GENERALIZATION SOME FUZZY SEPARATION AXIOMS TO DITOPOLOGICAL TEXTURE SPACES

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Abstract. The authors characterize the notion of quasi coincident in texture spaces and study the generalization of fuzzy quasi separation axioms defined by [12] to the ditopological texture spaces.

1. Introduction

Ditopological Texture Spaces: The notion of texture space was firstly introduced by L.M. Brown in [1, 2] under the name of fuzzy structure, and then it was called as texture space by L. M. Brown and R. Ertürk in [5, 6]. Ditopological texture spaces were introduced by L. M. Brown as a natural extension of the work of first author on the representation of lattice-valued topologies by bitopologies in [4]. It is well known that the concept of ditopology is more general than general topology, fuzzy topology and bitopology. So, it will be more advantage to generalize some various general (fuzzy, bi)-topological concepts to the ditopological texture spaces. An adequate introduction to the theory of texture spaces and ditopological texture spaces, and the motivation for its study may be obtained from [7, 8, 9, 10, 11, 16].

Let S be a set, a *texturing* $\mathcal{S}$ of S is a subset of $\mathcal{P}(S)$ which is a point-separating, complete, completely distributive lattice containing S and $\emptyset$, and for which meet coincides with intersection and finite joins with union. The pair $(S, \mathcal{S})$ is then called a *texture*, or a *texture space*.

In a texture, arbitrary joins need not be coincide with unions, and clearly this will be so if and only if $\mathcal{S}$ is closed under arbitrary unions. In this case $(S, \mathcal{S})$ is said to be *plain texture*. It is known that, in a plain texture the cases of

(1) $s \notin Q_s$ for all $s \in S$,
(2) $P_s \not\subseteq Q_s$ for all $s \in S$,
(3) $A = \bigvee_i A_i = \bigcup A_i$ for all $A_i \in \mathcal{S}, i \in I$

are equivalent.

A texture space is called *coseparated* if $Q_s \subseteq Q_t \implies P_s \subseteq P_t$ for all $s, t \in S$ In general, a texturing of S need not be closed under set complementation, it will be that if there exists a mapping $\sigma : \mathcal{S} \to \mathcal{S}$ satisfying $A = \sigma(\sigma(A))$ for all $A \in \mathcal{S}$, and $A \subseteq B \implies \sigma(B) \subseteq \sigma(A)$ for all $A, B \in \mathcal{S}$. In this case σ is called a *complementation* on $(S, \mathcal{S})$, and $(S, \mathcal{S}, \sigma)$ is said to be a *complemented texture*.

We will call a complementation σ on $(S, \mathcal{S})$ *grounded* [16] if there is an involution $s \to s'$ on S so that $\sigma(P_s) = Q_{s'}$ and $\sigma(Q_s) = P_{s'}$ where $s' = \sigma(s')$ for all $s \in S$, and in this case the complemented texture space $(S, \mathcal{S}, \sigma)$ is called *complemented grounded texture space*. It is noted that a complemented plain texture is grounded [17].

For a texture $(S, \mathcal{S})$, most properties are conveniently defined in terms of the *p-sets*

$$P_s = \bigcap\{A \in \mathcal{S} \mid s \in A\}$$

and the *q-sets*,

$$Q_s = \bigvee\{A \in \mathcal{S} \mid s \notin A\}.$$

Theorem 1.1. [7] *In any texture space* $(S, \mathcal{S})$*, we have the following statements:*

(1) $s \notin A \implies A \subseteq Q_s \implies s \notin A^\flat$ *for all* $s \in S, A \in \mathcal{S}$.
(2) $A^\flat = \{s \in S \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.
(3) $(\bigvee_{i \in I} A_i)^\flat = \bigcup_{i \in I} A_i^\flat$ *for all* $s \in S, A \in \mathcal{S}$.
(4) A *is the smallest element of* $\mathcal{S}$ *containing* $A^\flat$
(5) *For* $A, B \in \mathcal{S}$*, if* $A \not\subseteq B$ *then there exists* $s \in S$ *with* $A \not\subseteq Q_s$ *and* $P_s \not\subseteq B$
(6) $A = \bigcap\{Q_s \mid P_s \not\subseteq A\}$ *for all* $A \in \mathcal{S}$.
(7) $A = \bigvee\{P_s \mid A \not\subseteq Q_s\}$ *for all* $A \in \mathcal{S}$.

The followings are some basic examples of textures.

Examples 1.2. (1) If X is a set and $\mathcal{P}(X)$ the powerset of X, then $(X, \mathcal{P}(X))$ is the *discrete texture* on X. For $x \in X$, $P_x = \{x\}$ and $Q_x = X \setminus \{x\}$.

(2) Setting $\mathbb{I} = [0, 1]$, $\mathcal{I} = \{[0, r), [0, r] \mid r \in \mathbb{I}\}$ gives the *unit interval texture* $(\mathbb{I}, \mathcal{I})$. For $r \in \mathbb{I}$, $P_r = [0, r]$ and $Q_r = [0, r)$.

(3) The texture $(L, \mathcal{L})$ is defined by $L = (0, 1]$, $\mathcal{L} = \{(0, r] \mid r \in \mathbb{I}\}$. For $r \in L$, $P_r = (0, r] = Q_r$.

Since a texturing $\mathcal{S}$ need not be closed under the operation of taking the set complement, the notion of topology is replaced by that of *dichotomous topology* or *ditopology*, namely a pair (τ, κ) of subsets of $\mathcal{S}$, where the set of *open sets* τ satisfies

(1) $S, \emptyset \in \tau$,
(2) $G_1, G_2 \in \tau \implies G_1 \cap G_2 \in \tau$ and
(3) $G_i \in \tau, i \in I \implies \bigvee_i G_i \in \tau$,

and the set of *closed sets* κ satisfies

(1) $S, \emptyset \in \kappa$,
(2) $K_1, K_2 \in \kappa \implies K_1 \cup K_2 \in \kappa$ and
(3) $K_i \in \kappa, i \in I \implies \bigcap K_i \in \kappa$.

Hence a ditopology is essentially a "topology" for which there is no *a priori* relation between the open and closed sets. For $A \in \mathcal{S}$ we define the *closure* $[A]$ and the *interior* $]A[$ of A under (τ, κ) by the equalities

$$[A] = \bigcap \{K \in \kappa \mid A \subseteq K\} \text{ and }]A[= \bigvee \{G \in \tau \mid G \subseteq A\}.$$

On the other hand if the texture space $(S, \mathcal{S})$ is complemented, then we say that (τ, κ) is a complemented ditopology on $(S, \mathcal{S}, \sigma)$ if τ and κ are related by $\kappa = \sigma[\tau]$. In this case we have $\sigma([A]) =]\sigma(A)[$ and $\sigma(]A[) = [\sigma(A)]$.

It is well known that; in classic theory, $A \cap B = \emptyset \iff A \subseteq X \setminus B$ for

$A, B \subseteq X$, and in fuzzy set theory, $A \cap B = \emptyset \implies A \subseteq X \setminus B$ for $A, B \subseteq X$.

So it could be defined an alternative binary implication in fuzzy set theory such as: Let I^X be the family of fuzzy sets and $A, B \in I^X$, then we say that;

Definition 1.3. [15] (1) A is *quasi-coincident* with B (denoted by AqB) $\iff$ there exists an $x \in X$ such that $A(x) + B(x) > 1$,

(2) A is *not quasi-coincident* with B (denoted by $A \not q B$) $\iff$ $A(x) + B(x) \le 1$ for all $x \in X$.

2. Q-SEPARATION AXIOMS

In [9], L. M. Brown, R. Ertürk and Ş. Dost have generalized the fuzzy separation axioms in the sense of B. Hutton and I. Reilly [14] to the ditopological texture spaces and they have obtained important results in separation axioms theory. But in applications, it was seen that T_0 axiom which is the one of generalized separation axioms is not so useful and it was given that many equivalent conditions that T_0 axiom. Some of them are followings:

Theorem 2.1. [9] *Let $(S, \mathcal{S}, \tau, \kappa)$ be a ditopological texture space, then the following statements are characteristic properties of T_0 axiom in a ditopological texture space:*

(1) $P_s \not\subseteq P_t \implies \exists C_j \in \tau \cup \kappa; j \in J$ *with* $P_t \subseteq \bigvee_{j \in J} C_j \subseteq Q_s$, *for all* $s, t \in S$

(2) $Q_s \not\subseteq Q_t \implies \exists C_j \in \tau \cup \kappa; j \in J$ *with* $P_t \subseteq \bigcap_{j \in J} C_j \subseteq Q_s$, *for all* $s, t \in S$

(3) *For* $A \in \mathcal{S}$ *there exists* $C_j \in \tau \cup \kappa, j \in J, i \in I_j$ *with* $A = \bigvee_{j \in J} \bigcap_{i \in I_j} C_i^j$

(4) $Q_s \not\subseteq Q_t \implies \exists C \in \tau \cup \kappa$ *with* $P_s \not\subseteq C \not\subseteq Q_t$
(5) $[P_s] \subseteq [P_t]$ *and* $]Q_s[\subseteq]Q_t[$ *implies* $Q_s \subseteq Q_t$

(6) *For all $s \in S$ we have $Q_s = \bigvee\limits_{j \in J} C_j$ for $C_j \in \tau \cup \kappa$*

(7) *If $(S, \mathcal{S})$ is coseparated, then the following condition also characterizes the T_0 property: For all $s \in S$ we have $P_s = \bigcap_{j \in J} C_j$ for $C_j \in \tau \cup \kappa$.*

In this study, the new generalization separation axioms will be defined. These generalization axioms obtained that the fuzzy quasi separation axioms, in the sense of M. H. Ghanim, O. A. Tantawy [12]. In applications, the new generalized T_0 is more useful than generalized T_0 axiom in the sense of [9]. To do this we will give some basic results obtained by authors in a complemented grounded ditopological texture space.

Definition 2.2. Let $(S, \mathcal{S}, \sigma)$ be a complemented grounded texture space. Then we have the following statements:

(1) $AqB \iff A \not\subseteq \sigma(B) \iff \exists s \in S$ with $A \not\subseteq Q_s$ and $P_s \not\subseteq \sigma(B)$ by Theorem 1.1(5),

(2) $Aq\sigma(B) \iff A \not\subseteq B$,

(3) $A \subseteq B \iff A \not{q}\sigma(B)$.

As a result of the above definition we obtain the followings:

(1) AqB and if $A \subseteq C, B \subseteq D$ then CqD,

(2) $A \not{q}B$ and if $C \subseteq A, D \subseteq B$ then $C \not{q}D$.

(3) If $U, V \in \tau$ and if $UqV \implies [U]qV$ and so $\implies [U]q[V]$.

Now we want to give new definitions of quasi (Q-) separation axioms in a complemented grounded ditopological texture space $(S, \mathcal{S}, \tau, \kappa, \sigma)$ which are generalized fuzzy quasi separation axioms in the sense of [12].

Definition 2.3. The ditopological space (τ, κ) is called

(1) *Quasi-T_0 (QT_0) space* if for each $P_s \not\subseteq P_t$ $(s, t \in S)$ there exists a $U \in \tau$ such that P_sqU, $U \subseteq Q_{\sigma(t)}$ or there exists a $K \in \kappa$ such that $P_t \not{q}K$, $P_s \not\subseteq \sigma(K)$.

(2) *Strong quasi-T_0 (SQT_0) space* if P_s is a closed set for all $s \in S$.

(3) *Quasi-T_1 (QT_1) space* if for each $P_s \not\subseteq P_t$ $(s, t \in S)$ there exist $U \in \tau, K \in \kappa$ such that P_sqU, $U \subseteq Q_{\sigma(t)}$ and $P_t \not{q}K$, $P_s \not\subseteq \sigma(K)$.

(4) *Quasi-T_2 (QT_2) space* if for each $P_s \not\subseteq P_t$ $(s, t \in S)$ there exist $U \in \tau, K \in \kappa$ such that P_sqU, $U \subseteq Q_{\sigma(t)}$ and $P_t \not{q}K$, $P_s \not\subseteq \sigma(K)$ and $U \subseteq K$.

(5) *Quasi-$T_{2(1/2)}$ ($QT_{2(1/2)}$) space* if for each $P_s \not\subseteq P_t$ $(s, t \in S)$ there exist $U \in \tau, K \in \kappa$ such that P_sqU, $U \subseteq Q_{\sigma(t)}$, $P_t \not{q}K$, $P_s \not\subseteq \sigma(K)$ and $[U] \subseteq \sigma([\sigma(K)])$.

(6) Quasi-regular(Q-regular) if $\forall s \in S$ $F \in \kappa$ with $P_sq\sigma(F)$, there exist $U \in \tau, K \in \kappa$ such that P_sqU, $F \not{q}K$ and $[\sigma(K)] \subseteq \sigma([U])$.

   A Q-regular space which is strong QT_0 is called *quasi-T_3 (QT_3) space.*

(7) *Quasi-normal(Q-normal) space* if $\forall$ $F_1, F_2 \in \kappa$ with $F_1q\sigma(F_2)$ there exits $U \in \tau, K \in \kappa$ such that F_1qU, $F_2 \not{q}K$ and $[\sigma(K)] \subseteq \sigma([U])$.

   A Q-normal space which is strong QT_0 is called *quasi-T_4 (QT_4) space.*

Now, we will give some implications between these separation axioms as follows:

Corollary 2.4. *Let $(S, \mathcal{S}, \tau, \kappa, \sigma)$ be a complemented grounded texture space, then*

(1) $QT_{2(1/2)} \implies QT_2 \implies QT_1 \implies SQT_0 \implies QT_0$

(2) $QT_4 \implies QT_3 \implies QT_{2(1/2)}$

Proof. The first and the second implications of (1) are clear.

To show that the third implications of (1), $(QT_1 \implies SQT_0)$, we must show $P_s = [P_s]$ for all $s \in S$. To do this, assume that there exists an $s \in S$ such that $[P_s] \not\subseteq P_s$. Then there exists $t \in S$ such that $[P_s] \not\subseteq Q_t$ and $P_t \not\subseteq P_s$. Since (τ, κ) is (QT_1), we have $U \in \tau, K \in \kappa$ such that $P_t q U$, $U \subseteq Q_{\sigma(s)}$ and $P_s \notin K$, $P_t \not\subseteq \sigma(K)$. Hence $(U \subseteq Q_{\sigma(s)} \implies P_s \subseteq \sigma(U) = F \in \kappa, \quad [P_s] \subseteq F)$ and since $[P_s] \not\subseteq Q_t$, we have $F \not\subseteq Q_t$ that is $P_t \subseteq F$. On the other hand since $P_t q U \implies P_t \not\subseteq \sigma(U) = F$. These two case give a contradiction. That is $[P_s] \subseteq P_s$ and so $[P_s] = P_s$.

To show that the last implication of (1), $(SQT_0 \implies QT_0)$, take $s, t \in S$ with $P_s \not\subseteq P_t$. Then there exists $r \in S$ with $P_s \not\subseteq Q_r$ and $P_r \not\subseteq P_t$. Since (τ, κ) is (SQT_0), we have $P_t \in \kappa$ for all $t \in S$; that is $P_r \not\subseteq P_t = \bigcap\{K \in \kappa \mid P_t \subseteq K\}$. Hence there exists $K \in \kappa$ such that $P_r \not\subseteq K$ and $P_t \subseteq K$; that is $(\exists K \in \kappa; P_s \not\subseteq K$ and $P_t \subseteq K \implies \sigma(K) \subseteq \sigma(P_t) = Q_{\sigma(t)}$ and $\sigma(K) \not\subseteq \sigma(P_s) = Q_{\sigma(s)})$. Now if we take $U = \sigma(K) \in \tau$, then we obtain that $U \subseteq Q_{\sigma(t)}$ and $P_s \not\subseteq K \iff P_s q \sigma(K)$. So we have $P_s q U$ and $U \subseteq Q_{\sigma(t)}$, that is $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is a QT_0 space.

Now we will show the implications in (2). Firstly to show that the second implication of (2), $QT_3 \implies QT_{2(1/2)}$, take $s, t \in S$ with $P_s \not\subseteq P_t = \sigma(Q_{\sigma(t)})$, that is $P_s q Q_{\sigma(t)} = \sigma(P_t)$. By SQT_0 of ditopological space (τ, κ), it is known that $F = P_t \in \kappa$ and so we have $P_s q \sigma(F)$. Now since (τ, κ) is quasi-regular, there exist $U \in \tau, K \in \kappa$ such that $P_s q U$, $F = P_t \notin K$ and $[\sigma(K)] \subseteq \sigma([U])$. Then we have $[U] \subseteq \sigma([\sigma(K)]$. On the other hand since $[\sigma(K)] \subseteq \sigma([U])$, we have $P_s q U \implies P_s \not\subseteq \sigma(U) \implies P_s \not\subseteq \sigma([U]) \implies P_s \not\subseteq \sigma(K)$ and $P_t \notin K \implies P_t \subseteq \sigma(K) \subseteq [\sigma(K)] \subseteq \sigma([U]) \implies P_t \subseteq \sigma([U]) \implies U \subseteq \sigma(P_t) = Q_{\sigma(t)}$ That is $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is a $QT_{2(1/2)}$ space.

The first implication of (2), $QT_4 \implies QT_3$ is clear. $\square$

Corollary 2.5. *If $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is QT_0 then it is T_0 in the sense of L. M. Brown, R. Ertürk and Ş. Dost [9].*

Proof. To show that this implication, we will use the "Theorem 2.1.(1). Let $s, t \in S$ be with $P_s \not\subseteq P_t$. Then there exists $U \in \tau$ such that $P_s q U$, $U \subseteq Q_{\sigma(t)}$ or there exists a $K \in \kappa$ such that $P_t \notin K$ and $P_s \not\subseteq \sigma(K)$ by QT_0 of (τ, κ). So we have $P_s \not\subseteq \sigma(U)$ and $P_t \subseteq \sigma(U)$. If it taken $\sigma(U) = F \in \kappa$, we obtain that $P_s \not\subseteq F$ and $P_t \subseteq F \implies P_t \subseteq F \subseteq Q_s$. That is $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is T_0.

If $K \in \kappa$, such that $P_t \notin K$ and $P_s \not\subseteq \sigma(K) \implies P_t \subseteq \sigma(K)$ and $\sigma(K) \subseteq Q_s$. If it taken $U = \sigma(K) \in \tau$, the we have $P_t \subseteq U \subseteq Q_s$. Hence $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is QT_0. $\square$

The following example shows that the converse implication of above corollary is not true in generally.

Example 2.6. Let $S=\{a,b\}$, $\mathcal{S} = \mathcal{P}(S)$, $\tau = \{\emptyset, \{a\}, S\}$, $\kappa = \{\emptyset, \{b\}, S\}$. Then $(S, \mathcal{S}, \sigma)$ is a grounded complemented texture space since it is plain and each plain texture is grounded. Hence $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is a complemented ditopological texture space for the complemented $\sigma(\{a\})=\{b\}$, $\sigma(\{b\})=\{a\}$. This ditoplogical texture space $(S, \mathcal{S}, \tau, \kappa, \sigma)$ is T_0 in the sense of L. M. Brown, R. Ertürk, Ş. Dost but it is not QT_0.

Examples 2.7. (1) For $\mathbb{I} = [0,1]$ define $\mathcal{I} = \{[0,t] \mid t \in [0,1]\} \cup \{[0,t) \mid t \in [0,1]\}$, $\sigma([0,t]) = [0, 1-t)$ and $\sigma([0,t)) = [0, 1-t]$, $t \in [0,1]$ and $\tau = \{[0,t) \mid t \in [0,1]\} \cup \{\mathbb{I}\}$, $\kappa = \{[0,t] \mid t \in [0,1]\} \cup \{\emptyset\}$. Then $(\mathbb{I}, \mathcal{I}, \tau, \kappa, \sigma)$ is a grounded texture , which we will refer to as the *unit interval texture*. This time $(I, \mathcal{I})$ is a plain texture. Then the ditopological texture space $(\mathbb{I}, \mathcal{I}, \tau, \kappa, \sigma)$ is a QT_0 and T_0 space.

Definition 2.8. [5],[8] Let $(S_i, \mathcal{S}_i, \sigma_i)$ be textures $i \in I$, set $S = \prod_{i \in I} S_i$ and $A \subseteq \mathcal{S}_k$ for some $k \in I$. We write

$$E(k, A) = \prod_{i \in I} Y_i \text{ where } Y_i = \begin{cases} A, \text{ if } i = k \\ S_i, \text{ otherwise.} \end{cases}$$

Then the *product texturing* $\mathcal{S} = \bigotimes_{i \in I} \mathcal{S}_i$ on S consists of arbitrary intersections of elements of the set

$$\mathcal{E} = \left\{ \bigcup_{j \in J} E(j, A_j) \mid J \subseteq I, A_j \in \mathcal{S}_j \text{ for } j \in J \right\}.$$

and $\sigma : \mathcal{S} \longrightarrow \mathcal{S}$, $\sigma(A) = \bigcap_{s \in A} \bigcup_{i \in I} E(i, \sigma_i(P_{s_i}))$ is a complementation on $\mathcal{S}$.

We have noted that by [8]; If $(S, \mathcal{S}, \sigma)$ is a complemented product texture of $(S_i, \mathcal{S}_i, \sigma_i)$, then for $i \in I$ and $A_i \in \mathcal{S}_i$, $s = (s_i)$ it can be obtained the following equalities:

$$(a)\, P_s = \bigcap_{i \in I} E(i, P_{s_i}) = \prod_{i \in I} P_{s_i}, \quad (b)\, Q_s = \bigcup_{i \in I} E(i, Q_{s_i}),$$

$$(c)\, \sigma(E(i, A_i)) = E(i, \sigma_i(A_i)), \quad (d)\, \sigma(P_s) = \bigcup_{i \in I} E(i, \sigma_i(P_{s_i}))$$

Corollary 2.9. $(S_i, \mathcal{S}_i, \sigma_i)$ *are complemented grounded textures for each* $i \in I$ *iff the product texture* $(S, \mathcal{S}, \sigma)$ *of* $(S_i, \mathcal{S}_i, \sigma_i)$ *is complemented grounded.*

Definition 2.10. [5],[8] Let $(S_i, \mathcal{S}_i)$ $i \in I$ be textures with $S_i \cap S_j = \emptyset$ for $i \neq j$. Let $S = \bigcup_{i \in I} S_i$ and $\mathcal{S} = \{A \mid A \subseteq S, A \cap S_i \in \mathcal{S}_i, \forall i \in I\}$. Then $(S, \mathcal{S})$ is a texture which is called sum of disjoint textures $(S_i, \mathcal{S}_i)$ $i \in I$ and if $(S_i, \mathcal{S}_i, \sigma_i)$ are the complemented textures for all $i \in I$ then the complementation

$$\sigma : \mathcal{S} \longrightarrow \mathcal{S}, \sigma(A) \cap S_i = \sigma_i(A \cap S_i), i \in I$$

makes $(S, \mathcal{S}, \sigma)$ is a complemented texture.

We have noted that by [8]; If $(S, \mathcal{S}, \sigma)$ is a complemented sum texture of $(S_i, \mathcal{S}_i, \sigma_i)$, then for $j \in I$ it can be obtained the following equalities:

$$(a)P_s = P_{s_j} \times \{j\}, \quad (b)Q_s = (Q_{s_j} \times \{j\}) \cup (\bigcup_{k \in I \setminus \{j\}} S_k \times \{k\})$$

Corollary 2.11. $(S_i, \mathcal{S}_i, \sigma_i)$ $i \in I$ *is complemented grounded texture for each* $i \in I$ *iff the sum texture* $(S, \mathcal{S}, \sigma)$ *of* $(S_i, \mathcal{S}_i, \sigma_i)$ *is complemented grounded.*

Theorem 2.12. *Let* $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$, $j \in J$, *be non-empty complemented ditopolojical grounded texture spaces and* $(S, \mathcal{S}, \sigma, \tau, \kappa)$ *their product. Then* $(S, \mathcal{S}, \sigma, \tau, \kappa)$ *is* QT_0 *if and only if* $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ *is* QT_0 *for all* $j \in J$.

Proof. ($\Longleftarrow$) Suppose that $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ is QT_0 for all $j \in J$ an take $s = (s_j), t = (t_j) \in S$ with $P_s \nsubseteq P_t$. Since $P_s \nsubseteq P_t = \bigcap_{j \in J} E(j, P_{t_j})$, there exists $j \in J$ with $P_s \nsubseteq E(j, P_{t_j})$ and so we have $E(j, P_{s_j}) \nsubseteq E(j, P_{t_j})$, and hence it will be $P_{s_j} \nsubseteq P_{t_j}$. By the QT_0 of $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$, there exists $U_j \in \tau_j$ such that $P_{s_j} q U_j$ and $U_j \subseteq Q_{\sigma_j(t_j)}$ or there exists $F_j \in \kappa_j$ such that $P_{t_j} \not{q} F_j$ and $P_{s_j} \nsubseteq \sigma_j(F_j)$. Now if we choose $U = E(j, U_j)$, then $U \in \tau$. Since $P_{s_j} \nsubseteq \sigma_j(U_j)$, it can be obtained $P_s \nsubseteq \sigma(U) = E(j, \sigma_j(U_j))$ and this gives $P_s q U$. On the other hand since $U_j \subseteq Q_{\sigma_j(t_j)}$, we have $E(j, U_j) \subseteq \bigcup_{j \in J} E(j, Q_{\sigma_j(t_j)})$, and that is $U \subseteq Q_{\sigma(t)}$.

For the other case of "or", if we choose $F = E(j, F_j) \in \kappa$, it can be obtained the required properties similarly above case. So $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is QT_0.

($\Longrightarrow$) Let the complemented grounded product ditopological texture space $(S, \mathcal{S}, \sigma, \tau, \kappa)$ be QT_0. Take any $j \in J$ and $s_j, t_j \in S_j$ with $P_{s_j} \nsubseteq P_{t_j}$. For $k \in J \setminus \{j\}$, choose $u_k \in S_k^\flat$, which is possible since $S_k \neq \emptyset$. Now let $s = (s_i), t = (t_i) \in S$ defined by

$$s_i = \begin{cases} s_j, & \text{if } i = j \\ u_i, & \text{if } i \neq j \end{cases} \qquad t_i = \begin{cases} t_j, & \text{if } i = j \\ u_i, & \text{if } i \neq j \end{cases}$$

It is verify that $P_s \nsubseteq P_t$, since $P_{s_j} \nsubseteq P_{t_j}$. By the QT_0 of $(S, \mathcal{S}, \sigma, \tau, \kappa)$, there exists $B \in \tau$ such that $P_s q B$ and $B \subseteq Q_{\sigma(t)}$ or there exists $F \in \kappa$ such that $P_t \not{q} F$ and $P_s \nsubseteq \sigma(F)$. Firstly, suppose the case $B \in \tau$ with $P_s q B$ and $B \subseteq Q_{\sigma(t)}$, that is $\sigma(B) \in \kappa$ and $P_s \nsubseteq \sigma(B)$, $P_t \subseteq \sigma(B)$. By the definition of product cotopology [8], we have $j_1, j_2, \ldots, j_n \in J$ and $\sigma_{j_k}(B_{j_k}) \in \kappa_{k_j}$, $1 \leq k \leq n$, so that $\sigma(B) \subseteq \bigcup_{k=1}^{n} E(j_k, \sigma_{j_k}(B_{j_k}))$ and $P_s \nsubseteq \bigcup_{k=1}^{n} E(j_k, \sigma_{j_k}(B_{j_k}))$. Thus we have $P_s \nsubseteq E(j_k, \sigma_{j_k}(B_{j_k}))$ for all k. On the other hand, $P_t \subseteq \bigcup_{k=1}^{n} E(j_k, \sigma_{j_k}(B_{j_k}))$, for some k, and hence it can be obtained $P_{s_j} \nsubseteq \sigma_{j_k}(B_{j_k})$ and $P_{t_j} \subseteq \sigma_{j_k}(B_{j_k})$. By the definition of s and t, $j_k = j$, we have $B_j \in \tau_j$ satisfying $P_{s_j} \nsubseteq \sigma_j(B_j)$ and $B_j \subseteq Q_{\sigma_j(t_j)}$.

The other case of "or", for the closed set case, can be shown similarly above. Thus, the complemented grounded ditopological texture space $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ is QT_0 for all $j \in J$.

$\square$

Theorem 2.13. *Let* $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$, $j \in J$, *be non-empty complemented ditopolojical grounded texture spaces and* $(S, \mathcal{S}, \sigma, \tau, \kappa)$ *their sum. Then* $(S, \mathcal{S}, \sigma, \tau, \kappa)$ *is* QT_0 *if and only if* $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ *is* QT_0 *for all* $j \in J$.

Proof. ($\Longleftarrow$) Suppose that $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ is QT_0 for all $j \in J$ and take $s, t \in S$ with $P_s \nsubseteq P_t$. Let $j, k \in J$ be the unique indices satisfying $s \in S_j$, $t \in S_k$. If $j \neq k$ then $P_{s_j} \nsubseteq P_{t_k}$. Hence $P_{t_k} \nsubseteq S_j$. $S_j \in \tau_j$ has the required properties.

Indeed, $P_s \nsubseteq \sigma(S_j) = \emptyset$ and $S_j \subseteq Q_{t_k} = \sigma(P_t)$. If $j = k$ then $P_{s_k} \nsubseteq P_{t_k}$. By the QT_0 of $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$, there exists $B_k \in \tau_k$ such that $P_{s_k} q B_k$ and $B_k \subseteq Q_{\sigma_k(t_k)}$ or there exists $F_k \in \kappa_k$ such that $P_{t_k} \, \slashed{q} F_k$ and $P_{s_k} \nsubseteq \sigma_k(F_k)$. Now, if we choose $B = B_k$, then $B \in \tau$ and $P_s = P_{s_k} q B_k$, $B = B_k \subseteq Q_{\sigma_k(t_k)}$. In the case of $F_k \in \kappa_k$ we have dual proof. So $(S, \mathcal{S}, \sigma, \tau, \kappa)$ is QT_0.

($\Longrightarrow$) Let the complemented grounded sum ditopological texture space $(S, \mathcal{S}, \sigma, \tau, \kappa)$ be QT_0. Take any $k \in J$ and $s_k, t_k \in S_k$ with $P_{s_k} \nsubseteq P_{t_k}$. Then for $s = (s_k, k), t = (t_k, k) \in S$ we have $P_s \nsubseteq P_t$. By the QT_0 of $(S, \mathcal{S}, \sigma, \tau, \kappa)$, there exists $B \in \tau$ such that $P_s q B$ and $B \subseteq Q_{\sigma(t)}$ or there exists $C \in \kappa$ such that $P_t \, \slashed{q} C$ and $P_s \nsubseteq \sigma(C)$. Firstly, suppose the case of $B \in \tau$ with $P_s q B$ and $B \subseteq Q_{\sigma(t)}$, that is $\sigma(B) \in \kappa$ and $P_s \nsubseteq \sigma(B)$, $P_t \subseteq \sigma(B)$. By the definiton of sum ditopological texture space there exists $B_j \in \tau_j, j \in J$ such that $\sigma_j(B_j) \in \kappa_j$ where, $\sigma(B) \subseteq \sigma_j(B_j) \times \{j\} \cup \bigcup_{j \in J \setminus \{j\}} (S_i \times \{i\})$ and $P_s \nsubseteq \sigma_j(B_j) \times \{j\} \cup \bigcup_{j \in J \setminus \{j\}} (S_i \times \{i\})$. If $j = k$ then $P_{s_k} \times \{k\} \nsubseteq \sigma_k(B_k) \times \{k\}$ and $P_{t_k} \times \{k\} \nsubseteq \sigma_k(B_k) \times \{k\}$ and so we have $P_{s_k} q B_k$ and $B_k \times \{k\} \subseteq \sigma_k(B_k) = Q_{\sigma(t)}$. If $j \neq k$ then $P_{s_k} \times \{k\} \nsubseteq S_j \times \{j\}$. It has the required properties.

The other case of "or", for the closed set case, can be shown similarly above. Thus, the complemented grounded ditopological texture spaces $(S_j, \mathcal{S}_j, \sigma_j, \tau_j, \kappa_j)$ are QT_0 for all $j \in J$.

$\square$

REFERENCES

1. L. M. Brown, *Ditopological fuzzy structures I*, Fuzzy Systems a A. I M. 3 (1), 1993.
2. L. M. Brown, *Ditopological fuzzy structures II*, Fuzzy Systems a A. I M. 3 (2), 1993.
3. L. M. Brown, Quotients of textures and of ditopological texture spaces, Topology Proceedings 29 (2) (2005), 337–368.
4. L. M. Brown and M. Diker, Ditopological texture spaces and intuitionistic sets, Fuzzy sets and systems 98, (1998), 217–224.
5. L. M. Brown and R. Ertürk, Fuzzy Sets as Texture Spaces, I. Representation Theorems, Fuzzy Sets and Systems 110 (2) (2000), 227–236.
6. L. M. Brown and R. Ertürk, Fuzzy sets as texture spaces, II. Subtextures and quotient textures, Fuzzy Sets and Systems 110 (2) (2000), 237–245.
7. L. M. Brown, R. Ertürk and Ş. Dost, Ditopological texture spaces and fuzzy topology, I. Basic Concepts, Fuzzy Sets and Systems 147 (2) (2004), 171–199.
8. L. M. Brown, R. Ertürk and Ş. Dost, Ditopological texture spaces and fuzzy topology, II. Topological Considerations, Fuzzy Sets and Systems 147 (2) (2004), 201–231.
9. L. M. Brown, R. Ertürk and Ş. Dost, Ditopological texture spaces and fuzzy topology, III. Separation Axioms, Fuzzy Sets and Systems 157 (14) (2006), 1886–1912.
10. L. M. Brown and M. Gohar, Compactness in ditopological texture spaces, Hacettepe J. Math. and Stat., V. 38(1) (2009), 21-43.
11. M. Demirci, Textures and C-spaces, Fuzzy Sets and Systems 158 (11) (2007), 1237–1245.
12. M. H. Ghanim, O. A. Tantawy and Fawzia M. Selim, On lower separation axioms, Fuzzy Sets and Systems 85 (1997), 385–389.
13. G. Gierz, K. H. Hofmann, D. Keimel, J. Lawson, M. Mislove and, D. Scott, *A compendium of continuous lattices* (Springer–Verlag, Berlin, Heidelberg, New York, 1980).
14. B. Hutton and I. Reilly, Separation axioms in fuzzy topological spaces, Fuzzy Sets and Systems 3 (1980), 93–104.
15. P. Pao Ming and L. Ying Ming, Fuzzy topology I, neighbourhood structure of a fuzzy point and Moore-Smith convergence, J. Math.Anal. Appl., 76 (1980), 571-599.

16. S. Özçağ and L. M. Brown, Dicompletion of plain diuniformities, Topology and its Applications, submitted.

17. I. Tiryaki and L. M. Brown, Plain textures and fuzzy sets over a poset, preprint.

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