldots i_m}f(x)|.$$

We then observe that

$$\int_0^{t_{m-1}}C_{i_1\ldots i_m}|\gamma'_{i_m}(t_m)|\,\mathrm{d}t_m\le\int_0^{t_{m-1}}C_{i_1\ldots i_{m-1}}|\boldsymbol{\gamma}'(t_m)|\,\mathrm{d}t_m,$$

where

$$C_{i_1\ldots i_{m-1}}:=\Big\{\sum_{i_m}(C_{i_1\ldots i_m})^2\Big\}^{1/2}\text{ and }|\boldsymbol{\gamma}'(t)|=\Big\{\sum_{i_m}|\gamma'_{i_m}(t_m)|^2\Big\}^{1/2}.$$

Continuing to similarly apply Cauchy-Schwarz inequalities, we eventually find that

$$|f(y)-\sum_{|\boldsymbol{\beta}|\le m}\frac{1}{\beta!}\partial^{\boldsymbol{\beta}}f(x)(y-x)^{\boldsymbol{\beta}}|$$

$$\le C\int_0^1\cdots\Big(\int_0^{t_{m-2}}\Big(\int_0^{t_{m-1}}|\boldsymbol{\gamma}'(t_m)|\,\mathrm{d}t_m\Big)\,|\boldsymbol{\gamma}'(t_{m-1})|\,\mathrm{d}t_{m-1}\Big)\cdots\,\mathrm{d}t_1,$$

where

$$C:=\Big\{\sum_{i_1}(C_{i_1})^2\Big\}^{1/2},C_{i_1}:=\Big\{\sum_{i_2}(C_{i_1i_2})^2\Big\}^{1/2},\ldots,$$
$$C_{i_1\cdots i_{m-2}}:=\Big\{\sum_{i_{m-1}}(C_{i_1\cdots i_{m-1}})^2\Big\}^{1/2}.$$

On the one hand, we have

$$C=\Big\{\sum_{i_1\cdots i_m}(C_{i_1\cdots i_m})^2\Big\}^{1/2}=\Big\{\sum_{i_1\cdots i_m}\sup_{z\in\boldsymbol{\gamma}([0,1])}|\partial_{i_1\cdots i_m}f(z)-\partial_{i_1\cdots i_m}f(x)|^2\Big\}^{1/2}$$
$$=m!\sum_{|\boldsymbol{\alpha}|=m}\frac{1}{\boldsymbol{\alpha}!}\Big\{\sup_{z\in\boldsymbol{\gamma}([0,1])}|\partial^{\boldsymbol{\alpha}}f(z)-\partial^{\boldsymbol{\alpha}}f(x)|^2\Big\}^{1/2}.$$

On the other hand, let

$$\lambda(t):=\int_0^t|\boldsymbol{\gamma}'(\tau)|\,\mathrm{d}\tau,\ 0\le t\le 1,$$

so that we may write

$$\int_0^{t_{m-1}} |\boldsymbol{\gamma}'(t_m)|\,\mathrm{d}t_m = \int_0^{t_{m-1}} \lambda'(t_m)\,\mathrm{d}t_m = \lambda(t_{m-1}),$$

$$\begin{aligned}\int_0^{t_{m-2}} \Big(\int_0^{t_{m-1}} |\boldsymbol{\gamma}'(t_m)|\,\mathrm{d}t_m\Big)|\boldsymbol{\gamma}'(t_{m-1})|\,\mathrm{d}t_{m-1}& \\ = \int_0^{t_{m-2}} \lambda(t_{m-1})\lambda'(t_{m-1})\,\mathrm{d}t_{m-1} &= \frac{1}{2!}(\lambda(t_{m-2}))^2,\end{aligned}$$

and so on, until we finally obtain

$$\begin{aligned}\int_0^1 \Big(\int_0^{t_1} \cdots \Big(\int_0^{t_{m-2}} \Big(\int_0^{t_{m-1}} |\boldsymbol{\gamma}'(t_m)|\,\mathrm{d}t_m\Big)\,|\boldsymbol{\gamma}'(t_{m-1})|\,\mathrm{d}t_{m-1}\Big)\cdots\Big)\,|\boldsymbol{\gamma}'(t_1)|\,\mathrm{d}t_1 \\ = \frac{1}{(m-1)!}\int_0^1 (\lambda(t_1))^{m-1}\lambda'(t_1)\,\mathrm{d}t_1 = \frac{1}{m!}\lambda(1)^m = \frac{1}{m!}L(\boldsymbol{\gamma})^m.\end{aligned}$$

Hence the estimate announced in part (ii) is established. The next step consists in getting rid of the dependence on the path $\boldsymbol{\gamma}$ in this estimate, thanks to the strong geodesic property:

(iii) *Let Ω be an open subset of $\mathbb{R}^n$ that satisfies the strong geodesic property and let a function $f \in \mathcal{C}^m(\overline{\Omega})$, $m \geq 1$, be given, the space $\mathcal{C}^m(\overline{\Omega})$ being that of Definition 2.1. Then, given any point $x_0 \in \overline{\Omega}$ and any number $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon)$ such that*

$$|\overline{f}(y) - \sum_{|\boldsymbol{\beta}|\leq m} \frac{1}{\boldsymbol{\beta}!}\overline{\partial^{\boldsymbol{\beta}} f}(x)(y-x)^{\boldsymbol{\beta}}| \leq \varepsilon|y-x|^m \text{ for all } x, y \in \overline{\Omega} \cap B(x_0;\delta),$$

where $\overline{f} \in \mathcal{C}^0(\overline{\Omega})$ and $\overline{\partial^{\boldsymbol{\beta}} f} \in \mathcal{C}^0(\overline{\Omega})$, $1 \leq |\boldsymbol{\beta}| \leq m$, denote the continuous extensions of the functions $f \in \mathcal{C}^0(\Omega)$ and $\partial^{\boldsymbol{\beta}} f \in \mathcal{C}^0(\Omega)$.

Given any point $x_0 \in \overline{\Omega}$ and any number $\varepsilon > 0$, there exists $\delta = \delta(x_0, \varepsilon)$ such that

$$\Big\{\sum_{|\boldsymbol{\alpha}|=m} \frac{1}{\boldsymbol{\alpha}!}|\partial^{\boldsymbol{\alpha}} f(z) - \partial^{\boldsymbol{\alpha}} f(x)|^2\Big\}^{1/2} \leq \frac{\varepsilon}{(2C_\Omega)^m} \text{ for all } x, z \in \Omega \cap B(x_0; (1+4C_\Omega)\delta)$$

since the extensions $\overline{\partial^{\boldsymbol{\alpha}} f}$, $|\boldsymbol{\alpha}| = m$, are locally uniformly continuous on $\overline{\Omega}$ (the constant C_Ω is that appearing in Definition 4.1).

Given any points $x, y \in \Omega \cap B(x_0;\delta)$, there exists a path $\boldsymbol{\gamma}$ joining x to y in Ω whose length satisfies

$$L(\boldsymbol{\gamma}) < 2d_\Omega(x,y) \leq 2C_\Omega|x-y| \leq 4C_\Omega\delta,$$

since Ω satisfies the strong geodesic property. Consequently, $\boldsymbol{\gamma}(z) \in B(x_0; (1+$

$4C_\Omega)\delta)$ for all $z \in \boldsymbol{\gamma}([0,1])$, since

$$|z - x_0| \le |z - x| + |x - x_0| \le L(\boldsymbol{\gamma}) + \delta < (1 + 4C_\Omega)\delta.$$

The estimate established in part (ii) thus implies that

$$|f(y) - \sum_{|\boldsymbol{\beta}| \le m} \frac{1}{\beta!} \partial^{\boldsymbol{\beta}} f(x)(y-x)^{\boldsymbol{\beta}}| \le \varepsilon |y-x|^m \text{ for all } x, y \in \Omega \cap B(x_0;\delta).$$

Given any points $x, y \in \overline{\Omega} \cap B(x_0;\delta)$, there exist points $x^k, y^k \in \Omega \cap B(x_0;\delta)$ such that $x^k \to x$ and $y^k \to x$ as $k \to \infty$. From the continuity on $\overline{\Omega}$ of the extensions $\overline{f}$ and $\overline{\partial^{\boldsymbol{\alpha}} f}$, $1 \le |\boldsymbol{\alpha}| \le m$, we thus infer that

$$\begin{aligned} \Big|\overline{f}(y) - \sum_{|\boldsymbol{\beta}| \le m} \overline{\partial^{\boldsymbol{\beta}} f}(x)(y-x)^{\boldsymbol{\beta}}\Big| &= \lim_{k\to\infty} \Big| f(y^k) - \sum_{|\boldsymbol{\beta}| \le m} \partial^{\boldsymbol{\beta}} f(x^k)(y^k - x^k)^{\boldsymbol{\beta}}\Big| \\ &\le \varepsilon \lim_{k\to\infty} |y^k - x^k|^m = \varepsilon |y-x|^m \text{ for all } x, y \in \overline{\Omega} \cap B(x_0;\delta). \end{aligned}$$

(iv) Let there be given a function f in the space $\mathcal{C}^\ell(\overline{\Omega})$, $\ell \ge 1$, according to Definition 2.1. According to a deep result of Whitney [27], f *is also the restriction to* Ω *of a function in the space* $\mathcal{C}^\ell(\mathbb{R}^n)$ *if, for each multi-index* $\boldsymbol{\alpha}$ *satisfying* $0 \le |\boldsymbol{\alpha}| \le \ell$*, there exist functions* $f_{\boldsymbol{\alpha}} \in \mathcal{C}^0(\overline{\Omega})$ *with the following property: For any points* $x, y \in \overline{\Omega}$ *and any multi-index* $\boldsymbol{\alpha}$ *satisfying* $0 \le |\boldsymbol{\alpha}| \le \ell$*, let*

$$R_{\boldsymbol{\alpha}}(y;x) := f_{\boldsymbol{\alpha}}(y) - \sum_{|\boldsymbol{\beta}| \le \ell - |\boldsymbol{\alpha}|} \frac{1}{\boldsymbol{\beta}!} f_{\boldsymbol{\alpha}+\boldsymbol{\beta}}(x)(y-x)^{\boldsymbol{\beta}}.$$

Then, given any point $x_0 \in \overline{\Omega}$ *and any number* $\varepsilon > 0$*, there exists* $\delta = \delta(x_0, \varepsilon)$ *such that*

$$|R_{\boldsymbol{\alpha}}(y;x)| \le \varepsilon |y-x|^{\ell - |\boldsymbol{\alpha}|} \text{ for all } x, y \in \overline{\Omega} \cap B(x_0;\delta) \text{ and } 0 \le |\boldsymbol{\alpha}| \le \ell.$$

To verify that this is indeed the case, let $x_0 \in \overline{\Omega}$ and $\varepsilon > 0$ be given. Then the estimate of part (iii) applied to each function $\overline{\partial^{\boldsymbol{\alpha}} f}, 0 \le |\boldsymbol{\alpha}| \le \ell$, shows that there exists $\delta_{\boldsymbol{\alpha}} = \delta_{\boldsymbol{\alpha}}(x_0, \varepsilon)$ such that

$$\begin{aligned} &\Big|\overline{\partial^{\boldsymbol{\alpha}} f}(y) - \sum_{|\boldsymbol{\beta}| \le \ell - |\boldsymbol{\alpha}|} \frac{1}{\boldsymbol{\beta}!} \overline{\partial^{\boldsymbol{\beta}}(\partial^{\boldsymbol{\alpha}} f)}(x)(y-x)^{\boldsymbol{\beta}}\Big| \le \\ &\le \varepsilon |y-x|^{\ell - |\boldsymbol{\alpha}|} \text{ for all } x, y \in \overline{\Omega} \cap B(x_0;\delta_{\boldsymbol{\alpha}}). \end{aligned}$$

Since $\partial^{\boldsymbol{\beta}}(\partial^{\boldsymbol{\alpha}} f)(x) = \partial^{\boldsymbol{\beta}+\boldsymbol{\alpha}} f(x)$ for all $x \in \Omega$, it likewise follows that $\overline{\partial^{\boldsymbol{\beta}}(\partial^{\boldsymbol{\alpha}} f)}(x) = \overline{\partial^{\boldsymbol{\beta}+\boldsymbol{\alpha}} f(x)}$ for all $x \in \overline{\Omega}$. Therefore Whitney's theorem can be applied, with $f_{\boldsymbol{\alpha}} := \overline{\partial^{\boldsymbol{\alpha}} f}$ and $\delta := \min\{\delta_{\boldsymbol{\alpha}};\, 0 \le |\boldsymbol{\alpha}| \le \ell\}$. $\square$

Remarks. (1) The identity

$$f(y) = f(x) + \cdots + \frac{1}{(m-1)!}\partial_{i_1 \dots i_{m-1}} f(x)(y_{i_1} - x_{i_1}) \cdots (y_{i_{m-1}} - x_{i_{m-1}})$$
$$+ \int_0^1 \cdots \Big(\int_0^{t_{m-2}} \Big(\int_0^{t_{m-1}} \partial_{i_1 \dots i_m} f(\boldsymbol{\gamma}(t_m)) \gamma'_{i_m}(t_m)\, \mathrm{d}t_m \Big) \gamma'_{i_{m-1}}(t_{m-1})\, \mathrm{d}t_{m-1} \Big) \cdots \mathrm{d}t_1$$

established for any function $f \in \mathcal{C}^m(\Omega)$ in the course of the proof of part (ii) may be viewed as a *Taylor formula along a path* and, in the same vein, the estimate likewise established in part (ii) may be viewed as a *generalized mean-value theorem along a path* (it is easily verified that both formulas reduce to standard ones when $\boldsymbol{\gamma}(t) = (1-t)x + ty$, $0 \le t \le 1$).

(2) The following example, kindly communicated to us by Sorin Mardare, shows that Theorem 4.2 no longer holds if Ω is only assumed to satisfy the weaker geodesic property of Definition 2.2: Let $\Omega = \{(x_1, x_2) \in \mathbb{R}^2; x_2 < \sqrt{|x_1|}\}$ and let the function $f : \Omega \to \mathbb{R}$ be defined for $(x_1, x_2) \in \Omega$ by $f(x_1, x_2) := (x_2)^2$ if $x_1 > 0$ and $x_2 > 0$, and by $f(x_1, x_2) := 0$ otherwise. Then it is easily verified that the open and connected set Ω satisfies the geodesic property but does not satisfy the strong geodesic property and that the function f belongs to the space $\mathcal{C}^1(\overline{\Omega})$ of Definition 2.1. A simple argument by contradiction shows, however, that there is no function in the space $\mathcal{C}^1(\mathbb{R}^2)$ whose restriction to Ω would be the function f. □

We are now in a position to prove the announced extension result. The notations are the same as in Theorem 3.3.

Theorem 4.3 *Let Ω be a simply-connected open subset of $\mathbb{R}^n$ that satisfies the strong geodesic property and let there be given a matrix field $(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ that satisfies*

$$R^p_{\cdot ijk} = 0 \text{ in } \Omega.$$

Then there exist a connected open subset $\widetilde{\Omega}$ of $\mathbb{R}^n$ containing $\overline{\Omega}$ and a matrix field $(\widetilde{g}_{ij}) \in \mathcal{C}^2(\widetilde{\Omega}; \mathbb{S}^n_>)$ such that

$$\widetilde{g}_{ij}(x) = g_{ij}(x) \text{ for all } x \in \Omega \text{ and } \widetilde{R}^p_{\cdot ijk} = 0 \text{ in } \widetilde{\Omega},$$

where the functions $\widetilde{R}^p_{\cdot ijk} \in \mathcal{C}^0(\widetilde{\Omega})$ denote the mixed components of the Riemann curvature tensor associated with the field $(\widetilde{g}_{ij})$.

PROOF. Since Ω *a fortiori* satisfies the geodesic property and Ω is simply-connected, there exists by Theorem 3.3 a mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ that satisfies

$$\overline{\partial_i \boldsymbol{\Theta}}(x) \cdot \overline{\partial_j \boldsymbol{\Theta}}(x) = \overline{g_{ij}}(x) \text{ for all } x \in \overline{\Omega}.$$

Since Ω satisfies the strong geodesic property, there in turn exists by Theorem 4.2 a mapping $\widetilde{\mathbf{\Theta}} \in \mathcal{C}^3(\mathbb{R}^n;\mathbb{R}^n)$ that satisfies

$$\widetilde{\mathbf{\Theta}}(x) = \mathbf{\Theta}(x) \text{ for all } x \in \Omega.$$

Let then

$$\widetilde{g}_{ij}(x) := \partial_i \widetilde{\mathbf{\Theta}}(x) \cdot \partial_j \widetilde{\mathbf{\Theta}}(x) \text{ for all } x \in \mathbb{R}^n,$$

and define the set

$$U := \{x \in \mathbb{R}^n; (\widetilde{g}_{ij}(x)) \in \mathbb{S}^n_>\},$$

which is open in $\mathbb{R}^n$ and contains $\overline{\Omega}$ (since $\widetilde{g}_{ij}(x) = \overline{g_{ij}}(x)$ for all $x \in \overline{\Omega}$). Finally, define the set $\widetilde{\Omega}$ as the connected component of U that contains $\overline{\Omega}$; hence the set $\widetilde{\Omega}$ is open and connected.

Furthermore, the mixed components $\widetilde{R}^p_{\cdot ijk}$ of the Riemann curvature tensor associated with the field $(\widetilde{g}_{ij})$ are well defined in the set $\widetilde{\Omega}$ since the matrices $(\widetilde{g}_{ij}(x))$ are by construction invertible for all $x \in \widetilde{\Omega} \subset U$.

Because $\widetilde{g}_{ij}(x) = \partial_i \widetilde{\mathbf{\Theta}}(x) \cdot \partial_j \widetilde{\mathbf{\Theta}}(x)$ for all $x \in \widetilde{\Omega}$ and the restriction $\widetilde{\mathbf{\Theta}}|_{\widetilde{\Omega}} \in \mathcal{C}^3(\widetilde{\Omega};\mathbb{R}^n)$ is an immersion, the relations $\widetilde{R}^p_{\cdot ijk} = 0$ in $\widetilde{\Omega}$ are simply the well-known necessary conditions that a Riemannian metric induced by an immersion satisfies. $\square$

5 Continuity of a manifold with boundary as a function of its metric tensor

Let Ω be a connected and simply-connected open subset of $\mathbb{R}^n$. Define the set

$$\mathcal{C}^2_0(\Omega;\mathbb{S}^n_>) := \{\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\Omega;\mathbb{S}^n_>); R^p_{\cdot ijk} = 0 \text{ in } \Omega\},$$

and let a point $x_0 \in \Omega$ be chosen *once and for all*. Then by Corollary 3.2, there exists a well-defined mapping

$$\mathcal{F}_0 : \mathcal{C}^2_0(\Omega;\mathbb{S}^n_>) \to \mathcal{C}^3(\Omega;\mathbb{R}^n)$$

that associates with any matrix field $\mathbf{C} = (g_{ij}) \in \mathcal{C}^2_0(\Omega;\mathbb{S}^n_>)$ the unique immersion $\mathbf{\Theta} \in \mathcal{C}^3(\Omega;\mathbb{R}^n)$ that satisfies

$$\boldsymbol{\nabla}\mathbf{\Theta}(x)^T \boldsymbol{\nabla}\mathbf{\Theta}(x) = \mathbf{C}(x) \text{ for all } x \in \Omega,$$

$$\mathbf{\Theta}(x_0) = 0 \text{ and } \boldsymbol{\nabla}\mathbf{\Theta}(x_0) = \mathbf{C}(x_0)^{1/2}.$$

A natural question then arises: *Do there exist topologies on* $\mathcal{C}^2(\Omega;\mathbb{S}^n)$ *and* $\mathcal{C}^3(\Omega;\mathbb{R}^n)$ *such that the mapping* $\mathcal{F}_0$ *is continuous?* (it being understood that the set $\mathcal{C}^2_0(\Omega;\mathbb{S}^n_>)$ is equipped with the induced topology). In order to address this question in a proper manner, we first need to introduce further notions and notations.

The relation $K \Subset \Omega$ indicates that K is a compact subset of Ω. If $\mathbf{\Theta} \in \mathcal{C}^\ell(\Omega; \mathbb{R}^n)$, $\ell \geq 0$, and $K \Subset \Omega$, let

$$\|\mathbf{\Theta}\|_{\ell,K} := \sup_{\begin{cases} x \in K \\ |\boldsymbol{\alpha}| \leq \ell \end{cases}} |\partial^{\boldsymbol{\alpha}} \mathbf{\Theta}(x)|.$$

For any integer $\ell \geq 0$, the space $\mathcal{C}^\ell(\Omega; \mathbb{R}^n)$ becomes a locally convex topological space when its topology is defined by the family of semi-norms $\|\cdot\|_{\ell,K}$, $K \Subset \Omega$, and a sequence $(\mathbf{\Theta}^m)_{m\geq 0}$ converges to $\mathbf{\Theta}$ with respect to this topology if and only if $\lim_{m\to\infty} \|\mathbf{\Theta}^m - \mathbf{\Theta}\|_{\ell,K} = 0$ for all $K \Subset \Omega$. Furthermore, this topology is *metrizable*: Let $(K_i)_{i\geq 0}$ be any sequence of compact subsets of Ω such that $\Omega = \bigcup_{i=0}^\infty K_i$ and $K_i \subset \operatorname{int} K_{i+1}$ for all $i \geq 0$. Then

$$\lim_{m\to\infty} \|\mathbf{\Theta}^m - \mathbf{\Theta}\|_{\ell,K} = 0 \text{ for all } K \Subset \Omega \Leftrightarrow \lim_{m\to\infty} d_\ell(\mathbf{\Theta}^m, \mathbf{\Theta}) = 0,$$

where

$$d_\ell(\mathbf{\Phi}, \mathbf{\Theta}) := \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{\|\mathbf{\Phi} - \mathbf{\Theta}\|_{\ell,K_i}}{1 + \|\mathbf{\Phi} - \mathbf{\Theta}\|_{\ell,K_i}}.$$

For details, see, e.g., Yosida [28, Chapter 1].

The space $\mathcal{C}^\ell(\Omega; \mathbb{S}^n)$ is equipped with the same distance d_ℓ once it has been identified with the space $\mathcal{C}^\ell(\Omega; \mathbb{R}^{\frac{n(n+1)}{2}})$.

We now show that the *continuity of the mapping* $\mathcal{F}_0$ *when the spaces* $\mathcal{C}^2(\Omega; \mathbb{S}^n)$ *and* $\mathcal{C}^3(\Omega; \mathbb{R}^n)$ *are equipped with the above Fréchet topologies* is a simple consequence of a continuity result recently established by Ciarlet & Laurent [9]. If d is a metric on a set X, the associated metric space is denoted $\{X; d\}$.

Theorem 5.1 *Let* Ω *be a connected and simply-connected open subset of* $\mathbb{R}^n$. *Then the mapping*

$$\mathcal{F}_0 : \{\mathcal{C}_0^2(\Omega; \mathbb{S}^n_>); d_2\} \to \{\mathcal{C}^3(\Omega; \mathbb{R}^n); d_3\}$$

is continuous.

PROOF. Since $\{\mathcal{C}_0^2(\Omega; \mathbb{S}^n_>); d_2\}$ and $\{\mathcal{C}^3(\Omega; \mathbb{R}^n); d_3\}$ are both metric spaces, it suffices to show that convergent sequences are mapped through $\mathcal{F}_0$ into convergent sequences. Let there be given matrix fields $\mathbf{C} \in \mathcal{C}_0^2(\Omega; \mathbb{S}^n_>)$ and $\mathbf{C}^m \in \mathcal{C}_0^2(\Omega; \mathbb{S}^n_>)$, $m \geq 0$, that satisfy $\lim_{m\to\infty} d_2(\mathbf{C}^m, \mathbf{C}) = 0$, or equivalently, such that

$$\lim_{m\to\infty} \|\mathbf{C}^m - \mathbf{C}\|_{2,K} = 0 \text{ for all } K \Subset \Omega.$$

Let $\boldsymbol{\Theta} := \mathcal{F}_0(\mathbf{C})$ so that, in particular, $\boldsymbol{\nabla\Theta}^T\boldsymbol{\nabla\Theta} = \mathbf{C}$ in Ω. Then, by Theorem 3 from Ciarlet & Laurent [9], there exist immersions $\widetilde{\boldsymbol{\Theta}}^m \in \mathcal{C}^3(\Omega;\mathbb{R}^n)$, $m \geq 0$, satisfying $(\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^m)^T\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^m = \mathbf{C}^m$ in Ω and

$$\lim_{m\to\infty} \|\widetilde{\boldsymbol{\Theta}}^m - \boldsymbol{\Theta}\|_{3,K} = 0 \text{ for all } K \Subset \Omega.$$

For each $m \geq 0$, define the mapping $\boldsymbol{\Theta}^m : \Omega \to \mathbb{R}^n$ by

$$\boldsymbol{\Theta}^m(x) := \mathbf{Q}_0^m(\widetilde{\boldsymbol{\Theta}}^m(x) - \widetilde{\boldsymbol{\Theta}}^m(x_0)) \text{ for all } x \in \Omega,$$

where

$$\mathbf{Q}_0^m := \mathbf{C}^m(x_0)^{1/2}\boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^m(x_0)^{-1} \in \mathbb{O}^n.$$

Then it is immediately verified that $\boldsymbol{\Theta}^m = \mathcal{F}_0(\mathbf{C}^m)$ for each $m \geq 1$. Furthermore,

$$\lim_{m\to\infty} \widetilde{\boldsymbol{\Theta}}^m(x_0) = \mathbf{0} \text{ and } \lim_{m\to\infty} \mathbf{Q}_0^m = \mathbf{I},$$

since

$$\boldsymbol{\Theta}(x_0) = \mathbf{0},\ \lim_{m\to\infty} \mathbf{C}^m(x_0)^{1/2} = \mathbf{C}(x_0)^{1/2},\ \lim_{m\to\infty} \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}^m(x_0)^{-1} = \boldsymbol{\nabla\Theta}(x_0)^{-1}.$$

Consequently, the relation

$$\begin{aligned}\boldsymbol{\Theta}^m(x) - \boldsymbol{\Theta}(x) = &\mathbf{Q}_0^m(\widetilde{\boldsymbol{\Theta}}^m(x) - \boldsymbol{\Theta}(x))\\ &+ (\mathbf{Q}_0^m - \mathbf{I})\boldsymbol{\Theta}(x) - \mathbf{Q}_0^m\widetilde{\boldsymbol{\Theta}}^m(x_0) \text{ for all } x \in \Omega\end{aligned}$$

implies that

$$\lim_{m\to\infty}\left(\sup_{x\in K}|\boldsymbol{\Theta}^m(x) - \boldsymbol{\Theta}(x)|\right) = 0 \text{ for all } K \Subset \Omega,$$

and the relations

$$\begin{aligned}\partial^{\boldsymbol{\alpha}}(\boldsymbol{\Theta}^m - \boldsymbol{\Theta})(x) = &\mathbf{Q}_0^m\partial^{\boldsymbol{\alpha}}(\widetilde{\boldsymbol{\Theta}}^m(x) - \boldsymbol{\Theta}(x))\\ &+ (\mathbf{Q}_0^m - \mathbf{I})\partial^{\boldsymbol{\alpha}}\boldsymbol{\Theta}(x) \text{ for all } x \in \Omega,\ 1 \leq |\boldsymbol{\alpha}| \leq 3,\end{aligned}$$

combined with the invariance of the Euclidean norm under the action of the orthogonal group, imply that

$$\lim_{m\to\infty}\sup_{x\in K}|\partial^{\boldsymbol{\alpha}}(\boldsymbol{\Theta}^m - \boldsymbol{\Theta})(x)| = 0,\ 1 \leq |\boldsymbol{\alpha}| \leq 3, \text{ for all } K \Subset \Omega.$$

Hence $\lim_{m\to\infty}\|\boldsymbol{\Theta}^m - \boldsymbol{\Theta}\|_{3,K} = 0$ for all $K \Subset \Omega$, and the proof is complete. $\square$

Let now Ω be a simply-connected open subset of $\mathbb{R}^n$ that satisfies the geodesic property (cf. Definition 2.2). Define the set

$$\mathcal{C}_0^2(\overline{\Omega};\mathbb{S}^n_>) := \{\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\overline{\Omega};\mathbb{S}^n_>);\ R^p_{\cdot ijk} = 0 \text{ in } \Omega\},$$

and let again a point $x_0 \in \Omega$ be chosen *once and for all.* Then by Corollary 3.4, there exists a well-defined mapping

$$\overline{\mathcal{F}}_0 : \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>) \to \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$$

that associates with any matrix field $\mathbf{C} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>)$ the unique mapping $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ that satisfies

$$\overline{\boldsymbol{\nabla}\boldsymbol{\Theta}}(x)^T \overline{\boldsymbol{\nabla}\boldsymbol{\Theta}}(x) = \overline{\mathbf{C}}(x) \text{ for all } x \in \overline{\Omega},$$
$$\boldsymbol{\Theta}(x_0) = \mathbf{0} \text{ and } \boldsymbol{\nabla}\boldsymbol{\Theta}(x_0) = \mathbf{C}(x_0)^{1/2}.$$

If in addition the set Ω is *bounded*, the spaces $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)$ and $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$, endowed with their natural norms, become Banach spaces and thus in this case the set $\mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>)$ becomes a metric space when it is equipped with the induced topology. So another natural question arises: *Is the mapping $\overline{\mathcal{F}}_0$ continuous when the set $\mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>)$ and the space $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ are equipped with these topologies*?

The *third objective* of this paper is to provide the following affirmative answer to this question. Note that, for this purpose, only the weaker notion of "geodesic property" introduced in Definition 2.2 is needed.

Theorem 5.2 *Let Ω be a simply-connected and bounded open subset of $\mathbb{R}^n$ that satisfies the geodesic property, let the spaces $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{M}^n)$ and $\mathcal{C}^\ell(\overline{\Omega}; \mathbb{R}^n)$, $\ell \geq 1$, be equipped with their usual norms, defined by*

$$\|\mathbf{F}\|_{\ell,\overline{\Omega}} = \sup_{\begin{cases} x\in\overline{\Omega} \\ |\boldsymbol{\alpha}|\leq \ell \end{cases}} |\overline{\partial^{\boldsymbol{\alpha}}\mathbf{F}}(x)| \text{ for all } \mathbf{F} \in \mathcal{C}^\ell(\overline{\Omega}; \mathbb{M}^n),$$
$$\|\boldsymbol{\Theta}\|_{\ell,\overline{\Omega}} = \sup_{\begin{cases} x\in\overline{\Omega} \\ |\boldsymbol{\alpha}|\leq \ell \end{cases}} |\overline{\partial^{\boldsymbol{\alpha}}\boldsymbol{\Theta}}(x)| \text{ for all } \boldsymbol{\Theta} \in \mathcal{C}^\ell(\overline{\Omega}; \mathbb{R}^n),$$

and let the set $\mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>)$ be equipped with the metric induced by the norm $\|\cdot\|_{2,\overline{\Omega}}$. Then the mapping

$$\overline{\mathcal{F}}_0 : \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>) \to \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$$

is continuous. It is even locally Lipschitz-continuous over the set $\mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>)$, in the sense that, given any matrix field $\widehat{\mathbf{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>)$, there exist constants $c(\widehat{\mathbf{C}}) > 0$ and $\delta(\widehat{\mathbf{C}}) > 0$ such that

$$\|\boldsymbol{\Theta} - \widetilde{\boldsymbol{\Theta}}\|_{3,\overline{\Omega}} \leq c(\widehat{\mathbf{C}}) \|\mathbf{C} - \widetilde{\mathbf{C}}\|_{2,\overline{\Omega}} \text{ for all } \mathbf{C}, \widetilde{\mathbf{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>) \cap B(\widehat{\mathbf{C}}; \delta(\widehat{\mathbf{C}})),$$

where $\boldsymbol{\Theta} := \overline{\mathcal{F}}_0(\mathbf{C})$, $\widetilde{\boldsymbol{\Theta}} := \overline{\mathcal{F}}_0(\widetilde{\mathbf{C}})$, and $B(\widehat{\mathbf{C}}; \delta(\widehat{\mathbf{C}}))$ denotes the open ball of center $\widehat{\mathbf{C}}$ and radius $\delta(\widehat{\mathbf{C}})$ in the space $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)$.

PROOF. The proof is broken into four steps, numbered (i) to (iv).

(i) *Preliminaries.* We recall that the image $\mathbf{\Theta} = \overline{\mathcal{F}}_0(\mathbf{C}) \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ of an arbitrary element $\mathbf{C} = (g_{ij}) \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>)$ is constructed in the following manner (see the proof of Theorem 3.3 and Corollary 3.4):

First, the matrix fields $\mathbf{\Gamma}_i = (\Gamma^k_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$ are defined in Ω by letting

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}(\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}), \text{ where } (g^{k\ell}) = (g_{ij})^{-1},$$

and the matrix $\mathbf{C}(x_0)^{1/2} \in \mathbb{S}^n_>$ is defined as the unique square root of the matrix $\mathbf{C}(x_0)$.

Second, the matrix field $\mathbf{F} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{M}^n)$ is defined as the unique one that satisfies

$$\partial_i \mathbf{F}(x) = \mathbf{F}(x)\mathbf{\Gamma}_i(x),\ x \in \Omega, \text{ and } \mathbf{F}(x_0) = \mathbf{C}(x_0)^{1/2}.$$

Third, the vector field $\mathbf{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ is defined as the unique one that satisfies

$$\mathbf{\nabla}\mathbf{\Theta}(x) = \mathbf{F}(x),\ x \in \Omega, \text{ and } \mathbf{\Theta}(x_0) = \mathbf{0}.$$

Accordingly, our strategy will consist in establishing the local Lipschitz-continuity of each one of the above factor mapping separately (a composite mapping is locally Lipschitz-continuous if all its component mappings share this property).

(ii) *Given any matrix field* $\widehat{\mathbf{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>)$*, there exist constants* $c_1(\widehat{\mathbf{C}}) > 0$ *and* $\delta(\widehat{\mathbf{C}}) > 0$ *such that*

$$|\mathbf{C}(x_0)^{1/2} - \widetilde{\mathbf{C}}(x_0)^{1/2}| + \max_i \|\mathbf{\Gamma}_i - \widetilde{\mathbf{\Gamma}}_i\|_{1,\overline{\Omega}} \le c_1(\widehat{\mathbf{C}})\|\mathbf{C} - \widetilde{\mathbf{C}}\|_{2,\overline{\Omega}}$$

for all matrix fields $\mathbf{C}, \widetilde{\mathbf{C}} \in \mathcal{C}_0^2(\overline{\Omega}; \mathbb{S}^n_>) \cap B(\widehat{\mathbf{C}}; \delta(\widehat{\mathbf{C}}))$*, where the matrix fields* $\widetilde{\mathbf{\Gamma}}_i = (\widetilde{\Gamma}^k_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$ *are defined in* Ω *by*

$$\widetilde{\Gamma}^k_{ij} := \frac{1}{2} \widetilde{g}^{k\ell}(\partial_i \widetilde{g}_{j\ell} + \partial_j \widetilde{g}_{\ell i} - \partial_\ell \widetilde{g}_{ij}), \text{ where } (\widetilde{g}_{ij}) := \widetilde{\mathbf{C}} \text{ and } (\widetilde{g}^{k\ell}) := (\widetilde{g}_{ij})^{-1}.$$

The following observations are used in the ensuing argument. Let X and Y be normed vector spaces and let A be a subset of X. As exemplified in the statement of the theorem, a mapping $\chi : A \to Y$ is said to be *locally Lipschitz-continuous over* A if, given any $\widehat{u} \in A$, there exist constants $c(\widehat{u}) > 0$ and $\delta(\widehat{u}) > 0$ such that

$$\|\chi(u) - \chi(\widetilde{u})\|_Y \le c(\widehat{u})\|u - \widetilde{u}\|_X \text{ for all } u, \widetilde{u} \in A \cap B(\widehat{u}; \delta(\widehat{u})).$$

Let now U be an open subset of X. Then the *mean-value theorem* (for a proof, see, e.g., Schwartz [26, Theorem 3.5.2]) asserts that any mapping $\chi \in \mathcal{C}^1(U;Y)$ satisfies

$$\|\chi(u) - \chi(\widetilde{u})\|_Y \leq \sup_{v \in]u,\widetilde{u}[} \|D\chi(v)\|_{\mathcal{L}(X;Y)} \|u - \widetilde{u}\|_X$$

for any $u, \widetilde{u} \in U$ such that the open segment $]u, \widetilde{u}[$ is contained in U, where $D\chi(v) \in \mathcal{L}(X;Y)$ denotes the Fréchet derivative of χ at v.

Consequently, *a mapping $\chi \in \mathcal{C}^1(U;Y)$ is locally Lipschitz-continuous over the open set U* (hence *a fortiori* over any subset of U) if at least one of the following additional hypotheses is satisfied: The mapping χ is the restriction to U of a continuous linear mapping from X into Y; or the space X is finite-dimensional; or, given any $\widehat{u} \in U$, there exists $\delta(\widehat{u}) > 0$ such that

$$\sup_{v \in B(\widehat{u};\delta(\widehat{u}))} \|D\chi(v)\|_{\mathcal{L}(X;Y)} < +\infty.$$

We now apply these observations to the present situation. To begin with, notice that it makes sense to study the differentiability of mappings defined over the set $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$, because this set is open in the Banach space $\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)$.

Since the mapping $\mathbf{C} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n) \to \mathbf{C}(x_0) \in \mathbb{S}^n$ is linear and continuous, hence of class $\mathcal{C}^\infty$, and since the mapping $\mathbf{C} \in \mathbb{S}^n_> \to \mathbf{C}^{1/2} \in \mathbb{S}^n_>$ is of class $\mathcal{C}^\infty$ (cf. Lemma 2.4), the mapping

$$\mathbf{C} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>) \to \mathbf{C}(x_0)^{1/2} \in \mathbb{S}^n_>$$

is also of class $\mathcal{C}^\infty$. Hence it is locally Lipschitz-continuous since the space $\mathbb{S}^n$ is finite-dimensional.

Consider next any one of the mappings $\mathbf{C} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>) \to \Gamma^k_{ij} \in \mathcal{C}^1(\overline{\Omega})$. First, each linear mapping

$$\mathbf{C} = (g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>) \to (\partial_i g_{j\ell} + \partial_j g_{\ell i} - \partial_\ell g_{ij}) \in \mathcal{C}^1(\overline{\Omega})$$

is clearly continuous, hence of class $\mathcal{C}^\infty$. Second, each function $g^{k\ell}$ is a quotient by $\det(g_{ij})$ of a homogeneous polynomial $h^{k\ell}((g_{ij}))$ of degree $(n-1)$ in terms of the functions g_{ij}, and each mapping

$$(g_{ij}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>) \to \left(\left(h^{k\ell}((g_{ij})) \right), \det(g_{ij}) \right) \in (\mathcal{C}^2(\overline{\Omega}))^{n^2+1}$$

is of class $\mathcal{C}^\infty$ since each one of its components is a sum of continuous multilinear mappings. Since

$$\det(g_{ij}) \in U := \{ f \in \mathcal{C}^2(\overline{\Omega});\ \overline{f}(x) > 0 \text{ for all } x \in \overline{\Omega} \},$$

it suffices to establish that the mapping

$$\varphi : f : U \subset \mathcal{C}^2(\overline{\Omega}) \to \frac{1}{f} \in \mathcal{C}^2(\overline{\Omega})$$

is of class $\mathcal{C}^\infty$ (again, this question makes sense since the set U is open in $\mathcal{C}^2(\overline{\Omega})$).

To this end, we remark that the mapping $\psi : U \times U \subset \mathcal{C}^2(\overline{\Omega}) \times \mathcal{C}^2(\overline{\Omega}) \to \mathcal{C}^2(\overline{\Omega})$ defined by $\psi(f, g) = fg$ for all $(f, g) \in U \times U$ is of class $\mathcal{C}^\infty$ since it is bilinear and continuous and that, at any point $(f, g) \in U \times U$, its Fréchet partial derivative $A := \dfrac{\partial\psi}{\partial g}(f, g) \in \mathcal{L}(\mathcal{C}^2(\overline{\Omega}); \mathcal{C}^2(\overline{\Omega}))$, which is given by $Ah = fh$ for all $h \in \mathcal{C}^2(\overline{\Omega})$, is an isomorphism (this property readily follows from the fact that $f \in U$).

Observing that the above mapping φ is simply the *implicit function* that satisfies the equation $\psi(f, \varphi(f)) = 1$ for all $f \in U$, we conclude from the *implicit function theorem* (see, e.g., Dieudonné [14, Theorems 10.2.1 and 10.2.3] or Schwartz [26, Theorems 3.8.5 and 3.8.15]) that φ is indeed of class $\mathcal{C}^\infty$.

Each mapping

$$\chi : \mathbf{C} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>) \to \chi(\mathbf{C}) = \Gamma^k_{ij} \in \mathcal{C}^1(\overline{\Omega})$$

being thus Fréchet differentiable (for brevity, the dependence with respect to the indices i, j, k is dropped in the notation used for such a mapping), it is an easy matter to compute the Gâteaux derivative $D\chi(\mathbf{C})\mathbf{\Delta C} \in \mathcal{C}^1(\overline{\Omega})$ corresponding to a variation $\mathbf{\Delta C} = (\Delta g_{pq}) \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n)$ at $\mathbf{C} = (g_{ij})$, viz., as the linear part with respect to $\mathbf{\Delta C}$ in the difference $\chi(\mathbf{C} + \mathbf{\Delta C}) - \chi(\mathbf{C})$. It is found in this fashion that $D\chi(\mathbf{C})\mathbf{\Delta C}$ is a sum of polynomials of degree $(n-1)$ in terms of the functions g_{ij} and of degree one in terms of the functions $\partial_k g_{\ell m}$, times some component Δg_{pq}, and divided by $\deg(g_{ij})$ or $(\det(g_{ij}))^2$. Hence given any two constants $M > 0$ and $d > 0$, there exists a constant $c(M, d) > 0$ such that

$$\|D\chi(\mathbf{C})\|_{\mathcal{L}(\mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n); \mathcal{C}^1(\overline{\Omega}))} \leq c(M, d)$$

for any matrix field $\mathbf{C} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>)$ that satisfies

$$\|\mathbf{C}\|_{2,\overline{\Omega}} \leq M \text{ and } \det \overline{\mathbf{C}}(x) \geq d \text{ for all } x \in \overline{\Omega}.$$

We thus conclude that the mapping χ is locally Lipschitz-continuous. Hence each mapping

$$\mathbf{C} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>) \to \mathbf{\Gamma}_i = (\Gamma^k_{ij}) \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$$

is also locally Lipschitz-continuous.

Note that we have thus established at no extra cost that the mapping

$$\mathbf{C} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{S}^n_>) \to ((\mathbf{\Gamma}_i), \mathbf{C}(x_0)^{1/2}) \in ((\mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n))^n \times \mathbb{S}^n_>)$$

is of class $\mathcal{C}^\infty$, even though we only needed the $\mathcal{C}^1$-differentiability in the above argument.

(iii) *The matrix fields* $\mathbf{\Gamma}_i, \widetilde{\mathbf{\Gamma}}_i \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$ *being defined in terms of the matrix fields* $\mathbf{C}, \widetilde{\mathbf{C}} \in \mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n_>)$ *as in* (i), *let the matrix fields* $\mathbf{F}, \widetilde{\mathbf{F}} \in \mathcal{C}^2(\overline{\Omega}; \mathbb{M}^n)$ *satisfy*

$$\partial_i \mathbf{F}(x) = \mathbf{F}(x)\mathbf{\Gamma}_i(x) \text{ for all } x \in \Omega \text{ and } \mathbf{F}(x_0) = \mathbf{C}(x_0)^{1/2},$$
$$\partial_i \widetilde{\mathbf{F}}(x) = \widetilde{\mathbf{F}}(x)\widetilde{\mathbf{\Gamma}}_i(x) \text{ for all } x \in \Omega \text{ and } \widetilde{\mathbf{F}}(x_0) = \widetilde{\mathbf{C}}(x_0)^{1/2},$$
$$\widetilde{\mathbf{F}}(x)^T \widetilde{\mathbf{F}}(x) = \widetilde{\mathbf{C}}(x) \text{ for all } x \in \Omega.$$

Then, given any matrix field $\widehat{\mathbf{C}} \in \mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n_>)$*, there exists a constant* $c_2(\widehat{\mathbf{C}}) > 0$ *such that*

$$\|\mathbf{F} - \widetilde{\mathbf{F}}\|_{2,\overline{\Omega}} \leq c_2(\widehat{\mathbf{C}}) \left(|\mathbf{C}(x_0)^{1/2} - \widetilde{\mathbf{C}}(x_0)^{1/2}| + \max_i \|\mathbf{\Gamma}_i - \widetilde{\mathbf{\Gamma}}_i\|_{1,\overline{\Omega}} \right)$$

for all matrix fields $\mathbf{C}, \widetilde{\mathbf{C}} \in \mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n_>) \cap B(\widehat{\mathbf{C}}; \delta(\widehat{C}))$*, where* $\delta(\widehat{\mathbf{C}}) > 0$ *is the constant found in* (ii).

Since the open set Ω satisfies the geodesic property and is bounded, its geodesic diameter D_Ω is finite (cf. Lemma 2.3). By definition of D_Ω, there thus exists a constant Λ such that, given any $x \in \Omega$, there exists a path $\boldsymbol{\gamma}$ joining x_0 to x whose length satisfies $L(\boldsymbol{\gamma}) \leq \Lambda$. Fix $x \in \Omega$ and consider such a path $\boldsymbol{\gamma} = (\gamma_i)$. Then the matrix field $\mathbf{Z} := (\mathbf{F} - \widetilde{\mathbf{F}}) \circ \boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{M}^n)$ satisfies

$$\mathbf{Z}'(t) = \gamma_i'(t)\mathbf{Z}(t)\mathbf{\Gamma}_i(\gamma(t)) + \gamma_i'(t)\widetilde{\mathbf{F}}(\boldsymbol{\gamma}(t))(\mathbf{\Gamma}_i(\gamma(t)) - \widetilde{\mathbf{\Gamma}}_i(\boldsymbol{\gamma}(t))),\ 0 \leq t \leq 1,$$

so that, by Lemma 2.5,

$$\begin{aligned} |\mathbf{Z}(1)| &\leq |\mathbf{Z}(0)| \exp\left(\int_0^1 |\gamma_i'(\tau)\mathbf{\Gamma}_i(\boldsymbol{\gamma}(\tau))| \,\mathrm{d}\tau \right) \\ &+ \int_0^1 |\gamma_i'(s)\widetilde{\mathbf{F}}(\boldsymbol{\gamma}(s))(\mathbf{\Gamma}_i(\boldsymbol{\gamma}(s)) - \widetilde{\mathbf{\Gamma}}_i(\boldsymbol{\gamma}(s)))| \exp\left(\int_s^1 |\gamma_i'(\tau)\mathbf{\Gamma}_i(\boldsymbol{\gamma}(\tau))| \,\mathrm{d}\tau \right) \mathrm{d}s. \end{aligned}$$

We know from part (ii) that, for any $\mathbf{C} \in \mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n_>) \cap B(\widehat{\mathbf{C}}; \delta(\widehat{\mathbf{C}}))$, the associated matrix fields $\mathbf{\Gamma}_i \in \mathcal{C}^1(\overline{\Omega}; \mathbb{M}^n)$ satisfy

$$\max_i \|\mathbf{\Gamma}_i\|_{1,\overline{\Omega}} \leq c_1(\widehat{\mathbf{C}})\delta(\widehat{\mathbf{C}}) + \max_i \|\widehat{\mathbf{\Gamma}}_i\|_{1,\overline{\Omega}} =: a_1(\widehat{\mathbf{C}}).$$

Consequently,

$$\int_s^1 |\gamma_i'(\tau)\mathbf{\Gamma}_i(\boldsymbol{\gamma}(\tau))| \,\mathrm{d}t \leq \int_0^1 \Big(\sum_i |\gamma_i'(\tau)|^2 \Big)^{1/2} \Big(\sum_i |\mathbf{\Gamma}_i(\boldsymbol{\gamma}(\tau))|^2 \Big)^{1/2} \mathrm{d}\tau \leq \sqrt{n}\, \Lambda a_1(\widehat{\mathbf{C}})$$

for any $0 \le s \le 1$, and likewise,

$$\int_0^1 |\gamma_i'(s)\widetilde{\mathbf{F}}(\boldsymbol{\gamma}(s))(\boldsymbol{\Gamma}_i(\boldsymbol{\gamma}(s)) - \widetilde{\boldsymbol{\Gamma}}_i(\gamma(s)))|\,\mathrm{d}s$$
$$\le \sqrt{n}\,\Lambda\Big(\sup_{x\in\Omega}|\widetilde{\mathbf{F}}(x)|\Big)\max_i \sup_{x\in\Omega}|\boldsymbol{\Gamma}_i(x) - \widetilde{\boldsymbol{\Gamma}}_i(x)|.$$

The relation $\widetilde{\mathbf{F}}^T(x)\widetilde{\mathbf{F}}(x) = \widetilde{\mathbf{C}}(x)$ for all $x \in \Omega$ next implies that, for any $\widetilde{\mathbf{C}} \in \mathcal{C}_0^2(\overline{\Omega};\mathbb{S}^n_>) \cap B(\widehat{\mathbf{C}};\delta(\widehat{\mathbf{C}}))$,

$$\sup_{x\in\Omega}|\widetilde{\mathbf{F}}(x)| \le \Big(\sup_{x\in\Omega}|\widetilde{\mathbf{C}}(x)|\Big)^{1/2} \le \Big(\|\widetilde{\mathbf{C}}\|_{2,\overline{\Omega}}\Big)^{1/2}$$
$$\le \Big(\delta(\widehat{\mathbf{C}}) + \|\widehat{\mathbf{C}}\|_{2,\widehat{\Omega}}\Big)^{1/2} =: a_2(\widehat{\mathbf{C}}).$$

Noting that $\mathbf{Z}(1) = (\mathbf{F} - \widetilde{\mathbf{F}})(x)$ and that $\mathbf{Z}(0) = \mathbf{C}(x_0)^{1/2} - \widetilde{\mathbf{C}}(x_0)^{1/2}$, we have thus shown that

$$\sup_{x\in\Omega}|(\mathbf{F} - \widetilde{\mathbf{F}})(x)|$$
$$\le \exp(\sqrt{n}\,\Lambda a_1(\widehat{\mathbf{C}}))\Big(|\mathbf{C}(x_0)^{1/2} - \widetilde{\mathbf{C}}(x_0)^{1/2}| + \sqrt{n}\,\Lambda a_2(\widehat{\mathbf{C}})\max_i \|\boldsymbol{\Gamma}_i - \widetilde{\boldsymbol{\Gamma}}_i\|_{1,\overline{\Omega}}\Big).$$

Finally, the relations

$$\partial_i(\mathbf{F} - \widetilde{\mathbf{F}}) = (\mathbf{F} - \widetilde{\mathbf{F}})\boldsymbol{\Gamma}_i + \widetilde{\mathbf{F}}(\boldsymbol{\Gamma}_i - \widetilde{\boldsymbol{\Gamma}}_i) \text{ in } \Omega$$

imply that

$$\sup_{x\in\Omega}|\partial_i(\mathbf{F} - \widetilde{\mathbf{F}})(x)| \le a_1(\widehat{\mathbf{C}})\sup_{x\in\Omega}|(\mathbf{F} - \widetilde{\mathbf{F}})(x)| + a_2(\widehat{\mathbf{C}})\|\boldsymbol{\Gamma}_i - \widetilde{\boldsymbol{\Gamma}}_i\|_{1,\overline{\Omega}}$$

and the relations

$$\partial_{ij}(\mathbf{F} - \widetilde{\mathbf{F}}) = \partial_j(\mathbf{F} - \widetilde{\mathbf{F}})\boldsymbol{\Gamma}_i + (\mathbf{F} - \widetilde{\mathbf{F}})\partial_j\boldsymbol{\Gamma}_i + \widetilde{\mathbf{F}}\widetilde{\boldsymbol{\Gamma}}_i(\boldsymbol{\Gamma}_i - \widetilde{\boldsymbol{\Gamma}}_i) + \widetilde{\mathbf{F}}\partial_j(\boldsymbol{\Gamma}_i - \widetilde{\boldsymbol{\Gamma}}_i)$$

similarly imply that

$$\sup_{x\in\Omega}|\partial_{ij}(\mathbf{F} - \widetilde{\mathbf{F}})(x)| \le a_1(\widehat{\mathbf{C}})\Big(\sup_{x\in\Omega}|\partial_j(\mathbf{F} - \widetilde{\mathbf{F}})(x)| + \sup_{x\in\Omega}|(\mathbf{F} - \widetilde{\mathbf{F}})(x)|\Big)$$
$$+ a_2(\widehat{\mathbf{C}})(1 + a_1(\widehat{\mathbf{C}}))\|\boldsymbol{\Gamma}_i - \widetilde{\boldsymbol{\Gamma}}_i\|_{1,\overline{\Omega}}.$$

The last three inequalities combined thus produce the announced upper bound for the norm $\|\mathbf{F} - \widetilde{\mathbf{F}}\|_{2,\overline{\Omega}}$.

(iv) *Let there be given matrix fields* $\mathbf{F}, \widetilde{\mathbf{F}} \in \mathcal{C}^2(\overline{\Omega};\mathbb{M}^n)$ *and vector fields*

$\boldsymbol{\Theta}, \widetilde{\boldsymbol{\Theta}} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ *that satisfy*

$$\begin{aligned}\boldsymbol{\nabla}\boldsymbol{\Theta}(x) &= \mathbf{F}(x) \ \text{ for all } x \in \Omega \text{ and } \boldsymbol{\Theta}(x_0) = \mathbf{0},\\ \boldsymbol{\nabla}\widetilde{\boldsymbol{\Theta}}(x) &= \widetilde{\mathbf{F}}(x) \ \text{ for all } x \in \Omega \text{ and } \widetilde{\boldsymbol{\Theta}}(x_0) = \mathbf{0}.\end{aligned}$$

Then there exists a constant $c_3 > 0$ *independent of these fields such that*

$$\|\boldsymbol{\Theta} - \widetilde{\boldsymbol{\Theta}}\|_{3,\overline{\Omega}} \leq c_3 \|\mathbf{F} - \widetilde{\mathbf{F}}\|_{2,\overline{\Omega}}.$$

Let $\mathbf{g}_i(x)$ and $\widetilde{\mathbf{g}}_i(x)$ denote the i-th column vectors of the matrices $\mathbf{F}(x)$ and $\widetilde{\mathbf{F}}(x)$. Given any $x \in \Omega$, there exists a path $\boldsymbol{\gamma}$ joining x_0 to x with $L(\boldsymbol{\gamma}) \leq \Lambda$ (cf. (iii)). Fix $x \in \Omega$ and consider such a path $\boldsymbol{\gamma} = (\gamma_i)$. Then the vector field $\mathbf{z} := (\boldsymbol{\Theta} - \widetilde{\boldsymbol{\Theta}}) \circ \boldsymbol{\gamma} \in \mathcal{C}^1([0,1]; \mathbb{R}^n)$ satisfies

$$\mathbf{z}'(t) = \gamma_i'(t)\Big(\mathbf{g}_i(\boldsymbol{\gamma}(t)) - \widetilde{\mathbf{g}}_i(\boldsymbol{\gamma}(t))\Big), \ 0 \leq t \leq 1.$$

Since

$$\boldsymbol{\Theta}(x) - \widetilde{\boldsymbol{\Theta}}(x) = \mathbf{z}(1) = \mathbf{z}(1) - \mathbf{z}(0) = \int_0^1 \mathbf{z}'(t)\,\mathrm{d}t,$$

we conclude that

$$\begin{aligned}|\boldsymbol{\Theta}(x) - \widetilde{\boldsymbol{\Theta}}(x)| &\leq \int_0^1 |\gamma_i'(t)\Big(\mathbf{g}_i(\boldsymbol{\gamma}(t)) - \widetilde{\mathbf{g}}_i(\boldsymbol{\gamma}(t))\Big)|\,\mathrm{d}t\\ &\leq L(\boldsymbol{\gamma}) \sup_{x\in\Omega} \Big(\sum_i |(\mathbf{g}_i - \widetilde{\mathbf{g}}_i)(x)|^2\Big)^{1/2}\\ &\leq \sqrt{n}\,\Lambda \sup_{x\in\Omega} |(\mathbf{F} - \widetilde{\mathbf{F}})(x)| \leq \sqrt{n}\,\Lambda \|\mathbf{F} - \widetilde{\mathbf{F}}\|_{2,\overline{\Omega}}.\end{aligned}$$

Since, in addition,

$$\|\partial_i(\boldsymbol{\Theta} - \widetilde{\boldsymbol{\Theta}})\|_{2,\overline{\Omega}} = \|\mathbf{g}_i - \widetilde{\mathbf{g}}_i\|_{2,\overline{\Omega}} \leq \|\mathbf{F} - \widetilde{\mathbf{F}}\|_{2,\overline{\Omega}},$$

the announced upper bound on the norm $\|\boldsymbol{\Theta} - \widetilde{\boldsymbol{\Theta}}\|_{3,\overline{\Omega}}$ follows from the last two inequalities. □

Remarks. (1) Contrary to the proof of Theorem 3.3, which relied on the existence theory on the open set Ω recalled in Theorem 3.1, that of Theorem 5.2 does not rely on the continuity, established in Theorem 5.1, of the mapping $\mathcal{F}_0$ corresponding to matrix and vector fields defined on the open set Ω.

(2) Since $\mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n_>)$ is not an open subset of the vector space $\mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n)$, the Fréchet differentiability of the mapping $\overline{\mathcal{F}}_0$ cannot be defined in the usual manner. Otherwise this would have been a convenient way of establishing that $\overline{\mathcal{F}}_0$ is pointwise Lipschitz-continuous. □

The mapping $\overline{\mathcal{F}}_0$ whose continuity is established in Theorem 5.2 corresponds to the situation covered by Corollary 3.4, i.e., where the vector fields $\boldsymbol{\Theta} = \overline{\mathcal{F}}_0(\mathbf{C}) \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ are required to satisfy the relations $\boldsymbol{\Theta}(x_0) = \mathbf{0}$ and $\boldsymbol{\nabla\Theta}(x_0) = \mathbf{C}(x_0)^{1/2}$ at some fixed point $x_0 \in \Omega$. We conclude our study of continuity by examining the case where the vector fields $\boldsymbol{\Theta}$ are no longer subjected to such requirements.

Let $\dot{\mathcal{C}}^3(\overline{\Omega}; \mathbb{R}^n) := \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)/\mathcal{R}$ denote the quotient set of the space $\mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ by the equivalence relation $\mathcal{R}$, where $(\boldsymbol{\Phi}, \boldsymbol{\Theta}) \in \mathcal{R}$ means that there exist a vector $\mathbf{a} \in \mathbb{R}^n$ and a matrix $\mathbf{Q} \in \mathbb{O}^n$ such that $\overline{\boldsymbol{\Phi}}(x) = \mathbf{a} + \mathbf{Q}\overline{\boldsymbol{\Theta}}(x)$ for all $x \in \overline{\Omega}$.

If the open set Ω is simply-connected and satisfies the geodesic property, Theorem 3.3 establishes the existence of a well-defined mapping

$$\overline{\mathcal{F}} : \mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n_>) \to \dot{\mathcal{C}}^3(\overline{\Omega}; \mathbb{R}^n)$$

that associates with any matrix field $\mathbf{C} \in \mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n_>)$ the equivalence class $\dot{\boldsymbol{\Theta}} \in \dot{\mathcal{C}}^3(\overline{\Omega}; \mathbb{R}^n)$ of all vector fields $\boldsymbol{\Theta} \in \mathcal{C}^3(\overline{\Omega}; \mathbb{R}^n)$ that satisfy

$$\overline{\boldsymbol{\nabla\Theta}}(x)^T \overline{\boldsymbol{\nabla\Theta}}(x) = \overline{\mathbf{C}}(x) \text{ for all } x \in \overline{\Omega}.$$

When both sets $\mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n_>)$ and $\dot{\mathcal{C}}^3(\overline{\Omega}; \mathbb{R}^n)$ are equipped with their natural topologies, the continuity of the mapping $\overline{\mathcal{F}}$ can be deduced from that of the mapping $\overline{\mathcal{F}}_0$, according to the following result.

Corollary 5.3 *Let Ω be a simply-connected and bounded open subset of $\mathbb{R}^n$ that satisfies the geodesic property, let the set $\mathcal{C}^2_0(\overline{\Omega}; \mathbb{R}^n_>)$ be equipped with the metric induced by the norm $\|\cdot\|_{2,\overline{\Omega}}$, and let the set $\dot{\mathcal{C}}^3(\overline{\Omega}; \mathbb{R}^n)$ be equipped with the distance $\dot{d}_3$ defined by*

$$\dot{d}_3(\dot{\boldsymbol{\psi}}, \dot{\boldsymbol{\Theta}}) := \inf_{\begin{cases}\boldsymbol{\kappa}\in\dot{\boldsymbol{\psi}}\\ \boldsymbol{\chi}\in\dot{\boldsymbol{\Theta}}\end{cases}} \|\boldsymbol{\kappa} - \boldsymbol{\chi}\|_{3,\overline{\Omega}} \text{ for all } \dot{\boldsymbol{\psi}}, \dot{\boldsymbol{\Theta}} \in \dot{\mathcal{C}}^3(\overline{\Omega}; \mathbb{R}^n).$$

Then the mapping

$$\overline{\mathcal{F}} : \mathcal{C}^2_0(\overline{\Omega}; \mathbb{S}^n_>) \to \dot{\mathcal{C}}^3(\overline{\Omega}; \mathbb{R}^n)$$

is locally Lipschitz-continuous.

PROOF. First, it is easily verified that the function $\dot{d}_3$, which can be equivalently defined by

$$\dot{d}_3(\dot{\boldsymbol{\psi}}, \dot{\boldsymbol{\Theta}}) = \inf_{\begin{cases}\mathbf{a}\in\mathbb{R}^n\\ \mathbf{Q}\in\mathbb{O}^n\end{cases}} \|\boldsymbol{\psi} - (\mathbf{a} + \mathbf{Q}\boldsymbol{\Theta})\|_{3,\overline{\Omega}},$$

is a *bona fide* distance on the set $\dot{\mathcal{C}}^3(\overline{\Omega};\mathbb{R}^n)$. Next, given any matrix field $\widehat{\mathbf{C}} \in \mathcal{C}_0^2(\overline{\Omega};\mathbb{S}^n_>)$, Theorem 5.2 shows that there exist constants $c(\widehat{\mathbf{C}}) > 0$ and $\delta(\widehat{\mathbf{C}}) > 0$ such that

$$\|\overline{\mathcal{F}}_0(\mathbf{C}) - \overline{\mathcal{F}}_0(\widetilde{\mathbf{C}})\|_{3,\overline{\Omega}} \le c(\widehat{\mathbf{C}})\|\mathbf{C} - \widetilde{\mathbf{C}}\|_{2,\overline{\Omega}}$$

for all $\mathbf{C}, \widetilde{\mathbf{C}} \in \mathcal{C}_0^2(\overline{\Omega};\mathbb{S}^n_>) \cap B(\widehat{\mathbf{C}};\delta(\widehat{\mathbf{C}}))$. Hence the conclusion follows from the inequality

$$\dot{d}_3(\overline{\mathcal{F}}(\mathbf{C}), \overline{\mathcal{F}}(\widetilde{\mathbf{C}})) \le \|\overline{\mathcal{F}}_0(\mathbf{C}) - \overline{\mathcal{F}}_0(\widetilde{\mathbf{C}})\|_{3,\overline{\Omega}},$$

itself a consequence of the definition of the distance $\dot{d}_3$. $\square$

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