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Algebraicity of local holomorphisms between real-algebraic submanifolds of complex spaces

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Contents

1. Introduction

1.1. *Algebraic submanifolds and maps.* A real (resp. complex) submanifold $M\subset\mathbf{C}^n$ is *real-algebraic* (resp. *complex-algebraic*), if it is contained in a real-algebraic (resp. complex-algebraic) subset of the same dimension (see §2.1). By a *local holomorphism* between real submanifolds $M\subset\mathbf{C}^n$ and $M'\subset\mathbf{C}^{n'}$ we mean a holomorphic map f from a domain $U\subset\mathbf{C}^n$ with $U\cap M\neq\varnothing$ into $\mathbf{C}^{n'}$ with $f(M\cap U)\subset M'$. If in addition f^{-1} exists and is a local holomorphism between M' and M, we call f a *local biholomorphism.* In this paper we study the following question:

When is a local holomorphism between real-algebraic submanifolds complex-algebraic?

Here a map f is called *complex-algebraic*, if its graph is a complex-algebraic submanifold of $\mathbf{C}^n\times\mathbf{C}^{n'}$. We use this setting throughout the paper.

1.2. *A short history of the question.* Poincaré [Po] was one of the first who studied algebraicity properties of local biholomorphisms between hypersurfaces. He proved that a local biholomorphism between open pieces of 3-spheres in $\mathbf{C}^2$ is a rational map. This result was extended by Tanaka [Ta] to higher-dimensional spheres. An important step in understanding this phenomenon was done by Webster [W1] who proved the algebraicity of local biholomorphisms f between Levi-nondegenerate algebraic hypersurfaces M and M'. Optimal conditions on M and M' for the algebraicity of local biholomorphisms were found recently by Baouendi and Rothschild [BR1] in the case when M and M' are hypersurfaces, and later extended by Baouendi, Ebenfelt and Rothschild [BER1] to the case when M and M' are submanifolds of higher codimensions. The question, when the so-called *normal component* of a local biholomorphism f (rather than f itself) is algebraic, was recently answered by Mir [Mi] in the case when M and M' are hypersurfaces.

A typical example of a biholomorphism is an extension of a biholomorphic map between smooth domains $\Omega,\Omega'\subset\mathbf{C}^n$ to a boundary point of Ω. A consequence of the algebraicity of such a biholomorphism is a holomorphic extension as *a correspondence* to the boundary $M=\partial\Omega$ (see e.g. [DP1], [BHR], [H3], [HJ], [DP2] and [Z3] for recent applications of extensions as correspondences). Furthermore, algebraic properties of single biholomorphisms can be used to study automorphism groups of domains defined by polynomial inequalities (see e.g. [Z1], [HZ]).

Similarly boundary extensions of *proper holomorphic maps* lead to the study of local holomorphisms between hypersurfaces of different dimensions. An important step here was done by Huang [H1] (see also [H2]) who proved the algebraicity of holomorphisms between strongly pseudoconvex hypersurfaces. A generalization in another direction was obtained by Sharipov and Sukhov [SS] (see also Sukhov [Su2]) for different dimensions under certain conditions on the Levi forms (see §1.5).

One of the goals of Theorem 1.1 is to unify the algebraicity results of Webster [W1], Huang [H1], Sharipov–Sukhov [SS] and Baouendi–Ebenfelt–Rothschild [BER1]. On the other hand, Theorem 1.1 covers also new situations, e.g. maps between non-pseudoconvex Levi-nondegenerate hypersurfaces of different dimensions (see §§1.4–1.6).

1.3. *The main result.* We first formulate the most general algebraicity result and then go to applications and special cases. Given a local holomorphism $f: U \to U' \subset \mathbf{C}^{n'}$ and $x \in U \cap M$, let $\varrho(z, \bar{z})$ and $\varrho'(z', \bar{z}')$ be local defining algebraic vector-valued functions for $M \subset \mathbf{C}^n$ and $M' \subset \mathbf{C}^{n'}$ defined in U and U' respectively with $d\varrho$ and $d\varrho'$ of maximal ranks. Important invariants of M and M' are the corresponding families of *Segre varieties* $Q_w := \{z \in U : \varrho(z, \bar{w}) = 0\}$ for w near x, and $Q'_{w'} := \{z' \in U' : \varrho'(z', \bar{w}') = 0\}$ for w' near $f(x)$ (see §2.3 for more details).

Given M, M', f and x as before, we attach to them two invariantly defined germs r_x and r_x^2 of analytic subsets of $\mathbf{C}^{n'}$ at $f(x)$ as follows (see below for the definitions in terms of Segre varieties):

$$r_x := \{w' \text{ near } f(x) : \varrho'(f(z), \bar{w}') = 0 \text{ for every } z \text{ near } x \text{ with } \varrho(z, \bar{x}) = 0\} \tag{1}$$

and

$$r_x^2 := \{z' \text{ near } f(x) : \varrho'(z', \bar{w}') = 0 \text{ for every } w' \in r_x \text{ near } f(x)\}. \tag{2}$$

Since the defining functions are unique up to multiplication by an invertible matrix function, both r_x and r_x^2 are independent of the choice of the defining functions. If U and U' are sufficiently small neighborhoods of x and $f(x)$ respectively, then $r_x = \{w' : Q'_{w'} \supset f(Q_x)\}$ and r_x^2 is the intersection of all Segre varieties $Q'_{w'}$ containing $f(Q_x)$ (cf. §2.4). The germ r_x generalizes the *essential variety* A_x attached to a real-analytic CR-submanifold that has been used by many authors (see e.g. [DW], [BJT], [DF2], [BR2], [Fo1], [H2], [DP1], [H3], [HJ] and [DP2]). Namely one has $A_x = r_x$ in the case $M = M'$ and $f = \mathrm{id}$.

The second invariant r_x^2 seems to be new. It may seem natural to apply this procedure one more time by replacing r_x with r_x^2 in (2). However, the germ "r_x^3" obtained in this way coincides with r_x.

Recall that M is called *generic* in $\mathbf{C}^n$ at x_0, if ϱ can be chosen near x_0 with $\partial\varrho$ of the same rank as $d\varrho$. Then the *complex tangent space* $T_x^c M := T_x M \cap i T_x M$ is of constant dimension for x near x_0 (i.e. M is CR). A generic submanifold M is said to be *of finite type* (in the sense of Bloom–Graham [BlGr] and Kohn [K]) at x if $T_x M$ is spanned by smooth $T^c M$-vector fields on M together with their higher-order commutators. In this paper we always mean this notion of finite type unless otherwise specified. A connected real-analytic submanifold is generic and of finite type at some point if and only if it is generic and of finite type on an open dense subset (see §2.2).

THEOREM 1.1. *Let* $f: U \to \mathbf{C}^{n'}$ *be a local holomorphism between connected real-algebraic submanifolds* $M \subset \mathbf{C}^n$ *and* $M' \subset \mathbf{C}^{n'}$ *satisfying the following conditions*:

(i) *M is generic and of finite type at some point*;

(ii) *For every x from a nonempty open subset of $M \cap U$, $f(x)$ is isolated in* $r_x \cap r_x^2$.

Then f is complex-algebraic.

Remarks. The basic property $\varrho(z, \bar{w})=0 \Rightarrow \varrho'(f(z), \overline{f(w)})=0$ for local holomorphisms implies $f(x) \in r_x$. Since $\varrho(x, \bar{x})=0$, the identity $\varrho'(f(x), \bar{w}')=0$ holds for all $w' \in r_x$. Hence $f(x) \in r_x^2$. Given M, M' and f, the conditions of Theorem 1.1 can be verified effectively using the defining functions ϱ and ϱ'.

The condition of finite type holds, in particular, if the Levi cone of M has a nonempty interior (see §1.5). If $M \subset \mathbf{C}^n$ is a hypersurface, it is automatically generic and (i) is equivalent to the condition that M is not Levi-flat (i.e. the Levi form does not vanish identically).

The conditions in Theorem 1.1 are optimal in the following sense. If (i) does not hold, there always exist nonalgebraic local holomorphisms between M and M' as follows from Theorem 1.3. In §2.4 we show that $r_x \cap r_x^2$ is always contained in M'. If (ii) does not hold, any nonalgebraic holomorphic map from a neighborhood of x in $\mathbf{C}^n$ into $r_x \cap r_x^2$ is a local holomorphism between M and M'. An example in §2.4 shows that condition (ii) in Theorem 1.1 cannot be replaced by the weaker condition that $f(x)$ is isolated in $r_x \cap r_x^2$ for some x, even in the case when M and M' are hypersurfaces.

The proof of Theorem 1.1 is given in §5. The remainder of §1 contains several basic consequences of Theorem 1.1 including the known results.

1.4. *Conditions in terms of analytic discs in M'.* By an *analytic disc* in a subset $A \subset \mathbf{C}^{n'}$ we mean a nonconstant holomorphic map h from the unit disc $\Delta \subset \mathbf{C}$ into $\mathbf{C}^{n'}$ with $h(\Delta) \subset A$. If condition (ii) in Theorem 1.1 does not hold for some $x \in M \cap U$, then $r_x \cap r_x^2$ defines a nontrivial germ of an analytic subset through $f(x)$. In fact, this germ is always contained in M' (Corollary 2.8). Hence we obtain the following corollary:

COROLLARY 1.2. *Let* $f: U \to \mathbf{C}^{n'}$ *be a local holomorphism between connected real-algebraic submanifolds* $M \subset \mathbf{C}^n$ *and* $M' \subset \mathbf{C}^{n'}$ *satisfying the following conditions*:

(i) *M is generic and of finite type at some point*;

(ii) *For every x from a nonempty open subset of $M \cap U$, $M' \cap r_x$ does not contain analytic discs through $f(x)$.*

Then f is complex-algebraic.

Corollary 1.2 contains, as a special case, the implication (i) $\Rightarrow$ (ii) in the following criterion:

THEOREM 1.3. *Let $M\subset\mathbf{C}^n$ and $M'\subset\mathbf{C}^{n'}$ be connected real-algebraic submanifolds. Then the following are equivalent*:

(i) *M is generic and of finite type at some point and M' contains no analytic discs*;

(ii) *Every local holomorphism between M and M' is algebraic.*

Proof. It remains to show (ii) $\Rightarrow$ (i). If the subset of generic points of M is not dense, the local intrinsic complexification $\widetilde{M}$ of M near some $x\in M$ is a complex submanifold of $\mathbf{C}^n$ of positive codimension. Hence in a neighborhood of x there exists a nonalgebraic holomorphic map f which sends $\widetilde{M}$ into one point of M' and thus obviously satisfies $f(M)\subset M'$.

If the subset of generic but not finite-type points of M is not dense, then by Lemma 3.4.1 in [BER1], there exists a nonconstant complex-algebraic holomorphic function h in a neighborhood of some point $x\in M$ such that $h(M)\subset\mathbf{R}$. Hence every real-analytic nonalgebraic curve $\gamma\colon(h(a)-1, h(a)+1)\subset\mathbf{R}\to M'$ defines a nonalgebraic holomorphism $\gamma\circ h$ between M and M'.

Finally, if M' contains an analytic disc, a neighborhood of a point $x\in M$ can be sent to this disc by a nonalgebraic holomorphic map. □

Condition (i) holds e.g. if M and M' are hypersurfaces (of possibly different dimensions) of finite type in the sense of D'Angelo [D]. In particular, Corollary 1.3 contains the algebraicity theorem of Huang [H1].

Theorem 1.3 can be applied to proper mappings between not necessarily smooth bounded domains of different dimensions. By a result of Diederich and Fornaess [DF1], if M is contained in a compact real-analytic subvariety of $\mathbf{C}^n$, it does not contain analytic discs. We obtain the following corollary:

COROLLARY 1.4. *Let $D\subset\mathbf{C}^n$ $(n\geqslant 2)$ and $D'\subset\mathbf{C}^{n'}$ be bounded domains whose boundaries are contained in compact real-algebraic subsets. Then every proper holomorphic map $f\colon D\to D'$ is algebraic provided it extends holomorphically to a neighborhood of at least one boundary point $x\in\partial D$.*

For $n=1$ the statement obviously fails. See [TH], [Tu2], [Fo2], [Su3], [Su4] for related results in the case of quadrics.

After Theorem 1.3 was written in the first version of this paper the author received the preprint by Coupet, Meylan and Sukhov [CMS], where they proved a weaker version of Theorem 1.3 with the condition of finite type replaced by the stronger condition of "Segre transversality". They also gave a sharp estimate on the "transcendence degree" of f in a more general situation, where f is not necessarily algebraic. The nature of the method in [CMS] is algebraic in contrast to the analytic method of this paper.

1.5. *Levi-form conditions.* Suppose that $M\subset\mathbf{C}^n$ is a CR-submanifold (i.e. $\dim T_x^cM$ is independent of $x\in M$) and let

$$L_x\colon T_x^cM\times T_x^cM\to\mathbf{C}T_xM/\mathbf{C}T_x^cM$$

be the Levi form (see §2.6) after the standard identification $T^cM\cong T^{1,0}M$. To every linear subspace $V\subset T_x^cM$ we associate its *Levi-orthogonal complement*:

$$V^{\perp}:=\{u\in T_x^cM: L(u,v)=0 \text{ for all } v\in V\}. \tag{3}$$

In §2.6 we prove that the Whitney tangent cone $T_{f(x)}r_x$ is contained in $(df(T_x^cM))^{\perp}$. Then, for every analytic disc $h\colon\Delta\to r_x$ through $f(x)$, the Whitney tangent cone $T_{f(x)}h(\Delta)$ is contained in $(df(T_x^cM))^{\perp}$. We say that a disc h is in the direction of a linear subspace $W\subset\mathbf{C}^{n'}$, if $T_{f(x)}h(\Delta)\subset W$. This is equivalent to the condition that the first nonvanishing derivative of h at $h^{-1}(f(x))$ is contained in W. The following is a special case of Corollary 1.2 that does not involve r_x (the Levi orthogonality is understood with respect to the Levi form of M').

Corollary 1.5. *Let $f\colon U\to\mathbf{C}^{n'}$ be a local holomorphism between connected real-algebraic CR-submanifolds $M\subset\mathbf{C}^n$ and $M'\subset\mathbf{C}^{n'}$ satisfying the following conditions*:

(i) *M is generic and of finite type at some point*;

(ii) *For every x from a nonempty open subset of $M\cap U$, M' does not contain analytic discs through $f(x)$ in the direction of $(df(T_x^cM))^{\perp}$.*

Then f is complex-algebraic.

A further special case of Corollary 1.5 can be formulated without analytic discs. We say that a CR-submanifold $M'\subset\mathbf{C}^{n'}$ is *strongly pseudoconvex* in the direction of a linear subspace $W\subset T_{x'}^cM'$ if $L_{x'}(u,u)=0$ for $u\in W$ implies $u=0$. This notion coincides with the usual strong pseudoconvexity if M' is a hypersurface and $W=T_{x'}^cM'$.

Corollary 1.6. *Let $f\colon U\to\mathbf{C}^{n'}$ be a local holomorphism between connected real-algebraic CR-submanifolds $M\subset\mathbf{C}^n$ and $M'\subset\mathbf{C}^{n'}$ satisfying the following conditions*:

(i) *M is generic and of finite type at some point*;

(ii) *For some $x\in M\cap U$, M' is strongly pseudoconvex in the direction of $(df(T_x^cM))^{\perp}$.*

Then f is complex-algebraic.

Note that if condition (ii) holds for some $x\in M\cap U$, it holds also for all x nearby. Corollary 1.6 is a consequence of Corollary 1.5 because, if h is an analytic disc in M' through x', the Levi form $L_{x'}$ is vanishing on the Whitney tangent cone of h.

Using the wedge extension by Tumanov [Tu1] and the reflection principle in the form of Sukhov [Su1] we obtain from Corollary 1.6:

THEOREM 1.7. *Let* $f\colon M\to M'$ *be a CR-map of class* C^1 *between connected real-algebraic CR-submanifolds* $M\subset\mathbf{C}^n$ *and* $M'\subset\mathbf{C}^{n'}$. *Suppose that the following is satisfied*:

(i) M *is generic and of finite type at some point*;

(ii) $(df(T_x^c M))^\perp=\{0\}$ *for some* $x\in M$.

Then f *is complex-algebraic.*

Theorem 1.7 generalizes a result of Sharipov and Sukhov [SS], where (i) was replaced by the stronger assumption that the Levi cone of M (i.e. the convex hull of the set of vectors $L_x(u,u)$ for all $u\in T_x^c M$) has a nonempty interior.

1.6. *Essential finiteness and holomorphic nondegeneracy.* Baouendi, Jacobowitz and Trèves [BJT] introduced the notion of the *essential finiteness* of a real-analytic CR-submanifold $M\subset\mathbf{C}^n$. Using the construction (1) for the identity map $f\colon\mathbf{C}^n\to\mathbf{C}^n$, their definition can be reformulated as

Definition 1.8. Let $M=M'\subset\mathbf{C}^n$ be a real-analytic CR-submanifold and $f=\mathrm{id}$. Then M is *essentially finite* at $x\in M$ if x is isolated in r_x.

In fact, if M is essentially finite, the property of $f(x)$ to be isolated in r_x holds for every f with $f(Q_x)$ open in $Q'_{f(x)}$. Using the elementary properties of Segre varieties (see Proposition 2.5) we obtain the following consequence of Theorem 1.1.

THEOREM 1.9. *Let* $f\colon U\to\mathbf{C}^{n'}$ *be a local holomorphism between connected real-algebraic CR-submanifolds* $M\subset\mathbf{C}^n$ *and* $M'\subset\mathbf{C}^{n'}$ *satisfying the following conditions*:

(i) M *is generic and of finite type at some point*;

(ii) M' *is essentially finite at some point*;

(iii) $df(T_x^c M)=T_{f(x)}^c M'$ *for some* $x\in M\cap U$.

Then f *is complex-algebraic.*

By Proposition 1.3.1 in [BER1], M' is essentially finite at some point if and only if it is holomorphically nondegenerate. Hence the conditions of Theorem 1.9 are in particular satisfied in the important case where $\dim M=\dim M'$, $M,M'\subset\mathbf{C}^n$ are generic, of finite type, holomorphically nondegenerate, and f is locally biholomorphic. In this case Theorem 1.9 is equivalent to the algebraicity result of Theorem 3 of Baouendi, Ebenfelt and Rothschild [BER1]. Example 1.1 in [Z2] gives a situation, where Theorem 1.9 is applicable but f is not locally biholomorphic.

We conclude with an application of Theorem 1.9 for smooth CR-maps. Suppose that $M,M'\subset\mathbf{C}^n$ are generic submanifolds of the same dimension. Recall that a CR-map $f\colon M\to M'$ is called *not totally degenerate* at a point x if the Jacobian of the formal map between the Segre varieties Q_x and $Q'_{f(x)}$, induced by the complexified formal Taylor series of f at x, is not identically vanishing. Meylan [Me] proved the following version of the reflection principle for generic manifolds of arbitrary codimension.

THEOREM 1.10 (Meylan). *Let $M, M' \subset \mathbf{C}^n$ be connected generic real-analytic submanifolds of the same dimension, and $f: M \to M'$ be a CR-map of class C^∞ that extends holomorphically to a wedge with the edge M. Suppose that f is not totally degenerate at a point $x \in M$ and M' is essentially finite at $f(x)$. Then f extends holomorphically to a neighborhood of x.*

We use Theorem 1.10 together with Theorem 1.9 in the following result.

THEOREM 1.11. *Let $M, M' \subset \mathbf{C}^n$ be connected generic real-algebraic submanifolds of the same dimension, and $f: M \to M'$ be a CR-map of class C^∞ satisfying the following conditions*:

(i) *M is of finite type at x*;

(ii) *f is not totally degenerate at x*;

(iii) *M' is essentially finite at $f(x)$.*

Then f extends to a complex-algebraic holomorphic map in a neighborhood of x.

Proof. By Tumanov's theorem [Tu1], f extends holomorphically to a wedge with the edge M. Since f is of class C^∞, the extension is also of class C^∞ (see e.g. [BER4, Theorem 7.5.1]). By Theorem 1.10, f extends to a holomorphic map in a neighborhood of x, also denoted by f. Then condition (ii) means that the restriction $f|Q_x$ is generically of maximal rank. By Proposition 2.5 below, condition (iii) in Theorem 1.9 becomes satisfied after possible change of x. Then the required statement follows from Theorem 1.9. □

1.7. *Basic methods.* Our method of proving Theorem 1.1 is based on a modified geometric reflection principle. One main difference from the methods of [W1] and of [BER1] is that the graph of f is obtained as an intersection of *two different algebraic families*, each given by a suitable reflection of jets, rather than a reflection of one jet that is in general not sufficient in the situation of Theorem 1.1. We use families of Segre varieties, their jets and iterations of them. The latter are closely related to the Segre sets (see [BER1]).

The basic elementary property of Segre varieties used in most applications is their invariance under local holomorphisms (see Proposition 2.4). As pointed out in §1.3, this fact immediately implies $f(x) \in r_x$. The known idea is to extend r_x to an "analytic family" by the formula $r_w := \{w' : Q'_{w'} \supset f(Q_w)\}$ with $w \in \mathbf{C}^n$ near x without loosing the inclusion $f(w) \in r_w$. If f is a biholomorphism and M' is essentially finite, the set r_x is finite near $f(x)$, and hence $f(w)$ is determined by the above inclusion up to a finite number of possibilities. This finite determinacy is still valid after replacing $f(Q_w)$ by its k-jet if k is sufficiently large. See [DW], [BR1], [BER1], [BER2], [BER3], [Z2] and [BER5] for higher-order jet reflections of this kind.

The main difficulty in the situation of Theorem 1.1 is that the above finite determinacy is no longer valid because the set r_w may have positive dimension. This makes it impossible to apply the above method. Another approach to this problem was proposed by Forstnerič [Fo1] in the case where $f(w)$ is determined by the condition $f(w) \in r_w \cap M'$ for $w \in M$. This determinacy is a consequence of the condition (A) (see [Fo1, §2, definition (2.4)]), essentially meaning that $f(x)$ is isolated in $r_x \cap M'$. Compare this with Corollary 1.2, where condition (A) is replaced by a weaker condition of nonexistence of analytic discs in the same set. If M' is a strongly pseudoconvex hypersurface, condition (A) is always satisfied. For general hypersurfaces, condition (A) is stronger than e.g. condition (i) in Theorem 1.3 as the example $M' = \{\operatorname{Re} w = z\bar{z}(z+\bar{z})\} \subset \mathbf{C}^2$ shows.

Our idea here is to establish an algebraic relation between the jets of f at *three* (instead of two) different points w, z and w_1 (see Propositions 5.1 and 5.5). The k-jets $j_w^k f$ and $j_z^k f$ parametrize certain algebraic families (denoted by A and B) such that $f(w_1)$ is algebraically determined up to finitely many possibilities by the *intersection of these families* (see the proof of Proposition 5.1). We show that this finite determinacy is guaranteed by condition (ii) in Theorem 1.1. In fact, one family arises from r_x and the other from r_x^2. The last family is not analytic in general and we have to move to a generic point to avoid this difficulty. Another difficulty is due to the fact that r_x^2 extends to a family of analytic subsets depending *antiholomorphically* on x (in contrast to r_x). This is overcome by taking the conjugate and moving the parameters of these two families independently. This is basically the reason that two different jets appear as parameters. The condition of finite type is not used on this step.

An immediate consequence of Proposition 5.1 is the algebraicity of f along the Segre varieties. The second main step is the differentiation and the iteration of the identity provided by Proposition 5.1. This leads to the algebraicity of f along the Segre sets Q_w^s (see Definition 2.9). Due to the criterion of Baouendi, Ebenfelt and Rothschild [BER1] (see Theorem 2.10), condition (i) in Theorem 1.1 guarantees that $\dim Q_w^s = n$ for some s. For this s, the algebraicity of $f|Q_w^s$ is equivalent to the algebraicity of f as required in Theorem 1.1.

1.8. *Organization of the paper.* In §2 we recall some basic constructions that are used in the paper and their properties. In §3 we develop some technical tools for the proof of Theorem 1.1. In §4 we prove basic properties of jets of holomorphic mappings and jets of complex submanifolds that are needed for the proof of Proposition 5.1. In §5.2 the proof of Theorem 1.1 is completed.

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2. Some background material

2.1. *Real- and complex-algebraic sets and their dimensions.* A subset $A\subset\mathbf{R}^m$ (resp. $A\subset\mathbf{C}^n$) is real-algebraic (resp. complex-algebraic) if it is the zero set of (finitely many) real (resp. complex) polynomials. A real-algebraic subset of $\mathbf{C}^n$ is defined via the identification $\mathbf{C}^n\cong\mathbf{R}^{2n}$. The real dimension of a real-algebraic set $A\subset\mathbf{R}^m$ is the maximal dimension of a real-analytic submanifold of $\mathbf{R}^m$ that is contained in A. The complex dimension of a complex-algebraic set is defined similarly. See e.g. [BeRi] and [Mu] for basic properties of real- and complex-algebraic sets respectively.

2.2. *CR-points of finite type.* Let $M\subset\mathbf{C}^n$ be a connected real-analytic submanifold. Recall that M is CR at a point $x\in M$ if $\dim T^c_xM=\min\{\dim T^c_yM : y\in M\}$. If $\varrho(z,\bar z)$ is a defining function, M is CR at x if and only if $\operatorname{rank}(\partial\varrho(x))=\max\{\operatorname{rank}(\partial\varrho(y)) : y\in M\}$. Either M is nowhere generic or it is generic exactly at the CR-points. The idea of the proof of the following lemma is essentially borrowed from [BR1, Lemma 4.9]. In the case when M is a CR-manifold, a simpler proof can be found in [BER4, Theorem 1.5.10].

LEMMA 2.1. *The set of all points where M is either not CR or M is CR but not of finite type is real-analytic.*

Proof. Since the statement is local, we may assume that M is globally defined by a real-analytic defining function $\varrho(z,\bar z)$. The set of points where M is not CR is real-analytic because this is exactly the set where the rank of $\partial\varrho$ is smaller than maximal. If $X\in\Gamma(M,T_x\mathbf{C}^n)$ is a vector field along M, the condition $X(x)\in T^c_xM$ defines a linear system $A(x)X(x)=0$, where A is a real-analytic matrix function.

Set $d:=\min\{\dim T^c_yM : y\in M\}$. Then, for every CR-point $x\in M$, i.e. if $\dim T^c_xM=d$, there exists a coordinate permutation in $\mathbf{C}^n=\mathbf{C}^d\times\mathbf{C}^{n-d}$ such that the projection of T^c_xM to $\mathbf{C}^d$ is bijective. Writing $X(y)=Y(y)+Z(y)\in\mathbf{C}^d\times\mathbf{C}^{n-d}$, we conclude that, for every $Y(y)$ and y near x, the system $A(y)(Y(y)+Z(y))=0$ has a unique solution $Z(y)$. Applying Cramer's rule to the standard basis $Y_1,...,Y_d$ of constant vector fields in $\mathbf{C}^d$ we obtain a collection $Y_1+Z_1(y),...,Y_d+Z_d(y)$ of vector fields in T^cM that span T^c_xM for $y=x$, and whose coefficients are ratios of real-analytic functions with denominators nonvanishing at x. After multiplying by the common denominator we obtain a collection $\mathcal{X}$ of real-analytic vector fields $X_1,...,X_d$ that span T^c_xM. If for some CR-point $y\in M$, $\mathcal{X}$ spans T^c_yM, then M is of finite type at y if and only if $\mathcal{X}$ spans T_yM together with higher-order commutators. The set $\widetilde S$ of points $y\in M$ where $\mathcal{X}$ and the higher-order commutators do not span T_yM is clearly real-analytic. Then the subset $S\subset M$ in the lemma is the union of the set of all "non-CR-points" and the intersection of the sets $\widetilde{S}$ for different permutations of the coordinates in $\mathbf{C}^n$. Hence S is real-analytic. $\square$

2.3. *Segre varieties and their properties.* For the reader's convenience we collect here some well-known facts about Segre varieties. Let $M\subset\mathbf{C}^n$ be a real-analytic submanifold, $x\in M$ be arbitrary and $\varrho(z,\bar z)$ be an analytic vector-valued defining function.

Convention. Throughout this paper we use the notation $\overline{V}$, where V is a complex manifold. The complex manifold $\overline{V}$ has the same coordinate charts with conjugated coordinates. The conjugation defines a canonical antiholomorphic map $V\to\overline{V}$, $z\mapsto\bar z$.

Definition 2.2. For every pair (U,ϱ), define the *complexification* $\mathcal{M}$ by

$$\mathcal{M}=\mathcal{M}(U,\varrho):=\{(z,\overline{w})\in U\times\overline{U}:\varrho(z,\overline{w})=0\}$$

and, for every $w\in U$, the *Segre variety* $Q_w\subset U$ by

$$Q_w=Q_w(U,\varrho):=\{z\in U:(z,\overline{w})\in\mathcal{M}\}=\{z\in U:\varrho(z,\overline{w})=0\}.$$

In general, $\mathcal{M}$ is a complex-analytic subset of $U\times\overline{U}$. If M is a CR-submanifold, then $\mathcal{M}$ is a complex submanifold near $(x,\bar x)$. The same holds for the Segre varieties Q_w near x with w also near x.

Segre varieties were introduced by Segre [Se] and played an important role in the reflection principle in several complex variables (see e.g. [Pi], [L2], [W1], [W2], [DW], [BJT] and more recent papers). In the case when $n=1$ and $M\subset\mathbf{C}$ is a real-analytic curve, the Segre variety Q_w is a one-point set, whose only element is the classical anti-holomorphic Schwarz reflection of w about M. This reflection is involutive and has M as the fixed point set. These two properties have the following counterpart in several complex variables (a consequence of the identity $\varrho(z,\overline{w})=\overline{\varrho(w,\bar z)}$):

PROPOSITION 2.3. *For every $z,w\in U$, $z\in Q_w$ is equivalent to $w\in Q_z$, and $z\in Q_z$ is equivalent to $z\in M\cap U$.*

Segre varieties are invariant under local holomorphisms:

PROPOSITION 2.4. *Let $M\subset\mathbf{C}^n$ and $M'\subset\mathbf{C}^{n'}$ be real-analytic submanifolds, (U,ϱ) and (U',ϱ') be their defining functions, and $f:U\to U'$ be a local holomorphism between M and M'. Then $f(Q_w\cap U(x))\subset Q'_{f(w)}$ for all $w\in U(x)$ with $U(x)\subset U$ an appropriate neighborhood of x.*

A more precise description of the Segre varieties in the case where M is a CR-submanifold is given in the following proposition (a consequence of the lower semicontinuity of the rank of a holomorphic map):

Proposition 2.5. *In the above notation let* $M\subset\mathbf{C}^n$ *be a CR-submanifold and* $d:=\dim T_x^c M$. *Then the following is satisfied*:

(1) *There exist neighborhoods* $U(x), V(x)\subset U$ *such that for every* $w\in U(x)$, $Q_w\cap V(x)$ *is a* (*possibly empty*) *connected complex submanifold of dimension* d;

(2) $T_x Q_x = T_x^c M$;

(3) *If* $f: U\to\mathbf{C}^{n'}$ *is a holomorphic map and* $(z,\bar{w})\in\mathcal{M}$ *is close to* $(x,\bar{x})$, *then*

$$\dim df(T_x^c M)\leqslant \operatorname{rank}_z(f|Q_w)\leqslant \max_{y\in M\cap U}\dim df(T_y^c M).$$

If M is in addition generic, Q_w in statement (1) is always nonempty for w close to x.

2.4. *Reflections of subsets in terms of Segre varieties.* The germs r_x and r_x^2 defined in §1.3 are special cases of the following construction:

Definition 2.6. Let $M\subset\mathbf{C}^n$ be a real-analytic submanifold and (U,ϱ) be a defining function. For every subset $A\subset U$ define *the inner reflection* by

$$r_M(A) := \{w\in U : Q_w\supset A\} = \bigcap_{z\in A} Q_z \tag{4}$$

and *the outer reflection* by

$$R_M(A) := \{w\in U : Q_w\cap A\neq\varnothing\} = \bigcup_{z\in A} Q_z.$$

The equalities here are consequences of Proposition 2.3. The Segre varieties Q_w themselves are also special cases of these reflections: $Q_w = r_M(\{w\}) = R_M(\{w\})$. If U and U' are sufficiently small (more precisely, if Q_x is contained in a connected submanifold Q of the same dimension with $f(Q)\subset U'$, and if all irreducible components of $r_M(f(Q_x))$ pass through $f(x)$), the definitions of r_x and r_x^2 in §1.3 can be reformulated as follows:

$$r_x = r_M(f(Q_x)) \quad\text{and}\quad r_x^2 = r_M(r_x) \quad \text{(in the sense of germs).} \tag{5}$$

Clearly $r_M(A)$ is a closed analytic subset of U, even if A is not ($R_M(A)$ is in general not analytic). If M is in addition algebraic, then $r_M(A)$ is a local algebraic subset of $\mathbf{C}^n$, i.e. a subset locally defined by vanishing of algebraic functions. Notice that this property is different from being locally defined by vanishing of polynomials, as is shown by the example of analytically irreducible components of the cubic $\{(z_1,z_2)\in\mathbf{C}^2 : z_2^2 = z_1^2(1-z_1)\}$ at the origin.

In the situation of Theorem 1.1 the germ $r_x\subset\mathbf{C}^{n'}$ is defined by holomorphic functions whose coefficients are real-analytic in $x\in M$. Hence $\dim r_x$ is upper semicontinuous in x. However, $\dim r_x^2$ is not necessarily upper semicontinuous, and therefore condition Theorem 1.1 (ii) cannot be weakened by requiring that $f(x)$ is isolated in $r_x \cap r_x^2$ for some x. This can be demonstrated by the following example, communicated to the author by J. Merker.

Example. Define

$$M := \{2\operatorname{Re} z_1 = |z_2|^2\} \subset \mathbf{C}^2,$$
$$M' := \{2\operatorname{Re} z_1 = |z_2|^2 + 2\operatorname{Re}(z_3 + z_2^2 \bar{z}_3)\bar{z}_4\} \subset \mathbf{C}^4$$

and $f(z_1, z_2) := (z_1, z_2, \varphi(z_1), 0)$, where φ is an arbitrary holomorphic function on $\mathbf{C}$ with $\varphi(0)=0$. Clearly $f(M) \subset M'$, and M is generic and of finite type. Then, for $x=(x_1, x_2) \in M$, $Q_x = \{(z_1, z_2) : z_1 + \bar{x}_1 = z_2 \bar{x}_2\}$, $f(Q_x) = \{(t\bar{x}_2 - \bar{x}_1, t, \varphi(t\bar{x}_2 - \bar{x}_1), 0) : t \in \mathbf{C}\}$ and

$$r_{(x_1,x_2)} = \{w \in \mathbf{C}^4 : t\bar{x}_2 - \bar{x}_1 + \bar{w}_1 = t\bar{w}_2 + (\varphi(t\bar{x}_2 - \bar{x}_1) + t^2 \bar{w}_3))\bar{w}_4 \text{ for all } t \in \mathbf{C}\}.$$

In particular,

$$r_0 = \{(0,0,w_3,0) : w_3 \in \mathbf{C}\} \cup \{(0,0,0,w_4) : w_4 \in \mathbf{C}\} \quad \text{and} \quad r_0^2 = \{(0, z_2, 0, 0) : z_2 \in \mathbf{C}\}.$$

Hence $f(0)=0$ is isolated in $r_0 \cap r_0^2$ but f is not necessarily algebraic.

On the other hand, if f (and therefore φ) is not algebraic, $\varphi'''(-\bar{x}_1) \neq 0$ holds for some $x=(x_1, x_2) \in M$. Then the differentiations in t up to the third order of the equation in the above formula for $t=0$ yield $r_{(x_1,x_2)} = \{(x_1, x_2, w_3, 0) : w_3 \in \mathbf{C}\}$ and $r_{(x_1,x_2)}^2 = \{(z_2\bar{x}_2 - \bar{x}_1, z_2, z_3, 0) : (z_2, z_3) \in \mathbf{C}^2\}$. Hence $f(x)$ is not isolated in $r_x \cap r_x^2 = r_x$. Thus condition (ii) in Theorem 1.1 is not satisfied.

For the inner reflection, we use the following elementary properties:

Lemma 2.7. *For every subset $A \subset U$, $A \cap r_M(A) \subset M$ and $A \subset r_M(r_M(A))$.*

Proof. If $a \in A$ and $a \in r_M(A) = \bigcap_{z \in A} Q_z$, then clearly $a \in Q_a$. Then Proposition 2.3 implies $a \in M$. Since $a \in A$ is arbitrary, this means $A \subset M$. Furthermore, by the construction of $r_M(A)$,

$$r_M(r_M(A)) = \bigcap\{Q_z : z \in r_M(A)\} = \bigcap\{Q_z : Q_z \supset A\} \supset A. \qquad \square$$

Applying Lemma 2.7 to $A := r_x$ and using (5) we obtain:

COROLLARY 2.8. *The germ $r_x \cap r_x^2$ is always contained in M'.*

2.5. *A finite-type criterion of Baouendi, Ebenfelt and Rothschild.* The outer reflection can be used to define the Segre sets in the sense of Baouendi, Ebenfelt and Rothschild [BER1] in a slightly different but equivalent way:

Definition 2.9. Let $M \subset \mathbf{C}^n$ be a real-analytic submanifold and (U, ϱ) be a defining function. Then the Segre sets are defined inductively by $Q_w^1 := Q_w$, $Q_w^{s+1} := R_M(Q_w^s) = \bigcup \{Q_z : z \in Q_w^s\}$.

In general the Segre sets are not analytic. However, if U is sufficiently small, they are finite unions of (not necessarily closed) complex submanifolds. A useful tool for our purposes is given by the following criterion (see [BER1], [BER4]):

THEOREM 2.10 (Baouendi, Ebenfelt, Rothschild). *Let $M \subset \mathbf{C}^n$ be a generic real-analytic submanifold and $x \in M$. Then M is of finite type at x if and only if there exists an integer $2 \leqslant s \leqslant \operatorname{codim} M + 1$ such that for every defining function (U, ϱ) with $x \in U$, the s-th Segre set Q_x^s contains an open subset of $\mathbf{C}^n$.*

2.6. *Levi orthogonality and the inner reflection.* Let $M \subset \mathbf{C}^n$ be a CR-submanifold and $x \in M$ be an arbitrary point. Recall that a $(1,0)$-vector field on M is a vector field X in the complexification $\mathbf{C}T^cM = \mathbf{C} \otimes_{\mathbf{R}} T^cM$ such that $JX = iX$, where $J: \mathbf{C}T^cM \to \mathbf{C}T^cM$ is the complexification of the CR-structure $J: T^cM \to T^cM$, and iX is the multiplication by i in the component $\mathbf{C}$ of the tensor product.

The Levi form of M at x is the Hermitian (vector-valued) form

$$L = L_{M,x}: T_x^{1,0}M \times T_x^{1,0}M \to \mathbf{C}T_xM / \mathbf{C}T_x^cM, \tag{6}$$

defined by

$$L_{M,a}(u,v) := \frac{1}{2i}\pi([X, \bar{Y}]), \tag{7}$$

where X and Y are $(1,0)$-vector fields on M with $X(a) = u$, $Y(a) = v$ and

$$\pi: \mathbf{C}T_xM \to \mathbf{C}T_xM / \mathbf{C}T_x^cM \tag{8}$$

is the canonical projection. Notice that the right-hand side of (7) depends only on u and v, and not on their extensions X and Y as $(1,0)$-vector fields.

If (U, ϱ) is a defining function for M, the Levi form satisfies the following identity (see e.g. [Bo]):

$$-dp(L_{M,x}(u,v)) = \sum_{j,k} \frac{\partial^2 \varrho}{\partial z_j \, \partial \bar{z}_k}(x) \, u_j \bar{v}_k. \tag{9}$$

We use (9) to derive a connection between the Levi orthogonality as defined by (3) and the inner reflection in §2.4. For every subset $A\subset U$ and a point $a\in A$, denote by T_aA the Whitney tangent cone, i.e. the set of all possible limits of sequences $(a_m-a)/c_m$ with $c_m\in\mathbf{R}_+$ and $a_m\in A$ such that $a_m\to a$ as $m\to\infty$. Then the connection between (4) and (3) can be stated as

PROPOSITION 2.11. *Let $M\subset\mathbf{C}^n$ be a real-analytic CR-submanifold, $x\in M$ and $x\in A\subset Q_x$. Then $T_x r_M(A)\subset(T_xA)^\perp$.*

Proof. Since $T_xA\subset T_xQ_x$ and $T_xQ_x=T_x^cM$ by Proposition 2.5, the Levi-orthogonal complement $(T_xA)^\perp$ makes sense. We have to prove that $L(u,v)=0$ for every $u\in T_xA$ and $v\in T_x r_M(A)$.

For this, consider sequences $b_m, c_k\in\mathbf{R}_+$, $a_m\in A$ and $r_k\in r_M(A)$ such that $a_m\to x$, $(a_m-x)/b_m\to u$ as $m\to\infty$, and $(r_k-x)/c_k\to v$, $r_k\to x$ as $k\to\infty$. By definition of $r_M(A)$, $\varrho(a_m,\bar r_k)=0$ for all m and k. This implies

$$\partial\varrho(x,\bar r_k)(u)=\lim_{m\to\infty}\frac{\varrho(a_m,\bar r_k)-\varrho(x,\bar r_k)}{b_m}=0$$

for every k, and, further,

$$\partial\bar\partial\varrho(x,\bar x)(u,v)=\lim_{k\to\infty}\frac{\partial\varrho(x,\bar r_k)(u)-\partial\varrho(x,\bar x)(u)}{b_m}=0.$$

The required statement follows from (9). □

3. Holomorphic and algebraic families

3.1. *Notation.* By saying "analytic" we shall always mean "complex-analytic", and by "dimension" the dimension over $\mathbf{C}$. All submanifolds and analytic subsets without further specification are supposed to be complex. The notion of algebraic subsets and submanifolds extends in an obvious way to subsets of algebraic varieties more general than $\mathbf{C}^n$, e.g. of the jet spaces introduced in §4.

3.2. *Definitions of families and their basic properties.* In the following let U and V be connected complex manifolds and $F\subset U\times V$ be an arbitrary submanifold. It will be useful for our purposes to have the following notion.

Definition 3.1. We call the triple (U,V,F) a *holomorphic family*, if there exist connected complex manifolds V_1 and V_2, a biholomorphism $\Phi: V\to V_1\times V_2$ and a holomorphic map $\varphi: U\times V_1\to V_2$ such that

$$(\mathrm{id}_U\times\Phi)(F)=\{(u,v_1,v_2)\in U\times V_1\times V_2 : v_2=\varphi(u,v_1)\}.$$

If in addition U, V, V_1, V_2 are open subsets of some smooth algebraic varieties, and φ and Φ are algebraic maps, we call F an *algebraic family.* We write $F_u:=\{v\in V:(u,v)\in F\}\subset V$, $u\in U$, for the fibers of F.

Elementary examples. For arbitrary U and V, $F:=U\times V$ defines an obvious holomorphic family (U,V,F). Another extremal case is $F:=U\times\{v\}$, where $v\in V$ is arbitrary. More generally, if $V=V_1\times V_2$ for some manifolds V_1 and V_2, we obtain a holomorphic family (U,V,F) by setting $F:=U\times V_1$. These are examples of families with constant fibers. Another type of examples can be obtained by taking a holomorphic map $f:U\to V$ and setting $F:=\{(u,f(u)):u\in U\}\subset U\times V$. Here the fibers $F_u=\{f(u)\}$ are one-point sets but their dependence on u is obviously not necessarily constant. Families of this type can be generalized to families of linear subspaces by setting $F:=\{(u,v_1,A(u)v_1)\}\subset U\times V_1\times V_2$, where V_1 and V_2 are linear spaces, and A is a holomorphic map from U into the space $\mathcal{L}(V_1,V_2)$ of linear operators.

Remark. It follows immediately from the definition that all fibers are closed connected submanifolds of the same dimension. However, the converse does not hold as is shown in the following example. Set $U:=\mathbf{C}$, $V:=\mathbf{C}^2$ and define

$$F:=\{(u,v_1,v_2):u^2v_2=v_1^2+u\}.$$

Clearly F is a smooth algebraic hypersurface in $U\times V$ and every fiber F_u, $u\in U$, is a smooth algebraic hypersurface in V. However, (U,V,F) is not a holomorphic family, e.g. because of Lemma 3.2 below. Moreover, there do not exist open neighborhoods $\widetilde{U}\subset U$ and $\widetilde{V}\subset V$ of the origins such that $(\widetilde{U},\widetilde{V},F\cap(\widetilde{U}\times\widetilde{V}))$ is a holomorphic family.

The following is an elementary consequence of the rank theorem.

Lemma 3.2. *In the above notation fix a point $x=(u,v)\in F\subset U\times V$. Then the following are equivalent*:

(i) *There exist open subsets $\widetilde{U}\subset U$, $\widetilde{V}\subset V$ such that $x\in\widetilde{U}\times\widetilde{V}$ and $(\widetilde{U},\widetilde{V},F\cap(\widetilde{U}\times\widetilde{V}))$ is a holomorphic family*;

(ii) *The natural projection $\pi_{F,U}:F\to U$ is submersive at x, i.e. $d\pi_{F,U}:T_xF\to T_uU$ is surjective.*

The same statement holds also in the algebraic category.

We use the following elementary criterion of genericity that is proved here for the reader's convenience.

Proposition 3.3. *Let $M\subset\mathbf{C}^n$ be a real-analytic submanifold given by a defining function $\varrho(z,\bar z)$ in a neighborhood of $x\in M$. Then M is generic at x if and only if* *there exist open neighborhoods* $U(x), V(x) \subset U$ *such that* $(U(x), \overline{V(x)}, F)$ *is a holomorphic family, where*

$$F := \mathcal{M} \cap (U(x) \times \overline{V(x)}) = \{(z, \bar{w}) \in U(x) \times \overline{V(x)} : \varrho(z, \bar{w}) = 0\}.$$

If M *is in addition algebraic, the family* $(U(x), \overline{V(x)}, F)$ *can be chosen algebraic.*

Proof. Let M be generic at x, i.e. $T_xM + iT_xM = T_xU$. Then every vector $u \in T_xU$ can be written in the form $u = a + ib$ with $a, b \in T_xM$, i.e. $d\varrho(a, \bar{a}) = d\varrho(b, \bar{b}) = 0$ or equivalently $(a, \bar{a}), (b, \bar{b}) \in T_{(x,\bar{x})}\mathcal{M}$. Since $\mathcal{M}$ is a complex submanifold, $\xi := (u, \bar{a} + i\bar{b}) \in T_{(x,\bar{x})}\mathcal{M}$ with $d\pi_U(\xi) = u$, where $\pi_U : \mathcal{M} \to U$ is the projection. Since $u \in T_xU$ is arbitrary, π_U is submersive at $(x, \bar{x})$ and the claim follows from Lemma 3.2.

Conversely, if $(U(x), \overline{V(x)}, F)$ is a holomorphic family, then $\pi_U : \mathcal{M} \to U$ is submersive at $(x, \bar{x})$ by Lemma 3.2. This means that for every $u \in T_xU$ there exists a preimage $\xi = (u, \bar{v}) \in T_{(x,\bar{x})}\mathcal{M}$. Then we can write it in the form $\xi = (a, \bar{a}) + i(b, \bar{b})$, where $a := (u+v)/2$ and $b := (u-v)/2i$. Since ξ is tangent to $\mathcal{M}$, we obtain

$$0 = d\varrho(\xi) = \partial\varrho(a, \bar{a}) + i\partial\varrho(b, \bar{b}) + \bar{\partial}\varrho(a, \bar{a}) + i\bar{\partial}\varrho(b, \bar{b}) = d\varrho(a, \bar{a}) + id\varrho(b, \bar{b}). \tag{10}$$

Since $\mathcal{M}$ is complex, $i\xi$ is also tangent to it and we similarly obtain $dp(a) - idp(b) = 0$. Together with (10) this yields $a, b \in T_xM$, i.e. $u = a + ib \in T_xM + iT_xM$. Since $u \in T_x\mathbf{C}^n$ is arbitrary, this shows that $T_xM + iT_xM = T_xU$, i.e. the required genericity.

Finally, if M is algebraic, ϱ can be chosen holomorphic algebraic. This implies the algebraicity of $\mathcal{M}$, and the statement follows from the algebraic version of Lemma 3.2. □

We shall use the following general construction.

LEMMA 3.4. *Let* U *and* V *be algebraic submanifolds of some* $\mathbf{C}^N$, $S \subset U \times V$ *be an algebraic subset and* $M \subset U \times V$ *a nonempty real-analytic submanifold with* $M \subset S$. *Then there exist an algebraic submanifold* $N \subset U$, *a finite collection of open subsets* $V_1, ..., V_s \subset V$ *and a nonempty open subset* $W \subset M$ *with the following properties*:

(i) $(N, V_j, S \cap (N \times V_j))$ *is an algebraic family for every* $j = 1, ..., s$;

(ii) $W \subset N \times V$;

(iii) *For every* $(u, v) \in W$, *every local irreducible component at* v *of the fiber* $S_u \subset V$ *intersects* V_j *for some* $j = 1, ..., s$.

Proof. We prove the statement by induction on $\dim S$. It is clear for S discrete. Denote by $\pi := \pi_{S,U} : S \to U$ the standard projection. Without loss of generality, M is connected. Let $S_1 \subset S$ be the (algebraic) Zariski closure of M. After replacing U with a submanifold, and M and S with $M \cap (U \times V)$ and $S \cap (U \times V)$ respectively, we may assume that U coincides with $\pi(S_1)$ and is connected and smooth. Let $d(x) := \dim_x \pi^{-1}(\pi(x))$

denote the fiber dimension. Since $d(x)$ is upper semicontinuous, we may assume that it is constant and equal to d on S_1 and is not larger on S by further shrinking U and V. Then $\dim \pi^{-1}(u)=d$ for every $u\in U$. In particular, $\dim S=\dim U+d$.

Every irreducible component of S where the generic value of $d(x)$ is lower than d must be of lower dimension than $\dim S$. Denote by S' the union of S-components of the highest dimension containing M, and by S'' the union of lower-dimensional S-components containing M. By further shrinking U and V we may assume that $S=S'\cup S''$ and $M\subset S'\cap S''$. Fix $x\in M$. Since S' is pure-dimensional with π-fibers of the same dimension, there exist open connected neighborhoods $O(x)\subset S'$, $U(\pi(x))\subset U$ and $E(0)\subset \mathbf{C}^d$ such that the restriction $\pi|O(x)$ can be written as a composition $P\circ H$, where $H\colon O(x)\to U(\pi(x))\times E(0)$ is a finite branched holomorphic covering and $P\colon U(\pi(x))\times E(0)\to U(\pi(x))$ is the natural projection (see [G, Volume II, Theorem L.8]).

For the dimension reason there exists a point $u_0\in U(\pi(x))$ such that $u_0\times E(0)$ is not contained in the branch locus of H. Since M is Zariski dense in S_1, $\pi(M)$ is Zariski dense in U, and therefore u_0 can be chosen in $\pi(M)\cap U(\pi(x))$, i.e. $(u_0,v_0)\in M$ for some $v_0\in V$. Let (u_0,e_0) be a point outside the branch locus and $H^{-1}(u_0,e_0)=\{(u_0,v_1),...,(u_0,v_l)\}\subset O(x)\subset S'$, where l is the branch number of H. Then the points $(u_0,v_j)\in S'$ satisfy the condition (ii) of Lemma 3.2. Hence there exist neighborhoods V_j of v_j $(j=1,...,l)$ such that $(U,V_j,S'\cap(U\times V_j))$ become algebraic families after appropriate shrinking of U. By the construction, a version of property (iii) is satisfied for these families, where $(u,v)\in M$ is close to (u_0,v_0) and the fibers S'_u (rather than S_u) are considered. Since $S=S'\cup S''$, it remains to construct additional families for the fibers S''_u. But this can be done by using the induction because $\dim S''<\dim S$. □

Given two holomorphic families (U,V,F) and (V,W,G), we define their *composition* $(U,V\times W,H)$ as follows:

$$H:=\{(u,v,w)\in U\times V\times W:(u,v)\in F,\,(v,w)\in G\}. \tag{11}$$

As a direct consequence of the definition, the composition of holomorphic (resp. algebraic) families is always holomorphic (resp. algebraic).

The notion of genericity can be trivially extended to the case when M is a real submanifold of an arbitrary complex manifold X. This condition implies, in particular, that M cannot be contained in a proper analytic subset of X. We use this simple observation in the following lemma.

LEMMA 3.5. *Let (U,V,F) be a holomorphic family, $M\subset F$ a nonempty generic real-analytic submanifold, V' another complex manifold and $f\colon V\to V'$ a holomorphic map. Then there exist domains $\widetilde U\subset U$, $\widetilde V\subset V$ and $\widetilde V'\subset V'$ such that the following is satisfied*:

(1) $M\cap(\widetilde{U}\times\widetilde{V})\neq\varnothing$;
(2) $f(\widetilde{V})\subset\widetilde{V}'$;
(3) $(\widetilde{U},\widetilde{V}',\widetilde{F})$ *is a holomorphic family, where*

$$\widetilde{F}:=\{(u,f(v)):u\in\widetilde{U},\,v\in\widetilde{V},\,(u,v)\in F\}.$$

Remark. An algebraic version of Lemma 3.5 holds with the same proof but we do not use it in this paper.

Proof. Set $g:=(\mathrm{id}_U\times f):F\to U\times V'$. Since $M\subset F$ is generic, it is not contained in the analytic subset defined by the degeneration of the rank of g. Hence there exist domains $\widetilde{U}\subset U$ and $\widetilde{V}\subset V$ such that property (1) holds, g is of constant rank on $F\cap(\widetilde{U}\times\widetilde{V})$ and $F':=g(F\cap(\widetilde{U}\times\widetilde{V}))$ is a submanifold in $U\times V'$. Fix $x\in M\cap(\widetilde{U}\times\widetilde{V})$. By Lemma 3.2, $\pi_{F,\widetilde{U}}$ is submersive at x. Hence $\pi_{F',\widetilde{U}}$ is submersive at $g(x)$. By the other direction of Lemma 3.2, we can replace $\widetilde{U}$ with a smaller domain and find a domain $\widetilde{V}'\subset V'$ such that $g(x)\in\widetilde{U}\times\widetilde{V}'$ and $(\widetilde{U},\widetilde{V}',F'\cap(\widetilde{U}\times\widetilde{V}'))$ is a holomorphic family. It remains to replace $\widetilde{V}$ with $\widetilde{V}\cap f^{-1}(\widetilde{V}')$. □

3.3. *Constructions with families and their fibers.* Let U,U',V be connected complex manifolds, and $F\subset U\times V$, $F'\subset U'\times V$ arbitrary subsets. Define

$$A(F,F'):=\{(u,u')\in U\times U':F_u\subset F'_{u'}\}. \tag{12}$$

In general $A(F,F')$ is not analytic, even if both F and F' are analytic. This is the case, however, if F is a holomorphic family as defined before.

Lemma 3.6. *Let (U,V,F) be a holomorphic* (*resp. algebraic*) *family and $F'\subset U'\times V$ be a closed analytic* (*resp. algebraic*) *subset. Then $A(F,F')\subset U\times U'$ is a closed analytic* (*resp. algebraic*) *subset.*

Proof. Let V_1,V_2, $\Phi:V\to V_1\times V_2$ and $\varphi:U\times V_1\to V_2$ be as in Definition 3.1. Let $(u_0,u_0')\in U\times U'$, $(v_0)_1\in V_1$ be arbitrary points and $(v_0)_2:=\varphi(u_0,(v_0)_1)$. It is sufficient to prove that $A(F,F')$ is analytic (resp. algebraic) in a neighborhood of (u_0,u_0'). Since F' is analytic (resp. algebraic), it is defined by the vanishing of holomorphic (resp. algebraic) functions $f_1,...,f_s$ in a neighborhood of $(u_0',v_0)\in U'\times V$, where $v_0:=((v_0)_1,(v_0)_2)$. Then for $(u,u')\in U\times U'$ close to (u_0,u_0'), $(u,u')\in A(F,F')$ is equivalent to the vanishing of $f_i(u',\Phi^{-1}(v_1,\varphi(u,v_1)))$ for all $v_1\in V_1$ near $(v_0)_1$, because V_1 is connected. The statement follows by the analyticity (resp. algebraicity) of this condition of vanishing. □

For arbitrary subsets $A\subset U\times U'$ and $G\subset U'\times V$ define

$$B_u(A,G):=\bigcap_{u'\in A_u}G_{u'},\quad u\in U, \tag{13}$$

$$B(A,G):=\{(u,v)\in U\times V:v\in B_u(A,G)\} \tag{14}$$

(in the case $A_u=\varnothing$ we set $B_u(A,G):=V$).

LEMMA 3.7. *Let* (U, U', A) *be a holomorphic* (*resp. algebraic*) *family and* $G \subset U' \times V$ *a closed analytic* (*resp. algebraic*) *subset. Then* $B(A, G) \subset U \times V$ *is a closed analytic* (*resp. algebraic*) *subset.*

Proof. By Definition 3.1, there exist a biholomorphism (resp. algebraic biholomorphism) $\Phi: U' \to U'_1 \times U'_2$ and a holomorphic (resp. algebraic) map $\varphi: U \times U'_1 \to U'_2$ such that

$$(\mathrm{id}_U \times \Phi)(A) = \{(u, u'_1, u'_2) \in U \times U'_1 \times U'_2 : u'_2 = \varphi(u, u'_1)\}.$$

Let $(u_0, v_0) \in U \times V$ and $(u'_0)_1 \in U'_1$ be arbitrary, $(u'_0)_2 := \varphi(u, (u'_0)_1)$. It is sufficient to show that $B(A, G)$ is analytic (resp. algebraic) in a neighborhood of (u_0, v_0). Since G is analytic (resp. algebraic), it is locally defined by the vanishing of holomorphic (resp. algebraic) functions $f_1, ..., f_s$ in a neighborhood of $(u'_0, v_0) \in U' \times V$, where $u'_0 := ((u'_0)_1, (u'_0)_2)$. Then for (u, v) close to (u_0, v_0), the condition $(u, v) \in B(A, G)$ is equivalent to the vanishing of $f_j(\Phi^{-1}(u'_1, \varphi(u, u'_1)), v)$ for all $j = 1, ..., s$, and $u'_1 \in U'_1$ close to $(u'_0)_1$, because U'_1 is connected. The last condition is analytic (resp. algebraic) which proves the statement. □

4. Jets of holomorphic maps and jets of complex submanifolds

4.1. *Constructions of jets.* Let X, X' be connected complex manifolds, $x \in X$ and let $k \geqslant 0$ be an integer. Recall that a k-jet at x of a holomorphic map from X into X' is an equivalence class of holomorphic maps from a neighborhood of x in X into X' with fixed partial derivatives at x up to order k. Denote by $J^k_x(X, X')$ the space of all such k-jets. The union $J^k(X, X') := \bigcup_{x \in X} J^k_x(X, X')$ carries a natural fiber bundle structure over X. For f a holomorphic map from a neighborhood of x in X into X', denote by $j^k_x f \in J^k_x(X, X')$ the corresponding k-jet. If X and X' are smooth algebraic varieties, $J^k_x(X, X')$ and $J^k(X, X')$ are also of this type.

Furthermore, we shall need the k-jets of d-dimensional submanifolds. Let $\mathcal{C}^d_x(X)$ be the set of all germs at x of d-dimensional submanifolds of X. We say that two germs $V, V' \in \mathcal{C}^d_x(X)$ are k-equivalent, if, in a local coordinate neighborhood of x of the form $U_1 \times U_2$, V and V' can be given as graphs of holomorphic maps $\varphi, \varphi': U_1 \to U_2$ such that $j^k_{x_1} \varphi = j^k_{x_1} \varphi'$, where $x = (x_1, x_2) \in U_1 \times U_2$. Denote by $J^{k,d}_x(X)$ the space of all k-equivalence classes at x and by $J^{k,d}(X)$ the union $\bigcup_{x \in X} J^{k,d}(X)$ with the natural fiber bundle structure over X. Furthermore, for $V \in \mathcal{C}^d_x(X)$, denote by $j^k_x(V) \in J^{k,d}_x(X)$ the corresponding k-jet. For $g \in J^{k,d}(X)$ we use the notation $g(0) := x$, if g is a k-jet at x. If X is a smooth algebraic variety, $J^{k,d}_x(X)$ and $J^{k,d}(X)$ are also of this type.

4.2. *Jet compositions.* Let $g \in J^{k,d}(X)$ and $j \in J^k(X, X')$ be k-jets at some $x \in X$ represented by a d-dimensional submanifold $V \in \mathcal{C}^d_x(X)$ and a local holomorphic map

$f\colon U(x)\to X'$ respectively. Our goal here will be to define a composing map $(j,g)\mapsto j\circ g$ sending $(j^k_x f, j^k_x V)$ into $j^k_{f(x)}f(V)$.

Warning. Even in the simplest nontrivial case $X=X'=\mathbf{C}^2$, $k=d=1$, a map c: $J^1_x(\mathbf{C}^2,\mathbf{C}^2)\times J^{1,1}_x(\mathbf{C}^2)\to J^{1,1}_x(\mathbf{C}^2)$ with $j^k_{f(x)}f(V)=c(j^k_x f, j^k_x V)$, whenever $f(V)$ is smooth at x, need not exist. Indeed, take $x=(0,0)$, $f(z,w):=(z^2,w)$ and $V:=\{w=az^2\}$, where $a\in\mathbf{C}$ is arbitrary. Then $f(V)=\{w'=az'\}$. Clearly $j^1_0 f(V)$ depends on a, but $j^k_0 f$ and $j^k_0 V$ do not.

In view of this we construct a composing map for $k\geqslant 1$, defined on the subset

$$\mathcal{R}:=\{(j,g)\in J^k(X,X')\times J^{k,d}(X): j(0)=g(0), \dim(l(j)\circ l(g))=d\}, \tag{15}$$

where $l(j)$ and $l(g)$ denote the linear parts of j and g respectively.

LEMMA 4.1. *There exists exactly one holomorphic mapping $c\colon\mathcal{R}\to J^{k,d}(X')$ such that $j^k_{f(x)}f(V)=c(j^k_x f, j^k_x V)$, whenever x, V and f are as before and $(j^k_x f, j^k_x V)\in\mathcal{R}$. If X and X' are algebraic, then c is also algebraic.*

Proof. Fix $(j_0,g_0)\in\mathcal{R}$ and their representatives f_0 and V_0. There exists a coordinate neighborhood $U_1\times U_2$ near $x_0:=g_0(0)=j_0(0)$ such that $V_0=U_1\times\{(x_0)_2\}$. Without loss of generality, $x_0=(0,0)$.

Since $\dim(l(j_0)\circ l(g_0))=d$, there exists a coordinate neighborhood $U'_1\times U'_2$ near $x'_0:=j_0(x_0)$ such that the map $u_1\mapsto(f_0)_1(u_1,0)$ is locally invertible at 0, where $(f_0)_1$ denotes the first component. Then the map $h_f\colon u_1\mapsto f_1(u_1,0)$ is locally invertible at $j(0)$ for every $j\in J^k(X,X')$ sufficiently close to j_0 and its representative f.

Denote by h_f^{-1} the local inverse near $j(0)$. Without loss of generality, g is represented by the graph V of a holomorphic map $s_V\colon U_1\to U_2$. Then $f(V)$ equals the graph of the map $\varphi_{f,V}:=f_2\circ(h_f^{-1}, s_V\circ h_f^{-1})$. The k-jet $j^k_{j(x)}f(V)$ is given by the derivatives of $\varphi_{f,V}$ at $j(0)$ up to order k. Hence $j^k_{j(x)}f(V)$ depends only on j and g, and we can write $j^k_{j(x)}f(V)=:c(j,g)$. Clearly $c(j,g)$ is holomorphic in $(j,g)\in\mathcal{R}$. If X and X' are algebraic, all constructions can be done algebraically. □

4.3. *The inclusion relation for the jets of complex submanifolds.* Our next goal will be to define an inclusion relation and prove its analyticity (resp. algebraicity), if the ambient space X is an analytic (resp. an algebraic) submanifold of $\mathbf{C}^n$.

Definition 4.2. Let $0\leqslant d\leqslant d'$ be integers, $g\in J^{k,d}(X)$ and $g'\in J^{k,d'}(X)$ be arbitrary k-jets. We say that g *is contained in* g' and write $g\subset g'$, if $g(0)=g'(0)$ and there exist representatives $V\in g$ and $V'\in g'$ such that $V\subset V'$.

Lemma 4.3. *Define*

$$I := \{(g, g') \in J^{k,d}(X) \times J^{k,d'}(X) : g \subset g'\}. \tag{16}$$

Then $I \subset J^{k,d}(X) \times J^{k,d'}(X)$ *is a closed analytic subset. If* X *is an algebraic submanifold of* $\mathbf{C}^n$*, then* I *is an algebraic subset of* $J^{k,d}(\mathbf{C}^n) \times J^{k,d'}(\mathbf{C}^n)$.

Proof. Let $(g_0, g_0') \in J^{k,d}(X) \times J^{k,d'}(X)$ be arbitrary. It is sufficient to prove that I is analytic (resp. algebraic) in a neighborhood of (g_0, g_0'). Without loss of generality, $g_0(0) = g_0'(0) =: x_0$ and $d' > d > 0$. Let $V_0 \in \mathcal{C}^d_{x_0}(X)$, $V_0' \in \mathcal{C}^{d'}_{x_0}(X)$ be representatives of g_0 and g_0' respectively. Then there exists a coordinate neighborhood $U_1 \times U_2 \times U_3$ of x_0 in X such that V_0 is locally the graph of a holomorphic map $U_1 \to U_2 \times U_3$ and V_0' is locally the graph of a holomorphic map $U_1 \times U_2 \to U_3$.

Every jet $g \in J^{k,d}(X)$ that is sufficiently close to g_0 can be represented by the graph of a unique polynomial map $\varphi_g : U_1 \to U_2 \times U_3$ of degree k. Similarly every jet $g' \in J^{k,d'}(X)$ that is close to g_0' can be represented by the graph of a unique polynomial map $\psi_{g'} : U_1 \times U_2 \to U_3$ of degree k. The coefficients of polynomials can be seen as coordinates in the corresponding jet spaces. Every other representative of g (resp. of g') is locally a graph of a holomorphic map $\widetilde{\varphi}_g : U_1 \to U_2 \times U_3$ (resp. $\tilde{\psi}_{g'} : U_1 \times U_2 \to U_3$) such that

$$j^k_{x_1} \varphi_g = j^k_{x_1} \widetilde{\varphi}_g \quad (\text{resp. } j^k_{(x_1', x_2')} \psi_{g'} = j^k_{(x_1', x_2')} \tilde{\psi}_{g'}), \tag{17}$$

where $x = (x_1, x_2, x_3) := g(0)$ (resp. $x' = (x_1', x_2', x_3') := g'(0)$).

By Definition 4.2, $g \subset g'$ means $x = x'$ and the existence of $\widetilde{\varphi}_g$ and $\tilde{\psi}_{g'}$ such that

$$(\widetilde{\varphi}_g(u_1))_3 = \tilde{\psi}_{g'}(u_1, (\widetilde{\varphi}_g(u_1))_2).$$

By differentiating at x_1 up to the order k, we obtain

$$j^k_{x_1}(\widetilde{\varphi}_g)_3 = (j^k_{(x_1, x_2)} \tilde{\psi}_{g'}) \circ (\mathrm{id}, j^k_{x_1}(\widetilde{\varphi}_g)_2).$$

By (17), this is equivalent to

$$j^k_{x_1}(\varphi_g)_3 = (j^k_{(x_1, x_2)} \psi_{g'}) \circ (\mathrm{id}, j^k_{x_1}(\varphi_g)_2). \tag{18}$$

We have showed that the inclusion $g \subset g'$ implies $x = x'$ and (18). Conversely, suppose that (18) holds. This means that the graph V of the map

$$u_1 \mapsto ((\varphi_g(u_1))_2, \psi_{g'}(u_1, (\varphi_g(u_1))_2))$$

is a representative of g. Clearly V is a subset of the graph of $\psi_{g'}$. Hence $g \subset g'$.

Thus we have proved that the inclusion $g\subset g'$ is equivalent to $x=x'$ and (18). Since (18) is an algebraic condition on the coefficients of φ_g and $\psi_{g'}$, I is algebraic in the given coordinates near (g_0, g_0'). This finishes the proof. □

4.4. *Operations with jets and families of analytic subsets.* Let (U,V,F) and (U',V',F') be holomorphic or algebraic families such that V is an open subset of V'. Analogously to $A(F,F')$ define

$$A^k(F,F') := \{(u,v,u')\in U\times V\times U' : (u,v)\in F,\ (u',v)\in F',\ j^k_v F_u \subset j^k_v F'_{u'}\}. \tag{19}$$

Then $A^k(F,F')$ is the preimage of I (defined by (16)) under the holomorphic (resp. algebraic) map $(u,v,u')\mapsto (j^k_v F_u, j^k_v F'_{u'})$, defined on the subset

$$\tilde{A}(F,F') := \{(u,v,u')\in U\times V\times U' : (u,v)\in F,\ (u',v)\in F'\}.$$

As a consequence of Lemma 4.3 we obtain

LEMMA 4.4. *Let (U,V,F) and (U',V',F') be holomorphic* (*resp. algebraic*) *families, where V is an open subset of V'. Then $A^k(F,F')\subset U\times V\times U'$ is a closed analytic* (*resp. algebraic*) *subset.*

It follows that $A^k_{u,v}(F,F')\supset A^{k+1}_{u,v}(F,F')$ for all integers $k\geqslant 0$. Since the fibers $F_u\subset V$, $u\in U$, are connected, inclusions of their k-jets for all k imply inclusions of the fibers, i.e. $A_u(F,F')=\bigcap_k A^k_{u,v}(F,F')$ for all $u\in U$ and $v\in F_u$. Therefore

$$\pi^{-1}(A(F,F')) = \bigcap_{k\geqslant 0} A^k(F,F'), \tag{20}$$

where $\pi: \tilde{A}(F,F')\to U\times U'$ denotes the natural projection.

LEMMA 4.5. *Let (U,V,F) and (U',V',F') be holomorphic families such that V is an open subset of V', and $(u,v)\in F$, $(u',v)\in F'$ be such that $F_u\subset F'_{u'}$. Then there exist domains $\widetilde{U}\subset U$, $\widetilde{U}'\subset U'$, $\widetilde{V}\subset \widetilde{V}'\subset V$ and an integer $k\geqslant 0$ such that the following is satisfied*:

(1) $(u,v)\in \widetilde{U}\times\widetilde{V}$ *and* $(u',v)\in\widetilde{U}'\times\widetilde{V}$;

(2) $(\widetilde{U},\widetilde{V},\widetilde{F})$ *and* $(\widetilde{U}',\widetilde{V}',\widetilde{F}')$ *are holomorphic families, where* $\widetilde{F}:=F\cap(\widetilde{U}\times\widetilde{V})$ *and* $\widetilde{F}':=F'\cap(\widetilde{U}'\times\widetilde{V}')$;

(3) $\pi^{-1}(A(\widetilde{F},\widetilde{F}'))=A^k(\widetilde{F},\widetilde{F}')$.

Proof. By (20), $\pi^{-1}(A(F,F'))$ equals the intersection of the decreasing sequence of analytic subsets $A^k(F,F')$, $k\geqslant 0$. Hence this sequence stabilizes after some $k\geqslant 0$ in the sense of germs at (u,v,u'). This means that we can choose an integer $k\geqslant 0$ and a neighborhood $W_1 = U_1 \times V_1 \times U_1'$ of (u, v, u') in $U \times V \times U'$ such that $\pi^{-1}(A(F, F')) \cap W_1 = A^k(F, F') \cap W_1$. By Lemma 3.2, there exists domains $\widetilde{U}' \subset U_1'$ and $\widetilde{V}' \subset V_1$ satisfying conditions (1) and (2). The proof is completed by applying Lemma 3.2 to $(u, v) \in F \cap (U_1 \times \widetilde{V}')$. □

For $0 \leqslant d \leqslant \dim V$, define

$$A^{k,d}(F) := \{(g, u) \in J^{k,d}(V) \times U : g(0) \in F_u,\ g \subset j^k_{g(0)} F_u\}. \tag{21}$$

Lemma 4.6. *Let (U, V, F) be a holomorphic (resp. algebraic) family. Then $A^{k,d}(F)$ is a closed analytic (resp. algebraic) subset of $J^{k,d}(V) \times U$.*

Proof. Set $d' := \dim F_u$, $u \in U$. The subset $A^{k,d}(F) \subset J^{k,d}(V) \times U$ is the preimage of I given by (16) under the holomorphic (resp. algebraic) map $(g, u) \mapsto (g, j^k_{g(0)} F_u)$, defined on the subset $\{(g, u) \in J^{k,d}(V) \times U : g(0) \in F_u\}$. □

5. Proof of Theorem 1.1

5.1. *Reflections of jets of holomorphic maps.* The following proposition is an important part of the proof of Theorem 1.1. We use the notation of §1. Roughly speaking, it shows under the assumptions of Theorem 1.1 that the restriction of f to a Segre variety of M is algebraically determined by two different k-jets of f. Set

$$\mathcal{M}^2 := \{\xi = (z_1, \bar{w}, z) \in U \times \bar{U} \times U : (z_1, \bar{w}) \in \mathcal{M},\ (z, \bar{w}) \in \mathcal{M}\}. \tag{22}$$

Proposition 5.1. *Let $f: U \to \mathbf{C}^{n'}$ be a local holomorphism between connected real-algebraic submanifolds $M \subset \mathbf{C}^n$ and $M' \subset \mathbf{C}^{n'}$. Suppose that M is generic and condition* (ii) *in Theorem* 1.1 *is satisfied. Then there exist a point $x \in M$, an open neighborhood Σ of $(x, \bar{x}, x)$ in $\mathcal{M}^2$ and an integer k such that for every nonempty open subset $\Sigma' \subset \Sigma$ there exists another nonempty open subset $\Sigma'' \subset \Sigma'$ and an algebraic holomorphic map $\Psi: \Omega \to \mathbf{C}^{n'}$ defined in an open subset $\Omega \subset U \times \overline{J^k(\mathbf{C}^n, \mathbf{C}^{n'})} \times J^k(\mathbf{C}^n, \mathbf{C}^{n'})$ with*

$$(z_1, \overline{j^k_w f}, j^k_z f) \in \Omega \quad \textit{and} \quad f(z_1) = \Psi(z_1, \overline{j^k_w f}, j^k_z f) \tag{23}$$

for all $(z_1, w, z) \in \Sigma''$.

Proof. By shrinking U, we may assume that condition (ii) in Theorem 1.1 holds for all $x \in M \cap U$. By Proposition 3.3, there exist neighborhoods $\widetilde{U}, \widetilde{V} \subset U$ of x such that $(\widetilde{U}, \overline{\widetilde{V}}, \mathcal{M} \cap (\widetilde{U} \times \overline{\widetilde{V}}))$ is an algebraic family.

By the construction, the image of $M \cap U$ under the diagonal mapping $z \mapsto (z, \bar{z})$ is a generic submanifold of $\mathcal{M}$. By Lemma 3.5, there exist domains $\widetilde{U}_1 \subset \widetilde{U}$, $\widetilde{V}_1 \subset \widetilde{V}$, $\widetilde{V}_1' \subset \widetilde{V}'$

and a point $x_0 \in M \cap \widetilde{U}_1 \cap \widetilde{V}_1$ such that $f(\widetilde{V}_1) \subset \widetilde{V}'_1$ and $(\widetilde{U}_1, \overline{\widetilde{V}'_1}, F)$ is a holomorphic family, where

$$F := \{(z, \bar{f}(\bar{w})) : z \in \widetilde{U}_1,\, w \in Q_z \cap \widetilde{V}_1\}. \tag{24}$$

Without loss of generality we can assume that $U = \widetilde{U}_1$ and $\mathcal{M}' \subset U' \times \overline{\widetilde{V}'_1}$. Set for simplicity $V := \widetilde{V}_1$, $V' := \widetilde{V}'_1$. By Definition 3.1, $f(Q_z) \subset V'$ is a connected submanifold whose dimension is constant independently of $z \in U$. Set $d := \dim f(Q_z)$. Then the jet $j^k_{w'} f(Q_z) \in J^{k,d}(V')$ is defined for $w' \in f(Q_z)$. We now consider the subsets $A(F, \mathcal{M}') \subset U \times U'$, $A^k(F, \mathcal{M}') \subset U \times \overline{V'} \times U'$ and $A^{k,d}(\mathcal{M}') \subset J^{k,d}(\overline{V'}) \times U'$ as defined by (12), (19) and (21) respectively. For the reader's convenience, we recall the constructions in our case:

$$\begin{aligned}
A(F, \mathcal{M}') &:= \{(z, z') \in U \times U' : f(Q_z) \subset Q'_{z'}\},\\
A^k(F, \mathcal{M}') &:= \{(z, \bar{w}', z') \in U \times \overline{V'} \times U' : w' \in f(Q_z),\, w' \in Q'_{z'},\, j^k_{w'}(f(Q_z)) \subset j^k_{w'} Q'_{z'}\},\\
A^{k,d}(\mathcal{M}') &:= \{(\bar{g}, z') \in J^{k,d}(\overline{V'}) \times U' : g(0) \in Q'_{z'},\, g \subset j^k_{g(0)} Q'_{z'}\}.
\end{aligned}$$

By Lemmata 3.6, 4.4 and 4.6, these subsets are closed and analytic, $A^{k,d}(\mathcal{M}')$ is even algebraic.

We write

$$\begin{aligned}
A_z(F, \mathcal{M}') &:= \{z' \in U' : f(Q_z) \subset Q'_{z'}\}, \quad z \in U,\\
A^k_{z,\bar{w}'}(F, \mathcal{M}') &:= \{z' \in U' : w' \in Q'_{z'},\, j^k_{w'}(f(Q_z)) \subset j^k_{w'} Q'_{z'}\}, \quad z \in U,\, w' \in f(Q_z),\\
A^{k,d}_{\bar{g}}(\mathcal{M}') &:= \{z' \in U' : g(0) \in Q'_{z'},\, g \subset j^k_{g(0)} Q'_{z'}\}, \quad \bar{g} \in J^{k,d}(\overline{V'}),
\end{aligned}$$

for the corresponding fibers.

It follows from Proposition 2.4 and the construction that

$$A^k_{z,\bar{f}(\bar{w})}(F, \mathcal{M}') = A^{k,d}_{\bar{g}(z,\bar{w})}(\mathcal{M}'), \tag{25}$$

where

$$\bar{g}(z, \bar{w}) := \overline{j^k_{f(w)} f(Q_z)} \in J^{k,d}(\overline{V'}). \tag{26}$$

Fix $x \in M$. By Lemma 4.5,

$$A^k_{z,\bar{f}(\bar{w})}(F, \mathcal{M}') = A_z(F, \mathcal{M}')$$

for k sufficiently large and $(z, \bar{w}) \in \mathcal{M}$ close to $(x, \bar{x})$. By replacing U with a smaller neighborhood of x, we may assume that this holds for all $(z, \bar{w}) \in \mathcal{M}$. Together with (25) the last identity implies

$$A^{k,d}_{g(z,\bar{w})}(\mathcal{M}') = \{z' \in U' : Q'_{z'} \supset f(Q_z)\}. \tag{27}$$

Since $f(Q_x)$ is a connected manifold, it follows from Definition 2.6 that

$$r_x = \text{germ of } A^{k,d}_{g(x,\bar{x})}(\mathcal{M}') \text{ at } f(x) \tag{28}$$

for all $x \in M \cap U$.

By replacing U, V with smaller neighborhoods of x, we may assume that (27) holds for all $(z, \bar{w}) \in \mathcal{M}$. Furthermore we can choose U and V such that all Segre varieties $Q_z \subset V$ become connected for $z \in U$. By the invariance of Segre varieties (Proposition 2.4) and the connectedness of Q_z, $Q'_{f(z)} \supset f(Q_z)$, i.e. the right-hand side of (27) contains the point $f(z) \in U'$. Hence

$$f(z) \in A^{k,d}_{\bar{g}(z,\bar{w})}(\mathcal{M}') \quad \text{for all } (z, \bar{w}) \in \mathcal{M}. \tag{29}$$

Set

$$\widetilde{\mathcal{M}} := \{(\bar{g}(z, \bar{w}), f(z)) : (z, \bar{w}) \in \mathcal{M}\} \quad \text{and} \quad \widetilde{M} := \{(\bar{g}(z, \bar{z}), f(z)) : z \in M\}.$$

Then $\widetilde{M}$ is a real-analytic submanifold of $J^{k,d}(\overline{V'}) \times U'$. By (29), it is contained in the algebraic subset $A^{k,d}(\mathcal{M}') \subset J^{k,d}(\overline{V'}) \times U'$. Hence we are in the situation of Lemma 3.4 with $S = A^{k,d}(\mathcal{M}')$. Let $U'_1, ..., U'_s \subset U'$, $W \Subset \widetilde{M}$ be open subsets and $N \subset J^{k,d}(\overline{V'})$ be an algebraic submanifold given by Lemma 3.4.

We write for simplicity

$$G^j := A^{k,d}(\mathcal{M}') \cap (N \times U'_j), \quad j = 1, ..., s.$$

In the following we use the notation of §3.3. Define

$$B^j_g := \overline{B_{\bar{g}}(G^j, \mathcal{M}')} = \bigcap \{Q'_{z'} : z' \in G^j_{\bar{g}}\}, \quad B_g := B^1_g \cap ... \cap B^s_g$$

and

$$B := \{(g, z') \in \overline{N} \times V' : z' \in B_g\}.$$

By Lemma 3.7, $B \subset \overline{N} \times V'$ is a closed algebraic subset.

Since $G^j \subset A^{k,d}(\mathcal{M}')$ for every $j = 1, ..., s$, it follows from (27) that

$$f(Q_z) \subset \bigcap \{Q'_{z'} : z' \in A^{k,d}_{g(\bar{z},w)}(\mathcal{M}')\} \subset B_{g(\bar{z},w)} \quad \text{for all } (\bar{z}, w) \in \mathcal{M}. \tag{30}$$

In particular, for every $x \in M \cap U$, $f(x) \in B_{g(\bar{x},x)}$.

On the other hand, property (iii) in Lemma 3.4 in our situation means that for every $x \in M \cap U$, every local irreducible component of $A^{k,d}_{g(\bar{x},x)}(\mathcal{M})$ at $f(x)$ intersects $G^j_{g(\bar{x},x)}$ for some $j = 1, ..., s$. Now we use the simple fact that the intersection of $Q'_{z'}$ for all z' in an irreducible analytic set A coincides with the intersection of $Q'_{z'}$ for z' in an open subset of A. Using this fact for the local irreducible components of $A^{k,d}_{g(\bar{x},x)}(\mathcal{M})$ at $f(x)$ we conclude that

$$r_x^2 \supset \text{germ of } B_{g(\bar{x},x)} \text{ at } f(x) \tag{31}$$

for all $x\in M\cap U$. Here the inclusion instead of the equality is due to the fact that the sets G^j may have some additional irreducible components.

By Proposition 2.3, $(z,\overline{w})\in\mathcal{M}$ is equivalent to $(w,\bar{z})\in\mathcal{M}$ for $z,w\in U\cap V$. Then we can replace U and V with $U\cap V$ and rewrite (30) as

$$f(Q_w)\subset B_{g(\overline{w},z)} \quad \text{for all } (z,\overline{w})\in\mathcal{M}. \tag{32}$$

We also replace U' and V' with $U'\cap V'$, set $J_1:=J^{k,d}(\overline{V'})$ and $J_2:=J^{k,d}(V')$, and define

$$C:=\{(\bar{g}_1,g_2,z')\in J_1\times J_2\times U' : z'\in A^{k,d}_{\bar{g}_1}(\mathcal{M})\cap B_{g_2}\}.$$

Let $\xi=(z_1,\overline{w},z)\in\mathcal{M}^2$ be arbitrary. Then, by (29) and (32),

$$f(z_1)\in C_{(\bar{g}(z_1,\overline{w}),g(\overline{w},z))} \tag{33}$$

and, by (28) and (31),

$$r_x\cap r_x^2 \supset \text{germ of } C_{(\bar{g}(x,\bar{x}),g(\bar{x},x))} \text{ at } f(x) \tag{34}$$

for all $x\in M\cap U$. Together with condition (ii) in Theorem 1.1 this implies that $f(x)$ is isolated in $C_{(\bar{g}(x,\bar{x}),g(\bar{x},x))}$. Since $C\subset J_1\times J_2\times U'$ is an algebraic subset, all fibers $C_{\bar{g}_1,g_2}$ are discrete by the upper semicontinuity of the fiber dimension in a neighborhood of every point of the form $(\bar{g}(x,\bar{x}),g(\bar{x},x),f(x))$, $x\in M\cap U$.

Without loss of generality, $\operatorname{rank}_z(f|Q_w)=d$ for all $(z,\overline{w})\in\mathcal{M}$. Fix $x\in M\cap U$. Then it follows from the rank theorem that there exists a coordinate neighborhood $U_1\times U_2\subset U$ of x such that $\dim U_1=d$ and

$$\dim d_z f(T_z\widetilde{Q}_{z,w})=d \tag{35}$$

for all $(z,\overline{w})\in\mathcal{M}$ close to $(x,\bar{x})$, where

$$\widetilde{Q}_{z,w}:=\{v\in Q_w : v_2=z_2\}.$$

Without loss of generality, $U=U_1\times U_2$. Then for every $(z,\overline{w})\in\mathcal{M}$, $(j^k_z f, j^k_z\widetilde{Q}_{z,w})$ is contained in $\mathcal{R}$, where $\mathcal{R}$ is defined by (15). By Lemma 4.1,

$$j^k_{f(z)}f(Q_w)=c(j^k_z f, j^k_z\widetilde{Q}_{z,w}), \tag{36}$$

where $c\colon \mathcal{R}\to J^{k,d}(X')$ is an algebraic holomorphic map.

Let $\Sigma'\subset\mathcal{M}^2$ be a nonempty open subset. By (33), the subset

$$\widetilde{\Sigma'} := \{(\bar g(z_1,\overline{w}), g(\overline{w},z), f(z_1)) : (z_1,\overline{w},z)\in\Sigma'\}$$

is contained in the algebraic subset $C\subset J_1\times J_2\times U'$. Let $C'\subset C$ denote the (algebraic) Zariski closure of $\widetilde{\Sigma'}$. Then there exists at least one point

$$(\bar g(z_1^0,\overline{w^0}), g(\overline{w^0},z^0), f(z_1^0))\in\widetilde{\Sigma'}$$

outside both the singular locus of C' and the branch locus of the projection to $J_1\times J_2$. Hence near this point, C' can be represented as a graph of an algebraic holomorphic map $\varphi\colon N_J\to U'$, where N_J is an algebraic submanifold in $J_1\times J_2$. Furthermore, in a neighborhood of $(\bar g(z_1^0,\overline{w^0}), g(\overline{w^0},z^0))$, φ can be extended to an algebraic holomorphic map $\psi\colon\Omega_J\to U'$, where Ω_J is an open subset in $J_1\times J_2$. In particular, this means that

$$f(z_1) = \psi(\bar g(z_1,\overline{w}), g(\overline{w},z)) \tag{37}$$

holds for all $(z_1,\overline{w},z)\in\Sigma'':=\{(z_1,\overline{w},z)\in\Sigma' : (\bar g(z_1,\overline{w}), g(\overline{w},z))\in\Omega_J\}$. Together with (26) and (36) this yields

$$f(z_1) = \psi\big(\overline{c(j_w^k f, j_w^k \widetilde{Q}_{w,z_1})}, c(j_z^k f, j_z^k\widetilde{Q}_{z,w})\big).$$

By setting

$$\Psi(z_1,\bar{j}_1,j_2) := \psi\big(\overline{c(j_1, j_w^k \widetilde{Q}_{w,z_1})}, c(j_2, j_z^k\widetilde{Q}_{z,w})\big),$$

where $w:=j_1(0)$ and $z:=j_2(0)$, we obtain an algebraic holomorphic map defined in a neighborhood of $p_0:=(z_1^0,\overline{j_{w^0}^k f}, j_{z^0}^k f)$ in the submanifold

$$S := \big\{(z_1,\bar{j}_1,j_2)\in U\times\overline{J^k(\mathbf{C}^n,\mathbf{C}^{n'})}\times J^k(\mathbf{C}^n,\mathbf{C}^{n'}) : (j_1, j_w^k\widetilde{Q}_{w,z_1})\in\mathcal{R},\ (j_2, j_z^k\widetilde{Q}_{z,w})\in\mathcal{R}\big\}. \tag{38}$$

We finish the proof by extending Ψ as an algebraic holomorphic map to a neighborhood Ω of p_0 in $U\times\overline{J^k(\mathbf{C}^n,\mathbf{C}^{n'})}\times J^k(\mathbf{C}^n,\mathbf{C}^{n'})$. □

5.2. *Iterated complexifications and jet reflections.* Our goal here will be to iterate the construction of the previous section. Let $M\subset\mathbf{C}^n$ be a real-analytic submanifold and $(U,\varrho(z,\bar z))$ be a defining function. For every integer $j\geqslant 0$ set

$$U_j := \begin{cases} U & \text{for } j \text{ even},\\ \overline{U} & \text{for } j \text{ odd},\end{cases}$$

and

$$\varrho_j(\xi_j,\xi_{j-1}) := \begin{cases} \varrho(\xi_j,\xi_{j-1}) & \text{for } j \text{ even,} \\ \bar\varrho(\xi_j,\xi_{j-1}) & \text{for } j \text{ odd.} \end{cases}$$

Definition 5.2. The *iterated complexifications* $\mathcal{M}^q$, $q\geqslant 1$, of M are defined as follows:

$$\mathcal{M}^q := \{\xi=(\xi_q,...,\xi_0)\in U_q\times...\times U_0 : \varrho_j(\xi_j,\xi_{j-1})=0 \text{ for all } 1\leqslant j\leqslant q\}. \tag{39}$$

Then the Segre sets $Q^q_w = Q^q_w(U,\varrho)$ (see Definition 2.9) are equal to the corresponding fibers or their conjugations as follows directly from the construction. We formulate this as a lemma here. The proof is left to the reader.

Lemma 5.3. *Set*

$$\mathcal{M}^q_w := \{\xi_q\in U_q : \exists(\xi_{q-1},...,\xi_1), (\xi_q,...,\xi_1,w)\in\mathcal{M}^q\}. \tag{40}$$

Then $Q^q_w=\mathcal{M}^q_w$ *for* q *even and* $Q^q_w=\overline{\mathcal{M}^q_w}$ *for* q *odd, where* $w\in U$ *is arbitrary.*

The main disadvantage of the Segre sets is the lack of analyticity in general for $q\geqslant 2$. For this reason we prefer to deal with $\mathcal{M}^q$ instead.

Lemma 5.4. *Let* $M\subset\mathbf{C}^n$ *be a generic real-analytic* (*resp. algebraic*) *submanifold. Then for every* $q\geqslant 1$ *and* $x\in M$, $\mathcal{M}^q$ *is a complex* (*resp. algebraic*) *submanifold of* $U_0\times...\times U_q$ *near the point* $x^q:=(x,\bar x,x,...)$. *Furthermore, every projection* $\pi_j:\mathcal{M}^q\to U_j$ *is of the maximal rank* n *at* x^q.

Proof. By Lemma 3.3, there exists neighborhoods $U(x)$ and $V(x)$ such that $(U(x),\overline{V(x)},\mathcal{M}\cap(U(x)\times\overline{V(x)}))$ is a holomorphic (resp. algebraic) family. By the induction on q we prove that there exists a product neighborhood

$$W(x^q)=W_1\times W_2\subset U_0\times(U_1\times...\times U_q)$$

such that $(W_1,W_2,\mathcal{M}^q\cap W(x^q))$ is a composition of q holomorphic (resp. algebraic) families in the sense of (11) and is therefore a holomorphic (resp. algebraic) family. This proves the first statement and the second for $j=0$.

For $j=q$, consider the canonical involution $\tau: U_q\times...\times U_0\to U_0\times...\times U_q$ defined by

$$\tau(\xi_q,...,\xi_0) := \begin{cases} (\xi_0,...,\xi_q) & \text{for } q \text{ even,} \\ (\bar\xi_0,...,\bar\xi_q) & \text{for } j \text{ odd.} \end{cases}$$

Then $\tau(\mathcal{M}^q)=\mathcal{M}^q$, and the last statement for $j=q$ follows from the statement for the case $j=1$. For $0<j<q$ it suffices to notice that $\mathcal{M}^q$ can be identified with the fiber product of $\mathcal{M}^j$ and $\mathcal{M}^{q-j}$ over U_j. □

Given $U'\subset\mathbf{C}^{n'}$ we define U'_j in a similar way as U_j. By differentiating the identity in (23) we obtain

Proposition 5.5. *Under the assumptions of Proposition* 5.1 *there exist a point* $x \in M$*, an open neighborhood* Σ *of* $(x, \bar{x}, x)$ *in* $\mathcal{M}^2$ *and an integer* k *such that for every nonempty open subset* $\Sigma' \subset \Sigma$ *there exists another nonempty open subset* $\Sigma'' \subset \Sigma'$ *and for every integer* $s \geqslant 0$ *an algebraic holomorphic map* $\Psi^s : \Omega^s \to J^s(U_2, U_2')$ *defined in an open subset* $\Omega^s \subset U \times \overline{J^{k+s}(\mathbf{C}^n, \mathbf{C}^{n'})} \times J^{k+s}(\mathbf{C}^n, \mathbf{C}^{n'})$ *with*

$$(z_1, \overline{j_w^{k+s} f}, j_z^{k+s} f) \in \Omega^s \quad \text{and} \quad j_{z_1}^k f = \Psi^s(z_1, \overline{j_w^{k+s} f}, j_z^{k+s} f) \tag{41}$$

for all $(z_1, w, z) \in \Sigma''$.

For $0 \leqslant j \leqslant q$ we use the notation

$$f_j(z) := \begin{cases} f(z) & \text{for } j \text{ even,} \\ \bar{f}(z) & \text{for } j \text{ odd.} \end{cases}$$

In the following proposition we iterate the identity in (41).

Proposition 5.6. *Under the assumptions of Proposition* 5.1 *for every* $q \geqslant 2$ *and* $s \geqslant 0$*, there exists an integer* $l \geqslant 1$*, a nonempty open subset* $E \subset \mathcal{M}^q$ *and an algebraic holomorphic map* $H_q^s : \Omega_q^s \to J^s(U_q, U_q')$ *defined in an open subset* $\Omega_q^s \subset E \times \overline{J^l(U_1, U_1')} \times J^l(U_0, U_0')$ *with*

$$(\xi, j_{\xi_1}^l \bar{f}, j_{\xi_0}^l f) \in \Omega_q^s \quad \text{and} \quad j_{\xi_q}^s f_q = H_q^s(\xi, j_{\xi_1}^l \bar{f}, j_{\xi_0}^l f) \tag{42}$$

for all $\xi \in E$.

Proof. We prove the proposition by induction on $q \geqslant 2$. For $q = 2$ the statement follows from Proposition 5.5. Fix $s \geqslant 0$ and $q \geqslant 3$. Let k, x and $\Sigma \subset \mathcal{M}^2$ be given by Proposition 5.5. Choose neighborhoods $U(x) \subset \widetilde{U}(x) \subset U$ with $\mathcal{M}^2 \cap (\widetilde{U}(x) \times \overline{\widetilde{U}(x)} \times \widetilde{U}(x)) \subset \Sigma$ and such that $Q_z \cap \widetilde{U}(x) \neq \varnothing$ for $z \in U(x)$. The induction for $q-1$ applied to the real submanifold $M \cap U(x) \subset \mathbf{C}^n$ yields a nonempty open subset $E_1 \subset \mathcal{M}^{q-1}$, an integer $l \geqslant 1$ and an algebraic holomorphic map $H_{q-1}^s : \Omega_{q-1}^s \to J^s(U_{q-1}, U_{q-1}')$ such that

$$j_{\xi_{q-1}}^s f_{q-1} = H_{q-1}^s(\xi, j_{\xi_1}^l \bar{f}, j_{\xi_0}^l f) \tag{43}$$

for all $\xi \in E_1$.

Furthermore, by the choice of $U(x)$, the set

$$\Sigma' := \{(\xi_2, \xi_1, \xi_0) \in \Sigma : \exists (\xi_q, ..., \xi_3), (\xi_q, ..., \xi_1) \in \overline{E_1}\}$$

is nonempty. Proposition 5.5 yields another nonempty subset $\Sigma'' \subset \Sigma'$ and an algebraic holomorphic map $\Psi : \Omega \to \mathbf{C}^{n'}$ satisfying (41) for all $(z_1, w, z) \in \Sigma''$. By the construction of Σ', the open subset

$$E := \{(\xi_q, ..., \xi_0) \in \mathcal{M}^q : (\xi_q, ..., \xi_1) \in \overline{E_1}, (\xi_2, \xi_1, \xi_0) \in \Sigma''\}$$

is also nonempty. Then for $(\xi_q, ..., \xi_0) \in E$, (43) can be rewritten as

$$j^s_{\xi_q} f_q = \overline{H^s_{q-1}}(\bar{\xi}_q, ..., \bar{\xi}_1, j^l_{\xi_2} f, j^l_{\xi_1} \bar{f}). \tag{44}$$

To finish the proof, it remains to express $j^l_{\xi_2} f$ in terms of $j^{l+k}_{\xi_1} \bar{f}$ and $j^{l+k}_{\xi_0} f$ using (41), and then to increase l by k. □

5.3. *The end of the proof.* Suppose that the assumptions of Theorem 1.1 are fulfilled. Without loss of generality, $M \subset U$ is generic and of finite type at every point. By the criterion of Baouendi, Ebenfelt and Rothschild (Proposition 2.10), there exists $q>0$ such that $\mathcal{M}^q_w$ contains an open subset of $\mathbf{C}^n$ for every $w \in U$.

Let $E \subset \mathcal{M}^{q+1}$ be given by Proposition 5.6 and fix $(\chi_{q+1}, ..., \chi_0) \in E$. Then the identity in (42) implies that the restriction of f to

$$\mathcal{M}^{q+1}_{\chi_1,\chi_0} := \{\xi_{q+1} : \exists (\xi_q, ..., \xi_2), (\xi_{q+1}, ..., \xi_2, \chi_1, \chi_0) \in \mathcal{M}^{q+1}\}$$

is algebraic. Since $\mathcal{M}^{q+1}_{\chi_1,\chi_0} = \overline{\mathcal{M}^q_{\chi_1}}$ and the latter contains an open subset of $\mathbf{C}^n$, f is itself algebraic. This finishes the proof of Theorem 1.1.

Note added in the proof. After this paper was submitted for publication, the author received the preprint "On the partial algebraicity of holomorphic mappings between real algebraic sets" by J. Merker, containing related results.

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