Volume 3 Number 10

Electronic Journal of Theoretical Physics

ISSN 1729-5254

EJTP

Majorana Issue

Editor
Ignazio Licata

http://www.ejtp.com April, 2006 E-mail:info@ejtp.com

Volume 3 Number 10

Electronic Journal of Theoretical Physics

ISSN 1729-5254

EJTP

Majorana Issue

Editor
Ignazio Licata

http://www.ejtp.com April, 2006 E-mail:info@ejtp.com

Editorial Board

Sergey Danilkin

Instrument Scientist, The Bragg Institute
Australian Nuclear Science and Technology Organization
PMB 1, Menai NSW 2234
Australia
Tel: +61 2 9717 3338
Fax: +61 2 9717 3606

e-mail: s.danilkin@ansto.gov.au

Robert V. Gentry

The Orion Foundation
P. O. Box 12067
Knoxville, TN 37912-0067
USA
e-mail: gentryrv[@orionfdn.org

rvgentry@ejtp.info

Attilio Maccari

Nonlinear phenomena, chaos and solitons in classic and quantum physics
Technical Institute "G. Cardano"
Via Alfredo Casella 3
00013 Mentana RM - ITALY

e-mail: solitone@yahoo.it

Beny Neta

Applied Mathematics
Department of Mathematics
Naval Postgraduate School
1141 Cunningham Road
Monterey, CA 93943, USA

e-mail: byneta@gmail.com

Haret C. Rosu

Advanced Materials Division
Institute for Scientific and Technological Research (IPICyT)
Camino a la Presa San José 2055
Col. Lomas 4a. sección, C.P. 78216
San Luis Potosí, San Luis Potosí, México

e-mail: hcr@titan.ipicyt.edu.mx

A. Abdelkader

Experimental Physics
Physics Department, AjmanUniversity
Ajman-UAE
e-mail: atef28@gmail.com

atef@ejtp.info

Leonardo Chiatti

Medical Physics Laboratory ASL VT
Via S. Lorenzo 101, 01100 Viterbo (Italy)
Tel : (0039) 0761 236903
Fax (0039) 0761 237904

e-mail: fisica1.san@asl.vt.it

chiatti@ejtp.info

Zdenek Stuchlik

Relativistic Astrophysics
Department of Physics, Faculty of Philosophy and Science,
Silesian University, Bezru covo n´am. 13, 746 01 Opava,
Czech Republic

e-mail: Zdenek.Stuchlik@fpf.slu.cz

Table of Contents

Editorial Note

MAJORANA IMPACT ON CONTEMPORARY PHYSICS

Ettore Majorana (1906 – 1938) passed through theoretical physics like a meteor. In fact, his "official" fundamental papers are just nine. They all were written in the short period from 1928 to 1933. They are audacious and strongly beautiful works which impose themselves over and over again on any generation of theoretical physicists as the paradigm of a style able to fuse - by a singular critical thinking – both the attention for the experimental data and the freedom of theoretical reasoning in a mathematical formulation reaching the essential core of the problem.

At the beginning, their fame was directly linked to the topics of the period and consequently they were perceived in a slightly different way by theoretical physicists and mathematicians. On a purely physical level, just consider, for instance, the Majorana-Brossel effect, the adiabatic spin-flip and the Heisenberg-Majorana exchange forces, while - on the more specifically physical mathematical one - the Lorentz group at infinite dimension and the Dirac matrices' representation in real form. It was only in the '50s –'60s that the importance of works such as *Teoria Relativistica di Particelle con Momento Intrinseco Arbitrario* and *Teoria Simmetrica dell'Elettrone e del Positrone* started to be fully comprehended. They both are still a source of inspiration for many Quantum Field Theory approaches, such as the representation of spinorial fields whose implications span the physics of neutrinos and much more "exotic" objects like anyons and Majorana zero modes, or the roles of Clifford algebras and non-commutative geometries.

During the last years, a new kind of interest for Majorana legacy has grown. The widening of theoretical physics' spheres has favoured an increasing awareness of the deep connection between symmetries and interactions, and a renewed conception of theoretical physics and mathematics relation. How Roger Penrose effectively wrote, the deeper our understanding of physical laws becomes, the more we penetrate into the abstract world of mathematical concepts. Which thing allowed the new generation of theorists to get out new topics from Majorana work and to approach theoretical physics according to what we can define as the Majorana style. In this way, the Majorana ideas have found elegant and fecund applications in new fields. That is the case of the Riemann-Majorana-Bloch Sphere, which from being a hidden structure in *Atomi Orientati in Campo Magnetico Variabile* showed to be precious in Quantum Computing and in studying the non-local correlations or the Majorana Oscillator, implicitly included in his Neutrino Theory.

This anthology has been thought not only as an owed celebrative act, but especially as a meeting of researchers on some presently debated aspects in physics in Majorana spirit.

As the editor, coordinating the work of friends and colleagues has been an exciting and compelling experience. I am really grateful to all of them for taking part so cordially and creatively in Majorana Centenary Special Issue. Special thanks for Erasmo Recami: Ettore Majorana and his work have been a constant of our long friendship. This issue could not be published without Ammar Sakaji – Editor in Chief of the Electronic Journal of Theoretical Physics – who promptly and enthusiastically agreed to the project and followed its growing up with his usual care and Sante Di Renzo Publisher in Rome for the hard copy version.

Ignazio Licata

EJTP **3, No. 10** (2006) 1–10 *Electronic Journal of Theoretical Physics*

The Scientific Work Of Ettore Majorana: An Introduction

Erasmo Recami *

Facoltà di Ingegneria, Università statale di Bergamo; and I.N.F.N.–Sezione di Milano, Milan, Italia

Received 22 February 2006, Published 28 May 2006

Abstract: A Brief bibliography of the scientific work of Ettore Majorana has been discussed.

Keywords: Ettore Majorana, Scientific work
PACS (2006): 01.30.Tt, 01.65.+g

1. Historical Prelude

Ettore Majorana's fame solidly rests on testimonies like the following, from the evocative pen of Giuseppe Cocconi. At the request of Edoardo Amaldi[1], he wrote from CERN (July 18, 1965):

"In January 1938, after having just graduated, I was invited, essentially by you, to come to the Institute of Physics at the University in Rome for six months as a teaching assistant, and once I was there I would have the good fortune of joining Fermi, Bernardini (who had been given a chair at Camerino a few months earlier) and Ageno (he, too, a new graduate), in the research of the products of disintegration of μ "mesons" (at that time called mesotrons or yukons), which are produced by cosmic rays [...]

"It was actually while I was staying with Fermi in the small laboratory on the second floor, absorbed in our work, with Fermi working with a piece of Wilson's chamber (which would help to reveal mesons at the end of their range) on a lathe and me constructing a jalopey for the illumination of the chamber, using the flash produced by the explosion of an aluminum ribbon shortcircuited on a battery, that Ettore Majorana came in search of Fermi. I was introduced to him and we exchanged few words. A dark face. And that was

* recami@mi.infn.it

it. An easily forgettable experience if, after a few weeks while I was still with Fermi in that same workshop, news of Ettore Majorana's disappearance in Naples had not arrived. I remember that Fermi busied himself with telephoning around until, after some days, he had the impression that Ettore would never be found.

"It was then that Fermi, trying to make me understand the significance of this loss, expressed himself in quite a peculiar way; he who was so objectively harsh when judging people. And so, at this point, I would like to repeat his words, just as I can still hear them ringing in my memory: 'Because, you see, in the world there are various categories of scientists: people of a secondary or tertiary standing, who do their best but do not go very far. There are also those of high standing, who come to discoveries of great importance, fundamental for the development of science' (and here I had the impression that he placed himself in that category). 'But then there are geniuses like Galileo and Newton. Well, Ettore was one of them. Majorana had what no one else in the world had [...]'"

And, with first-hand knowledge, Bruno Pontecorvo, adds: "Some time after his entry into Fermi's group, Majorana already possessed such an erudition and had reached such a high level of comprehension of physics that he was able to speak on the same level with Fermi about scientific problems. Fermi himself held him to be the greatest theoretical physicist of our time. He often was astounded [...]. I remember exactly these words that Fermi spoke: 'If a problem has already been proposed, no one in the world can resolve it better than Majorana.' " (See also [2].)

Ettore Majorana disappeared rather misteriously on March 26, 1938, and was never seen again [3]. The myth of his "disappearance" has contributed to nothing more than the notoriety he was entitled to, for being a true genius and a genius well ahead of his time.

Majorana was such a pioneer, that even his manuscripts known as the *Volumetti*, which comprise his *study* notes written in Rome between 1927, when he abandoned his studies in engineering to take up physics, and 1931, are a paragon not only of order, based on argument and even supplied with an index, but also of conciseness, essentiality and originality: So much so that those notebooks could be regarded as an excellent modern text of theoretical physics, even after about eighty years, and a "gold-mine" of seminal new theoretical, physical, and mathematical ideas and hints, quite stimulating and useful for modern research. Such scientific manuscripts, incidentally, have been published for the first time (in 2003) by Kluwer[4]. But Majorana's most interesting notebooks or papers –those that constituted his "reserach notes" will not see the light in the near future: it being too hard the task of selecting, interpreting and...electronically typing them! Each notebook was written during a period of about one year, starting from the years —as we said above— during which Ettore Majorana was completing his studies at the University of Rome. Thus the contents of these notebooks range from typical topics covered in academic courses to topics at the frontiers of research. Despite this unevenness in the level of sophistication, the style in which any particular

topic is treated is never obvious. As an example, we refer here to Majorana's study of the shift in the melting point of a substance when it is placed in a magnetic field or, more interestingly, his examination of heat propagation using the "cricket simile." Also remarkable is his treatment of contemporary physics topics in an original and lucid manner, such as Fermi's explanation of the electromagnetic mass of the electron, the Dirac equation with its applications, and the Lorentz group, revealing in some cases the literature preferred by him. As far as frontier research arguments are concerned, let us here recall only two illuminating examples: the study of quasi-stationary states, anticipating Fano's theory by about 20 years, and Fermi's theory of atoms, reporting analytic solutions of the Thomas-Fermi equation with appropriate boundary conditions in terms of simple quadratures, which to our knowledge were still lacking.

Let us recall that Majorana, after having switched to physics at the beginning of 1928, graduated with Fermi on July 6, 1929, and went on to colaborate with the famous group created by Enrico Fermi and Franco Rasetti (at the start with O.M.Corbino's important help); a theoretical subdivision of which was formed mainly (in the order of their entrance into the Institute) by Ettore Majorana, Gian Carlo Wick, Giulio Racah, Giovanni Gentile Jr., Ugo Fano, Bruno Ferretti, and Piero Caldirola. The members of the experimental subgroup were: Emilio Segré, Edoardo Amaldi, Bruno Pontecorvo, Eugenio Fubini, Mario Ageno, Giuseppe Cocconi, along with the chemist Oscar D'Agostino. Afterwards, Majorana qualified for university teaching of theoretical physics ("Libera Docenza") on November 12, 1932; spent about six months in Leipzig with W. Heisenberg during 1933; and then, for some unknown reasons, stopped participating in the activities of Fermi's group. He even ceased publishing the results of his research, except for his paper "Teoria simmetrica dell'elettrone e del positrone," which (ready since 1933) Majorana was persuaded by his colleagues to remove from a drawer and publish just prior to the 1937 Italian national competition for three full-professorships.

With respect to the last point, let us recall that in 1937 there were numerous Italian competitors for these posts, and many of them were of exceptional caliber; above all: Ettore Majorana, Giulio Racah, Gian Carlo Wick, and Giovanni Gentile Jr. (the son of the famous philosopher bearing the same name, and the inventor of "parastatistics" in quantum mechanics). The judging committee was chaired by E. Fermi and had as members E. Persico, G. Polvani, A. Carrelli, and O. Lazzarino. On the recommendation of the judging committee, the Italian Minister of National Education installed Majorana as professor of theoretical physics at Naples University because of his "great and well-deserved fame," independently of the competition itself; actually, "the Commission hesitated to apply the normal university competition procedures to him." The attached report on the scientific activities of Ettore Majorana, sent to the minister by the committee, stated:

"Without listing his works, all of which are highly notable both for their originality of the methods utilized as well as for the importance of the achieved results, we limit ourselves to the following:

"In modern nuclear theories, the contribution made by this researcher to the introduc-

tion of the forces called "Majorana forces" is universally recognized as the one, among the most fundamental, that permits us to theoretically comprehend the reasons for nuclear stability. The work of Majorana today serves as a basis for the most important research in this field.

"In atomic physics, the merit of having resolved some of the most intricate questions on the structure of spectra through simple and elegant considerations of symmetry is due to Majorana.

"Lastly, he devised a brilliant method that permits us to treat the positive and negative electron in a symmetrical way, finally eliminating the necessity to rely on the extremely artificial and unsatisfactory hypothesis of an infinitely large electrical charge diffused in space, a question that had been tackled in vain by many other scholars."

One of the most important works of Ettore Majorana, the one that introduces his "infinite-components equation" was not mentioned, since it had not yet been understood. It is interesting to note, however, that the proper light was shed on his theory of electron and anti-electron symmetry (today climaxing in its application to neutrinos and anti-neutrinos) and on his resulting ability to eliminate the hypothesis known as the "Dirac sea," a hypothesis that was defined as "extremely artificial and unsatisfactory," despite the fact that in general it had been uncritically accepted.

The details of Majorana and Fermi's first meeting were narrated by E. Segré [5]: "The first important work written by Fermi in Rome ['Su alcune proprietà statistiche dell'atomo' (On certain statistical properties of the atom)] is today known as the Thomas-Fermi method.... When Fermi found that he needed the solution to a non-linear differential equation characterized by unusual boundary conditions in order to proceed, in a week of assiduous work with his usual energy, he calculated the solution with a little hand calculator. Majorana, who had entered the Institute just a short time earlier and who was always very skeptical, decided that Fermi's numeric solution probably was wrong and that it would have been better to verify it. He went home, transformed Fermi's original equation into a Riccati equation, and resolved it without the aid of any calculator, utilizing his extraordinary aptitude for numeric calculation. When he returned to the Institute and skeptically compared the little piece of paper on which he had written his results to Fermi's notebook, and found that their results coincided exactly, he could not hide his amazement." We have indulged in the foregoing anecdote since the pages on which Majorana solved Fermi's differential equation have in the end been found, and it has been shown recently [6] that he actually followed two independent (and quite original) paths to the same mathematical result, one of them leading to an Abel, rather than a Riccati, equation.

Majorana delivered his lectures only during the beginning of 1938, starting on Jan.13 and ending with his disappearance (March 26). But his activity was intense, and his interest for teaching extremely high. For the benefit of his beloved students, and perhaps also for writinng down a book, he prepared careful notes for his lectures. And ten of such lectures appeared in print in 1987 (see ref.[7]): and arised the admired comments of

many (especially British) scholars. The remainig six lecture-notes, which had gone lost, have been rediscovered in 2005 by Salvatore Esposito and Antonino Drago, and will soon appear in print.

2. Ettore Majorana's Published Papers

Majorana published few scientific articles: nine, actually, besides his sociology paper entitled "Il valore delle leggi statistiche nella fisica e nelle scienze sociali" (The value of statistical laws in physics and the social sciences), which was however published not by Majorana but (posthumously) by G. Gentile Jr., in *Scientia* [**36** (1942) 55-56]. We already know that Majorana switched from engineering to physics in 1928 (the year in which he published his first article, written in collaboration with his friend Gentile) and then went on to publish his works in theoretical physics only for a very few years, practically only until 1933. Nevertheless, even his *published* works are a mine of ideas and techniques of theoretical physics that still remains partially unexplored. Let us list his nine published articles:

(1) "Sullo sdoppiamento dei termini Roentgen ottici a causa dell'elettrone rotante e sulla intensità delle righe del Cesio," in collaboration with Giovanni Gentile Jr., *Rendiconti Accademia Lincei* **8** (1928) 229-233.
(2) "Sulla formazione dello ione molecolare di He," *Nuovo Cimento* **8** (1931) 22-28.
(3) "I presunti termini anomali dell'Elio," *Nuovo Cimento* **8** (1931) 78-83.
(4) "Reazione pseudopolare fra atomi di Idrogeno," *Rendiconti Accademia Lincei* **13** (1931) 58-61.
(5) "Teoria dei tripletti *P'* incompleti," *Nuovo Cimento* **8** (1931) 107-113.
(6) "Atomi orientati in campo magnetico variabile," *Nuovo Cimento* **9** (1932) 43-50.
(7) "Teoria relativistica di particelle con momento intrinseco arbitrario," *Nuovo Cimento* **9** (1932) 335-344.
(8) "Über die Kerntheorie," *Zeitschrift für Physik* **82** (1933) 137-145; and "Sulla teoria dei nuclei," *La Ricerca Scientifica* **4**(1) (1933) 559-565.
(9) "Teoria simmetrica dell'elettrone e del positrone," *Nuovo Cimento* **14** (1937) 171-184.

The first papers, written between 1928 and 1931, concern atomic and molecular physics: mainly questions of atomic spectroscopy or chemical bonds (within quantum mechanics, of course). As E. Amaldi has written [1], an in-depth examination of these works leaves one struck by their superb quality: They reveal both a deep knowledge of the experimental data, even in the minutest detail, and an uncommon ease, without equal at that time, in the use of the symmetry properties of the quantum states in order to qualitatively simplify problems and choose the most suitable method for their quantitative resolution. Among the first papers, "Atomi orientati in campo magnetico variabile" (Atoms oriented in a variable magnetic field) deserves special mention. It is in this arti-

cle, famous among atomic physicists, that the effect now known as the *Majorana-Brossel effect* is introduced. In it, Majorana predicts and calculates the modification of the spectral line shape due to an oscillating magnetic field. This work has also remained a classic in the treatment of non-adiabatic spin-flip. Its results —once generalized, as suggested by Majorana himself, by Rabi in 1937 and by Bloch and Rabi in 1945— established the theoretical basis for the experimental method used to reverse the spin also of neutrons by a radio-frequency field, a method that is still practiced today, for example, in all polarized-neutron spectrometers. The Majorana paper introduces moreover the so-called *Majorana sphere* (to represent spinors by a set of points on the surface of a sphere), as noted not long ago by R. Penrose [8] and others.

Majorana's last three articles are all of such importance that none of them can be set aside without comment.

The article "Teoria relativistica di particelle con momento intrinseco arbitrario" (Relativistic theory of particles with arbitrary spin) is a typical example of a work that is so far ahead of its time that it became understood and evaluated in depth only many years later. Around 1932 it was commonly thought that one could write relativistic quantum equations only in the case of particles with zero or half spin. Convinced of the contrary, Majorana —as we know from his manuscripts— began constructing suitable quantum-relativistic equations [9] for higher spin values (one, three-halves, etc.); and he even devised a method for writing the equation for a generic spin-value. But still he published nothing, until he discovered that one could write a single equation to cover an infinite series of cases, that is, an entire infinite family of particles of arbitrary spin (even if at that time the known particles could be counted on one hand). In order to implement his programme with these "infinite components" equations, Majorana invented a technique for the representation of a group several years before Eugene Wigner did. And, what is more, Majorana obtained the infinite-dimensional unitary representations of the Lorentz group that will be re-discovered by Wigner in his 1939 and 1948 works. The entire theory was re-invented by Soviet mathematicians (in particular Gelfand and collaborators) in a series of articles from 1948 to 1958 and finally applied by physicists years later. Sadly, Majorana's initial article remained in the shadows for a good 34 years until D. Fradkin, informed by E. Amaldi, released [*Am. J. Phys.* **34** (1966) 314] what Majorana many years earlier had accomplished.

As soon as the news of the Joliot-Curie experiments reached Rome at the beginning of 1932, Majorana understood that they had discovered the "neutral proton" without having realized it. Thus, even before the official announcement of the discovery of the neutron, made soon afterwards by Chadwick, Majorana was able to explain the structure and stability of atomic nuclei with the help of protons and neutrons, antedating in this way also the pioneering work of D. Ivanenko, as both Segré and Amaldi have recounted. Majorana's colleagues remember that even before Easter he had concluded that protons and neutrons (indistinguishable with respect to the nuclear interaction) were bound by the

"exchange forces" originating from the exchange of their spatial positions alone (and not also of their spins, as Heisenberg would propose), so as to produce the alpha particle (and not the deuteron) saturated with respect to the binding energy. Only after Heisenberg had published his own article on the same problem was Fermi able to persuade Majorana to meet his famous colleague in Leipzig; and finally Heisenberg was able to convince Majorana to publish his results in the paper "Über die Kerntheorie." Majorana's paper on the stability of nuclei was immediately recognized by the scientific community –a rare event, as we know, from his writings– thanks to that timely "propaganda" made by Heisenberg himself. We seize the present opportunity to quote two brief passages from Majorana's letters from Leipzig. On February 14, 1933, he writes his mother (the italics are ours): "The environment of the physics institute is very nice. I have good relations with Heisenberg, with Hund, and with everyone else. *I am writing some articles in German. The first one is already ready....*" The work that is already ready is, naturally, the cited one on nuclear forces, which, however, remained *the only paper* in German. Again, in a letter dated February 18, he tells his father (we italicize): "*I will publish in German, after having extended it, also my latest article which appeared in Nuovo Cimento.*" Actually, Majorana published nothing more, either in Germany or after his return to Italy, except for the article (in 1937) of which we are about to speak. It is therefore of importance to know that Majorana was engaged in writing other papers: in particular, he was expanding his article about the infinite-components equations.

As we said, from the existing manuscripts it appears that Majorana was also formulating the essential lines of his symmetric theory of electrons and anti-electrons during the years 1932-1933, even though he published this theory only years later, when participating in the forementioned competition for a professorship, under the title "Teoria simmetrica dell'elettrone e del positrone" (Symmetrical theory of the electron and positron), a publication that was initially noted almost exclusively for having introduced the Majorana representation of the Dirac matrices in real form. A consequence of this theory is that a neutral fermion has to be identical with its anti-particle, and Majorana suggested that neutrinos could be particles of this type. As with Majorana's other writings, this article also started to gain prominence only decades later, beginning in 1957; and nowadays expressions like Majorana spinors, Majorana mass, and Majorana neutrinos are fashionable. As already mentioned, Majorana's publications (still little known, despite it all) is a potential gold-mine for physics. Recently, for example, C. Becchi pointed out how, in the first pages of the present paper, a clear formulation of the quantum action principle appears, the same principle that in later years, through Schwinger's and Symanzik's works, for example, has brought about quite important advances in quantum field theory.

3. Ettore Majorana's Unpublished Papers

Majorana also left us several unpublished scientific manuscripts, all of which have been catalogued [10] and kept at the "Domus Galilaeana" of Pisa, Italy. Our analysis

of these manuscripts has allowed us to ascertain that all the existing material seems to have been written by 1933; even the rough copy of his last article, which Majorana proceeded to publish in 1937 —as already mentioned— seems to have been ready by 1933, the year in which the discovery of the positron was confirmed. Indeed, we are unaware of what he did in the following years from 1934 to 1938, except for a series of 34 letters written by Majorana between March 17, 1931, and November 16, 1937, in reply to his uncle Quirino —a renowned experimental physicist and at a time president of the Italian Physical Society— who had been pressing Majorana for theoretical explanations of his own experiments. By contrast, his sister Maria recalled that, even in those years, Majorana —who had reduced his visits to Fermi's Institute, starting from the beginning of 1934 (that is, after his return from Leipzig)— continued to study and work at home many hours during the day and at night. Did he continue to dedicate himself to physics? From a letter of his to Quirino, dated January 16, 1936, we find a first answer, because we get to learn that Majorana had been occupied "since some time, with quantum electrodynamics"; knowing Majorana's modesty and love for understatements, this no doubt means that by 1935 Majorana had profoundly dedicated himself to original research in the field of quantum electrodynamics.

Do any other unpublished scientific manuscripts of Majorana exist? The question, raised by his letters from Leipzig to his family, becomes of greater importance when one reads also his letters addressed to the National Research Council of Italy (CNR) during that period. In the first one (dated January 21, 1933), Majorana asserts: "At the moment, I am occupied with the elaboration of a theory for the description of arbitrary-spin particles that I began in Italy and of which I gave a summary notice in *Nuovo Cimento*...." In the second one (dated March 3, 1933) he even declares, referring to the same work: "I have sent an article on nuclear theory to Zeitschrift für Physik. I have the manuscript of a new theory on elementary particles ready, and will send it to the same journal in a few days." Considering that the article described here as a "summary notice" of a new theory was already of a very high level, one can imagine how interesting it would be to discover a copy of its final version, which went unpublished. [Is it still, perhaps, in the *Zeitschrift für Physik* archives? Our own search ended in failure]. One must moreover not forget that the above-cited letter to Quirino Majorana, dated January 16, 1936, revealed that his nephew continued to work on theoretical physics even subsequently, occupying himself in depth, at least, with quantum electrodynamics.

Some of Majorana's other ideas, when they did not remain concealed in his own mind, have survived in the memories of his colleagues. One such reminiscence we owe to Gian Carlo Wick. Writing from Pisa on October 16, 1978, he recalls: "...The scientific contact [between Ettore and me], mentioned by Segré, happened in Rome on the occasion of the 'A. Volta Congress' (long before Majorana's sojourn in Leipzig). The conversation took place in Heitler's company at a restaurant, and therefore without a blackboard...; but even in the absence of details, what Majorana described in words was a 'relativistic theory of charged particles of zero spin based on the idea of field quantization' (second

quantization). When much later I saw Pauli and Weisskopf's article [*Helv. Phys. Acta* **7** (1934) 709], I remained absolutely convinced that what Majorana had discussed was the same thing...."

We attach to this paper a short bibliography. Far from being exhaustive, it provides only some references about the topics touched upon in this Introduction.

Acknowledgments

The author wishes to thank Ignazio Licata for his kind invitation to contribute to this book. He is moreover grateful to A. van der Merwe, Salvatore Esposito and Ettore Majoranna Jr. for their very stimulating cooperation along the years; and acknowledges useful discussions with many colleagues (in particular Dharam Ahluwalia, Roberto Battiston and Enrico Giannetto).

References

[1] E.Amaldi, *La Vita e l'Opera di E. Majorana* (Accademia dei Lincei; Rome, 1966); "Ettore Majorana: Man and scientist," in *Strong and Weak Interactions. Present problems*, A.Zichichi ed. (Academic; New York, 1966).

[2] B.Pontecorvo, *Fermi e la fisica moderna* (Editori Riuniti; Rome, 1972); and in *Proceedings International Conference on the History of Particle Physics, Paris, July 1982*, *Physique* **43** (1982).

[3] E.Recami: *Il caso Majorana: Epistolario, Documenti, Testimonianze*, 2nd edition (Oscar, Mondadori; Milan, 1991), pp.230; 4th edition (Di Renzo; Rome, 2002), pp.273.

[4] S.Esposito, E.Majorana Jr., A. van der Merwe and E.Recami (editors): *Ettore Majorana — Notes on Theoretical Pysics* (Kluwer Acad. Press; Dordrecht and Boston, 2003); 512 pages.

[5] E.Segré: *Enrico Fermi, Fisico* (Zanichelli; Bologna (1971); and *Autobiografia di un Fisico* (Il Mulino; Rome, 1995).

[6] S.Esposito: "Majorana solution of the Thomas-Fermi equation", *Am. J. Phys.* **70** (2002) 852-856; "Majorana transformation for differential equations", *Int. J. Theor. Phys.* **41** (2002) 2417-2426.

[7] *Ettore Majorana – Lezioni all'Università di Napoli*, ed. by B.Preziosi (Bibliopolis; Napoli, 1987).

[8] R.Penrose, "Newton, quantum theory and reality," in *300 Years of Gravitation*, S.W.Hawking and W.Israel eds. (University Press; Cambridge, 1987); J.Zimba and R.Penrose, *Stud. Hist. Phil. Sci.* **24** (1993) 697; R. Penrose: Ombre della Mente (Shadows of the Mind) (Rizzoli; 1996), pp.338–343 and 371–375; and subsequent papers by C.Leonardi, F.Lillo, et al.

[9] R.Mignani, E.Recami e M.Baldo: "About a Dirac–like equation for the photon, according to E.Majorana", *Lett. Nuovo Cimento* **11** (1974) 568, and subsequent papers by E.Giannetto and by S.Esposito.

[10] M.Baldo, R.Mignani, and E.Recami, "Catalogo dei manoscritti scientifici inediti di E.Majorana," in *Ettore Majorana – Lezioni all'Università di Napoli*, B.Preziosi ed. (Bibliopolis; Naples, 1987) pp.175-197; and E.Recami, "Ettore Majorana: L'opera edita ed inedita," *Quaderni di Storia della Fisica (of the Giornale di Fisica)* (S.I.F., Bologna) **5** (1999) 19-68.

EJTP **3, No. 10** (2006) 11–16 *Electronic Journal of Theoretical Physics*

On the Hamiltonian Form of Generalized Dirac Equation for Fermions with Two Mass States

S. I. Kruglov *

Physical and Environmental Sciences Department,
University of Toronto at Scarborough,
1265 Military Trail, Toronto, Ontario, Canada M1C 1A4

Received 18 March 2006, Published 28 May 2006

Abstract: Dynamical and non-dynamical components of the 20-component wave function are separated in the generalized Dirac equation of the first order, describing fermions with spin 1/2 and two mass states. After the exclusion of the non-dynamical components, we obtain the Hamiltonian Form of equations. Minimal and non-minimal electromagnetic interactions of particles are considered here.

Keywords: Quantum Electrodynamics, Dirac Equation, Barut Equation, Electromagnetic Interaction of Particles
PACS (2006): 12.20.-m,03.65.Pm,94.20.wj

1. Introduction

We continue to investigate the first order generalized Dirac equation (FOGDE), describing fermions with spin 1/2 and two mass states. This 20-component wave equation was obtained in [1] on the base of Barut's [2] second order equation describing particles with two mass states. Barut suggested the second order wave equation for the unified description of e, μ leptons. He treated this equation as an effective equation for partly "dressed" fermions using the non-perturbative approach to quantum electrodynamics. Some investigations of Barut's second order wave equation and FOGDE were performed in [3], [4], [5], [6], [7].

The purpose of this paper is to obtain the Hamiltonian Form of the 20-component wave equation of the first order.

The paper is organized as follows. In Sec. 2, we introduce the generalized Dirac

* kruglov@rogers.com

equation of the first order. The dynamical and non-dynamical components of the 20-component wave function are separated, and quantum-mechanical Hamiltonian is derived in Sec. 3. In Sec. 4, we make a conclusion. In Appendix, we give some useful matrices entering the Hamiltonian. The system of units $\hbar = c = 1$ is chosen, Latin letters run 1, 2, 3, and Greek letters run 1, 2, 3, 4, and notations as in [8] are used.

2. Field Equation of the First Order

The Barut second order field equation describing spin-1/2 and two mass states of particles may be rewritten as [1]:

$$\left(\gamma_\mu \partial_\mu - \frac{a}{m}\partial_\mu^2 + m\right)\psi(x) = 0, \tag{1}$$

where $\partial_\mu = \partial/\partial x_\mu = (\partial/\partial x_m, \partial/\partial(it))$, $\psi(x)$ is a Dirac spinor, m is a parameter with the dimension of the mass, and a is a massless parameter. We imply a summation over repeated indices. The commutation relations $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$ are valid for the Dirac matrices. Masses of fermions are given by

$$m_1 = \pm m\left(\frac{1-\sqrt{4a+1}}{2a}\right), \quad m_2 = \pm m\left(\frac{1+\sqrt{4a+1}}{2a}\right). \tag{2}$$

Signs in Eq. (2) should be chosen to have positive values of m_1, m_2.

Eq. (1) can be represented in the first order form [1]:

$$(\alpha_\mu \partial_\mu + m)\,\Psi(x) = 0, \tag{3}$$

where the 20-dimensional wave function $\Psi(x)$ and 20×20-matrices α_μ are

$$\Psi(x) = \{\psi_A(x)\} = \begin{pmatrix} \psi(x) \\ \psi_\mu(x) \end{pmatrix} \qquad (\psi_\mu(x) = -\frac{1}{m}\partial_\mu\psi(x)), \tag{4}$$

$$\alpha_\mu = \left(\varepsilon^{\mu,0} + a\varepsilon^{0,\mu}\right) \otimes I_4 + \varepsilon^{0,0} \otimes \gamma_\mu. \tag{5}$$

The I_4 is a unit 4×4-matrix, and $\otimes$ is a direct product of matrices. The elements of the entire algebra obey equations as follows (see, for example, [9]):

$$\left(\varepsilon^{M,N}\right)_{AB} = \delta_{MA}\delta_{NB}, \qquad \varepsilon^{M,A}\varepsilon^{B,N} = \delta_{AB}\varepsilon^{M,N}, \tag{6}$$

where $A, B, M, N = 0, 1, 2, 3, 4$.

After introducing the minimal electromagnetic interaction by the substitution $\partial_\mu \to D_\mu = \partial_\mu - ieA_\mu$ (A_μ is the four-vector potential of the electromagnetic field), and the non-minimal interaction with the electromagnetic field by adding two parameters κ_1, κ_2, we come [1] to the matrix equation:

$$\left[\alpha_\mu D_\mu + \frac{i}{2}\left(\kappa_0 P_0 + \kappa_1 P_1\right)\alpha_{\mu\nu}\mathcal{F}_{\mu\nu} + m\right]\Psi(x) = 0, \tag{7}$$

where $P_0 = \varepsilon^{0,0} \otimes I_4$, $P_1 = \varepsilon^{\mu,\mu} \otimes I_4$ are the projection operators, $P_0^2 = P_0$, $P_1^2 = P_1$, $P_0 + P_1 = 1$, and $\alpha_{\mu\nu} = \alpha_\mu\alpha_\nu - \alpha_\nu\alpha_\mu$. Parameters κ_0 and κ_1 characterize fermion anomalous electromagnetic interactions.

The tensor form of Eq. (7) is given by

$$(\gamma_\nu D_\nu + i\kappa_0\gamma_\mu\gamma_\nu\mathcal{F}_{\mu\nu} + m)\,\psi(x) + (aD_\mu + i\kappa_0\gamma_\nu\mathcal{F}_{\nu\mu})\,\psi_\mu(x) = 0, \tag{8}$$

$$(D_\mu + i\kappa_1\gamma_\nu\mathcal{F}_{\mu\nu})\,\psi(x) + (m\delta_{\mu\nu} + i\kappa_1 a\mathcal{F}_{\mu\nu})\,\psi_\nu(x) = 0, \tag{9}$$

where $\mathcal{F}_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the strength of the electromagnetic field. Eq. (8), (9) represent the system of equations for Dirac spinor $\psi(x)$ and vector-spinor $\psi_\nu(x)$ interacting with electromagnetic fields.

3. Quantum-Mechanical Hamiltonian

In order to obtain the quantum-mechanical Hamiltonian, we rewrite Eq. (7) as follows:

$$\begin{aligned} i\alpha_4\partial_t\Psi(x) = \Big[&\alpha_a D_a + m + eA_0\alpha_4 + \\ &+\frac{i}{2}\left(\kappa_0 P_0 + \kappa_1 P_1\right)\alpha_{\mu\nu}\mathcal{F}_{\mu\nu}\Big]\Psi(x). \end{aligned} \tag{10}$$

One can verify with the help of Eq. (6) that the matrix α_4 obeys the matrix equation

$$\alpha_4^4 - (1+2a)\alpha_4^2 + a^2\Lambda = 0, \tag{11}$$

where Λ:

$$\Lambda = \left(\varepsilon^{0,0} + \varepsilon^{4,4}\right) \otimes I_4, \tag{12}$$

is the projection operator, $\Lambda^2 = \Lambda$. It should be noted that the matrix Λ can be considered as the unit matrix in the 8-dimensional sub-space of the wave function [1]. The operator Λ, acting on the wave function $\Psi(x)$, extracts the dynamical components $\Phi(x) = \Lambda\Psi(x)$. We may separate[1] the dynamical and non-dynamical components of the wave function $\Psi(x)$ by introducing the second projection operator:

$$\Pi = 1 - \Lambda = \varepsilon^{m,m} \otimes I_4, \tag{13}$$

so that $\Pi^2 = \Pi$. This operator defines non-dynamical components $\Omega = \Pi\Psi(x)$. Multiplying Eq. (10) by the matrix

$$\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2} = \left(\varepsilon^{0,4} + \frac{1}{a}\varepsilon^{4,0}\right) \otimes I_4 - \frac{1}{a}\varepsilon^{4,4} \otimes \gamma_4,$$

and taking into consideration Eq. (11), we obtain the equation as follows:

$$i\partial_t\Phi(x) = eA_0\Phi(x) + \left[\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2}\right]\Big[\alpha_a D_a + m + K\Big]\Psi(x), \tag{14}$$

[1] In the work [1], the dynamical and non-dynamical components of the wave function were not separated.

where

$$K = \frac{i}{2}\left(\kappa_0 P_0 + \kappa_1 P_1\right)\alpha_{\mu\nu}\mathcal{F}_{\mu\nu}$$
$$= i\mathcal{F}_{\mu\nu}\left[\kappa_0\left(\varepsilon^{0,0}\otimes\gamma_\mu\gamma_\nu + \varepsilon^{0,\nu}\otimes\gamma_\mu\right) + \kappa_1\left(\varepsilon^{\mu,0}\otimes\gamma_\nu + a\varepsilon^{\mu,\nu}\otimes I_4\right)\right]. \tag{15}$$

It should be mentioned that because $\Lambda+\Pi=1$, the equality $\Psi(x)=\Phi(x)+\Omega(x)$ is valid. To eliminate the non-dynamical components $\Omega(x)$ from Eq. (14), we multiply Eq. (10) by the matrix Π, and using the equality $\Pi\alpha_4=0$, we obtain

$$\Pi\left(\alpha_a D_a + K\right)\left(\Phi(x)+\Omega(x)\right) + m\Omega(x) = 0. \tag{16}$$

With the help of equation $\Pi\alpha_a\Pi = 0$, one may find from Eq. (16), the expression as follows:

$$\Omega(x) = -\left(m+\Pi K\right)^{-1}\Pi\left(\alpha_a D_a + K\right)\Phi(x). \tag{17}$$

With the aid of Eq. (17), Eq. (14) takes the form

$$i\partial_t\Phi(x) = \mathcal{H}\Phi(x), \tag{18}$$

$$\mathcal{H} = eA_0 + \left[\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2}\right]\left[\alpha_a D_a + m + K\right]$$
$$\times\left[1-\left(m+\Pi K\right)^{-1}\Pi\left(\alpha_b D_b + K\right)\right], \tag{19}$$

Eq. (18) represents the Hamiltonian form of the equation for 8-component wave function $\Phi(x)$. It is obvious that for the relativistic description of fermionic fields, possessing two mass states, it is necessary to have 8-component wave function (two bispinors). The Hamiltonian (19) can be simplified by using products of matrices given in Appendix.

Now we consider the particular case of fermions minimally interacting with electromagnetic fields, $\kappa_0=\kappa_1=0$, $K=0$. In this case, Eq. (18) becomes

$$i\partial_t\Phi(x) = \Bigg[eA_0 + \frac{m}{a}\left(a\varepsilon^{0,4}\otimes I_4 + \varepsilon^{4,0}\otimes I_4 - \varepsilon^{4,4}\otimes\gamma_4\right)$$
$$+\frac{1}{a}\left(\varepsilon^{4,0}\otimes\gamma_m\right)D_m - \frac{1}{m}\left(\varepsilon^{4,0}\otimes I_4\right)D_m^2\Bigg]\Phi(x). \tag{20}$$

In component form, Eq. (20) is given by the system of equations

$$i\partial_t\psi(x) = eA_0\psi(x) + m\psi_4(x),$$
$$i\partial_t\psi_4(x) = \left(eA_0 - \frac{m}{a}\gamma_4\right)\psi_4(x) + \left(\frac{m}{a} + \frac{1}{a}\gamma_m D_m - \frac{1}{m}D_m^2\right)\psi(x). \tag{21}$$

Eq. (21) can also be obtained from Eq. (8), (9), at $\kappa_0=\kappa_1=0$, after the exclusion of non-dynamical (auxiliary) components $\psi_m(x) = -(1/m)D_m\psi(x)$. So, only components with time derivatives enter Eq. (21) and Eq. (18).

4. Conclusion

We have analyzed the 20-component matrix equation of the first order, describing fermions with spin 1/2 and two mass states which is convenient for different applications. There are two parameters characterizing non-minimal electromagnetic interactions of fermions including the interaction of the anomalous magnetic moment of particles. The Hamiltonian form of the equation was obtained, and it was shown that the wave function, entering the Hamiltonian equation, contains 8 components what is necessary for describing fermionic field with two mass states in the formalism of the first order. The Hamiltonian (19) can be used for a consideration of the non-relativistic limit which is convenient for the physical interpretation of constants κ_0, κ_1 introduced. This can be done with the help of the Foldy - Wouthuysen procedure [10].

The approach developed may be applied for a consideration of two families of leptons or quarks, but this requires further investigations.

Appendix

With the help of Eq. (6), one can obtain expressions as follows:

$$\left[\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2}\right]\alpha_m D_m = \left(\frac{1}{a}\varepsilon^{4,0}\otimes\gamma_m + \varepsilon^{4,m}\otimes I_4\right)D_m, \tag{22}$$

$$\Pi\alpha_m D_m = \left(\varepsilon^{m,0}\otimes I_4\right)D_m, \tag{23}$$

$$\Pi K = i\kappa_1\mathcal{F}_{m\nu}\left(\varepsilon^{m,0}\otimes\gamma_\nu + a\varepsilon^{m,\nu}\otimes I_4\right), \tag{24}$$

$$\begin{aligned}\left[\frac{(1+2a)}{a^2}\alpha_4 - \frac{\alpha_4^3}{a^2}\right]K &= i\frac{\kappa_0}{a}\mathcal{F}_{\mu\nu}\left(\varepsilon^{4,0}\otimes\gamma_\mu\gamma_\nu + \varepsilon^{4,\nu}\otimes\gamma_\mu\right)\\ &+i\kappa_1\mathcal{F}_{4\nu}\left(\varepsilon^{0,0}\otimes\gamma_\nu - \frac{1}{a}\varepsilon^{4,0}\otimes\gamma_4\gamma_\nu + a\varepsilon^{0,\nu}\otimes I_4 - \varepsilon^{4,\nu}\otimes\gamma_4\right).\end{aligned} \tag{25}$$

One may verify that the equations

$$\mathcal{F}_{nm}\mathcal{F}_{mi} = B_nB_i - B^2\delta_{ni}, \qquad \mathcal{F}_{nm}\mathcal{F}_{mi}\mathcal{F}_{ik} = -B^2\mathcal{F}_{nk} \tag{26}$$

are hold, where $B^2 = B_m^2$, $B_m = (1/2)\epsilon_{mnk}\mathcal{F}_{nk}$ is the strength of the magnetic field. Eq. (26) allow us to obtain the relation for the matrix $\Sigma \equiv m + \Pi K$:

$$\Sigma^4 - 4m\Sigma^3 + \left(6m^2 - b\right)\Sigma^2 + 2m\left(b - 2m^2\right)\Sigma + m^4 - bm^2 = 0, \tag{27}$$

where $b = a^2\kappa_1^2B^2$. From Eq. (27), we find the inverse matrix Σ^{-1}:

$$\begin{aligned}\Sigma^{-1} &= \frac{1}{m^2\left(b-m^2\right)}\left[\Sigma^3 - 4m\Sigma^2 + \left(6m^2-b\right)\Sigma + 2m\left(b-2m^2\right)\right]\\ &= \frac{1}{m} + \frac{1}{m^2\left(b-m^2\right)}\Big[i\kappa_1\left(m^2-b\right)\mathcal{F}_{m\nu} + am\kappa_1^2\mathcal{F}_{mk}\mathcal{F}_{k\nu}\\ &\quad -ia^2\kappa_1^3\mathcal{F}_{mk}\mathcal{F}_{kn}\mathcal{F}_{n\nu}\Big]\left(\varepsilon^{m,0}\otimes\gamma_\nu + a\varepsilon^{m,\nu}\otimes I_4\right).\end{aligned} \tag{28}$$

References

[1] S. I. Kruglov, Ann. Fond. Broglie **29**, 1005 (2004); Errata-ibid (in press); arXiv: quant-ph/0408056.

[2] A. O. Barut, Phys. Lett. **73B**, 310 (1978); Phys. Rev. Lett. **42**, 1251 (1979).

[3] A. O. Barut, P. Cordero, and G. C. Ghirardi, Nuovo. Cim. **A66**, 36 (1970).

[4] A. O. Barut, P. Cordero, and G. C. Ghirardi, Phys. Rev. **182**, 1844 (1969).

[5] R. Wilson, Nucl. Phys. **B68**, 157 (1974).

[6] V. V. Dvoeglazov, Int. J. Theor. Phys. **37**, 1009 (1998); Ann. Fond. Broglie **25**, 81 (2000); Hadronic J. **26**, 299 (2003).

[7] S. I. Kruglov, Quantization of fermionic fields with two mass states in the first order formalism, to appear in the proceedings of 18th Workshop on Hadronic Mechanics Honoring the 70th Birthday of Prof. R.M. Santilli (Karlstad, Sweden, 20-22 Jun 2005); arXiv: hep-ph/0510103.

[8] A. I. Ahieser, and V. B. Berestetskii, Quantum Electrodynamics (New York: Wiley Interscience, 1969).

[9] S.I.Kruglov, Symmetry and Electromagnetic Interaction of Fields with Multi-Spin (Nova Science Publishers, Huntington, New York, 2001).

[10] L.L. Foldy, and S. A. Wouthuysen, Phys. Rev. **78**, 29 (1950).

EJTP **3, No. 10** (2006) 17–31 *Electronic Journal of Theoretical Physics*

Majorana Equation and Exotics: Higher Derivative Models, Anyons and Noncommutative Geometry

Mikhail S. Plyushchay *

Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile

Received 20 April 2006, Published 28 May 2006

Abstract: In 1932 Ettore Majorana proposed an infinite-component relativistic wave equation for particles of arbitrary integer and half-integer spin. In the late 80s and early 90s it was found that the higher-derivative geometric particle models underlie the Majorana equation, and that its (2+1)-dimensional analogue provides with a natural basis for the description of relativistic anyons. We review these aspects and discuss the relationship of the equation to the exotic planar Galilei symmetry and noncommutative geometry. We also point out the relation of some Abelian gauge field theories with Chern-Simons terms to the Landau problem in the noncommutative plane from the perspective of the Majorana equation.

Keywords: Majorana Equation, Dirac Equation, Noncommutative Geometry, Gauge Theories
PACS (2006): 02.40.Gh, 03.65.Pm, 11.15.-q, 11.15.Ex

1. Introduction

Ettore Majorana was the first to study the infinite-component relativistic fields. In the pioneering 1932 paper [1], on the basis of the linear differential wave equation of a Dirac form, he constructed a relativistically invariant theory for arbitrary integer or half-integer spin particles. It was the first recognition, development and application of the infinite-dimensional unitary representations of the Lorentz group. During a long period of time, however, the Majorana results remained practically unknown, and the theory was rediscovered in 1948 by Gel'fand and Yaglom [2] in a more general framework of the group theory representations. In 1966 Fradkin revived the Majorana remarkable work (on the suggestion of Amaldi) by translating it into English and placing it in the context of the later research [3]. In a few years the development of the concept of the infinite-

* mplyushc@lauca.usach.cl

component fields [4]–[8] culminated in the construction of the dual resonance models and the origin of the superstring theory [9]–[15].

After the revival, the Majorana work inspired an interesting line of research based on a peculiar property of his equation: its time-like solutions describe *positive energy* states lying on a Regge type trajectory, but with unusual dependence of the mass, M, on the spin, s, $M_s \propto (const + s)^{-1}$. In 1970, Dirac [16] proposed a covariant *spinor set* of linear differential equations for the infinite-component field, from which the Majorana and Klein-Gordon equations appear in the form of integrability (consistency) conditions. As a result, the new Dirac relativistic equation describes a massive, spin-zero positive-energy particle. Though this line of research [17]–[22] did not find essential development, in particular, due to the problems arising under the attempt to introduce electromagnetic interaction, recently it was pushed [23]–[25] in the unexpected direction related to the anyon theory [26]–[38], exotic Galilei symmetry [39]–[42], and non-commutative geometry [43]–[46].

In pseudoclassical relativistic particle model associated with the quantum Dirac spin-1/2 equation, the spin degrees of freedom are described by the *odd* Grassmann variables [47]. In 1988 it was observed [48] that the (3+1)D particle analogue of the Polyakov string with rigidity [49] possesses the mass spectrum of the *squared* Majorana equation. The model of the particle with rigidity contains, like the string model [49], the higher derivative curvature term in the action. It is this higher derivative term that effectively supplies the system with the *even* spin degrees of freedom of noncompact nature and leads to the infinite-dimensional representations of the Lorentz group. Soon it was found that the quantum theory of another higher derivative model of the (2+1)D relativistic particle with torsion [32], whose Euclidean version underlies the Bose-Fermi transmutation mechanism [50], is described by the *linear* differential infinite-component wave equation of the Majorana form. Unlike the original Majorana equation, its (2+1)D analogue provides with the quantum states of any (real) value of the spin, and so, can serve as a basis for the construction of relativistic anyon theory [31]–[38]. It was shown recently [23, 24] that the application of the special non-relativistic limit ($c \to \infty$, $s \to \infty$, $s/c^2 \to \kappa = const$) [51, 52] to the model of relativistic particle with torsion produces the higher derivative model of a planar particle [40] with associated exotic (two-fold centrally extended) Galilei symmetry [39]. The quantum spectrum of the higher derivative model [40], being unbounded from below, is described by reducible representations of the exotic planar Galilei group. On the other hand, the application of the same limit to the (2+1)D analogue of the Dirac spinor set of anyon equations [38] gives rise to the Majorana-Dirac-Levy-Leblond type infinite-component wave equations [24], which describe irreducible representations of the exotic planar Galilei group corresponding to a free particle with non-commuting coordinates [41].

Here we review the described relations of the Majorana equation to the higher derivative particle models, exotic Galilei symmetry and associated noncommutative structure. We also discuss the relationship of the (2+1)D relativistic Abelian gauge field theories with Chern-Simons terms [55]–[62] to the Landau problem in the noncommutative plane

[25, 41, 53, 54] from the perspective of the Majorana equation.

2. Majorana Equation and Dirac Spinor Set of Equations

Majorana equation [1] is a linear differential equation of the Dirac form,

$$(P^\mu \Gamma_\mu - m)\,\Psi(x) = 0, \tag{2.1}$$

with $P^\mu = i\partial^\mu$ and matrices Γ_μ generating the Lorentz group via the anti-de Sitter SO(3,2) commutation relations similar to those satisfied by the usual γ-matrices[1],

$$[\Gamma_\mu, \Gamma_\nu] = iS_{\mu\nu}, \qquad [S_{\mu\nu}, \Gamma_\lambda] = i(\eta_{\nu\lambda}\Gamma_\mu - \eta_{\mu\lambda}\Gamma_\nu), \tag{2.2}$$

$$[S_{\mu\nu}, S_{\lambda\rho}] = i(\eta_{\mu\rho}S_{\nu\lambda} - \eta_{\mu\lambda}S_{\nu\rho} + \eta_{\nu\lambda}S_{\mu\rho} - \eta_{\nu\rho}S_{\mu\lambda}). \tag{2.3}$$

The original Majorana realization of the Γ_μ corresponds to the infinite-dimensional unitary representation of the Lorentz group in which its Casimir operators C_1 and C_2 and the Lorentz scalar $\Gamma_\mu\Gamma^\mu$ take the values

$$C_1 \equiv \frac{1}{2}S_{\mu\nu}S^{\mu\nu} = -\frac{3}{4}, \qquad C_2 \equiv \epsilon^{\mu\nu\lambda\rho}S_{\mu\nu}S_{\lambda\rho} = 0, \qquad \Gamma_\mu\Gamma^\mu = -\frac{1}{2}. \tag{2.4}$$

A representation space corresponding to (2.4) is a direct sum of the two irreducible SL(2,C) representations characterized by the integer, $j = 0, 1, \ldots$, and half-integer, $j = 1/2, 3/2, \ldots$, values of the SU(2) subalgebra Casimir operator, $M_i^2 = j(j+1)$, $M_i \equiv \frac{1}{2}\epsilon_{ijk}S^{jk}$. In both cases the Majorana equation (2.1) has time-like (massive), space-like (tachyonic) and light-like (massless) solutions. The spectrum in the light-like sector is

$$M_j = \frac{m}{j + \frac{1}{2}}, \qquad j = s + n, \qquad n = 0, 1, \ldots, \qquad s = 0 \quad \text{or} \quad \frac{1}{2}. \tag{2.5}$$

The change $\Gamma_\mu \to -\Gamma_\mu$ in accordance with (2.2), (2.3) does not effect on representations of the Lorentz group as a subgroup of the SO(3,2). For the Majorana choice with the diagonal generator Γ_0,

$$\Gamma_0 = j + \frac{1}{2}, \tag{2.6}$$

Eq. (2.1) has the time-like ($P^2 < 0$) solutions with positive energy.

In [16], Dirac suggested an interesting modification of the Majorana infinite-component theory that effectively singles out the lowest spin zero time-like state from all the Majorana equation spectrum. The key idea was to generate the Klein-Gordon and Majorana wave equations via the integrability conditions for some covariant set of linear differential equations. Dirac covariant spinor set of (3+1)D equations has the form

$$\mathcal{D}_a\Psi(x, q) = 0, \qquad \mathcal{D}_a = (P^\mu\gamma_\mu + m)_{ab}Q_b, \tag{2.7}$$

[1] We use the metric with signature $(-, +, +, +)$.

where γ-matrices are taken in the Majorana representation, and $Q_a = (q_1, q_2, \pi_1, \pi_2)$ is composed from the mutually commuting dynamical variables q_α, $\alpha = 1, 2$, and commuting conjugate momenta π_α, $[q_\alpha, \pi_\beta] = i\delta_{\alpha\beta}$, while $\Psi(x, q)$ is a single-component wave function. The SO(3,2) generators are realized here as quadratic in Q operators,

$$\Gamma_\mu = \frac{1}{4}\bar{Q}\gamma_\mu Q, \qquad S_{\mu\nu} = \frac{i}{8}\bar{Q}[\gamma_\mu, \gamma_\nu]Q,$$

where $\bar{Q} = Q^t\gamma^0$. The covariance of the set of equations (2.7) follows from the commutation relations $[S_{\mu\nu}, Q] = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]Q$, which mean that the Q_a is transformed as a Lorentz spinor, and so, the set of four equations (2.7) is the spinor set. Note also that $[\Gamma_\mu, Q] = \frac{1}{2}\gamma_\mu Q$, and the Q_a anticommute between themselves for a linear combination of the SO(3,2) generators. This means that the Q_a, Γ_μ and $S_{\mu\nu}$ generate a supersymmetric extension of the anti-de Sitter algebra.

The Klein-Gordon,

$$(P^2 + m^2)\Psi = 0, \tag{2.8}$$

and the Majorana equations (with the parameter m changed in the latter for $\frac{1}{2}m$) are the integrability conditions for the spinor set of equations (2.7) [16]. Taking into account that the $\Gamma_0 = \frac{1}{4}(q_1^2 + q_2^2 + \pi_1^2 + \pi_2^2)$ coincides up to the factor $\frac{1}{2}$ with the Hamiltonian of a planar isotropic oscillator, one finds that the possible eigenvalues of the Γ_0 are given by the sets $j = 0, 1, \ldots$ and $j = 1/2, 3/2, \ldots$ in correspondence with Eq. (2.6). The former case corresponds to the Γ_0 eigenstates given by the even in q_α wave functions, while the latter case corresponds to the odd eigenstates. Having in mind the Majorana equation spectrum (2.5) (with the indicated change of the mass parameter) and Eq. (2.8), one concludes that the spinor set of equations (2.7) describes the positive energy spinless states[2] of the fixed mass.

3. Higher Derivative Relativistic Particle Models

The model of relativistic particle with curvature [48, 63, 64, 65], being an analogue of the model of relativistic string with rigidity [49], is given by the reparametrization invariant action

$$A = -\int (m + \alpha k)ds, \tag{3.1}$$

where $ds^2 = -dx_\mu dx^\mu$, $\alpha > 0$ is a dimensionless parameter[3], and k is the worldline curvature, $k^2 = x''_\mu x''^\mu$, $x'_\mu = dx_\mu/ds$. In a parametrization $x_\mu = x_\mu(\tau)$, Lagrangian of the system is $L = -\sqrt{-\dot{x}^2}(m + k)$, where we assume that the particle moves with the velocity less than the speed of light, $\dot{x}^2 < 0$, $\dot{x}_\mu = dx_\mu/d\tau$, and then $k^2 = (\dot{x}^2\ddot{x}^2 - (\dot{x}\ddot{x}^2)^2)/(\dot{x}^2)^3 \geq 0$

[2] Staunton [20] proposed a modification of the Dirac spinor set of equations that describes the spin-1/2 representation of the Poincaré group

[3] For $\alpha < 0$ the equations of motion of the system have the only solutions corresponding to the curvature-free case $\alpha = 0$ of a spinless particle of mass m [48].

[48]. The Lagrangian equations of motion have the form of the conservation law of the energy-momentum vector,

$$\frac{d}{d\tau}P_\mu = 0, \qquad P_\mu = \frac{\partial L}{\partial \dot{x}^\mu} - \frac{d}{d\tau}\left(\frac{\partial L}{\partial \ddot{x}^\mu}\right). \tag{3.2}$$

The dependence of the Lagrangian on higher derivatives supplies effectively the system with additional translation invariant degrees of freedom described by the velocity $v_\mu \equiv \dot{x}_\mu$ and conjugate momentum [48]. This higher derivative dependence is responsible for a peculiarity of the system: though the particle velocity is less than the speed of light, the equations of motion (3.2) have the time-like ($P^2 < 0$), the light-like ($P^2 = 0$) and the space-like ($P^2 > 0$) solutions [48], whose explicit form was given in [48, 65]. This indicates on a possible relation of the model (3.1) to the infinite-component field theory associated with the Majorana equation. Unlike the Majorana system, however, the quantum version of the model (3.1) has the states of integer spin only, which lie on the nonlinear Regge trajectory of the form very similar to (2.5) [48],

$$M_l = \frac{m}{\sqrt{1+\alpha^{-2}l(l+1)}}, \qquad l = 0, 1, \ldots. \tag{3.3}$$

The choice of the laboratory time gauge $\tau = x^0$ separates here the positive energy time-like solutions.

Before we pass over to the discussion of a relativistic particle model more closely related to the original (3+1)D Majorana equation from the viewpoint of the structure of the spectrum, but essentially different from it in some important properties, it is worth to note that the higher derivative dependence of the action does not obligatorily lead to the tachyonic states. In Ref. [66] the model given by the action of the form (3.1) with parameter $m = 0$ was suggested. It was shown there that in the case of $\dot{x}^2 < 0$, the model is inconsistent (its equations of the motion have no solutions), but for $\dot{x}^2 > 0$ the model is consistent and describes massless states of the arbitrary, but fixed integer or half-integer helicities $\lambda = \pm j$, whose values are defined by the quantized parameter α, $\alpha^2 = j^2$. The velocity higher than the speed of light in such a model originates from the Zittervewegung associated with nontrivial helicity. System (3.1) with $m = 0$ possesses additional local symmetry [66, 67] (action (3.1) in this case has no scale parameter), and it is such a gauge symmetry that is responsible for separation of the two physical helicity components from the infinite-component Majorana type field (cf. the system given by the Dirac spinor set of equations (2.7)). Recently, the interest to such a higher derivative massless particle system has been revived [68, 69] in the context of the massless higher spin field theories [70, 71].

The (2+1)D relativistic model of the particle with torsion [32] is given by the action

$$A = -\int (m + \alpha\varrho)ds, \qquad \varrho = \epsilon^{\mu\nu\lambda}x'_\mu x''_\nu x'''_\lambda, \tag{3.4}$$

where α is a dimensionless parameter, and ϱ is the particle worldline trajectory torsion. Unlike the model (3.1), here the parameter α can take positive or negative values, and for

the sake of definiteness, we assume that $\alpha > 0$. Action (3.4) with $\alpha = 1/2$ appeared originally in the Euclidean version in the context of the Bose-Fermi transmutation mechanism [50, 29]. Like the model of the particle with curvature (3.1), the higher derivative system (3.4) possesses the translation invariant dynamical spin degrees of freedom $J_\mu = -\alpha e_\mu$, $e_\mu = \dot{x}_\mu/\sqrt{-\dot{x}^2}$, as well as the three types of solutions to the classical equations of motion, with $P^2 < 0$, $P^2 = 0$ and $P^2 > 0$ [32]. At the quantum level operators J_μ satisfy the SO(2,1) commutation relations

$$[J_\mu, J_\mu] = -i\epsilon_{\mu\nu\lambda}J^\lambda, \tag{3.5}$$

analogous to those for the (2+1)D γ-matrices. Note that in (2+1)D, there is a duality relation $J_\mu = -\frac{1}{2}\epsilon_{\mu\nu\lambda}S^{\nu\lambda}$ between the (2+1)D vector J_μ and the spin tensor $S_{\mu\nu}$ satisfying the commutation relations of the form (2.3). The parameter α is not quantized here, and it fixes the value of the Casimir operator of the algebra (3.5), $J^2 = -\alpha(\alpha - 1)$ [32]. For the gauge $\tau = x^0$, in representation where the operator J_0 is diagonal, its eigenvalues are $j_0 = \alpha + n$, $n = 0, 1, \ldots$. This means that the spin degrees of freedom of the system realize a bounded from below unitary infinite-dimensional representation D^+_α of the universal covering group of the (2+1)D Lorentz group [72, 73]. The physical states of the system are given by the quantum analogue of the constraint responsible for the reparametrization invariance of the action (3.4) [32],

$$(PJ - \alpha m)\Psi = 0. \tag{3.6}$$

One can treat Eq. (3.6) as a (2+1)D analogue of the original Majorana equation (2.1). The difference of the (2+1)D from the (3+1)D case proceeds from the isomorphism between SO(2,2) and SO(2,1)$\oplus$SO(2,1) algebras, and here the SO(2,1) generators J_μ simultaneously play the role analogous to that played by the SO(3,2) generators Γ_μ satisfying the commutation relations (2.2). In the time-like sector, the solutions of Eq. (3.6) describe the positive energy states of the spin $s_n = \alpha + n$ lying on the Majorana type trajectory [32]

$$M_n = \frac{m}{1 + \alpha^{-1}n}, \qquad n = 0, 1, \ldots. \tag{3.7}$$

4. Fractional Spin Fields

The (2+1)D analogue of the Majorana equation (3.6) being supplied with the Klein-Gordon equation (2.8) describes the fields carrying irreducible representation of the Poincaré ISO(2,1) group of any, but fixed spin $s = \alpha > 0$ [32], and so, can serve as a basis for relativistic anyon theory [26]–[31]. Instead of these two equations, one can obtain the same result starting from the linear differential (2+1)D Majorana-Dirac wave equations suggested in [34][4]. In such a case it is supposed that besides the index n associated with the infinite-dimensional half-bounded unitary representation D^+_α, the infinite-component

[4] Jackiw and Nair [33] proposed an alternative theory based on the (2+1)D Majorana equation supplied with the equation for topologically massive vector gauge field.

field carries in addition a spinor index, and that it satisfies Eq. (3.6) as well as the Dirac equation

$$(P\gamma - m)\Psi = 0. \tag{4.1}$$

As a consequence of Eqs. (3.6), (4.1), the Majorana-Dirac field satisfies not only the Klein-Gordon equation, but also the equations

$$(J\gamma + \alpha)\Psi = 0, \qquad \epsilon_{\mu\nu\lambda} J^\mu \gamma^\nu P^\lambda \Psi = 0, \tag{4.2}$$

and one finds that it describes the positive energy states of the mass m and spin $s = \alpha - \frac{1}{2}$ [34].

The alternative way to describe an anyon field of the fixed mass and spin consists in the construction of the (2+1)D analogue of the Dirac spinor set of equations (2.7) generating the Majorana and Klein-Gordon equations in the form of integrability conditions. The construction needs the application of the so called deformed Heisenberg algebra with reflection intimately related to parabosons [74, 75],

$$[a^-, a^+] = 1 + \nu R, \qquad R^2 = 1, \qquad \{a^\pm, R\} = 0, \tag{4.3}$$

where ν is a real deformation parameter. Here operator $N = \frac{1}{2}\{a^+, a^-\} - \frac{1}{2}(\nu + 1)$ plays the role of a number operator, $[N, a^\pm] = \pm a^\pm$, allowing us to present a reflection operator R in terms of $a^\pm$: $R = (-1)^N = \cos \pi N$. For $\nu > -1$ algebra (4.3) admits infinite-dimensional unitary representations realized on a Fock space[5]. In terms of operators $a^\pm$ the SO(2,1) generators (3.5) are realized in a quadratic form,

$$J_0 = \frac{1}{4}\{a^+, a^-\}, \qquad J_\pm = J_1 \pm i J_2 = \frac{1}{2}(a^\pm)^2. \tag{4.4}$$

Here $J_\mu J^\mu = -s(s-1)$ with $s = \frac{1}{4}(1 \pm \nu)$ on the even/odd eigensubspaces of the reflection operator R, i.e. as in the (3+1)D case we have a direct sum of the two infinite-dimensional irreducible representations of the (2+1)D Lorentz group. These quadratic operators together with linear operators

$$L_1 = \frac{1}{\sqrt{2}}(a^+ + a^-), \qquad L_2 = \frac{i}{\sqrt{2}}(a^+ - a^-), \tag{4.5}$$

extend the SO(2,1) algebra into the OSP(1|2) superalgebra:

$$\{L_\alpha, L_\beta\} = 4i(J\gamma)_{\alpha\beta}, \qquad [J_\mu, L_\alpha] = \frac{1}{2}(\gamma_\mu L)_\alpha, \tag{4.6}$$

where the (2+1)D γ-matrices are taken in the Majorana representation, $(\gamma_0)_\alpha{}^\beta = (\sigma_2)_\alpha{}^\beta$, $(\gamma_1)_\alpha{}^\beta = i(\sigma_1)_\alpha{}^\beta$, $(\gamma_2)_\alpha{}^\beta = i(\sigma_3)_\alpha{}^\beta$, and $(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha{}^\rho \epsilon_{\rho\beta}$. With these ingredients, the (2+1)D analogue of the Dirac spinor set of wave equations (2.7) is [38]

$$\left((P\gamma)_\alpha{}^\beta + m\epsilon_\alpha{}^\beta\right) L_\beta \Psi = 0. \tag{4.7}$$

[5] For negative odd integer values $\nu = -(2k+1)$, $k = 1, 2 \ldots$, the algebra has finite, (2k+1)-dimensional nonunitary representations [74].

From these two ($\alpha = 1, 2$) equations the (2+1)D Majorana and Klein-Gordon equations appear in the form of integrability conditions.

The spinor set of equations (4.7) was used, in particular, for investigation of the Lorentz symmetry breaking in the (3+1)D massless theories with fractional helicity states [76].

5. Exotic Galilei Group and Noncommutative Plane

A special non-relativistic limit (c is a speed of light) [51, 52]

$$c \to \infty, \qquad s \to \infty, \qquad \frac{s}{c^2} = \kappa, \tag{5.1}$$

applied to the spinor set of equations (4.7) results in the infinite-component Dirac-Majorana-Lévy-Leblond type wave equations [24]

$$i\partial_t \phi_k + \sqrt{\frac{k+1}{2\theta}}\,\frac{P_+}{m}\,\phi_{k+1} = 0, \tag{5.2}$$

$$P_- \phi_k + \sqrt{\frac{2(k+1)}{\theta}}\,\phi_{k+1} = 0, \tag{5.3}$$

where $k = 0, 1, \ldots$, $P_\pm = P_1 \pm iP_2$, and

$$\theta = \frac{\kappa}{m^2}. \tag{5.4}$$

The first equation (5.2) defines the dynamics. The second equation relates different components of the field allowing us to present them in terms of the lowest component,

$$\phi_k = (-1)^k \left(\frac{\kappa}{2}\right)^{\frac{k}{2}} \left(\frac{P_-}{m}\right)^k \phi_0. \tag{5.5}$$

Though a simple substitution of the second equation into the first one shows that every component ϕ_k satisfies the Shrödinger equation of a free planar particle, the nontrivial nature of the system is encoded in its symmetry. The (2+1)D Poincaré symmetry of the original relativistic system in the limit (5.1) is transformed into the exotic planar Galilei symmetry characterized by the noncommutative boosts [39, 41],

$$[\mathcal{K}_1, \mathcal{K}_2] = -i\kappa. \tag{5.6}$$

The system of the two infinite-component equations (5.2), (5.3) can be presented in the equivalent form

$$i\partial_t \phi = H\phi, \qquad V_-\phi = 0, \tag{5.7}$$

with

$$H = P_i v_i - \frac{1}{2} m v_+ v_-\,, \qquad V_- = v_- - \frac{P_-}{m}\,. \tag{5.8}$$

The translation invariant operators $v_\pm = v_1 \pm i v_2$, $[v_i, v_j] = -i\kappa^{-1}\epsilon_{ij}$, is the non-relativistic limit (5.1) of the noncompact Lorentz generators, $-(c/s)J_\pm \to v_\pm$. The symmetry of the

quantum mechanical system (5.7) is given by the Hamiltonian H, the space translation generators P_i, and by the rotation and boost generators,

$$\mathcal{J} = \epsilon_{ij} x_i P_j + \frac{1}{2}\kappa v_+ v_-, \qquad \mathcal{K}_i = m x_i - t P_i + \kappa \epsilon_{ij} v_j. \tag{5.9}$$

These integrals generate the algebra of the two-fold centrally extended planar Galilei group [39, 41] characterized by the non-commutativity of the boosts (5.6).

The first equation from (5.7) is nothing else as a non-relativistic limit of the (2+1)D Majorana equation (3.6) [24]. The system described by it (without the second equation from (5.7)) corresponds to the classical system given by the higher derivative Lagrangian

$$L = \frac{1}{2} m \dot{x}_i{}^2 + \kappa \epsilon_{ij} \dot{x}_i \ddot{x}_j, \tag{5.10}$$

which, in its turn, corresponds to the non-relativistic limit (5.1) of the relativistic model of the particle with torsion (3.4) [23]. It is interesting to note that the system (5.10) (for the first time considered by Lukierski, Stichel and Zakrzewski [40], in ignorance of its relation to the relativistic higher derivative model (3.4)), reveals the same dynamics as a charged non-relativistic planar particle in external homogeneous magnetic and electric fields [77]. The spectrum of the Hamiltonian (5.8),

$$E_n(P) = \frac{1}{2m} P_i^2 - m\kappa^{-1} n, \qquad n = 0, 1, \ldots, \tag{5.11}$$

is not restricted from below, and the system (5.10), similarly to its relativistic analogue (3.4), describes a reducible representation of the exotic Galilei group. The role of the second equation from (5.7), whose component form is given by Eq. (5.3), consists in singling out the highest (at fixed P_i^2) energy state from (5.11) with $n = 0$, and fixing an irreducible infinite-dimensional unitary representation of the exotic planar Galilei group [24, 77]. The system being reduced to the surface given by this second equation (classically equivalent to the set of the two second class constraints $V_i = 0$, $i = 1, 2$) corresponds to the exotic planar particle considered by Duval and Horvathy [41, 79], which is described by the free particle Hamiltonian and an exotic symplectic two-form,

$$H = \frac{1}{2m} P_i^2\,, \qquad \omega = dP_i \wedge dx_i + \frac{1}{2}\theta \epsilon_{ij} dP_i \wedge dP_j. \tag{5.12}$$

The system (5.12) reveals a noncommutative structure encoded in the nontrivial commutation relations of the particle coordinates,

$$[x_i, x_j] = i\theta \epsilon_{ij}. \tag{5.13}$$

This noncommutative structure is the non-relativistic limit (5.1) [51] of the commutation relations

$$[x_\mu, x_\nu] = -is\epsilon_{\mu\nu\lambda} \frac{P^\lambda}{(-P^2)^{3/2}} \tag{5.14}$$

associated with the minimal canonical approach for relativistic anyon of spin s [78]. Note that as was observed by Schonfeld [55] (see also [80]), the commutation relations (5.14) are

dual to the (Euclidean) commutation relations for the mechanical momentum of a charged particle in the magnetic monopole field. The latter system also admits a description by the higher derivative Lagrangian [80],

$$L_{CM} = \frac{1}{2}m\dot{\vec{r}}^2 - eg\frac{|\vec{r}|}{(\vec{r}\times\dot{\vec{r}})^2}(\vec{r}\times\dot{\vec{r}})\cdot\ddot{\vec{r}}. \tag{5.15}$$

There is a close relationship between the charge-monopole non-relativistic system (5.15) and the model of relativistic particle with torsion (3.4). Indeed, in a parametrization $x_\mu = x_\mu(\tau)$, the torsion term from (3.4) takes the (Minkowski) form of the higher derivative charge-monopole coupling term, but in the *velocity* space with $v^\mu \equiv \dot{x}^\mu$,

$$L_{tor} = -\alpha\frac{\sqrt{-v^2}}{(\epsilon_{\gamma\rho\sigma}v^\rho\dot{v}^\sigma)^2}\epsilon_{\mu\nu\lambda}v^\mu\dot{v}^\nu\ddot{v}^\lambda. \tag{5.16}$$

For system (5.15) the relation $\vec{J}\vec{n} + eg = 0$ is the analogue of the (2+1)D Majorana equation (3.6), where $\vec{n} = \vec{r}/|\vec{r}|$ and $\vec{J}$ is the charge-monopole angular momentum.

The exotic planar particle described by the symplectic structure (5.12), or by the Dirac-Majorana-Lévy-Leblond type equations (5.2), (5.3), can be consistently coupled to an arbitrary external electromagnetic field at the *classical* level [41, 25]. However, at the quantum level the Hamiltonian reveals a nonlocal structure in the case of inhomogeneous magnetic field [25]. Another peculiarity reveals even in the case of homogeneous magnetic field corresponding to the Landau problem for a particle in a noncommutative plane [25, 54, 81], where the initial particle mass m is changed for the effective mass [41]

$$m^* = m(1 - eB\theta), \tag{5.17}$$

see below. As a result, the system develops three essentially different phases corresponding to the subcritical, $eB\theta < 1$, critical, $eB\theta = 1$, and overcritical, $eB\theta > 1$, values of the magnetic field [25, 54].

6. Gauge Theories with Chern-Simons Terms and Exotic Particle

In the case of the choice of finite-dimensional non-unitary representations of the deformed Heisenberg algebra with reflection (4.3) corresponding to the negative odd values of the deformation parameter, $\nu = -(2k+1)$, $k = 1, 2, \ldots$, the (2+1)D spinor set of equations (4.7) describes a spin-j field with $j = k/2$ and both signs of the energy [82, 83, 37]. In particular, in the simplest cases of $\nu = -3$ and $\nu = -5$, Eq. (4.7) gives rise, respectively, to the Dirac spin-1/2 particle theory and to the topologically massive electrodynamics [55, 56]. The latter system is described by the Lagrangian

$$L_{TME} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{m}{4}\epsilon^{\mu\nu\lambda}A_\mu F_{\nu\lambda}, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{6.1}$$

Let us suppress the dependence on the spatial coordinates x_i by making a substitution $A^\mu(x) \to \sqrt{m}\, r^\mu(t)$. Then (6.1) takes a form of the Lagrangian of a non-relativistic

charged particle in the homogeneous magnetic field $B = m^2 e^{-1}$, $L = \frac{1}{2} m \dot{r}_i^2 + \frac{1}{2} eB\epsilon_{ij} r_i \dot{r}_j$, while the variable r^0 disappears[6].

In Ref. [62], Deser and Jackiw proposed an extension of the topologically massive electrodynamics by adding to Lagrangian (6.1) the higher derivative term of the Chern-Simons form,

$$L_{DJ} = L_{TME} + L_{ECS}, \qquad L_{ECS} = \kappa m^{-1} \epsilon^{\mu\nu\lambda} F_{\mu\sigma} \partial^\sigma F_{\nu\lambda}, \tag{6.2}$$

where κ is a dimensionless numerical parameter. Making the same substitution as before, and changing $r_i \to x_i$, we reduce the (2+1)D field Lagrangian (6.2) to the mechanical Lagrangian for a particle in a plane,

$$L = \frac{1}{2} m \dot{x}_i^2 + \frac{1}{2} eB\epsilon_{ij} x_i \dot{x}_j + \kappa \epsilon_{ij} \dot{x}_i \ddot{x}_j, \tag{6.3}$$

that describes the higher derivative model (5.10) coupled to the external homogeneous magnetic field. The system (6.3), like the free higher derivative system (5.10) underlying the special non-relativistic limit (5.1) of the (2+1)D Majorana equation, has a spectrum unbounded from below. This drawback can be removed by supplying the coupled system with the appropriately modified constraint (5.3) [25]. Classically, this is equivalent to the change of the higher derivative Lagrangian (6.3) for the first order exotic Duval-Horvathy Lagrangian [41]

$$L_{ex} = P_i \dot{x}_i - \frac{1}{2m} P_i^2 + \frac{1}{2} \theta \epsilon_{ij} P_i \dot{P}_j + \frac{1}{2} eB\epsilon_{ij} x_i \dot{x}_j, \tag{6.4}$$

corresponding in a free case to the symplectic structure (5.12)[7]. It generates the equations of motion with the effective mass (5.17), $P_i = m^* \dot{x}_i$, $\dot{P}_i = eB\epsilon_{ij} \dot{x}_j$.

The interacting exotic particle system (6.4) can also be obtained by a reduction of another (2+1)-dimensional Abelian gauge field theory given by the Lagrangian with several Chern-Simons terms,

$$L_H = -\epsilon^{\mu\nu\lambda} \Phi_\mu \partial_\nu A_\lambda - \frac{1}{2} \lambda \Phi_\mu \Phi^\mu - \frac{1}{2} \kappa m^{-1} \epsilon^{\mu\nu\lambda} \Phi_\mu \partial_\nu \Phi_\lambda - \frac{1}{2} \beta m \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda, \tag{6.5}$$

where λ, κ and β are dimensionless parameters. The system with Lagrangian (6.5) was investigated by Hagen [59], see also [60]. Suppressing the dependence of the fields Φ_μ and A_μ on the spatial coordinates by making the substitutions $A^\mu(x) \to \sqrt{\lambda m}\, r^\mu(t)$ and $\Phi^\mu(x) \to \pi^\mu(t)/\sqrt{m\lambda}$ (we assume $\lambda > 0$), and denoting $\lambda\beta m^2 = eB$ and $\kappa/(\lambda m^2) = \theta$, we reduce (6.5) to the first order Lagrangian

$$L = \epsilon_{ij} \pi_i \dot{r}_j - \frac{1}{2m} \pi_i^2 + \frac{1}{2m} \pi_0^2 + \frac{1}{2} \theta \epsilon_{ij} \pi_i \dot{\pi}_j + \frac{1}{2} eB\epsilon_{ij} r_i \dot{r}_j.$$

[6] This corresponds to the nature of the A^0 field, which can be removed by imposing the Weyl gauge $A^0 = 0$.

[7] For the system (6.3), one can get rid of the unbounded from below spectrum by changing the sign in the first, kinetic term. In this case the problem reappears at $\kappa = 0$.

Hence, the π_0 plays the role of the auxiliary variable, and can be omitted using its equation of motion $\pi_0 = 0$[8]. Then, changing the notations $r_i \to x_i$ and $\pi_i \to \epsilon_{ij}P_j$, we arrive at the Lagrangian (6.4).

Therefore, the both systems (6.3) and (6.4), corresponding (in a free case) to the special non-relativistic limit (5.1) of the (2+1)D Majorana equation (3.6) and Dirac spinor set of equations (4.7), can be treated as reduced versions of the relativistic Lagrangians (6.2) and (6.5) of the (2+1)D Abelian gauge field theories with Chern-Simons terms.

7. Conclusion

To conclude, we point out two interesting open problems related to the Majorana equation.

It is known that the spin-statistics connection for the infinite-component fields described by the Majorana type equations is absent [5, 7, 8]. On the other hand, the question on such a connection for the fields of fixed mass and spin described by the Dirac covariant set of equations is open. The question on the spin-statistics relation for the fractional spin field theories constructed on the basis of the (2+1)D analogue of the Majorana equation also still waits for the solution.

As we saw, the original (3+1)D Majorana equation and the Dirac spinor set of equations constructed on its basis, as well as their (2+1)D analogues, have a hidden supersymmetric structure encoded in the Majorana spectrum (2.5). Hence, it would be very natural to try to construct a supersymmetric extension of these theories. Such an attempt was undertaken in Ref. [83] for the case of the (2+1)D analogue of the Dirac spinor set of equations. Within the framework of a restricted approach taken there, the supersymmetric extension was obtained only for a few special cases corresponding to finite-dimensional representations of the underlying deformed Heisenberg algebra with reflection (4.3)[9]. A supersymmetric extension could help to resolve the problem of the electromagnetic coupling, including the quantum case of the non-relativistic exotic particle in the noncommutative plane.

Acknowledgements. The work was supported in part by FONDECYT grant 1050001.

[8] Disappearance of π_0 (r_0) is rooted in the independence of Lagrangian (6.5) of the time derivative of Φ^0 (A^0).

[9] See also ref. [35] for the case of $\nu = 0$.

References

[1] E. Majorana, *Teoria relativistica di particelle con momento intrinseco arbitrario*, *Nuovo Cimento* **9** (1932) 335.

[2] I. M. Gel'fand and A. M. Yaglom, *Zh. Eksp. Teor. Fiz.* **18** (1948) 703, 1096, 1105.

[3] M. Fradkin, *Am. J. of Physics* **34** (1966) 314.

[4] C. Fronsdal, *Phys. Rev.* **156** (1967) 1665; *Phys. Rev.* **182** (1969) 1564; C. Fronsdal and R. White, *Phys. Rev.* **163** (1967) 1835.

[5] E. Abers, I. T. Grodsky and R. E. Norton, *Phys. Rev.* **159** (1967) 1222.

[6] Y. Nambu, *Phys. Rev.* **160** (1967) 1171.

[7] D. Tz. Stoyanov and I. T. Todorov, *J. Math. Phys.* **9** (1968) 2146.

[8] N. N. Bogolyubov, A. A. Logunov, A. I. Oksak and I. T. Todorov, *General Principles Of Quantum Field Theory* Dordrecht, Netherlands: Kluwer (1990).

[9] G. Veneziano, *Nuovo Cim. A* **57** (1968) 190.

[10] A. Neveu and J. H. Schwarz, *Nucl. Phys. B* **31** (1971) 86.

[11] P. Ramond, *Phys. Rev. D* **3**, 2415 (1971).

[12] S. Mandelstam, *Phys. Rept.* **13** (1974) 259.

[13] F. Gliozzi, J. Scherk and D. I. Olive, *Phys. Lett. B* **65** (1976) 282.

[14] M. B. Green and J. H. Schwarz, *Nucl. Phys. B* **181** (1981) 502.

[15] M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, Cambridge University Press, Cambridge, 1987.

[16] P. A. M. Dirac, *Proc. R. Soc. Lond. Ser. A* **322** (1971) 435, and ibid. **328** (1972) 1.

[17] L. P. Staunton, *Phys. Rev. D* **13** (1976) 3269; L. P. Staunton and S. Browne, *Phys. Rev. D* **12** (1975) 1026.

[18] L. C. Biedenharn and H. Van Dam, *Phys. Rev. D* **9** (1974) 471; **14** (1976) 405.

[19] A. O. Barut and I. H. Duru, *Phys. Rev. D* **10** (1974) 3448.

[20] L. P. Staunton, *Phys. Rev. D* **10** (1974) 1760.

[21] H. Leutwyler and J. Stern, *Annals Phys.* **112** (1978) 94.

[22] N. Mukunda, H. Van Dam and L. C. Biedenharn, *Phys. Rev. D* **22** (1980) 1938.

[23] P. A. Horváthy and M. S. Plyushchay, *JHEP* **0206** (2002) 033 [hep-th/0201228].

[24] P. A. Horvathy and M. S. Plyushchay, *Phys. Lett. B* **595** (2004) 547 [hep-th/0404137].

[25] P. A. Horvathy and M. S. Plyushchay, *Nucl. Phys. B* **714** (2005) 269 [hep-th/0502040].

[26] J. M. Leinaas and J. Myrheim, *Nuovo Cim. B* **37** (1977) 1.

[27] F. Wilczek and A. Zee, *Phys. Rev. Lett.* **51** (1983) 2250.

[28] Y. S. Wu and A. Zee, *Phys. Lett. B* **147** (1984) 325.

[29] R. MacKenzie, F. Wilczek, *Int. J. Mod. Phys. A* **3** (1988) 2827.

[30] D. V. Volkov, *JETP Lett.* **49** (1989) 541.

[31] M. S. Plyushchay, *Phys. Lett. B* **248** (1990) 107.

[32] M. S. Plyushchay, *Phys. Lett. B* **262** (1991) 71; *Nucl. Phys. B* **362** (1991) 54.

[33] R. Jackiw and V. P. Nair, *Phys. Rev. D* **43** (1991) 1933.

[34] M. S. Plyushchay, *Phys. Lett. B* **273** (1991) 250; *Int. J. Mod. Phys. A* **7** (1992) 7045.

[35] D. P. Sorokin and D. V. Volkov, *Nucl. Phys. B* **409** (1993) 547.

[36] Ch. Chou, V. P. Nair and A. P. Polychronakos, *Phys. Lett. B* **304** (1993) 105 [hep-th/9301037].

[37] J. L. Cortes and M. S. Plyushchay, *J. Math. Phys.* **35** (1994) 6049 [hep-th/9405193].

[38] M. S. Plyushchay, *Phys. Lett. B* **320** (1994) 91 [hep-th/9309148] .

[39] J.-M. Lévy-Leblond, *Galilei group and Galilean invariance*. In: *Group Theory and Applications* (Loebl Ed.), **II**, Acad. Press, New York, p. 222 (1972); A. Ballesteros, M. Gadella and M. del Olmo, *Journ. Math. Phys.* **33** (1992) 3379; Y. Brihaye, C. Gonera, S. Giller and P. Kosiński, hep-th/9503046.

[40] J. Lukierski, P. C. Stichel, W. J. Zakrzewski, *Annals of Physics (NY)* **260** (1997) 224 [hep-th/9612017].

[41] C. Duval and P. A. Horváthy, *Phys. Lett. B* **479** (2000) 284 [hep-th/0002233]; *Journ. Phys. A* **34** (2001) 10097 [hep-th/0106089]

[42] P. A. Horvathy, L. Martina and P. C. Stichel, *Phys. Lett. B* **564** (2003) 149 [arXiv:hep-th/0304215].

[43] N. Seiberg and E. Witten, JHEP **9909** (1999) 032 [hep-th/9908142].

[44] D. Bigatti and L. Susskind, *Phys. Rev. D* **62** (2000) 066004 [hep-th/9908056].

[45] M. R. Douglas and N. A. Nekrasov, *Rev. Mod. Phys.* **73** (2001) 977 [hep-th/0106048].

[46] A. Connes, M. R. Douglas and A. S. Schwarz, *JHEP* **9802** (1998) 003 [hep-th/9711162].

[47] F. A. Berezin and M. S. Marinov, *Annals Phys.* **104** (1977) 336.

[48] M. S. Plyushchay, *Mod. Phys. Lett. A* **3** (1988) 1299; *Int. J. Mod. Phys. A* **4** (1989) 3851.

[49] A. M. Polyakov, *Nucl. Phys. B* **268** (1986) 406.

[50] A. M. Polyakov, *Mod. Phys. Lett. A* **3** (1988) 325.

[51] R. Jackiw and V. P. Nair, *Phys. Lett. B* **480** (2000) 237 [hep-th/0003130].

[52] C. Duval and P. A. Horvathy, Phys. Lett. B **547** (2002) 306 [arXiv:hep-th/0209166].

[53] P. A. Horvathy, *Annals Phys.* **299** (2002) 128 [hep-th/0201007].

[54] M. S. Plyushchay, hep-th/0603034.

[55] J. F. Schonfeld, *Nucl. Phys. B* **185** (1981) 157.

[56] S. Deser, R. Jackiw and S. Templeton, *Phys. Rev. Lett.* **48** (1982) 975; *Annals Phys.* **140** (1982) 372.

[57] S. Deser and R. Jackiw, *Phys. Lett. B* **139** (1984) 371.

[58] A. M. Din and W. J. Zakrzewski, *Phys. Lett. B* **146** (1984) 341.

[59] C. R. Hagen, *Phys. Rev. Lett.* **58** (1987) 1074.

[60] S. Deser and R. Jackiw, *Phys. Rev. Lett.* **59** (1987) 1981.

[61] E. Witten, *Commun. Math. Phys.* **121** (1989) 351.

[62] S. Deser and R. Jackiw, *Phys. Lett. B* **451** (1999) 73 [hep-th/9901125].

[63] R. D. Pisarski, *Phys. Rev. D* **34** (1986) 670.

[64] V. V. Nesterenko, *J. Phys. A* **22** (1989) 1673.

[65] Y. A. Kuznetsov and M. S. Plyushchay, *Nucl. Phys. B* **389** (1993) 181.

[66] M. S. Plyushchay, *Mod. Phys. Lett. A* **4** (1989) 837; *Phys. Lett. B* **243** (1990) 383.

[67] E. Ramos and J. Roca, *Nucl. Phys. B* **436** (1995) 529 [hep-th/9408019]; **452** (1995) 705 [hep-th/9504071].

[68] J. Mourad, hep-th/0410009.

[69] L. Edgren, R. Marnelius and P. Salomonson, *JHEP* **0505** (2005) 002 [hep-th/0503136]; L. Edgren and R. Marnelius, hep-th/0602088.

[70] C. Fronsdal, *Phys. Rev. D* **18** (1978) 3624.

[71] M. A. Vasiliev, *Phys. Lett. B* **567** (2003) 139 [hep-th/0304049]; *Comptes Rendus Physique* **5** (2004) 1101 [hep-th/0409260].

[72] V. Bargmann, *Annals Math.* **48** (1947) 568.

[73] M. S. Plyushchay, *J. Math. Phys.* **34** (1993) 3954.

[74] M. S. Plyushchay, *Nucl. Phys. B* **491** [PM] (1997) 619 [hep-th/9701091];

[75] M. Plyushchay, *Int. J. Mod. Phys. A* **15** (2000) 3679 [hep-th/9903130].

[76] S. M. Klishevich, M. S. Plyushchay and M. Rausch de Traubenberg, *Nucl. Phys. B* **616** (2001) 419 [hep-th/0101190].

[77] M. A. del Olmo and M. S. Plyushchay, *Ann. Phys. (NY)* (2006), in press [hep-th/0508020].

[78] J. L. Cortes and M. S. Plyushchay, *Int. J. Mod. Phys. A* **11** (1996) 3331 [hep-th/9505117].

[79] P. A. Horvathy, hep-th/0602133.

[80] M. S. Plyushchay, *Nucl. Phys. B* **589** (2000) 413 [hep-th/0004032].

[81] S. Bellucci, A. Nersessian and C. Sochichiu, *Phys. Lett. B* **522** (2001) 345 [hep-th/0106138].

[82] M. S. Plyushchay, *Ann. Phys. (NY)* **245** (1996) 339 [hep-th/9601116].

[83] M. S. Plyushchay, *Mod. Phys. Lett. A* **12** (1997) 1153 [hep-th/9705034].

EJTP **3, No. 10** (2006) 33–38 *Electronic Journal of Theoretical Physics*

Wave Equations, Renormalization and Meaning of the Planck's Mass: Some Qualitative Considerations

Leonardo Chiatti *

Laboratorio di Fisica Medica ASL VT,
Via S. Lorenzo 101,
01100 Viterbo , Italy

Received 1 March 2006, Published 28 May 2006

Abstract: The five-dimensional version of the quantum relativistic Klein-Gordon wave equation is assumed to be a more fundamental description for the dynamics of the single particle without spin. The meaning of the renormalization procedure in QFT and the Planck's mass one are briefly discussed from this point of view.

Keywords: relativistic wave equations, renormalization, Planck mass
PACS (2006): 03.65.Pm; 11.10.Gh; 04.20.Cv

1. Introduction

The formulation of relativistically covariant wave equations was one of the first decisive steps towards the (not yet reached) unification of quantum mechanics and relativity. Maiorana has given various important, and may be not completely understood to date, contributes to this subject (refs. 1, 2, 3, 4). Therefore, it seems right to speak about the wave equations in a publication dedicated to his memory. We will express some elementary qualitative considerations on the relationship between this classical subject and some more recent questions of the modern quantum field theory (QFT) and the quantum gravity.

* fisica.san@asl.vt.it

2. The Klein-Gordon Equation

The prototype of the quantum-relativistic wave equations consists of well known four-dimensional Klein-Gordon equation (KG4):

$$\left[\hbar^2\Box^2 + m^2c^2\right]\psi(x_0, x_1, x_2, x_3) = 0 \tag{1}$$

where m is the particle mass. We will limit our line of reasoning to this particular wave equation, since it will be immediately applicable to any other similar equation.

The KG4 is a quantum translation of the ordinary dispersion law for free relativistic particles and then it is valid for this kind of particles. It may be derived from the general pentadimensional Klein-Gordon equation (KG5) :

$$\left[\hbar^2\Box^2 - (\hbar/c)^2\partial_\tau^2\right]\varphi(x_0, x_1, x_2, x_3, \tau) = 0 \tag{2}$$

if

$$\varphi(x_0, x_1, x_2, x_3, \tau) = \psi(x_0, x_1, x_2, x_3)e^{\frac{-imc^2\tau}{\hbar}} \tag{3}$$

in which τ is a scalar of Universe. In general, the KG5 does not satisfy the usual dispersion law for the free relativistic particle and is, therefore, applicable to real interacting particles or to virtual particles.

If one defines the mass operator as $\mathrm{i}(\hbar/\mathrm{c}^2)\partial_\tau$, the wavefunctions represented by the eq.(3) are eigenfunctions of this operator with eigenvalue m. These eigenfunctions satisfy the eq.(1) and are associated with free real particles having a definite mass.

A generic linear superposition:

$$\varphi(x_0, x_1, x_2, x_3, \tau) = \sum_j \psi_{m_j}(x_0, x_1, x_2, x_3)e^{\frac{-im_jc^2\tau}{\hbar}} \tag{4}$$

will be associated with a particle having a non definite mass (real interacting particle or virtual particle). The Fourier analysis of the (4) gives the following relationship between the dispersion of the mass, Δm and the dispersion of the τ variable, $\Delta\tau$:

$$(\Delta mc^2)(\Delta\tau) \approx \hbar \tag{5}$$

In the case of the real interacting particles, considered only within the limits of the interaction area, the dispersion $\Delta\mathrm{mc}^2$ is of the same order as the interaction energy E_{int}, thus one can write $\mathrm{E}_{int}(\Delta\tau) \approx \hbar$. Therefore, if we assume the KG5 to be the fundamental equation, the KG4 is obtained as approximation in the limit $(\Delta\mathrm{m})/\mathrm{m} << 1$, that is $\mathrm{E}_{int} << \mathrm{mc}^2$. In other terms, the KG4 is an approximation which is no more valid when the energy exchanged with other fields or particles during the interaction is sufficiently high to create one or more copies of the particle under consideration.

The parameter τ must not be considered as a fifth spacetime coordinate; it admits an immediate physical interpretation in the ordinary four-dimensional spacetime. In fact, if one assumes the existence of a set of paths $\{\mathrm{x}_\mu = \mathrm{x}_\mu(\tau)\ ;\ \mu = 0, 1, 2, 3\}$ in the ordinary spacetime, such as $\mathrm{dx}_\mu/\mathrm{cd}\tau = \gamma_\mu$ (so called Breit equation), the KG5 takes the form:

$$(iD_\tau)^2\varphi(x_0, x_1, x_2, x_3, \tau) = 0 \tag{6}$$

which is a constraint on the variation of φ along these paths. This is if the quadrivelocities (Dirac matrices) $\gamma_\mu$don't depend explicitly on τ. D is the operator of total derivation.

Therefore, these paths consist in series of infinitesimal "jumps" at light speed along the spacetime, a sort of generalization of the Zitterbewegung.

The paths orientation along the t axis is not, in general, definite. In other terms, both the past and the future lightcones having their origin in an arbitrarily chosen point on a certain path, do not always contain portions of that path. But, if the eq. (3) is valid:

$$\frac{d\tau}{dt} = [\tau, H] + \partial_t \tau = \partial_t \tau = \left(\frac{dx_0}{cd\tau}\right)^{-1} = \gamma_0 \tag{7}$$

since in this case H is the ordinary Klein-Gordon Hamiltonian, which not depend on τ. Consequently, $t = \gamma_0 \tau + t_0$, that is, the paths have a fixed time orientation given by the sign of the eigenvalue of γ_0. So we arrive to the conclusion: in respect of the interactions localizing the particle with an uncertainty greater than $\hbar$/mc (that is for interaction of energies $E_{int} << mc^2$), the particle is a time orientated process, its dynamics being described by the KG4. Otherwise it is not a t-orientated process (creations and annihilations of the particle take place, which diffuse it along the spatial radius $\hbar$/mc) and its dynamics is plausibly described by the KG5.

The equations (1), (2) may be immediately generalized to a curve spacetime, by expressing the Dalembertian operator in the appropriate coordinates. If one admits that the gravitation is described by the spacetime curvature, as it is, for example, in the general relativity, one has immediately a description of the gravitational field effect on the particle. This effect is manifested even for m = 0, as in any metric theory of gravitation. Nevertheless, in the terms of the KG5 the gravitation is merely the dependence of the quadrivelocities γ_μ on x_μ.

3. Renormalization: A heuristic Justification

From the above reasoning follows that during the high energy interactions ($E_{int} >> mc^2$) the particle dynamics would be described by using the KG5 (for example, by adding the proper terms describing the interaction with the external field). The formulation of the modern quantum field theory (QFT) starts with a translation of the KG4 into the second quantization formalism. This option enables to assume a definite time orientation for all the interacting particles; after all, the time coordinate used by the observer in order to coordinate the events is t, not τ. This results in necessity to renormalize the calculated physical observables[1]. In fact, at each instant t_0, the particle now appears to be dispersed in the cloud of its various locations at that time t_0 [the values of $x_i(\tau)$, with i = 1, 2, 3, corresponding to the various values of τ for which one has $t(\tau) = t_0$], extended upon a space region of size $\hbar$/mc. Naturally, the charge, the mass and the total quantum

[1] For a discussion about the possibility to treat the problem involving several interacting particles by using a formalism of first quantization, remaining at the ontological level of the KG5, one may see ref. 5.

numbers of the cloud and the single particle ones are the same. In fact, any time when the particle passes through the hyperplane t = constant in the opposite directions, the contributions to these observables elide each other.

An external interaction distorts the "form" of this cloud. Since the only physical reality consists of interaction events with the outside, the physically relevant values must be calculated by subtracting the "unperturbed" cloud from the "distorted" cloud one. Any renormalization procedure is based on this principle. So, the renormalization is far from being an "arbitrary" procedure or one introduced "ad hoc" only in order to arrive to physically defined results, but it is a necessity resulting from the option to have a t-oriented theory. In the terms of the QFT, it consists of subtracting of the unmodified terms, which express the particle "self interaction" by through its own fields, from those modified by the external interaction.

4. Limits of Applicability and Planck's Mass

The particular expression of the limit $E_{int} \approx mc^2$, in which the KG5 is substituted by the KG4, certainly depends on the field that mediates the interaction. For the electrostatic field, this limit is reached when the distance r between two particles of charge e goes below the value r_0 at which the potential energy is equal to the rest energy : $e^2/r_0 = mc^2$. If the interaction is mediated by a photon exchange, the limit is reached when the photon wavelength λ goes below the value λ_0 at which the photon energy is equal to the rest energy: $\hbar c/\lambda_0 = mc^2$, where $\lambda\!\!\!^{-}_0 = \lambda_0/2\pi$. Then, in the case of electromagnetic interaction we have two different values of the collision parameter at which one passes from the KG5 description to the KG4 one: $r_0 = e^2/mc^2$ (the so called classical radius) for the static interaction, $\lambda\!\!\!^{-}_0 = \hbar/mc$ (the so called Compton wavelength) for the radiative interaction. Their ratio is a fundamental constant of the Universe : the fine structure constant α, and then it can't be changed. Since α is much less than 1, the KG4 is not more valid, in general, at distances comparable to $\lambda\!\!\!^{-}_0$. In order to describe the dynamics at shorter distances, it is necessary to use the KG5 or, according to the consolidated practice, to pass to QFT formalism by using the second quantization.

Now let's consider the gravitational interaction. One has now the static limit for $Gm^2/r_0 = mc^2$, that is for $r_0 = Gm/c^2$. In the case of graviton exchange the limit is still expressed by $\hbar c/\lambda_0 = mc^2$, that is $\lambda\!\!\!^{-}_0 = \hbar/mc$, since gravitons and photons manifest the same dispersion laws.

The ratio of r_0 and $\lambda\!\!\!^{-}_0$ is now $Gm^2/\hbar c$ and it depends on m; one has $r_0 = \lambda\!\!\!^{-}_0$ for $m = M_{Planck} = (\hbar c/G)^{1/2}$. For the mass values less than Planck's mass is $r_0 < \lambda\!\!\!^{-}_0$, while for masses greater than Planck's mass is $r_0 > \lambda\!\!\!^{-}_0$. In general, $r_0/\lambda\!\!\!^{-}_0 = (m/\ M_{Planck})^2$.

From this point of view, the Planck's mass is a parameter which controls the passage between two different modes of violation of the KG4. In the terms of this reasoning, the fundamental description of the dynamics is given by the KG5, which is a relativistically invariant, continuous equation. Therefore, do not seem to emerge, for $E_{int} > M_{Planck}c^2$, transitions to finite or discrete geometries with the appearance of quantized spacetime

intervals or breaking of the relativistic invariance.

References

[1] E. Maiorana; Nuovo Cim. (9) , 335 (1932).

[2] E. Maiorana; Nuovo Cim. (14), 171 (1937).

[3] M. Baldo, R.Mignani, E. Recami; Lett. Nuovo Cim. (11), 568 (1974).

[4] D. Fradkin; Am. Journ. Phys. (34), 314 (1966).

[5] L. Chiatti; Le strutture archetipali del mondo fisico (*in italian*); Di Renzo, Roma 2005.

EJTP **3, No. 10** (2006) 39–88 *Electronic Journal of Theoretical Physics*

Nonlinear Field Equations and Solitons as Particles

Attilio Maccari *

Via Alfredo Casella 3,
00013 Mentana (Rome)-Italy

Received 17 March 2006 , Published 28 May 2006

Abstract: Profound advances have recently interested nonlinear field theories and their exact or approximate solutions. We review the last results and point out some important unresolved questions. It is well known that quantum field theories are based upon Fourier series and the identification of plane waves with free particles. On the contrary, nonlinear field theories admit the existence of coherent solutions (dromions, solitons and so on). Moreover, one can construct lower dimensional chaotic patterns, periodic-chaotic patterns, chaotic soliton and dromion patterns. In a similar way, fractal dromion and lump patterns as well as stochastic fractal excitations can appear in the solution. We discuss in some detail a nonlinear Dirac field and a spontaneous symmetry breaking model that are reduced by means of the asymptotic perturbation method to a system of nonlinear evolution equations integrable via an appropriate change of variables. Their coherent, chaotic and fractal solutions are examined in some detail. Finally, we consider the possible identification of some types of coherent solutions with extended particles along the de Broglie-Bohm theory. However, the last findings suggest an inadequacy of the particle concept that appears only as a particular case of nonlinear field theories excitations.

Keywords: Chaos, Fractal, Dromion, Soliton, Nonlinear Field Theories.
PACS (2006): 05.45.Gg, 05.45.Df, 03.75.Lm, 05.45.Yv, 09.05.Fg, 11.10.-z, 11.10.Lm

1. Introduction

Solitons and other coherent solutions of nonlinear partial differential equations (NPDEs) have been extensively studied and their importance have been recognized in quite different areas of natural sciences and especially in almost all fields of physics such as plasma physics, astrophysics, nonlinear optics, particle physics, fluid mechanics and solid state physics. Solitons have been observed with spatial scales from $10^{-9}m$ to $10^{9}m$, if we consider density waves in the spiral galaxies, the giant Red Spot in the atmosphere of

* solitone@yahoo.it

Jupiter, the various types of plasma waves, superfluid helium, shallow water waves, structural phase transitions, liquid crystals, laser pulses, acoustics, high temperature superconductors, molecular systems, nervous pulses, population dynamics, Einstein cosmological equations, elementary particles structure and so on ([4],[11],[15], [20],[28],[25],[30],[70]).

In particular, solitons of NPDEs in 1+1 dimensions (one spatial plus one temporal dimension) possess the following properties:

(1) they are spatially localized;
(2) they maintain their localization during the time, i.e. they are waves of permanent form;
(3) when a single soliton collides with another one, both of them retain their identities and velocities after collision.

Usually mathematicians call solitons only the solutions that satisfy all the three above mentioned properties (as we will see the third property is connected with the integrability of the NPDEs) and call solitary waves solutions that satisfy only the first two properties. However, in many physics papers, the concept of soliton has been applied in a more extensive way, even if conditions ii)-iii) are not satisfied, because in the real world this concept is so useful and fruitful that one cannot afford to consider only the perfect mathematical world of soliton equations and not to use it.

For many years these solutions have been thought impossible, because a dispersive and nonlinear medium was expected to alter the wave shape over time. The first soliton observation has been given by John Scott Russell (1834) that found a solitary wave in a water channel. In 1895 J. Korteweg and G. de Vries demonstrated that for shallow water waves in a straight channel one is effectively left with a 1+1 dimensional problem and derived the appropriate nonlinear equation for the Russell soliton [27].

Subsequently, solitons have been found in many other nonlinear equations (called S-integrable equations) integrable by the inverse scattering transformation (IST) or spectral transform [1].

On the other hand, equations integrable by an appropriate transformation on the dependent/independent variables that convert them into a linear equation (integrable by the Fourier method) have been called C-integrable equations.

In the last years it has been shown that S-integrable equations are only a limited sector of nonlinear equations with solitonic solutions, because it has been demonstrated that nonintegrable equations (for example the double-sine Gordon equation and the Hasegawa-Mima system [65] and C-integrable systems have soliton solutions that satisfy conditions i-iii).

Nontrivial solutions of nonlinear equations can be found with many different methods and the inverse scattering method (S-integrable equations) has not a prominent role and it would be not useful to limit the soliton concept to a particular integration technique. Besides, there is no agreement about the concept of integrability for a nonlinear partial differential equation.

In Sect. 2 we briefly review the most important NPDEs in 1+1 dimensions and their coherent solutions, while in Sect. 3 we review the IST technique for obtaining interesting

exact solutions of a nonlinear equation.

In Sect. 4 we consider S-integrable equations in 2+1 dimensions and in particular the Davey-Stewartson and Kadomtsev-Petviashvili equations. They exhibit soliton solutions that are now spatially localized in all the directions except one. However, in many nonlinear NPDEs in 2+1 dimensions, different type of coherent solutions (dromions, ring solitons as well as istantons, and breathers) are found. In particular, dromions are solutions exponentially localized in all directions, which propagate with constant velocity and are usually driven by straight line solitons, for example in the so called DS-I equation.

If we now consider the physics applications of the above treated concepts, we must begin from the fact that two different procedures can be applied in order to find physically relevant NPDEs and then determine their solutions: exact solutions of approximate model equations or approximate solutions of exact equations.

In the first case using appropriate reduction methods (and in particular the asymptotic reduction (AP) method, a very general reduction method that can be also applied to construct approximate solutions for weakly nonlinear ordinary differential equations [Mac1, Mac8, Mac12]) and introducing approximations directly into the equations describing the system under study some model nonlinear equations are obtained and their exact solutions are investigated. Usually the approximations concern with the temporal and/or spatial scale of the solutions with respect to some physical parameters. The most important nonlinear model equations are the above-mentioned S-integrable equations and this fact is not obviously a coincidence. Indeed, as it has been known for some time, very large classes of NPDEs, with a dispersive linear part, can be reduced, by a limiting procedure involving the wave modulation induced by weak nonlinear effects, to a very limited number of "universal" nonlinear evolution PDEs. These model equations appear in many applicative fields because this reduction technique is able to take into account weakly nonlinear effects. The model equations are integrable, since it is sufficient that the very large class from which they are obtainable contain just one integrable equation, because it is clear from this method that the property of integrability is inherited through the limiting technique. Using an appropriate reduction method provides a powerful tool to understand the integrability of known equations and to derive new integrable equations likely to be relevant in applicative contexts. By the AP method many new nonlinear S-integrable equations have been identified for the first time ([40],[41],[43],[45],[49],[51],[53],[54],[56]).

In Sect.5 we illustrate a powerful method for nonlinear model equations, the multilinear variable separation technique, that can be used in order to obtain solitons and other coherent solutions but also chaotic solutions, with their sensitive dependence on the initial conditions, and fractals, with their self-similar structures. Since lower dimensional arbitrary functions are present in the exact solutions of some two dimensional integrable models, we can use lower dimensional chaotic and/or fractal solutions in order to obtain solutions of higher dimensional integrable models.

However, a second procedure can be used in order to find coherent solutions for NPDEs (Sec. 6): approximate solutions for correct equations, The AP method can be applied directly to the original equations, which describe a given physical system. In this

way, no model equation of the above mentioned type is obtained, because approximate solutions are directly sought. It has been demonstrated that solitons and dromions exist as approximate solutions in the particular case of ion acoustic waves in a unmagnetized or magnetized plasma, electron waves and non-resonant interacting water waves in 2+1 dimensions [Mac4, Mac6, Mac14, Mac17]. In particular, we examine a nonlinear Dirac field and demonstrate that each dromion propagates with its own group velocity and during a collision maintains its shape, because a phase shift is the only change. They are solutions of a C-integrable nonlinear partial differential system of equations describing N-interacting waves ($N¿1$) for modulated amplitudes Ψ_j, $j=1,...N$. The AP method can be applied to soliton and/or dromion propagation in nonlinear dispersive media without the complexity of the IST technique. Moreover, in 3+1 dimensions there are no known examples of S-integrable equations while the AP method is easily applicable.

In Sec. 7, we illustrate another example of the use of the AP method and consider a spontaneous symmetry breaking model and in Sec. 8 demonstrate also in this case the existence of dromions which preserve their shape during collisions, the only change being a phase shift. Moreover, other coherent solutions (line solitons, multilumps, ring solitons, instantons and breathers) are derived.

In Sec. 9 we show the existence of lower dimensional chaotic patterns such as chaotic-chaotic and periodic-chaotic patterns as well as chaotic soliton and dromion patterns. At last, we derive fractal dromion and lump solutions as well as stochastic fractal solutions.

In the conclusion we examine some important questions. From the results exposed in the previous sections we see that quantum mechanics can be considered as a first order approximation of a nonlinear theory. Moreover, dromions would correspond to extended elementary particles, in such a way to perform the de Broglie-Bohm theories, however the various type of coherent solutions suggest that elementary particles are only a particular case of nonlinear excitations.

2. Solitons and Nonlinear Equations in 1+1 Dimensions

a) John Scott Russell

The first soliton observation was performed by John Scott Russell (1808-1882) in 1834 in a canal near Edinburgh [14]. A small boat in the channel suddenly stopped and a lump of bell-shaped water formed at the front of the boat and moved forward with approximately constant speed and shape for about two miles. He called it the Great Solitary Wave and with the aid of subsequent experiments derived an empirical law for its speed

$$V = \sqrt{g(h + A)}, \tag{2.1}$$

where g is the gravity acceleration, A the wave amplitude and h the channel profundity.

b)The Korteweg-de Vries (KdV equation)

For linear equations (valid in the limit of small amplitude solutions), solitons are not possible due to the superposition principle (the sum of two solutions is yet a solution) and to the dispersion (waves with different wavelength have different velocities).

If we consider a solution formed by the sum of two waves with different wavelength that are initially superposed, then after some time they will separate. In the dispersive linear equations localized solutions cannot exist.

On the contrary, in nonlinear media (for example shallow water or plasma) the wave packet spreadness can be exactly balanced with the nonlinear terms of the equation in such a way to originate solitons and other coherent solutions. In particular, for solitons in shallow water the appropriate nonlinear equation was found by Korteweg and de Vries in 1895 [27]

$$U_t + U_{xxx} - 6UU_x = 0, \tag{2.2}$$

where $U = U(x,t)$ stand for the wave amplitude, x represents the propagation direction and t the time. The soliton solution (bell shaped and localized in space) is

$$U(x,t) = \frac{-2A^2}{\cosh^2[A(x - x_0 - 4A^2t)]}, \tag{2.3}$$

where A is a positive constant. Note that "slim" solitons are "tall" and run faster. The relation among velocity, width and amplitude is a characteristic property of solitons, while on the contrary for traveling waves of linear equations all the three quantities are usually independent of each other.

We note that if in (2.2) the nonlinear term is absent, then localized solutions would be impossible due to the dispersion. On the other hand, if the second term is absent, solution can develop a singularity in a finite time.

c) The Fermi-Pasta-Ulam experiment

In the first half of the XX century, the KdV equation was substantially forgotten but suddenly it emerged in the statistical physics and then in plasma studies and in all the phenomena with weak nonlinearities and dispersion (ion and electron waves in magnetized or unmagnetized plasma, phonon packets in nonlinear crystals).

For example, we consider an oscillator chain coupled with nonlinear forces. It is well known that the motion equations can be decoupled if we consider the normal modes that are characterized by different frequencies and evolve independently with each other. The motion is multiply periodic and the mode energy is constant over the time.

If then we introduce nonlinear forces among the oscillators, dramatic changes would appear because we expect an energy transfer among the various frequencies (stochastic behavior), as it is forecasted by the ergodic hypothesis and the energy equipartition.

With the beginning of the computer age, Fermi, Pasta and Ulam wanted to verify this prediction and numerically integrated the nonlinear equations for the oscillators. With their great surprise, they found that the prediction was wrong, because the energy concentrated on a determinate mode over the time ('recurrence'). Starting with only one oscillator excited the energy distributed itself over the modes, but returned almost completely in the first excited one. Thermodynamic equilibrium was not reached and the excitation was stable.

d) The discovery of Zabusky and Kruskal

To elucidate this behavior, we must consider the KdV equation, that is the continuous approximation of the oscillators chain. For a linear chain of atoms with a quadratic

interaction, the motion equation is

$$m\ddot{y}_i = k\left(y_{i+1} - 2y_i + y_{i-1}\right) + k\alpha\left[\left(y_{i+1} - y_i\right)^2 - \left(y_i - y_{i-1}\right)^2\right], \tag{2.4}$$

where $y_i = y_i(t)$, $i = 1, ..N$, N is the total number of atoms and moreover we assume that $y_{N+1} = y_1$. By means of the fourth order Taylor expansion in a, where a is the lattice constant, the motion equation becomes

$$y_{t't'} = y_{x'x'} + \varepsilon y_{x'}y_{x'x'} + \beta y_{x'x'x'x'} + O(\varepsilon a^2, a^4), \tag{2.5}$$

where $\varepsilon = 2a\alpha$, $\beta = a^2/12$, $t' = \omega t$, $\omega = \sqrt{k/m}$, $x' = x/a$ and $x = ia$.

With the variable change $T = \varepsilon t/2$, $X = x - t$, equation (2.5) yields

$$\varepsilon\left(V_{TX} + V_X V_{XX}\right) + \beta V_{XXXX} = 0, \tag{2.6}$$

where $y(x', t') = V(X, T)$. Taking $U = V_X$, one arrives to the KdV equation (2.2).

In 1965 Norman Zabusky and Martin Kruskal [86] numerically studied the KdV equation and found elastic collisions among localized solutions (that they called solitons) that preserve their identities. Note however that an analytic (and then not numeric) expression for the elastic collision in the sine-Gordon equation (see subsect. *f*) was known from 1962 [69], but it was been ignored.

After the Zabusky-Kruskal's discovery there was an explosion of papers about nonlinear waves. It was demonstrated that the KdV equation is integrable by the IST technique (see Sect. 3) and in the subsequent years many other applicative (and integrable) equations were found.

e) The nonlinear Schrödinger equation

The most important nonlinear equation is perhaps the nonlinear Schrödinger (NLS) equation, that takes into account the slow modulation of a monochromatic plane wave with weak amplitude, in a strongly dispersive and weakly nonlinear medium:

$$i\Psi_t + \Psi_{xx} + s\left|\Psi\right|^2\Psi = 0, \tag{2.7}$$

where $\Psi(x, t)$ is a complex function and $s = \pm 1$.

The soliton solution exists only for $s = 1$ and is given by

$$\Psi(x,t) = \Psi_0 \mathrm{sech}\left[\Psi_0\frac{(x - at)}{\sqrt{2}}\right]\exp\left[i\left(\frac{a(x - bt)}{2}\right)\right], \tag{2.8}$$

where a and b are arbitrary constants, the envelope and phase velocity, respectively. The NLS equation is integrable by the IST method and has been applied in many fields (deep water, self-focusing of laser in dieletrics, optical fibers, vortices in fluid flow, etc.).

f) The sine-Gordon equation

The sine-Gordon (sG) equation,

$$U_{xx} - U_{tt} = sinU, \tag{2.9}$$

where $U = U(x,t)$ is a real function, was studied for the first time by Bianchi, Backlund and Darboux, because it describes pseudospherical surfaces with constant negative gaussian curvatiure. It was probably known to Gauss, being the reduction of the fundamental equation of differential geometry.

There are three types of coherent solutions:

$$i) \quad kink, \qquad U = 4\text{arc}\tan\left\{\exp\left[\frac{(x-vt-x_0)}{\sqrt{1-v^2}}\right]\right\} \tag{2.10}$$

$$ii) \quad antikink, \qquad U = 4\text{arc}\tan\left\{\exp\left[-\frac{(x-vt-x_0)}{\sqrt{1-v^2}}\right]\right\} \tag{2.11}$$

iii) breather (it is not a traveling wave, but a bound state formed by a kink-antikink couple)

$$U = 4\text{arc}\tan\left\{(\tan a)\sin\left[(\cos a)(t-t_0)\right]\text{sech}\left[(\sin a)(x-x_0)\right]\right\}, \tag{2.12}$$

being v (¡1) and a arbitrary constants.

The collision of a kink-antikink couple is described by

$$U = 4\text{arc}\tan\left\{v\frac{\sinh\left[\frac{x}{\sqrt{1-v^2}}\right]}{\cosh\left[\frac{vt}{\sqrt{1-v^2}}\right]}\right\}, \tag{2.13}$$

that is not also in this case a traveling wave.

We now expose a simple method for obtaining soliton solutions of the SG equation that is valid also for other nonlinear equations. We take a traveling wave with velocity v as solution of (2.9),

$$U = U(x-vt) = U(T). \tag{2.14}$$

Substituting in (2.9), we obtain

$$(1-v^2)U_{TT} = \sin U = -\frac{\partial V}{\partial U} = -\frac{\partial}{\partial U}\left(1+\cos U\right). \tag{2.15}$$

For v¡1, the equation (2.15) describes the motion of a particle, with mass $m = 1-v^2$, in a periodic potential. The kink solution corrisponds to a solution that passes from a maximum of the potential to the other in an infinite time (the antikink solution moves in the opposite direction). In the corresponding phase space, the soliton is constituted by the separatrices. There are also multisolitonic solutions characterized by a passage through varios maxima. On the contrary, if we assume v¿1, then $m = v^2 - 1$, and the potential becomes *V=1-cosU* and also in this case we get soliton solutions.

The sG equation, integrable by the IST, describes crystal dislocations (Frenkel-Kontorova solitons), magnetic walls, liquid crystals, magnetic fluxes in Josephson junctions, etc. Moreover, it is the Lorentz invariant and can be used in elementary particle physics, if we want to identify solitons with extended particles (in this case per v¿1, one obtain tachyons, particles with superluminal velocity).

g) Topological and non topological coherent solutions

In relativistic local field theories it is important the distinction between topological and non-topological solutions. In the first case, the boundary conditions at infinity are topologically the same for the vacuum as for the coherent solution. On the contrary, in topological solitons the boundary conditions at infinity are topologically different for the coherent solution than for a physical vacuum state.

We consider a simple example of topological solution, the kink solution of a nonlinear Klein-Gordon equation in 1+1 dimensions,

$$U_{xx} - U_{tt} = -\frac{dV}{dU}, \quad V(U) = \frac{\lambda}{4}\left[\left(\frac{m^2}{\lambda}\right) - U^2\right]^2. \tag{2.16}$$

The potential has two vacuum states

$$U = \pm\frac{m}{\sqrt{\lambda}}. \tag{2.17}$$

Since a moving solution is easily found by boosting (Lorentz transformation) a stationary solution, we consider only the latter and obtain

$$U = \pm\frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m}{\sqrt{2}}(x - x_0)\right], \tag{2.18}$$

with plus (minus) sign for the (anti) kink. These solutions are topological because they connect the two different vacuum states. A moving kink,

$$\Phi = \frac{m}{\sqrt{\lambda}}\tanh\left[\frac{m\gamma}{\sqrt{2}}(x - x_0 - vt)\right], \quad \gamma = (1 - v^2)^{-\frac{1}{2}}, \tag{2.19}$$

can be shown to collide with an anti-kink in a not shape conservative way.

3. Solving Methods for Nonlinear Equations

The IST method can be considered an extension to the linear case of the Fourier method for linear partial differential equations. Given a generic NPDE, there is no general method that can establish if soliton solutions exist and how can be constructed. However the IST is the most important in the solitons seeking. The method was set up by Gardner, Green, Kruskal e Miura [18] in 1967 in order to solve the KdV equation and it was subsequently applied to many other NPDEs. In 1971 Zakharov and Shabat [87] applied this method to the NLS equation, while in 1974 the equation sG was resolved by Ablowitz, Kaup, Newell and Segur [3].

In 1968, Lax [31] demonstrated that the S-integrability of an equation is equivalent to the identification of an operator (Lax) couple (L, A) in such a way that the equation is obtained, for example in the 1+1 dimensions case, as a compatibility condition of the system:

$$Lf = \lambda f, \tag{3.1}$$

$$f_t + Af = 0, \tag{3.2}$$

with $f = f(x,t)$. We consider for example the KdV equation,

$$U_t + U_{xxxx} - 3(U^2)_x = 0, \tag{3.3}$$

where the operators L and A are

$$L = -\partial_x^2 + U, \tag{3.4}$$

$$A = -4\partial_x^3 + 6U\partial_x + 3U_x. \tag{3.5}$$

A simple calculation shows that the compatibility of the equations (3.1) e (3.2) is equivalent to (3.3). The equation KdV is the first of a hierarchy of equations where L is always given by the Schr**ö**dinger equation, while the temporal evolution operator A changes. The principal drawback of the IST technique is that there is no method for finding the Lax couple (if any) of a given NPDE and then to discover integrable equations.

The IST technique can be considered the nonlinear generalization of the Fourier transform. We take for example the equation (3.1), i. e. the spectral problem for the Schrödinger operator,

$$\left(-\partial_x^2 + U(x)\right) f(K,x) = \lambda f(K,x) = K^2 f(K,x), \tag{3.6}$$

where $K^2 \geq 0$ corresponds to the continuous spectrum and K^2¡0 to the discrete spectrum. It is well known that we can define a reflection coefficient $R(K)$ and a transmission coefficient$T(K)$,

$$f(K,x) \to \exp(-iKx) + R(K)\exp(iKx), \quad per \quad x \to +\infty \tag{3.7}$$

$$f(K,x) \to T(K)\exp(-iKx), \quad per \quad x \to +\infty \tag{3.8}$$

We now consider the eigenfunctions corresponding to the discrete eigenvalues $K^2 = -p_n^2$ and define the normalization constant ρ_n, through the relation

$$\lim_{x\to\infty} \left(\exp(2p_n x)\left[f(ip_n,x)\right]^2 = \rho_n \quad .\right. \tag{3.9}$$

If we know the initial condition $U(x,t_0)$, we must insert it in (3.6) and calculate the spectral transform

$$S(K,t_0) = \left[R(K,t_0), -\infty < K < \infty, p_n, \rho_n(t_0), n = 1,2...N\right], \tag{3.10}$$

where $R(K,t_0)$ is the reflection coefficient, N the number of discrete eigenvalues $K^2 = -p_n^2$, with p_n¿0, and ρ_n the normalization constant (3.9).

At this point the function (3.10) is considered in the spectral space and it is demonstrated that the temporal evolution is

$$p_n(t) = p_n(t_0), \quad \rho_n(t) = \rho_n(t_0)\exp(8p_n^3(t-t_0)), \quad R(K,t) = R(K,t_0)\exp(8iK^3(t-t_0)). \tag{3.11}$$

We now antitransform in order to obtain $U(x,t)$, by a procedure that can be synthesized as follows. We define the function

$$M(z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} dK \exp(iKz)R(K) + \sum_{n=1}^{N} \rho_n \exp(-p_n z), \tag{3.12}$$

which satisfies the Gelfand-Levitan-Marchenko equation,

$$K(x,x') + M(x+x') + \int_x^{\infty} dx'' K(x,x'')M(x''+x') = 0, \quad x' \geq x, \tag{3.13}$$

where

$$U(x) = -2\frac{dK(x,x)}{dx}. \tag{3.14}$$

The spectral transform is formed by three steps: i) construction of the spectral transform (3.10); ii) evolution in the spectral space, (3.11); iii) antitransformation with (3.13-3.14). The IST method has been able to find the correct language for the description of many nonlinear equations. For example, in the KdV equation the discrete spectrum p_n corresponds to the localized solutions (solitons) and the continuous spectrum to solutions subject to dispersion (the so-called *background*).

In 1971 Zakharov e Faddeev demonstrated that the equation KdV is a hamiltonian system with infinite freedom degrees and found the relative angle-action variables. For this reason the KdV equation is called completely integrable.

A generalization of the Lax couple can be obtained if we take a NPDE as compatibility condition for a overdetermined system of PDE for a vectorial wave function:

$$\Psi_x = A(u,\lambda)\Psi, \quad \Psi_t = B(u,\lambda)\Psi, \tag{3.15}$$

$$A_t - B_x + [A,B] = 0. \tag{3.16}$$

Zakharov and Shabat [87] have demonstrated that the spectral problem can be reduced to the solution of a Hilbert-Riemann matricial problem.

The IST can be extended with some difficulties in the 2+1 dimensional case [1], while until now there is no known nonlinear equation in 3+1 dimensions, integrable through a spectral problem in 3 dimensions. Other important solving techniques are the Darboux, Backlund and Hirota methods ([13],[24],[66]).

4. Solitons and Coherent Solutions for Nonlinear Model Equations in 2+1 Dimensions

a) The Kadomtsev-Petviashvili equation

In the field of nonlinear equations in 2+1 dimensions, Kadomtsev and Petviashvili [26] derived a new S-integrable nonlinear equation considering the stability of KdV solitons with respect to transversal perturbations

$$\left(U_t + U_{xxx} + 3(U^2)_x\right)_x + sU_{yy} = 0, \tag{4.1}$$

where $s = \pm 1$. If $s = +1$, (equation KP-1), then we obtain the soliton

$$U(x,y,t) = 2a^2 \sec h \left\{ a \left[x + b\sqrt{3}y - \left(3b^2 + 4a^2\right) t + x_0 \right] \right\}, \tag{4.2}$$

that moves with arbitrary velocity in the plane (x, y). The soliton interaction is characterized by overtaking collisions as for the KdV equation [25]. If $s = -1$, we get the so-called KP-2 equation with a localized (but not exponentially) solution,

$$U(x,y,t) = 4\frac{\left(3a^2y^2 - (x + a^{-1} - 3a^2t)^2 + a^{-2}\right)}{\left(3a^2y^2 + (x + a^{-1} - 3a^2t)^2 + a^{-2}\right)^2}, \tag{4.3}$$

but instable. The KP equation has been applied to superficial water waves and to ion-acoustic plasma waves.

b) The Davey-Stewartson (DS) equation The S-integrable Davey-Stewartson (DS-I) equation [AnFr, DaSt]:

$$i\psi_t = (b-a)\psi_{xx} + (b+a)\psi_{yy} - \frac{s}{2}(b-a)\psi\varphi_1 - \frac{s}{2}(b+a)\psi\varphi_2, \tag{4.4a}$$

$$\varphi_{1,y} = (|\psi|^2)_x \quad \varphi_{2,x} = (|\psi|^2)_y, \tag{4.4b}$$

has been discovered in hydrodynamics and its canonical form corresponds to $a = 0, b = 1$. An alternative form is

$$i\psi_t = (b-a)\psi_{xx} + (b+a)\psi_{yy} + w\psi, \tag{4.5a}$$

$$w_{xy} = -\frac{s}{2}(b-a)(|\psi|^2)_{xx} - \frac{s}{2}(b+a)(|\psi|^2)_{yy}, \tag{4.5b}$$

that is obtained with the *ansatz*

$$w = -\frac{s}{2}(b-a)\varphi_1 - \frac{s}{2}(b+a)\varphi_2. \tag{4.6}$$

Another form (it is necessary a 45^0 rotation of the spatial axes) is

$$i\psi_t + \frac{1}{2}\left(\psi_{xx} + \psi_{yy}\right) + \alpha\left|\psi\right|^2\psi - v\psi = 0, \tag{4.7a}$$

$$v_{xx} - v_{yy} - 2\alpha(|\psi|^2)_{xx} = 0, \tag{4.7b}$$

where α is a real parameter. The equation (4.7) is the limit in shallow water of the Benney-Roskes equation [6], where $\psi = \psi(x,y,t)$is the amplitude of a surface wave packet and $v = v(x,y,t)$ characterizes the medium motion generated by the surface wave.

The equation DS-II is

$$i\psi_t = (b-a)\psi_{zz} + (b+a)\psi_{vv} - \frac{s}{2}(b-a)\psi\varphi_1 - \frac{s}{2}(b+a)\psi\varphi_2, \tag{4.8a}$$

$$\varphi_{1,v} = (|\psi|^2)_z \quad \varphi_{2,z} = (|\psi|^2)_v, \tag{4.8b}$$

where z=x+iy e v=x-iy and its canonical form corresponds to $a = 0, b = 1$.

Finally, the equation DS-III ([76],[88]), is given by

$$i\psi_t = (a-b)\psi_{xx} - (b+a)\psi_{yy} - \frac{s}{2}(a-b)\psi\varphi_1 + \frac{s}{2}(b+a)\psi\varphi_2, \tag{4.9a}$$

$$\varphi_{1,y} = (|\psi|^2)_x \quad \varphi_{2,x} = (|\psi|^2)_y, \tag{4.9b}$$

and its canonical form is obtained with $a = 1, b = 0$.

The S-integrable ([7],[16],[17]) DS equation is important in plasma physics [67] and in quantum field theory ([1],[75],[29],[68]). Other valuable properties of this equation are the Darboux transformations [66] , a special bilinear form [21] and soliton and dromion solutions ([7],[16],[17],[66]).

5. Variable Separation Method for Nonlinear Equations in 2+1 Dimensions

In the last years it has been developed a very interesting technique for obtaining exact (and in particular coherent) solutions of nonlinear model systems, the multilinear variable separation approach. This method was first established for the DS system [34] and then developed for many other nonlinear equations, for example the Nizhnik-Novikov-Veselov (NNV) equation [36], asymmetric NNV equation [35], DS equation [37], dispersive long wave equation ([81],[82]), Broer-Kaup-Kupershmidt system [85], nonintegrable or integrable KdV equations in 2+1 dimensions ([80],[32]) and a general (N+M)-component Ablowitz-Kaup-Newell-Segur system [33]. In particular, it can be demonstrated that the solution for many nonlinear equations can be written in the form

$$U = \frac{-2\Delta q_y p_x}{(a_0 + a_1 p + a_2 q + a_3 pq)^2}, \quad \Delta = a_0 a_3 - a_1 a_2 \quad , \tag{5.1}$$

where a_0, a_1, a_2, a_3 are arbitrary constants, p=p(x,t) is an arbitrary function and q=q(y,t) is an arbitrary function for some equations (for example the DS equation) or an arbitrary solution of the Riccati equation in other cases. Different selections of the functions p and q correspond to different selections of boundary conditions and then in some sense coherent solutions can be remote controlled by some other quantities which have nonzero boundary conditions. Subsequently the method has been used for deriving chaotic and fractal solutions. Indeed, the solution (5.1) for an integrable NPDE with two or more dimensions is characterized by some arbitrary functions of lower dimensionality. As consequence a generic chaotic and/or fractal solution with lower dimension can be used to construct solutions of the given NPDE ([38],[89],[83]). The variety of solutions of (2+1)-dimensional nonlinear equations results from the fact that arbitrary exotic behaviours can transmit along the special characteristic functions p and q. For the moment in the method there are only two characteristic functions and it is an open question how to introduce more characteristic functions.

6. A Nonlinear Dirac Equation

The asymptotic perturbation method can be used for constructing approximate solutions of NPDEs and has been applied to particle-like solutions for a nonlinear relativistic scalar complex field model in 3+1 dimensions [47] and non-resonant interacting waves for the nonlinear Klein-Gordon equation [48] . The method has been later extended in order to demonstrate the existence of solitons trapping and dromion bound states for the nonlinear Klein-Gordon equation with appropriate potentials ([57],[58]). Non trivial solutions can be also obtained for relativistic vectorial fields[59] and nonlinear Dirac equation [62].

In order to illustrate this powerful technique, we seek coherent or chaotic or fractal approximate solutions of a nonlinear Dirac equation. It is well known that, in relativistic quantum mechanics, a free electron is represented by a wave function $\Psi(\underline{x},t)$, with

$$i\hbar\Psi_t = -i\hbar c\underline{\alpha}.\nabla\Psi + \beta mc^2\Psi, \tag{6.1}$$

where c denotes the speed of light, m the mass of the electron and $\hbar$ is the Planck's constant [12]. The standard form of the $4X4$ matrices $\underline{\alpha},\beta$(in $2X2$ blocks) is

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \underline{\alpha} = \begin{pmatrix} 0 & \underline{\sigma} \\ \underline{\sigma} & 0 \end{pmatrix}, \tag{6.2}$$

where $\underline{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are the Pauli matrices.

We seek approximate localized solutions for a particular version of the nonlinear Dirac equation

$$i\gamma^\mu\partial_\mu\Psi - m\Psi + \lambda\gamma^0\left(\overline{\Psi}\Psi\right)\Psi = 0, \tag{6.3}$$

where $\lambda \quad << 1$ is a weak nonlinear parameter.

We use the asymptotic reduction (AP) method based on the spatio-temporal rescaling

$$\underline{\xi} = \varepsilon\underline{X}, \quad \tau = \varepsilon t, \tag{6.4}$$

and focus on a solution that, due to the weak nonlinearity (λ is a small parameter, $\lambda \to \varepsilon\lambda$), is close to a superposition of N several dispersive waves ($\varepsilon$is a bookkeeping device which will be set to unity in the final analysis).

In the linear limit the solution is

$$\sum_{j=1}^{N} A_j \exp(iz_j), \quad z_j = \underline{K}_j.\underline{X} - \omega_j t, \quad j = 1,...N, \quad N > 1, \tag{6.5}$$

where A_j are the complex amplitudes, $\underline{K}_j \equiv (K_{1,j}, K_{2,j}, K_{3,j})$ the wave vectors and the (circular) frequency ω_j is furnished by the dispersion relation $\omega_j = \omega_j(\underline{K}_j)$. The amplitudes of these N non-resonant dispersive waves (constant in the linear limit) are slowly modulated by the non linear term of the nonlinear Dirac equation (6.3).

We will demonstrate the existence of dromions which preserve their shape during collisions, the only change being a phase shift. In addition, some special coherent solutions (line solitons, dromions, multilump solutions, ring solitons, instanton solutions and breathers) are derived. Moreover, we will show the existence of lower dimensional chaotic patterns such as chaotic-chaotic patterns, periodic-chaotic patterns, chaotic line soliton patterns, chaotic dromion patterns, fractal dromion and lump patterns as well as stochastic fractal solutions.

The linearized version of (6.3) is the well-known Dirac equation for spin $1/2$ particles,

$$i\Psi_t = -i\underline{\alpha}.\nabla\Psi + \beta m\Psi, \tag{6.6}$$

satisfied by Fourier modes with constant amplitudes,

$$A_j \exp i(\underline{K}_j.\underline{X} - \omega_j t), \tag{6.7}$$

if the following dispersion relation is verified

$$\omega_j = \pm\sqrt{m^2 + \underline{K}^2}. \tag{6.8}$$

The group velocity $\underline{U}_j$ (the speed with which a wave packet peaked at that Fourier mode would move) is

$$\underline{U}_j = \frac{d\omega_j}{d\underline{K}_j} = \frac{\underline{K}_j}{\omega_j}. \tag{6.9}$$

In the following we consider a superposition of N dispersive waves, characterized by different group velocities not close to each other. Weak nonlinearity induces a slow variation of the amplitudes of these dispersive waves and the AP method derives the nonlinear system of equations for the Fourier modes amplitudes modulation, obviously in appropriate "slow" and "coarse-grained" variables defined by equations (6.4). Since the amplitudes of Fourier modes are not constant, higher order harmonics appear and in order to construct an approximate solution of the nonlinear equation (6.3) we introduce the asymptotic Fourier expansion for the positive-energy solutions (i.e. we consider the plus sign in (6.8))

$$\Psi(\underline{X}, t) = \sum_{\underline{n}(odd)} \varepsilon^{\gamma_{\underline{n}}} \begin{pmatrix} \varphi_{\underline{n}} \\ \\ \chi_{\underline{n}} \end{pmatrix} \exp\left[i \sum_{j=1}^{N} n_j \left(\underline{K}_j.\underline{X} - \omega_j t \right) \right], \tag{6.10}$$

where the index $\underline{n}$ stands for the set $\{n_j; j = 1, 2, ..., N\}$ withn_j *=0, 1, 3,..* and $\underline{n} \neq (0, ..0)$. The functions, $\varphi_{\underline{n}}\left(\underline{\xi}, \tau, \varepsilon\right)$, $\chi_{\underline{n}}\left(\underline{\xi}, \tau, \varepsilon\right)$ depend parametrically on ε and we assume that their limit for $\varepsilon \to 0$ exists and is finite. We moreover assume that there hold the conditions

$$\gamma_{\underline{n}} = \sum_{j=1}^{N} \frac{f(n_j) - 1}{2}, \tag{6.11a}$$

$$f(n_j) = n_j, \quad for \quad n_j > 0. \tag{6.11b}$$

This implies that we obtain the main amplitudes if one of the indices n_j has unit modulus and all the others vanish. In the following we use the notation

$$\varphi_j = \varphi_{\underline{n}}(\varepsilon \to 0) \quad if \quad n_j = 1 \quad and \quad n_m = 0 \quad for \quad j \neq m, \tag{6.12}$$

and similar notations for $\chi_{\underline{n}}$.

Taking into account (6.11-6.12), the Fourier expansion (6.10) can be written more explicitly in the following form

$$\Psi(\underline{X}, t) = \sum_{j=1}^{N} \left[\begin{pmatrix} \varphi_j \\ \chi_j \end{pmatrix} \exp(i(\underline{K}_j.\underline{X} - \omega_j t)) \right] + O(\varepsilon). \tag{6.13}$$

Substituting (6.13) in equation (6.3) and considering the different equations obtained for every harmonic, we obtain for $n_j = 1$, $n_m = 0$, if $j \neq m$ at the lowest order of approximation

$$\chi_j = c_j \varphi_j = \frac{\underline{\sigma K}_j}{m + \omega_j} \varphi_j, \tag{6.14}$$

and at the order of approximation of ε:

$$i\varphi_{j,\tau} = -i\underline{\sigma}\nabla\chi_j + \underline{\sigma K}\tilde{\chi}_j - \lambda \sum_{m=1}^{N} (\varphi_m^+ \varphi_m - \chi_m^+ \chi_m)\varphi_j, \tag{6.15a}$$

$$i\chi_{j,\tau} + \omega_j \tilde{\chi}_j = -i\underline{\sigma}\nabla\varphi_j - m\tilde{\chi}_j - \lambda \sum_{m=1}^{N} (\varphi_m^+ \varphi_m - \chi_m^+ \chi_m)\chi_j, \tag{6.15b}$$

where $\tilde{\chi}_j$ is the correction of order ε to χ_j. After some calculations, we arrive at a system of equations for the N modulated amplitudes φ_j,

$$\varphi_{j,\tau} + \underline{U}_j \nabla\varphi_j - i\lambda \sum_{m=1}^{N} b_m \left|\varphi_m\right|^2 \varphi_j = 0, \tag{6.16}$$

where $\underline{U}_j$ is the group velocity and b_m is a constant coefficient given by

$$b_m = \frac{2m}{m + \omega_m} \quad . \tag{6.17}$$

The system of equations (6.16) is C-integrable by means of an appropriate transformation of the dependent variables. We set

$$\varphi_j(\underline{\xi}, \tau) = \rho_j(\underline{\xi}, \tau) \exp\left[i\vartheta_j(\underline{\xi}, \tau)\right], \quad j = 1, ..N, \tag{6.18}$$

with $\rho_j = \rho_j(\underline{\xi}, \tau) > 0$ and $\vartheta_j = \vartheta_j(\underline{\xi}, \tau)$ real functions. Then equation (6.16) yields

$$\rho_{j,\tau} + \underline{U}_j \nabla\rho_j = 0, \tag{6.19a}$$

$$\vartheta_{j,\tau} + \underline{U}_j.\nabla\vartheta_j - \lambda \sum_{m=1}^{N} b_m \rho_m^2 = 0. \tag{6.19b}$$

The general solution for the Cauchy problem of (6.19a) reads

$$\rho_j(\underline{\xi},\tau) = \rho_j\left(\underline{\xi} - \underline{U}_j\tau\right), \tag{6.20}$$

where the N real functions $\rho_j\left(\underline{\xi}\right)$, which represent the initial shape, can be chosen arbitrarily. A simple particular case is the solution

$$\rho_j = \rho_j(\underline{a}_j\underline{\xi} + b_j\tau), \tag{6.21a}$$

where $b_j, \underline{a}_j$ are real constants which satisfy the relation

$$\underline{a}_j\underline{U}_j + b_j = 0. \tag{6.21b}$$

The general solution of (6.19b) is

$$\vartheta_j(\underline{\xi},\tau) = \delta_j(\underline{\xi} - \underline{U}_j\tau) + \lambda\sum_{m=1}^{N} b_m \int_0^\tau (\rho_m(\underline{\xi} - \underline{U}_j(\tau-\tilde{\tau}),\tilde{\tau}))^2 d\tilde{\tau}, \tag{6.22}$$

where the N arbitrary functions $\delta_j(\underline{\xi})$ are fixed by the initial data. The particular solution corresponding to (6.21b) is

$$\vartheta_j(\underline{\xi},\tau) = \delta_j(\underline{a}_j\underline{\xi} + \tilde{b}_j\tau) + \lambda\sum_{m=1}^{N} b_m \int_0^\tau (\rho_m(\underline{\xi} - \underline{U}_j(\tau-\tilde{\tau}),\tilde{\tau}))^2 d\tilde{\tau}, \tag{6.23a}$$

where

$$\tilde{\underline{a}}_j\underline{U}_j + \tilde{b}_j = 0. \tag{6.23b}$$

The approximate solution for the system of equations (6.3) is

$$\Psi(\underline{X},t) = \sum_{j=1}^{N} \begin{pmatrix} 1 \\ \\ c_j \end{pmatrix} \rho_j \exp\left[i(\vartheta_j + \underline{K}_j\underline{X} - \omega_j t)\right] + O(\varepsilon). \tag{6.24}$$

where c_j is given by (6.14). The corrections of order to the approximate solution depend on higher harmonics and can be easily calculated by the AP method.

a) Solitons. The C-integrable nature of the system (6.19) implies the existence of more interesting solutions, because of the existence of arbitrary functions in the seed solutions. It is possible the existence of N solitons, with fixed speeds but arbitrary shapes, which interact each other preserving their shapes and propagate with the relative group velocity. The collision of two solitons does not produce a change in the amplitude ρ_j of each of them, but only a change in the phase given by equation (6.22).

For instance, we take

$$\rho_j(\underline{\xi},\tau) = \frac{2A_j}{ch\left(2A_j(\underline{a}_j\underline{\xi} + b_j\tau)\right)}, \tag{6.25}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}, \tag{6.26}$$

where A_j, for $j = 1...N$, are real constants, and the phase ϑ_j is given by (6.22). Substituting (6.27) in (6.24) we obtain the approximate solution. Each soliton advances with a constant velocity (the group velocity) before and after collisions. Only the phase is changed during collisions owing to the presence of the other solitons.

b) Dromions. The existence of localized solutions is possible also for C-integrable systems, because dromion solutions are not exclusive characteristics of equations integrable by the inverse scattering method.

A particular solution of the model system (6.19) is given by

$$\rho_j(\underline{\xi}, \tau) = A_j \exp(-B_j \left|\underline{\xi} - U_j\tau\right|), \tag{6.27}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.28}$$

where A_j, B_j are real constants (note that the functions ρ_j (3.4a) are localized) and ϑ_j is given by equations (6.22).

c) Lumps. It is well known that in high dimension, in addition to the dromion solutions, other interesting localized solutions, formed by rational functions, are the multiple lumps. Obviously, there are many possible choices in order to obtain multilump solutions. For instance, we take

$$\rho_j = \frac{A_j}{B_j + C_j \left|\underline{\xi} - U_j\tau\right|^2}, \tag{6.29a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, ...N}, \tag{6.29b}$$

where A_j, B_j and C_j are arbitrary constants.

d) Ring solitons. The multiple ring solitons are solutions that are not equal to zero at some closed curves and decay exponentially away from the closed curves. A possible selection is

$$\rho_j = A_j \exp(-B_j f_j(R_j)), \tag{6.30a}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}. \tag{6.30b}$$

where

$$R_j = \left|\underline{\xi} - U_j\tau\right|, \tag{6.30c}$$

$$f_j(R) = (R - R_{0,j})^2, \tag{6.30d}$$

and A_j, B_j and $R_{0,j}$ are arbitrary constants. In Fig. 1 we show a collision between two ring solitons: the initial condition is showed in Fig. 1a, then the two ring solitons collide (Fig. 1b) and then separate (Fig. 1c). We can see that these solutions preserve their shapes but with a phase shift.

e) Instantons. If we choose a decaying function of time, we obtain also multiple instanton solutions, for example,

$$\rho_j = A_j \exp(\underline{a}_j \underline{\xi} - \lambda_j \tau), \tag{6.31a}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}, \tag{6.31b}$$

where A_j, $\alpha_{1,j}$ are arbitrary constants and

$$\lambda_j = \alpha_{1,j} U_{1,j} + \alpha_{2,j} U_{2,j}. \tag{6.31c}$$

f) Moving breather-like structures. Finally, if we choose some types of periodic functions of time in the above mentioned solutions, then we obtain breathers. For example, we take

$$\rho_j = A_j \cos(\underline{a}_j \underline{\xi} - \Omega_j \tau) \exp\left[-B_j \left|\underline{\xi} - U_j \tau\right|\right], \tag{6.32a}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}, \tag{6.32b}$$

where A_j, B_j, $\alpha_{1,j}$are arbitrary constants and

$$\Omega_j = \underline{a}_j \underline{U}_j. \tag{6.32c}$$

g) Chaotic-chaotic and chaotic-periodic patterns. If we select at least one of the arbitrary functions in order to contain some chaotic solutions of nonintegrable equations, then we obtain some type of space-time chaotic patterns, the so-called chaotic-chaotic (in all spatial directions) patterns. For example, we choose the arbitrary function as solution of the chaotic Lorenz system

$$X_T = -c(X - Y), \quad Y_T = X(a - Z) - Y, \quad Z_T = XY - bZ, \tag{6.33a}$$

with$a = 60$, $b = 8/3$, $c = 10$,or of the R**ö**ssler system

$$X_T = -Y - Z, \quad Y_T = X + aY, \quad Z_T = b + Z(X - c), \tag{6.33b}$$

with $a = 0.15$, $b = 0.2$, $c = 10$and $T = \xi - U_1 \tau$ (or $T = \xi - U_1 \tau$or $T = \zeta - U_3 \tau$). A phase and amplitude chaotic-chaotic pattern is given by

$$\rho_j(\underline{\xi}, \tau) = X(\xi - U_{1,j}\tau) Y(\eta - U_{2,j}\tau) Z(\zeta - U_{3,j}\tau) \tag{6.34a}$$

$$\delta_j = 0 \quad for \quad j = 1, \ldots .N, \tag{6.34b}$$

while ϑ_j is given by equations (6,22). An example is given in Fig. 2.

On the contrary, we obtain a phase chaotic-chaotic pattern, if we choose the function (6.34b) as solution of the Lorenz system. Finally, if we select a chaotic-periodic solution which is chaotic in one (or two) direction and periodic in the other(s) direction(s). then we obtain the so-called chaotic-periodic patterns.

h) Chaotic line soliton solutions

If we consider the soliton line solution (6.25-6.26) we can easily deduce a chaotic solution when we select A_j as solution of the Lorenz system,

$$\rho_j(\underline{\xi},\tau) = \frac{2A_j(\underline{\xi},\tau)}{ch\left(2(\underline{a}_j\underline{\xi}+b_j\tau)A_j(\underline{\xi},\tau)\right)}, \tag{6.35a}$$

$$\delta_j = 0 \quad for \quad \text{j=1 .. N}, \tag{6.35b}$$

where the phase ϑ_j is given as usual by (6.22), for $j = 1...N$, and the functions $A_j = A_j(T_j) = A_j(\underline{a}_j\underline{\xi}+b_j\tau)$ satisfy the third order ordinary differential equation equivalent to the Lorenz system,

$$A_{j,TTT} + (b+c+1)A_{j,TT} + (bc+b+A_j^2)A_{j,T} + c(b-ab+A_j^2)A_j - \frac{A_{j,TT}A_{j,T}+(c+1)A_{j,T}^2}{A_j} = 0. \tag{6.35c}$$

i) Chaotic dromion and lump patterns

If we consider the dromion solution (6.27-6.28), we can transform it into a chaotic pattern with an appropriate choice for A_j and/or B_j,

$$\rho_j(\underline{\xi},\tau) = A_j \exp(-B_j\left|\underline{\xi}-U_j\tau\right|), \tag{6.36a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.36b}$$

where $A_j = A_j(T_j) = A_j(\underline{a}_j\underline{\xi}+b_j\tau)$ and/or $B_j = B_j(T_j) = B_j(\underline{a}_j\underline{\xi}+b_j\tau)$ are solutions of the Lorenz equation and ϑ_j is given by equations (6.22). We obtain an amplitude (A_j chaotic) or a shape (B_j chaotic) or an amplitude and shape (A_j and B_j chaotic) dromion chaotic pattern. Similar considerations can be applied to the lump solutions (6.29).

j) Nonlocal fractal solutions. If we choose

$$\rho_j(\underline{\xi},\tau) = \prod_{m=1}^{3} T_{m,j}\left|T_{m,j}\right|\left\{\sin\left[\ln\left(T_{m,j}^2\right)\right] - \cos\left[\ln\left(T_{m,j}^2\right)\right]\right\} \tag{6.37}$$

with $\underline{T} = (T_1, T_2, T_3)$, $\underline{T}_j = \underline{\xi} - \underline{U}_j\tau$, we get a nonlocal fractal structure for small $\underline{T}_j$. It is well known that if we plot the structure of the solution at smaller regions we can obtain the same structures.

k) Fractal dromion and lump solutions. A fractal dromion (lump) solution is exponentially (algebraically) localized in large scale and possesses self-similar structure near the center of the dromion. We consider for example an amplitude fractal dromion

$$\rho_j(\underline{\xi},\tau) = A_j \exp(-B_j\left|\underline{\xi}-U_j\tau\right|), \tag{6.38a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.38b}$$

where ϑ_j is given by equations (6.22) and $A_j = A_j(T_j) = A_j(\underline{a}_j\underline{\xi} + b_j\tau)$ is given by

$$A_j = 2 + \sin\left\{\ln\left[T_j^2\right]\right\}. \tag{6.38c}$$

By a similar choice for B_j or $\delta_j$we obtain shape or phase fractal dromion.

l) Stochastic fractal dromion and lump excitations. It is well known the stochastic fractal property of the continuous but nowhere differentiable Weierstrass function

$$W(x) = \sum_{k=1}^{N} (c_1)^k \sin\left[(c_2)^k\, x\right], \quad N \to \infty, \tag{6.39a}$$

with c_2 odd and

$$c_1 c_2 > 1 + \frac{3\pi}{2}. \tag{6.39b}$$

A stochastic fractal solution is (see Fig. 3 for an example)

$$\rho_j(\underline{\xi}, \tau) = \prod_{m=1}^{3} A_{m,j}(\xi_m - U_{m,j}\tau), \tag{6.40a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.40b}$$

where ϑ_j is as usual given by equations (6.22), $\underline{A}_j = (A_{1,j}, A_{2,j}, A_{3,j})$, $\xi_1 = \xi$,$\xi_2 = \eta$,$\xi_3 = \zeta$, and $\underline{A}_j = \underline{A}_j(\underline{\xi} - \underline{U}_j\tau)$ is given by

$$\underline{A}_j = W(\underline{\xi} - \underline{U}_j\tau) + (\underline{\xi} - \underline{U}_j\tau)^2. \tag{6.40c}$$

m) Stochastic fractal dromion and lump excitations. In order to obtain a stochastic amplitude fractal dromion we choose

$$\rho_j(\underline{\xi}, \tau) = A_j \exp(-B_j \left|\underline{\xi} - U_j\tau\right|), \tag{6.41a}$$

$$\delta_j = 0 \quad for \quad \text{j=1, 2....N}, \tag{6.41b}$$

where ϑ_j is given by equations (6.22) $A_j = A_j(T_j) = A_j(\underline{a}_j\underline{\xi} + b_j\tau)$ is given by

$$A_j = W(T_j) + T_j^2 \tag{6.41c}$$

By similar methods we obtain shape or phase stochastic fractal dromion as well as stochastic fractal lump solutions.

7. Spontaneous Symmetry Breaking Model

We now illustrate in some detail the use of the AP method and consider a scalar complex field $\Phi = \Phi(x)$, $x = (x^0 = t, \underline{x})$, coupled with a massless vectorial gauge field $A_\mu = A_\mu(x)$, $A^\mu = (A^0, \underline{A})$, and seek approximate localized solutions for a spontaneous symmetry breaking (or hidden symmetry or Higgs) model ([19],[22],[23]) with Lagrangian [63]

$$L = \left[\left(\partial^\mu + iqA^\mu\right)\Phi\right]^* \left[\left(\partial^\mu + iqA^\mu\right)\Phi\right] - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{a^2}{2b^2}(\Phi^*\Phi)^2 + \frac{a^2}{2}(\Phi^*\Phi), \tag{7.1a}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.1b}$$

The Lagrangian (7.1) is invariant under the local transformation

$$\Phi(x) \to \Phi'(x) = \exp\left(-i\alpha(x)\Phi(x)\right), \tag{7.2a}$$

with accompanied by the gauge transformation on the potentials

$$A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \frac{1}{q}\partial^\mu\alpha(x). \tag{7.2b}$$

We note that this model contains four field degrees of freedom, two in the complex scalar Higgs field and two in the massless gauge field. The field equations are

$$\partial_\mu\partial^\mu A^\nu - \partial^\nu(\partial_\mu A^\mu) = J^\nu, \tag{7.3a}$$

$$J^\nu = iq\left[\Phi^*\left(\partial^\nu\Phi\right) - \left(\partial^\nu\Phi^*\right)\Phi\right] - 2q^2A^\nu\left|\Phi\right|^2, \tag{7.3b}$$

$$\partial_\mu\partial^\mu\Phi - \frac{a^2}{2}\Phi = -\frac{a^2}{b^2}\left|\Phi\right|^2\Phi, \tag{7.3c}$$

where $\partial_\mu\partial^\mu = \partial_t^2 - \nabla^2$, a, b and q are parameters. The potential for the scalar field is

$$V(\Phi) = -\frac{a^2}{2}\left|\Phi\right|^2 + \frac{a^2}{2b^2}\left|\Phi\right|^4, \tag{7.4}$$

and the physical vacuum is identified by the condition

$$\left|\Phi\right|^2 = b^2. \tag{7.5}$$

We consider the interaction and eventually the collisions among coherent solutions with different velocities that are not close to each other and use the asymptotic reduction (AP) method based on the spatio-temporal rescaling

$$\xi = \varepsilon^2 x, \tag{7.6}$$

where

$$\xi = (\xi^0, \underline{\xi}), \quad x = (x^0, \underline{x}). \tag{7.7}$$

and the small positive nondimensional parameter ε is artificially introduced to serve as bookkeeping device and will be set equal to unity in the final analysis. The linear evolution is most appropriately described in terms of Fourier modes, which have a constant

amplitude and a well defined group velocity (the speed with which a wave packet peaked at that Fourier mode would move in ordinary space). We study the modulation, in terms of the variables defined above, of the amplitude of the Fourier mode. The modulation (that would remain constant in the absence of nonlinear effects) is best described in terms of the rescaled variables, ξ, that account for the need to look on larger space and time scales, to obtain a not negligible contribution from the nonlinear term. The reduction method focuses on a solution that is small and is close to a superposition of N several dispersive waves, with different group velocities.

In the linear limit the solution is a linear combination of dispersive waves. For example for the scalar field the linear solution is

$$\sum_{j=1}^{N} C_j \exp\left(-is_j\right), \quad \tilde{s}_j = \tilde{k}_{j,\mu} x^\mu = \tilde{\omega}_j t - \underline{\tilde{K}}_j . \underline{x}, \tag{7.8}$$

where C_j are the complex amplitudes, $\tilde{k}_j^\mu = (\tilde{\omega}_j, \underline{\tilde{K}}_j)$, $\underline{\tilde{K}}_{\mathrm{j}} \equiv (\tilde{K}_{1,j}, \tilde{K}_{2,j}, \tilde{K}_{3,j})$ the wave vectors and the (circular) frequency $\tilde{\omega}_j$ is furnished by the dispersion relation $\tilde{\omega}_j = \tilde{\omega}_j(\underline{\tilde{K}}_j)$. The amplitudes of these N non-resonant dispersive waves (constant in the linear limit) are slowly modulated by the nonlinear terms. We derive a model system of equations for the slow modulation of the Fourier modes amplitudes and, subsequently, show that it is C-integrable. The Cauchy problem is resolved, just by quadratures, and explicit nontrivial solutions are constructed.

We introduce two real Higgs fields $U = U(\underline{x}, t)$ e $W = W(\underline{x}, t)$ and set

$$\Phi = \frac{1}{\sqrt{2}} \left(b + U + iW\right). \tag{7.9}$$

In the following we use the covariant 't Hooft gauge [tHo], which for this Abelian model is (M=qb)

$$\partial_\mu A^\mu = \lambda M W, \tag{7.10}$$

where λ is an arbitrary parameter. For any finite λ, we obtain R gauges, that are manifestly renormalisable, but involve unphysical Higgs fields such as W. We recall that in the limit $\lambda \to \infty$ we obtain the U (unitary) gauge, where only physical particles appear.

Using (7.1-7.8), the equations (7.9-7.11) yield

$$\left(\partial_\mu \partial^\mu + M^2\right) A^\nu - \left(1 - \frac{1}{\lambda}\right) \partial^\nu (\partial_\mu A^\mu) = J^\nu, \tag{7.11}$$

$$J^\nu = q \left[\frac{\left(\partial^\nu U - U \partial^\nu\right)\left(\partial_\mu A^\mu\right)}{\lambda M}\right] - q^2 A^\nu \left[U^2 + 2bU + \frac{\left(\partial_\mu A^\mu\right)^2}{\lambda^2 M^2}\right], \tag{7.12}$$

$$\left(\partial_\mu \partial^\mu + M^2\right) U = -\frac{a^2 U^2}{2b} - \frac{a^2}{2b}\left(1 + \frac{U}{b}\right)\left(U^2 + \frac{\left(\partial_\mu A^\mu\right)^2}{\lambda^2 M^2}\right). \tag{7.13}$$

We consider now a superposition of N dispersive waves, characterized by different values of the wave vector $\underline{K}_j$ and by group velocities not close to each other. Weak nonlinearity

induces a slow variation of the amplitudes of these dispersive waves and the AP method derives the nonlinear system of equations for the Fourier modes amplitudes modulation, obviously in appropriate "slow" and "coarse-grained" variables defined by equations (7.7). Since the amplitude of Fourier modes are not constant, higher order harmonics appear and in order to construct an approximate solution that is small of order ε and that is close in the limit of small ε to the linear solution (7.8), we introduce the asymptotic Fourier expansion

$$U(x^\mu) = \sum_{\underline{n}=-\infty}^{\infty} \exp\left(i\sum_{j=1}^{N} n_j \tilde{s}_j\right) \varepsilon^{\gamma_{\underline{n}}} \varphi_{\underline{n}}(\xi^\mu;\varepsilon), \quad \tilde{s}_j = \tilde{k}_{j,\mu} x^\mu = \tilde{\omega}_j t - \underline{\tilde{K}}_j.\underline{x}, \tag{7.14}$$

$$A^\nu(x^\mu) = \sum_{\underline{n}=-\infty}^{\infty} \exp\left(i\sum_{j=1}^{N} n_j s_j\right) \varepsilon^{\gamma_{\underline{n}}} \psi_{\underline{n}}(\xi^\mu;\varepsilon), \quad s_j = k_{j,\mu} x^\mu = \omega_j t - \underline{K}_j.\underline{x}, \tag{7.15}$$

where the index $\underline{n}$ stands for the set $\{n_j; j = 1, 2, ..., N\}$. In the expansion (7.14) $n_j = 0, \pm 1, \pm 2, ..$, while in the expansion (7.15) n_j may assume only odd values, $n_j = \pm 1, \pm 3...$ The functions, $\psi^\nu_{\underline{n}}(\xi;\varepsilon)$, $\varphi_{\underline{n}}(\xi;\varepsilon)$ depend parametrically on ε and we assume that their limit for $\varepsilon \to 0$ exists and is finite. We moreover assume that there hold the conditions

$$\gamma_{\underline{n}} = \gamma_{-\underline{n}}, \quad \gamma_{\underline{n}} = \sum_{j=1}^{N} |n_j|. \tag{7.16}$$

This implies that we obtain the main amplitudes if one of the indices n_j has unit modulus and all the others vanish. We use the following notations, for $j = 1, 2, ...N$,

$$\varphi_{\underline{n}}(\xi, \varepsilon \to 0) = \varphi_j(\xi), \quad if \quad n_j = 1 \quad and \quad n_m = 0 \quad for \quad j \neq m, \tag{7.17a}$$

$$\varphi_{\underline{n}}(\xi; \varepsilon \to 0) = \varphi_0(\xi), \quad if \quad n_j = 0, \tag{7.17b}$$

$$\varphi_{\underline{n}}(\xi; \varepsilon \to 0) = \varphi_{2,j}(\xi), \quad if \quad n_j = 2 \quad and \quad n_m = 0 \quad for \quad j \neq m, \tag{7.17c}$$

$$\varphi_{\underline{n}}(\xi; \varepsilon \to 0) = \varphi_{11,jm}(\xi), \quad if \quad n_j = n_m = 1 \quad and \quad n_l = 0 \quad for \quad l \neq j, m, \quad j \neq m, \tag{7.17d}$$

$$\varphi_{\underline{n}}(\xi; \varepsilon \to 0) = \varphi_{1-1,jm}(\xi), \quad if \quad n_j = 1, n_m = -1, \quad and \quad n_l = 0 \quad for \quad l \neq j, m \quad , \quad j \neq m, \tag{7.17e}$$

while for the vectorial field we set

$$\Psi^\nu_{\underline{n}}(\xi; \varepsilon \to 0) = \Psi^\nu_j(\xi). \tag{7.18}$$

Taking into account (7.17-7.18), the Fourier expansion (7.14-7.15) can be written more explicitly in the following form

$$A^{\nu}(x) = \varepsilon \sum_{j=1}^{N} \left[\exp(is_j)\Psi_j^{\nu}(\xi) + c.c.\right] + O(\varepsilon^3), \tag{7.19}$$

$$\begin{aligned} U(x) &= \varepsilon \sum_{j=1}^{N} \left[\exp(is_j)\varphi_j(\xi) + \varepsilon \exp(2is_j)\varphi_{2,j}(\xi) + c.c.\right] + \varepsilon^2\varphi_0(\xi) \\ &+\varepsilon^2 \sum_{j,m=1,j\neq m}^{N} \left[\exp(-is_j - is_m)\varphi_{11,jm}(\xi) + \exp(-is_j + is_m)\varphi_{1-1,jm}(\xi) + c.c.\right] + O(\varepsilon^3) \end{aligned} \tag{7.20}$$

where *c.c.* stands for complex conjugate.

The standard procedure is to consider the different equations obtained from the coefficients of the Fourier modes. Substituting (7.19-7.20) in equations (7.11-7.15) and considering the different equations obtained for every harmonic and for a fixed order of approximation in ε, we obtain for $n_j = 1$, $n_m = 0$, if $j \neq m$, to the order of ε, the following system of equations for the main modulated amplitudes,

$$(-\tilde{k}_{j,\mu}\tilde{k}_j^{\mu} + a^2)\varphi_j = 0, \tag{7.21a}$$

$$\left[(-k_{j,\sigma}k_j^{\sigma} + M^2)g^{\nu\mu} + \left(1 - \frac{1}{\xi}\right)(k_j^{\mu}k_j^{\nu})\right]\Psi_{j,\mu} = 0. \tag{7.21b}$$

From (7.21a-b) we obtain the dispersion relations

$$\tilde{\omega}_j^2 = \underline{\tilde{K}}_j^2 + a^2, \quad \omega_j^2 = \underline{K}_j^2 + M^2, \tag{7.22a}$$

with the associated group velocities

$$\tilde{V}_j = \frac{\underline{\tilde{K}}_j}{\tilde{\omega}_j}, \quad V_j = \frac{\underline{K}_j}{\omega_j}. \tag{7.22b}$$

Moreover, from (7.21b), as a consequence of the gauge invariance of the vectorial field (only three components of the fields are independent), we obtain

$$k_j^{\mu}\Psi_{j,\mu} = \omega_j\Psi_{j,0} - \underline{K\Psi} = 0. \tag{7.23}$$

We obtain for $n_j = 1$, $n_m = 0$, if $j \neq m$, to the order of ε^2,

$$\begin{aligned} &(-2i\tilde{k}_{j,\mu}\partial^{\mu})\varphi_j + \frac{3a^2}{2b}\left(2\varphi_0\varphi_j + 2\varphi_2\varphi_j^* + |\varphi_j|^2\varphi_j\right) \\ &+\frac{3a^2}{b^2}\sum_{m=1,m\neq j}^{N}\left(|\varphi_m|^2\varphi_j + \varphi_{11,jm}\varphi_j^* + \varphi_{1-1,jm}\varphi_j\right) = 0 \end{aligned} \tag{7.24a}$$

,

$$\left[(-2k_{j,\sigma}\partial^{\sigma})g^{\nu\mu}+\left(1-\frac{1}{\lambda}\right)(\partial^{\mu}k_{j}^{\nu})\right]\Psi_{j,\mu}+i\left(-2q^{2}\Psi_{j}^{\nu}\right)\left(\sum_{m=1}^{N}\left|\varphi_{m}\right|^{2}+b\varphi_{0}\right)=0, \quad (7.24b)$$

and for $n_j = 0$, to the order of ε^2,

$$\varphi_0 = A\sum_{m=1}^{N}|\varphi_m|^2, \quad A=-\frac{3}{b}, \quad (7.25)$$

and for $n_j = 2$, $n_m = 0$, if $j \neq m$, to the order of ε^2,

$$\varphi_{2,j} = B_2\varphi_j^2, \quad B_2 = -\frac{1}{2b} \quad (7.26)$$

and for $n_j = 1$, $n_m = 1$, $n_l = 0$ if $j, m \neq l$, to the order of ε^2,

$$\varphi_{11,jm} = \sum_{m=1}^{N} B_{11,jm}\varphi_j\varphi_m, \quad B_{11,jm} = \frac{3a^2}{b\left(a^2+2\tilde{k}_m^{\mu}\tilde{k}_{j,\mu}\right)}, \quad (7.27)$$

and for $n_j = 1$, $n_m = -1$, $n_l = 0$, if $j, m \neq l$, to the order of ε^2,

$$\varphi_{1-1,jm} = \sum_{m=1}^{N} B_{1-1,jm}\varphi_j\varphi_m^*, \quad B_{1-1,jm} = \frac{3a^2}{b\left(a^2-2\tilde{k}_m^{\mu}\tilde{k}_{j,\mu}\right)}. \quad (7.28)$$

Using (7.24a-b) then equations (7.21-7.22) yield

$$(\tilde{k}_{j,\mu}\partial^{\mu})\varphi_j + i\sum_{m=1}^{N}\left[\alpha_{jm}\left|\varphi_m\right|^2\right]\varphi_j = 0, \quad (7.29a)$$

$$\left[(-2k_{j,\sigma}\partial^{\sigma})g^{\nu\mu}+\left(1-\frac{1}{\lambda}\right)(\partial^{\mu}k_{j}^{\nu})\right]\Psi_{j,\mu}+i\beta\sum_{m=1}^{N}\left[\left|\varphi_m\right|^2\right]\Psi_j^{\nu}=0, \quad (7.29b)$$

where the coefficients α, β, are depending on the wave vectors of the scalar and vectorial fields,

$$\alpha_{jm} = -\frac{3a^2}{b^2}+\frac{9a^6}{b^2\left(a^4-4\left(k_m^{\mu}k_{j,\mu}\right)^2\right)}, \quad j\neq m, \quad (7.30a)$$

$$\alpha_{jj} = -\frac{9a^2}{2b^2}, \quad \beta = 4q^2. \quad (7.30b)$$

The system of equations (7.28-7.29) is C-integrable by means of an appropriate transformation of the dependent variables. We set

$$\varphi_j(\xi) = \rho_j(\xi)\exp\left[i\vartheta_j(\xi)\right], \quad j=1,..N, \quad (7.31a)$$

$$\Psi_j^{\nu}(\xi) = \chi_j^{\nu}(\xi)\exp\left[i\delta_j^{\nu}(\xi)\right], \quad j=1,..N, \quad (7.31b)$$

with $\rho_j = \rho_j(\xi), \chi_j^\nu = \chi_j^\nu(\xi) > 0$ and $\vartheta_j = \vartheta_j(\xi)$, $\delta_j^\nu = \delta_j^\nu(\xi)$ real functions. Then equation (7.28-7.29) yield

$$(\tilde{k}_{j,\mu}\partial^\mu)\rho_j = 0, \tag{7.32a}$$

$$(\tilde{k}_{j,\mu}\partial^\mu)\vartheta_j + \sum_{m=1}^{N}\left[\alpha_{jm}\rho_m^2\right] = 0, \tag{7.32b}$$

and

$$\left[(-2k_{j,\sigma}\partial^\sigma)g^{\nu\mu} + (1-\frac{1}{\lambda})(k_j^\nu\partial^\mu)\right]\chi_{j,\mu} = T_j^{\mu\nu}\chi_{j,\mu} = 0, \tag{7.33a}$$

$$\left[(-2k_{j,\sigma}\partial^\sigma)\chi_j^\nu g^{\nu\mu} + \left(1-\frac{1}{\lambda}\right)\chi_j^\nu(\partial^\mu k_j^\nu)\right]\delta_{j,\mu} + \beta\sum_{m=1}^{N}\left[|\varphi_m|^2\chi_j^\nu\right] = 0 \tag{7.33b}$$

We now consider a particular mode j with group velocity $\underline{V}_j = 0$, i. e. $\underline{K}_j = 0$ (see (7.22b)). This condition is equivalent to choose a frame where the solution of (7.33a) is not depending on the time (the proper frame). Equations (7.33a-b) yield

$$\chi_j^i = g_j^i(\underline{\xi}), \quad \delta_j^i = \tilde{\delta}_j^i(\underline{\xi}) + \beta\tau\sum_{m=1}^{N} g_j^i(\underline{\xi}), \quad for \quad i = 1,2,3 \tag{7.34}$$

where $g_j^i(\underline{\xi})$, $\tilde{\delta}_j^i(\underline{\xi})$ are arbitrary functions of the space variables. Note that $g_j^0(\underline{\xi})$ and $\tilde{\delta}_j^0(\underline{\xi})$ are fixed by the gauge condition (7.31). By a Lorentz boost we can construct the solution in a generic frame and in the following we use a frame moving con velocity $\underline{V}_j = (V_j, 0, 0)$with respect to the proper frame

$$\xi = \gamma_j(\xi' - V_j\tau'), \quad \eta = \eta', \quad \varsigma = \varsigma', \quad \tau = \gamma_j(\tau' - \frac{V_j\xi'}{c^2}), \quad \gamma_j = \left(1-\left(\frac{V_j}{c}\right)^2\right)^{-\frac{1}{2}}. \tag{7.35}$$

On a similar way we obtain the solution for the Higgs field,

$$\rho_j = f_j(\underline{\xi}), \quad \vartheta_j = \tilde{\vartheta}_j(\underline{\xi}) - \tau\sum_{m=1}^{N}\alpha_{jm}f_m^2(\underline{\xi}) \tag{7.36}$$

where $f_j(\underline{\xi})$, $\tilde{\vartheta}_j(\underline{\xi})$ are arbitrary functions of the space variables.

At last, an interesting particular solution for the Cauchy problem of (7.32a-7.33a) reads

$$\rho_j(\underline{\xi},\tau) = \rho_j\left(\tilde{V}_j^\sigma\xi_\sigma\right), \tag{7.37a}$$

$$\chi_j^\nu(\underline{\xi},\tau) = \chi_j^\nu\left(V_j^\sigma\xi_\sigma\right), \tag{7.37b}$$

where the $4N$ real functions $\rho_j\left(\underline{\xi}\right)$, $\chi_j^i\left(\underline{\xi}\right)$, *i=1, 2, 3*, which represent the initial shape, can be chosen arbitrarily and

$$k_{j,\mu}V_j^\mu = 0, \quad \tilde{k}_{j,\mu}\tilde{V}_j^\mu = 0. \tag{7.38}$$

Inserting (7.37b) in (7.33a) yields

$$\left[(-2k_{j,\sigma}V_j^{\sigma})g^{\nu\mu} + (1-\frac{1}{\lambda})(k_j^{\nu}V_j^{\mu})\right]\chi_{j,\mu} = T_j^{\mu\nu}\chi_{j,\mu} = 0, \tag{7.39}$$

and since

$$\det T = 8\left(2-\left(1-\frac{1}{\lambda}\right)\right)(k_{j,\mu}V_j^{\mu})^4 = 0, \tag{7.40}$$

we obtain

$$\chi_{j,\mu}V_j^{\mu} = 0. \tag{7.41}$$

The field $\chi_j^0(\xi)$ is fixed by the gauge condition (7.41).

In conclusion, the approximate solution for the system of equations (7.11-7.13) is

$$A^{\nu}(x) = 2\varepsilon\sum_{j=1}^{N}\chi_j^{\nu}\exp\left[i(\delta_j^{\nu}-k_{j,\mu}x^{\mu})\right] + O(\varepsilon^3), \tag{7.42}$$

$$U(x) = \varepsilon U_1(x) + \varepsilon^2 U_2(x) + O(\varepsilon^3), \tag{7.43a}$$

where

$$U_1(x) = 2\varepsilon\sum_{j=1}^{N}\Big[\rho_j\cos(\tilde{k}_{j,\mu}x^{\mu}-\vartheta_j)\Big], \tag{7.43b}$$

$$U_2(x) = 2\varepsilon^2\sum_{j=1}^{N}\left[B_2\rho_j^2\cos\left[2\left(\tilde{k}_{j,\mu}x^{\mu}-\vartheta_j\right)\right]\right] + \varepsilon^2\left(A\rho_j^2\right) \tag{7.43c}$$

$$+\varepsilon^2\sum_{j,m=1,j\neq m}^{N}\left[B_{11,jm}\rho_j\rho_m\cos\left[\left(\tilde{k}_{j,\mu}x^{\mu}-\vartheta_j\right)+\left(\tilde{k}_{m,\mu}x^{\mu}-\vartheta_m\right)\right]\right]$$

$$+\varepsilon^2\sum_{j,m=1,j\neq m}^{N}\left[B_{1-1,jm}\rho_j\rho_m\cos\left[\left(\tilde{k}_{j,\mu}x^{\mu}-\vartheta_j\right)-\left(\tilde{k}_{m,\mu}x^{\mu}-\vartheta_m\right)\right]\right].$$

The corrections of order to the approximate solution depend on higher harmonics and can be easily calculated by the AP method.

The validity of the approximate solution should be expected to be restricted on bounded intervals of the τ-variable and on time-scale $t = O(\frac{1}{\varepsilon^2})$. If one wishes to study solutions on intervals such that $\tau = O(\frac{1}{\varepsilon})$ then the higher terms will in general affect the solution and must be included.

8. Coherent Solutions

In the following we have written the solutions in the moving frame of reference, but for simplicity we have dropped the apices in the space and time variables.

i) Nonlinear wave. The most simple solution of the system (7.29) is the plane wave

$$\rho_j = A_j = \text{constant}, \qquad \vartheta_j = \underline{\tilde{K}}_j^{'}\underline{\xi} - \tilde{\omega}_j^{'}\tau \tag{8.1a}$$

$$\chi_j^{\mu} = B_j^{\mu} = \text{constant}, \qquad \delta_j^{\mu} = \underline{K}_j^{'}\underline{\xi} - \omega_{'j}^{'}\tau \tag{8.1b}$$

where the amplitudes and wave vectors are connected according to the nonlinear dispersion relation

$$\omega_j^{'} = \underline{V}_j.\underline{K}_j^{'} + \frac{\alpha}{\omega_j^{'}}\sum_{m=1}^{N} A_m^2, \quad \tilde{\omega}_j^{'} = \underline{\tilde{V}}_j.\underline{\tilde{K}}_j^{'} + \frac{1}{\tilde{\omega}_j}\sum_{m=1}^{N}\alpha_{jm}A_m^2, \tag{8.1c}$$

χ_j^0 and δ_j^0 are fixed by the gauge condition (7.23).

ii) Solitons. In the following we seek coherent solutions and use the gauge condition (7.23) which implies, being $\chi_j^{\nu} = (\chi_j^0, \underline{\chi}_j)$,

$$\chi_j^0 = \frac{\underline{K}_j.\underline{\chi}_j}{\omega_j}. \tag{8.2}$$

The C-integrable nature of the system (7.29) implies the existence of more interesting solutions, because of the existence of arbitrary functions in the seed solutions. It is possible the existence of N solitons, with fixed speeds but arbitrary shapes, which interact each other preserving their shapes and propagate with the relative group velocity. The collision of two solitons does not produce a change in the amplitude ρ_j of each of them, but only a change in the phase given by equation (7.36).

For instance, we take

$$\rho_j(\underline{\xi},\tau) = \frac{2\tilde{A}_j}{ch\left(2\tilde{A}_j\tilde{\gamma}_j(\xi - \tilde{V}_j\tau)\right)}, \tag{8.3a}$$

$$\chi_j^i(\underline{\xi},\tau) = \frac{2A_j}{ch\left(2A_j\gamma_j(\xi - V_j\tau)\right)}, \quad for \quad \text{i=1, 2, 3,} \tag{8.3b}$$

$$\tilde{\vartheta}_j = \delta_j^{\nu} = 0 \quad for \quad \text{j=1 .. N,} \tag{8.3c}$$

where A_j, for $j = 1...N$, are real constants,

$$\tilde{\gamma}_j = \left(1 - \left(\frac{\tilde{V}_j}{c}\right)^2\right)^{-\frac{1}{2}}, \tag{8.3d}$$

γ_j is given by (7.35b) and the phase ϑ_j and δ_j^{ν} are given by (7.34-7.36). Each soliton advances with a constant velocity (the group velocity) before and after collisions. Only the phase is changed during collisions owing to the presence of the other solitons. Substituting (8.3) in (7.42-7.43) we obtain the approximate solution good to the order of ε. Each

soliton advances with a constant velocity (the group velocity) before and after collisions. Only the phase is changed during collisions owing to the presence of the other solitons.

iii) Dromions. The existence of localized solutions is possible also for C-integrable systems, because dromion solutions are not exclusive characteristics of equations integrable by the inverse scattering method.

A particular solution of the model system is given by

$$\rho_j(\underline{\xi},\tau) = \tilde{A}_j \exp\left(-\tilde{B}_j\sqrt{\tilde{\gamma}_j^2(\xi-\tilde{V}_j\tau)^2+\eta^2+\varsigma^2}\right), \tag{8.4a}$$

$$\chi_j^i(\underline{\xi},t) = A_j^i \exp\left(-B_j^i\sqrt{\gamma_j^2(\xi-V_j\tau)^2+y^2+z^2)}\right), \quad for \quad \text{i=1, 2, 3} \tag{8.4b}$$

$$\tilde{\vartheta}_j = \delta_j^\nu = 0 \quad for \quad \text{j=1, 2....N}, \tag{8.4c}$$

where A_j, $\tilde{A}_j$,B_j, $\tilde{B}_j$ are real constants and ϑ_j and δ_j^ν are given by equations (7.34-7.36). Substituting the solution (8.4) in equation (7.42-7.43) and taking $N = 2$ we obtain for two dromions with different shapes and amplitudes the approximate solution

$$U(\underline{x},t) = 2\sum_{j=1}^{2}\tilde{A}_j \exp\left(-\tilde{B}_j\sqrt{\tilde{\gamma}_j^2(x-\tilde{V}_jt)^2+y^2+z^2)}\right)\cos(\tilde{k}_{\mu,j}x^\mu+\vartheta_j), \tag{8.5}$$

$$A^\nu(\underline{x},t) = 2\sum_{j=1}^{2}A_j^\nu \exp\left(-B_j^\nu\sqrt{\gamma_j^2(x-V_jt)^2+y^2+z^2)}\right)\cos(k_{\mu,j}x^\mu+\delta_j). \tag{8.6}$$

In Fig. 4 we show a collision between two dromions for the Higgs field (see (7.43b) and (8.5) with identical mass $M = 100$ *GeV/c2*, $B1 = B_2 = M$, $P_1 = 1600$ *TeV/c*, P_2*=2000 TeV/c*): the initial condition is showed in Fig. 4a, then the two dromions collide (Fig. 4b) and then separate (Fig. 4c). We can see that dromions preserve their shapes but with a phase shift.

iv) Lumps. It is well known that in high dimension, in addition to the dromion solutions, other interesting localized solutions, formed by rational functions, are the multiple lumps. Obviously, there are many possible choices in order to obtain multilump solutions. For instance, we take

$$\rho_j(\underline{\xi},\tau) = \frac{\tilde{A}_j}{\tilde{B}_j+\tilde{C}_j\sqrt{\tilde{\gamma}_j^2\left(\xi-\tilde{V}_j\tau\right)^2+\eta^2+\varsigma^2}}, \tag{8.7a}$$

$$\chi_j^i(\underline{\xi},\tau) = \frac{A_j^i}{B_j^i+C_j^i\sqrt{\gamma_j^2\left(\xi-V_j\tau\right)^2+\eta^2+\varsigma^2}}, \quad for\text{i=1, 2, 3}, \tag{8.7b}$$

$$\tilde{\vartheta}_j = \delta_j^\nu = 0 \quad for \quad \text{j=1, ...N}, \tag{8.7c}$$

where $\tilde{A}_j$, $\tilde{B}_j$, $\tilde{C}_j$, A_j^i, B_j^i and C_j^i are arbitrary constants and ϑ_j^ν and δ_j^ν are given by equations (7.34-7.36).

v) Ring solitons. The multiple ring solitons are solutions that are not equal to zero at some closed curves and decay exponentially away from the closed curves. A possible selection is

$$\rho_j(\underline{\xi},\tau) = \tilde{A}_j \exp(-\tilde{B}_j f_j(R_j(\underline{\xi},\tau))), \tag{8.8a}$$

$$\chi^i_j(\underline{\xi},\tau) = A^i_j \exp(-B^i_j f_j(\tilde{R}_j(\underline{\xi},\tau))), \quad for \quad \text{i=1, 2, 3,} \tag{8.8b}$$

$$\tilde{\vartheta}_j = \delta^\nu_j = 0 \quad for \quad \text{j=1 .. N.} \tag{8.8c}$$

where

$$\tilde{R}_j = \sqrt{\tilde{\gamma}^2_j \left(\xi - \tilde{V}_j\tau\right)^2 + \eta^2 + \varsigma^2}, \quad R_j = \sqrt{\gamma^2_j \left(\xi - V_j\tau\right)^2 + \eta^2 + \varsigma^2}, \tag{8.9a}$$

$$f_j(R_j) = (R_j - R_{0,j})^2, \quad f_j(\tilde{R}_j) = (R_j - \tilde{R}_{0,j})^2, \tag{8.9b}$$

and A_j, A^i_j, B_j, B^i_j, $R_{0,j}$and $\tilde{R}_{0,j}$ are arbitrary constants. In Fig. 5 we show a collision between two ring solitons for the Higgs field (see (7.43b) and (8.8a) with identical mass $M = 100\ GeV/c^2$, $B_1 = B_2 = M^2$, $P_1 = P_2$=*2000 TeV/c*, $R_1 = 6/M, R_2 = 12/M$): the initial condition is showed in Fig. 5a, then the two ring solitons collide (Fig. 5b) and then separate (Fig. 5c). We can see that these solutions preserve their shapes but with a phase shift.

vi) Instantons. If we choose a decaying function of time, we obtain also multiple instanton solutions, for example,

$$\rho_j(\underline{\xi},\tau) = \tilde{A}_j \exp\left[-\tilde{B}_j\tilde{\gamma}_j\left(\xi - \tilde{V}_j\tau\right)\right], \tag{8.10a}$$

$$\chi^i_j(\underline{\xi},\tau) = A^i_j \exp\left[-B^i_j\gamma_j\left(\xi - V_j\tau\right)\right], \quad for \quad \text{i=1, 2, 3,} \tag{8.10b}$$

$$\tilde{\vartheta}_j = \delta^\nu_j = 0 \quad for \quad \text{j=1 .. N,} \tag{8.10c}$$

where $\tilde{A}_j$, A^i_j, $\tilde{B}_j$, B^i_j are arbitrary constants.

vii) Moving breather-like structures. Finally, if we choose some types of periodic functions of time in the above mentioned solutions, then we obtain breathers. For example, we take

$$\rho_j(\underline{\xi},\tau) = \tilde{A}_j \cos(\tilde{\gamma}_j\left(\xi - \tilde{V}_j\tau\right)) \exp\left[-\tilde{B}_j\sqrt{\tilde{\gamma}^2_j\left(\xi - \tilde{V}_j\tau\right)^2 + \eta^2 + \varsigma^2}\right], \tag{8.11a}$$

$$\chi^i_j(\underline{\xi},\tau) = A^i_j \cos(\gamma_j\left(\xi - V_j\tau\right)) \exp\left[-B^i_j\sqrt{\gamma^2_j\left(\xi - V_j\tau\right)^2 + \eta^2 + \varsigma^2}\right], \quad for \quad \text{i=1, 2, 3,} \tag{8.11b}$$

$$\tilde{\vartheta}_j = \delta^\nu_j = 0 \quad for \quad \text{j=1 .. N,} \tag{8.11c}$$

where A_j, A^i_j, $\tilde{B}_j$, $B^i_j$are arbitrary constants.

9. Chaotic and Fractal Solutions

i) Chaotic-chaotic and chaotic-periodic patterns. If we select at least one of the arbitrary functions of Section 7 in order to contain some chaotic solutions of nonintegrable equations, then we obtain some type of space-time chaotic patterns, the so-called chaotic-chaotic (in all spatial directions) patterns. For example, we choose the arbitrary function as solution of the chaotic Lorenz system

$$X_T = -c(X - Y), \quad Y_T = X(a - Z) - Y, \quad Z_T = XY - bZ, \tag{9.1a}$$

with$a = 60$, $b = 8/3$, $c = 10$,or of the R**ö**ssler system

$$X_T = -Y - Z, \quad Y_T = X + aY, \quad Z_T = b + Z(X - c), \tag{9.1b}$$

with $a = 0.15$, $b = 0.2$, $c = 10$and $T = \gamma(\xi - V\tau)$ (or $T = \eta$ or $T = \varsigma$). A phase and amplitude chaotic-chaotic pattern is given by

$$\rho_j(\underline{\xi}, \tau) = X(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau))Y(\eta)Z(\zeta), \tag{9.2a}$$

$$\chi^i_j(\underline{\xi}, \tau) = X^i_j(\gamma_j(\xi - V_j\tau))Y^i_j(\eta)Z^i_j(\zeta), \quad for \quad \text{i=1, 2, 3,} \quad \chi^0_j = \frac{\underline{K}_j \cdot \underline{\chi}_j}{\omega_j} \tag{9.2b}$$

$$\vartheta_j = \tilde{\delta}^\nu_j = 0 \quad for \quad \text{j=1, 2....N,} \tag{9.2c}$$

while ϑ_j and δ^ν_j is given by equations (7.34-7.36). An example is given in Fig. 6, for the Higgs field ((7.43b) and (9.2a)) with $M = 100\ GeV/c^2$ and $\gamma = 50$.

On the contrary, we obtain a phase chaotic-chaotic pattern, if we choose the function (9.2c) as solution of the Lorenz system. Finally, if we select a chaotic-periodic solution which is chaotic in one (or two) direction and periodic in the other(s) direction(s). then we obtain the so-called chaotic-periodic patterns.

ii) Chaotic line soliton solutions

If we consider the soliton line solution (8.3), we can easily deduce a chaotic solution when we select A_j as solution of the Lorenz system,

$$\rho_j(\underline{\xi}, \tau) = \frac{2A_j(\underline{\xi}, \tau)}{ch\left(2\tilde{\gamma}_j(\xi - \tilde{V}_j\tau)A_j(\underline{\xi}, \tau)\right)}, \quad \vartheta_j = \tilde{\delta}^\nu_j = 0 \quad for \quad \text{j=1 .. N,} \tag{9.3a}$$

$$\chi^i_j(\underline{\xi}, \tau) = \frac{2A^i_j(\underline{\xi}, \tau)}{ch\left(2\gamma_j(\xi - V_j\tau)A^i_j(\underline{\xi}, \tau)\right)}, \quad for \quad \text{i=1, 2, 3,} \quad \chi^0_j = \frac{\underline{K}_j \cdot \underline{\chi}_j}{\omega_j} \tag{9.3b}$$

where the phases ϑ_j and δ^ν_j are given as usual by (7.34-7.36), for $j = 1...N$, and the functions $A^i_j = A^i_j(T_j) = A^i_j(\gamma_j(\underline{\xi} - V_j\tau))$ satisfy the third order ordinary differential equation equivalent to the Lorenz system (9.1),

$$\begin{aligned} &A_{j,TTT} + (b + c + 1)A_{j,TT} + (bc + b + A^2_j)A_{j,T} + c(b - ab + A^2_j)A_j \\ &-\tfrac{A_{j,TT}A_{j,T} + (c+1)A^2_{j,T}}{A_j} = 0. \end{aligned} \tag{9.4}$$

iii) Chaotic dromion and lump patterns

If we consider the dromion solution (8.4), we can transform it into a chaotic pattern with an appropriate choice for A_j and/or B_j,

$$\rho_j(\underline{\xi},\tau)=A_j\exp(-B_j\sqrt{\tilde{\gamma}_j^2\left(\xi-\tilde{V}_j\tau\right)^2+\eta^2+\varsigma^2}), \tag{9.5a}$$

$$\chi_j^i(\underline{\xi},\tau)=A_j^i\exp\left(-B_j^i\sqrt{\gamma_j^2\left(\xi-V_j\tau\right)^2+\eta^2+\varsigma^2}\right),\quad for\quad i=1,\ 2,\ 3,\quad \chi_j^0=\frac{\underline{K}_j\cdot\underline{\chi}_j}{\omega_j} \tag{9.5b}$$

$$\vartheta_j=\tilde{\delta}_j^\nu=0\quad for\quad \text{j=1, 2....N}, \tag{9.5c}$$

where the function $A_j=A_j(T_j)=A_j(\gamma_j(\xi-\tilde{V}_j\tau))$ and/or the other amplitude and shape functions $B_j=B_j(T_j)=B_j(\tilde{\gamma}_j(\xi-\tilde{V}_j\tau))$,for the scalar field, and $A_j^i=A_j^i(T_j)=A_j^i(\gamma_j(\xi-V_j\tau))$ and$B_j^i=B_j^i(T_j)=B_j^i(\gamma_j(\xi-V_j\tau))$ for the vectorial field are solutions of the Lorenz equation (9.4) and $\vartheta_j$and δ_j^ν are given by equations (7.34-7.36). We obtain an amplitude (A_j chaotic) or a shape (B_j chaotic) or an amplitude and shape (A_j and B_j chaotic) dromion chaotic pattern. Similar considerations can be applied to the lump solutions (8.7).

iv) Nonlocal fractal solutions. If we choose

$$\rho_j(\underline{\xi},\tau)=\prod_{m=1}^{3}\tilde{T}_{m,j}\left|\tilde{T}_{m,j}\right|\left\{\sin\left[\ln\left(\tilde{T}_{m,j}^2\right)\right]-\cos\left[\ln\left(\tilde{T}_{m,j}^2\right)\right]\right\} \tag{9.6a}$$

$$\chi_j^i(\underline{\xi},\tau)=\prod_{m=1}^{3}T_{m,j}\left|T_{m,j}\right|\left\{\sin\left[\ln\left(T_{m,j}^2\right)\right]-\cos\left[\ln\left(T_{m,j}^2\right)\right]\right\},\quad for\quad \text{i=1, 2, 3},\quad \chi_j^0=\frac{\underline{K}_j\cdot\underline{\chi}_j}{\omega_j} \tag{9.6b}$$

with $\underline{\tilde{T}}_j=(\tilde{T}_{1,j},\tilde{T}_{2,j},\tilde{T}_{3,j})$, $\underline{\tilde{T}}_j=(\tilde{\gamma}_j(\xi-\tilde{V}_j\tau),\eta,\varsigma)$, $\underline{T}_j=(T_{1,j},T_{2,j},T_{3,j})$, $\underline{T}_j=(\gamma_j(\xi-V_j\tau),\eta,\varsigma)$we get a nonlocal fractal structure for small $\underline{T}_j$and $\underline{\tilde{T}}_j$. It is well known that if we plot the structure of the solution at smaller regions we can obtain the same structures.

v) Fractal dromion and lump solutions. A fractal dromion (lump) solution is exponentially (algebraically) localized in large scale and possesses self-similar structure near the center of the dromion. We consider for example an amplitude fractal dromion

$$\rho_j(\underline{\xi},\tau)=\tilde{A}_j\exp(-\tilde{B}_j\sqrt{\tilde{\gamma}_j^2\left(\xi-\tilde{V}_j\tau\right)^2+\eta^2+\varsigma^2}), \tag{9.7a}$$

$$\chi_j^i(\underline{\xi},\tau)=A_j\exp(-B_j\sqrt{\gamma_j^2\left(\xi-V_j\tau\right)^2+\eta^2+\varsigma^2}),\quad for\quad \text{i=1, 2, 3},\quad \chi_j^0=\frac{\underline{K}_j\cdot\underline{\chi}_j}{\omega_j} \tag{9.7b}$$

$$\vartheta_j=\tilde{\delta}_j^\nu=0\quad for\quad \text{j=1, 2....N}, \tag{9.7c}$$

where ϑ_j and δ_j^ν are given by equations (7.34-7.36) and $A_j^i=A_j^i(T_j)=A_j^i(\gamma_j(\xi-V_j\tau))$, $A_j=A_j(T_j)=A_j(\tilde{\gamma}_j(\xi-\tilde{V}_j\tau))$ are given by

$$A_j^i(T_j)=A_j(T_j)=2+\sin\left\{\ln\left[T_j^2\right]\right\}. \tag{9.8}$$

By a similar choice for B_j or $\tilde{B}_j$, $\tilde{\delta}_j$, $\tilde{\vartheta}_j$we obtain shape or phase fractal dromion.

vi) Stochastic fractal dromion and lump excitations. It is well known the stochastic fractal property of the continuous but nowhere differentiable Weierstrass function

$$W(x) = \sum_{k=1}^{N} (c_1)^k \sin\left[(c_2)^k\, x\right], \quad N \to \infty, \tag{9.9a}$$

with c_2 odd and

$$c_1 c_2 > 1 + \frac{3\pi}{2}. \tag{9.9b}$$

A stochastic fractal solution is

$$\rho_j(\underline{\xi}, \tau) = \tilde{A}_{1,j}(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau))\tilde{A}_{2,j}(\eta)\tilde{A}_{3,j}(\varsigma), \tag{9.10a}$$

$$\chi_j^i(\underline{\xi}, \tau) = A_{1,j}^i(\gamma_j(\xi - V_j\tau))A_{2,j}^i(\eta)A_{3.j}^i(\varsigma), \quad for \quad \text{i=1, 2, 3}, \quad \chi_j^0 = \frac{\underline{K}_j \cdot \underline{\chi}_j}{\omega_j} \tag{9.10b}$$

$$\vartheta_j = \tilde{\delta}_j^\nu = 0 \quad for \quad \text{j=1, 2....N}, \tag{9.10c}$$

where ϑ_j and δ_j^ν are as usual given by equations (7.34-7.36), $\underline{A}_j = (A_{1,j}, A_{2,j}, A_{3,j})$, $\underline{\tilde{A}}_j = (\tilde{A}_{1,j}, \tilde{A}_{2,j}, \tilde{A}_{3,j})$, and $\underline{A}_j = \underline{A}_j(\gamma_j(\xi - V_j\tau), \eta, \varsigma)$, $\underline{\tilde{A}}_j = \underline{\tilde{A}}_j(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau), \eta, \varsigma)$ are given by

$$A_{1,j}(\gamma_j(\xi - V_j\tau)) = A_{1,j}(\gamma_j(\xi - V_j\tau)) = W(\gamma_j(\xi - V_j\tau)) + \gamma_j^2(\xi - V_j\tau)^2, \tag{9.11a}$$

$$A_{2,j}(\eta) = \tilde{A}_{2,j}(\eta) = W(\eta) + (\eta)^2 \quad , \quad A_{3,j}(\varsigma) = \tilde{A}_{3,j}(\varsigma) = W(\varsigma) + (\varsigma)^2. \tag{9.11b}$$

An example is given in Fig. 4 for the Higgs field (7.43b) and (9.10a) with $M = 100\ GeV/c^2$ and $\gamma = 50$.

vii) Stochastic fractal dromion and lump excitations. In order to obtain a stochastic amplitude fractal dromion we choose

$$\rho_j(\underline{\xi}, \tau) = \tilde{A}_j \exp(-\tilde{B}_j\sqrt{\tilde{\gamma}_j^2\left(\xi - \tilde{V}_j\tau\right)^2 + \eta^2 + \varsigma^2}), \tag{9.12a}$$

$$\chi_j^i(\underline{\xi}, \tau) = A_j^i \exp(-B_j^i\sqrt{\gamma_j^2\,(\xi - V_j\tau)^2 + \eta^2 + \varsigma^2}), \quad for \quad \text{i=1, 2, 3}, \quad \chi_j^0 = \frac{\underline{K}_j \cdot \underline{\chi}_j}{\omega_j} \tag{9.12b}$$

$$\vartheta_j = \tilde{\delta}_j^\nu = 0 \quad for \quad \text{j=1, 2....N}, \tag{9.12c}$$

where ϑ_j and δ_j^ν are given by equations (7.34-7.36) and $A_j^i = A_j^i(T_j) = A_j^i(\gamma_j(\xi - V_j\tau))$, $\tilde{A}_j = \tilde{A}_j(T_j) = \tilde{A}_j(\tilde{\gamma}_j(\xi - \tilde{V}_j\tau))$ are given by

$$A_j^i(T_j) = \tilde{A}_j(T_j) = W(T_j) + T_j^2, \tag{9.13}$$

By similar methods we obtain shape or phase stochastic fractal dromion as well as stochastic fractal lump solutions.

10. Conclusion

Many extensions of the work exposed in the precedent sections are possible, for example the investigation of nonlinear equations with solitons transporting superluminal signals [60], a simple technique for obtaining nonlinear equations with dromions of a given shape and velocity [61] and a modification of the Einstein general relativity equations that can produce various types of coherent solutions [64].

However, a major problem is the possibility of identification between dromions and elementary particles and indeed de Broglie [9], Bohm [8] and others ([71],[72],[73],[74]) hoped for the explanation of quantum mechanics through nonlinear classic effects.

Notably among others the Skyrme model ([77],[78],[79],[2]) describes nucleons and nucleon-nucleon interactions, while topological solitons give rise to quantization of charges. A localized and stable wave might be a good model for elementary, but we have seen that in nonlinear field equations there is a great variety of coherent solutions and chaotic and fractal patterns. If particles are excitations of nonlinear fields, it is clear that they are not the only possible excitations.

On the contrary, the quantization of the nonlinear solutions is complicated because there is no superposition principle. For example the shape of the dromion cannot be considered the shape of the wave function for the reason that a quantum soliton cannot be localized in space all the time and the uncertainty principle will cause a spreading. In the last years many methods have been proposed in order to realize the quantization that however seems to be possible in a satisfactory way only for weak nonlinear couplings.

In the next future, an exciting field of research will be the investigation of the physical interpretation of coherent, chaotic and fractal solutions in elementary particles physics. It is necessary to study further the behavior of the solutions, beyond the leading order in the expansion parameter, as well as the derivation of the model equations for the interactions among phase resonant waves.

References

[1] M. J. Ablowitz and P. A. Clarkson, *Solitons, Nonlinear Evolution Equations and Inverse Scattering*, Cambridge, Cambridge University Press (1990).

[2] G. S. Adkins, C. R. Nappi, E. Witten, *Nuclear Physics* **228B**, 552 (1983).

[3] M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur, *Physical Review Letters* **31**, 125, (1973).

[4] A. G. Abanov and P. B. Wiegmann, *Physical Review Letters* **86,** 1319 (2001).

[5] D. Anker, N. C. Freeman, *Proc. Roy. Soc. London A* **360**, 529 (1978).

[6] D. J. Benney, G. J. Roskes, *Studies Applied Mathematics* **47**, 377 (1969).

[7] M. Boiti, J. J.-P. Leon, M. Manna, F. Pempinelli, *Inverse Problems* **2**, 271 (1986).

[8] D. Bohm, *Causality and Chance in Modern Physics*, London, Routledge & Kegan (1957).

[9] L. de Broglie, *The current interpretation of Wave Mechanics - A Critical Study,* Amsterdam, Elsevier, (1964).

[10] A. Davey and K. Stewartson, *Proc. R. Soc. London* **A 338**, 101 (1974).

[11] A. S. Davydov, *Solitons in Molecular Systems*, Boston, Reidel (1991).

[12] P. A. M. Dirac, *Proceedings of the Royal Society A* **117**, 610 (1928).

[13] P. G. Drazin, R. S. Johnson, *Solitons : An Introduction*, Cambridge, Cambridge University Press (1989).

[14] G. S. Emmerson, *John Scott Russell*, London, John Murray, (1977).

[15] A. T. Filippov, *The Versatile Soliton*, Boston, Birkhauser (2000).

[16] A. S. Fokas and P. M. Santini, *Physical Review Letters* **63**, 1329 (1989).

[17] A. S. Fokas and P. M. Santini, *Physica* **D44**, 99 (1990).

[18] C. S. Gardner, J. M. Greene, M. D. Kruskal, R. M. Miura, *Physical Review Letters* **19**, 1095, (1967).

[19] J. Goldstone, *Nuovo Cimento* **19**, 154 (1961).

[20] A. Hasegawa, *Optical Solitons in Fibers*, New York, Springer-Verlag, (1990).

[21] J. Hietarinta, *Physics Letters A* **149**, 133 (1990).

[22] P. W. Higgs, *Physical Review Letters* **13**, 508 (1964).

[23] P. W. Higgs, *Physical Review* **145**, 1156 (1966).

[24] R. Hirota, *Journal of Mathematical Physics* **14**, 805 (1973).

[25] E. Infeld and N. Rowlands, *Nonlinear Waves, Solitons and Chaos*, Cambridge, Cambridge University Press (1992).

[26] B. B. Kadomtsev, V. I. Petviashvili, *Sov. Phys. Dokl.* **15**, 539 (1970) [*Dokl. Akad. Nauk SSSR* **192**, 753 (1970)].

[27] D. J. Korteweg, G. de Vries, *Phil. Mag.* **39**, 422 (1895).

[28] Y. S. Kivshar and B. A. Malomed, *Review of Modern Physics* **61**, 763 (1989).

[29] P. P. Kulish, V. D. Lipovsky, *Physics Letters A* **127**, 413 (1988)

[30] L. Lam, J. Prost, *Solitons in Liquid Crystals*, New York, Springer-Verlag, 1992.

[31] P. D. Lax, *Communications Pure and Applied Mathematics* **21**, 467, (1968).

[32] J. Lin, F.-m. Wu, *Chaos, Solitons and Fractals* **19**, 189 (2004).

[33] S.-Y. Lou, C.-L. Chen, X.-Y. Tang, *Journal of Mathematical Physics* **43**, 4078 (2002).

[34] S.-Y. Lou, J.-Z. Lu, *Journal of Physics A* **29**, 4029 (1996).

[35] S.-Y. Lou, H.-Y. Ruan, *Journal of Physics A* **34**, 305 (2001).

[36] S.-y. Lou, *Physics Letters A* **277**, 94 (2000).

[37] S.-y. Lou, *Physica Scripta* **65**, 7 (2000).

[38] S.-y. Lou, *Physics Letters A* **276**, 1 (2000).

[39] A. Maccari, *Il Nuovo Cimento* **111B**, 917, (1996).

[40] A. Maccari, *Journal of Mathematical Physics* **37**, 6207 (1996).

[41] A. Maccari, *Journal of Mathematical Physics* **38**, 4151 (1997).

[42] A. Maccari, *Journal of Plasma Physics* **60**, 275 (1998).

[43] A. Maccari, *Journal of Mathematical Physics* **39**, 6547, (1998).

[44] A. Maccari, *Journal of Physics* **A32**, 693 (1999).

[45] A. Maccari, *Journal of Mathematical Physics* **40**, 3971, (1999).

[46] A. Maccari, *International Journal of Non-Linear Mechanics,* **35**, 239, (2000).

[47] A. Maccari, *Physics Letters A* **276**, 79, (2000).

[48] A. Maccari, *Physica D* **135**, 331, (2000).

[49] A. Maccari, *Physics Letters A* **265**, 187, (2000).

[50] A. Maccari, *International Journal of Non-Linear Mechanics* **36**, 335, (2001).

[51] A. Maccari, *Journal of Mathematical Physics* **42**, 2689, (2001).

[52] A. Maccari, *Chaos, Solitons and Fractals* **14**, 105, (2002).

[53] A. Maccari, *Journal of Nonlinear Mathematical Physics* **9**, 11, (2002).

[54] A. Maccari, *Nonlinearity* **15**, 807, (2002).

[55] A. Maccari, *Chaos, Solitons and Fractals* **15**, 141 (2003).

[56] A. Maccari, *Journal of Mathematical Physics* **44**, 242, (2003).

[57] A. Maccari, *Chaos, Solitons and Fractals* **16**, 245, (2003).

[58] A. Maccari, *Chaos, Solitons and Fractals* **17**, 141, (2003).

[59] A. Maccari, *Journal of Mathematical Physics* **45**, 4506, (2004).

[60] A. Maccari, *Chaos, Solitons and Fractals* **23**, 1099, (2005).

[61] A. Maccari, *Physica Scripta* **71**, 20, (2005).

[62] A. Maccari, *Physics Letters A* **336**, 117, (2005).

[63] A. Maccari, *Chaos, Solitons and Fractals* **27**, 363, (2006).

[64] A. Maccari, *Chaos, Solitons and Fractals, Special issue: Sir Hermann Bondi 1919-2005* **28**, 846, (2006).

[65] M. Makino, T. Kamimura, T. Taniuti, *Journal Physics Society Japan* **50**, 980 (1981).

[66] V. B. Matveev, M. A. Salle, *Darboux Transformations and Solitons*, Springer, Berlin, (1991).

[67] K. Nishinari, K. Abe, J. Satsuma, *J. Phys. Soc. Jpn.* **62**, 2021 (1993).

[68] G. D. Pang, F. C. Pu, B. H. Zhao, *Physical Review Letters* **65**, 3227 (1990).

[69] K. K. Perring, T. H. R. Skyrme, *Nuclear Physics* **31**, 550 (1962).

[70] M. Remoissenet, *Waves Called Solitons: Concepts and Experiments*, New York, Springer-Verlag, 1992.

[71] Yu. P. Rybakov, *Foundations of Physics* **4,** 149 (1974).

[72] Yu. P. Rybakov, *Ann. Fond. L. de Broglie* **2,** 181 (1977)

[73] Yu. P. Rybakov, B. Saha, *Foundations of Physics* **25,** 1723 (1995).

[74] Yu. P. Rybakov, B. Saha, *Physics Letters A* **122,** 5 (1996)

[75] C. L. Schultz, M. J. Ablowitz, D. BarYaacov, *Physical Review Letters* **59**, 2825 (1987).

[76] E. I. Shulman, *Teor.Mat.Fiz.* **56**, 131 (1983) [*Theoric and Mathematical Physics* **56**, 720 (1983)].

[77] T. H. R. Skyrme, *Proceedings of the Royal Society of London* **260A**, 127, (1961).

[78] T. H. R. Skyrme, *Proceedings of the Royal Society of London* **262A**, 237, (1961).

[79] T. H. R. Skyrme, *Nuclear Physics* **31**, 556 (1962).

[80] X.-Y. Tang, S.-y. Lou, *Communications Theoretical Physics* **38**, 1 (2001).

[81] X.-Y. Tang, S.-y. Lou, *Chaos, Solitons and Fractals* **14**, 1451 (2002).

[82] X.-Y. Tang, C.-L. Chen, S.-y. Lou, *Journal of Physics A* **35**, L293 (2002).

[83] X.-Y. Tang, S.-Y. Lou, Y. Zhang, *Physical Review E*, **66**, 46601 (2002).

[84] G. 't Hooft, *Nuclear Physics B* **35**, 167 (1971).

[85] J.-P.Ying, S.-Y. Lou, *Z. Naturforsch., A: Physical Science* **56**, 619 (2001)

[86] N. J. Zabusky, M. D. Kruskal, *Physical Review Letters* **15**, 240, (1965).

[87] V. E. Zakharov, A. B. Shabat, *Zh. Eksp. Teor. Fiz.* **61**, 118, (1971) [*Sov. Phys. JETP* **34**, 62, (1972)]

[88] V. E. Zakharov, I. Shulman, *Physica D* **1**, 192, (1980)

[89] C.-L. Zheng, *Chinese Journal of Physics* **41**, 442 (2003).

Figure Captions

Fig. 1: A ring soliton. The initial condition is represented in Fig. 1a, then the two coherent solutions undergo a collision (Fig. 1b) and separate (Fig. 1c). Note that the z-variable has been suppressed in order to construct a more clear solution representation.

Fig. 2: An amplitude chaotic-chaotic pattern. Note that the z-variable has been suppressed in order to construct a more clear solution picture. Surface plot is shown in the region X = [-100, 100], Y= [-100, 100] .

Fig. 3: A stochastic fractal solution with the Weierstrass function. Note that the z-variable has been suppressed in order to construct a more clear solution picture. Surface plot is shown in the region X = [-0.21, 0.21], Y = [-0.21, 0.21].

Fig. 4: Evolution plots of two dromions with identical shapes and amplitudes. Note that the z-variable has been suppressed in order to construct a more clear solution picture. The initial condition is represented in Fig. 4a, then the two dromions undergo a collision (Fig. 4b) and separate (Fig. 4c).

Fig. 5: A ring soliton. The initial condition is represented in Fig. 5a, then the two coherent solutions undergo a collision (Fig. 5b) and separate (Fig. 5c). Note that the z-variable has been suppressed in order to construct a more clear solution representation.

Fig. 6: An amplitude chaotic-chaotic pattern. Note that the z-variable has been suppressed in order to construct a more clear solution picture. Surface plot is shown in the XY-region defined by X = [-100, 100], Y= [-100, 100] .

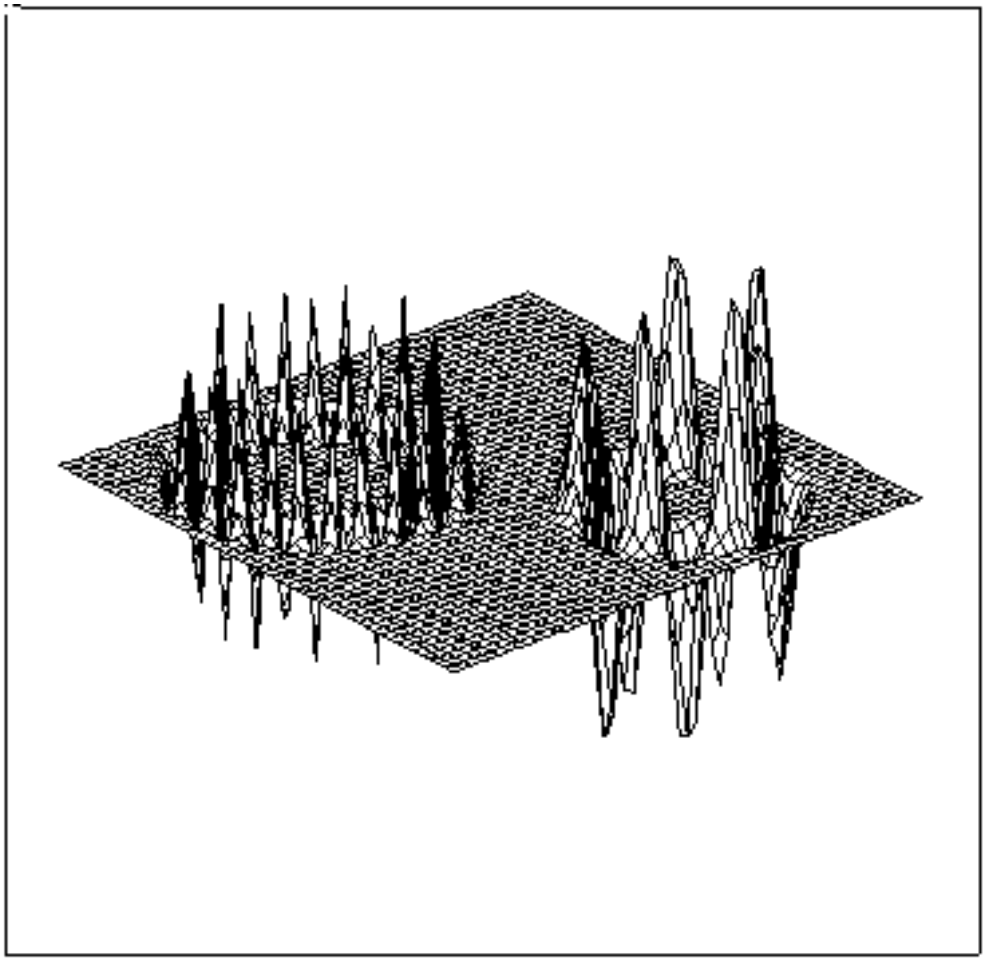


Figure 1a

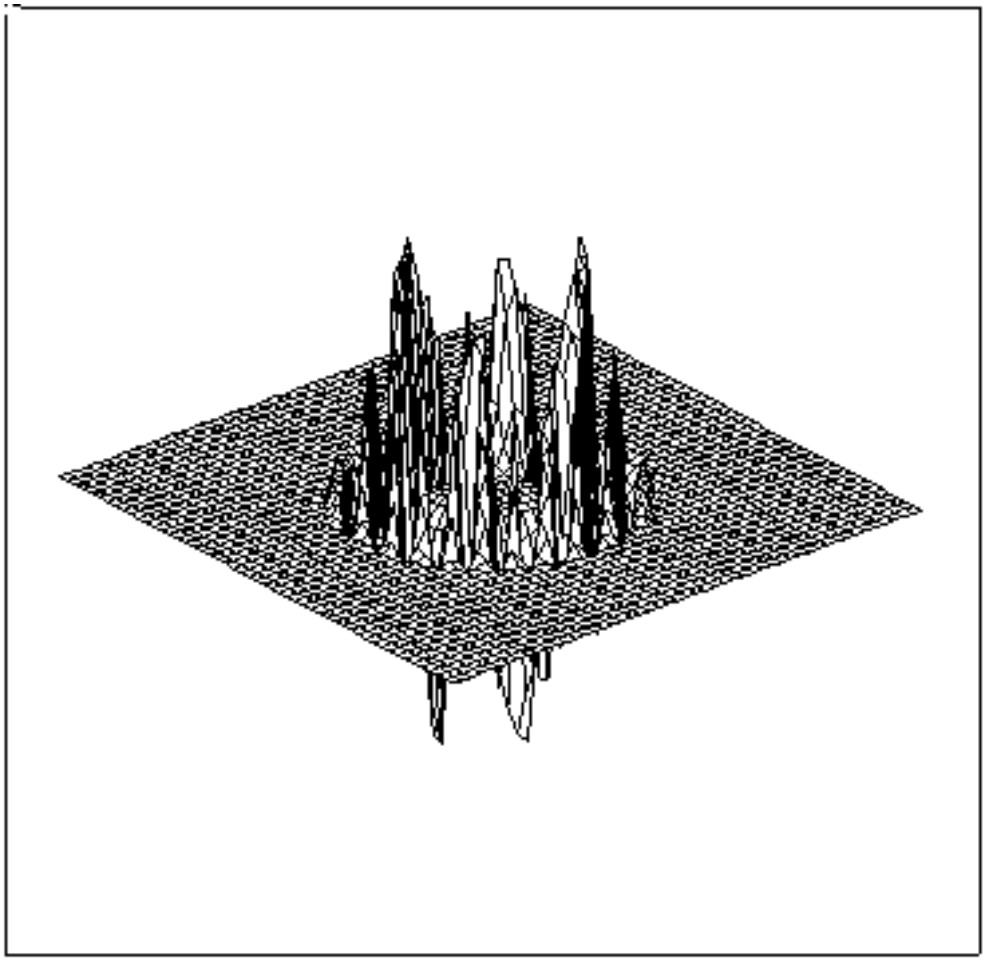


Figure 1b

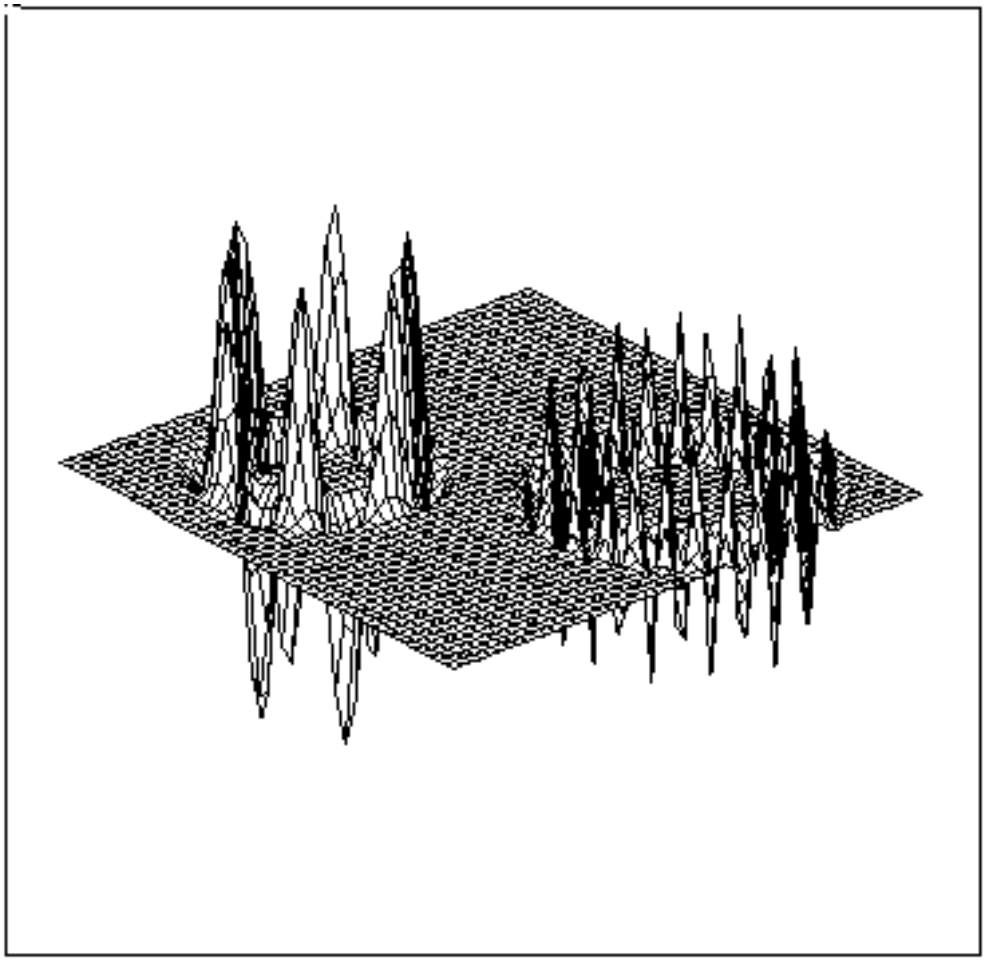


Figure 1c

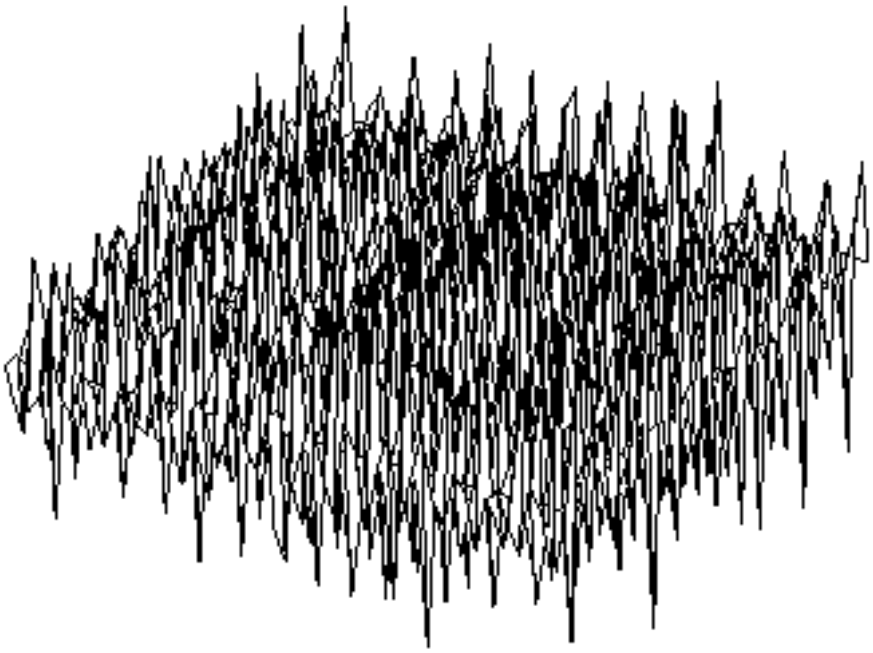


Figure 2

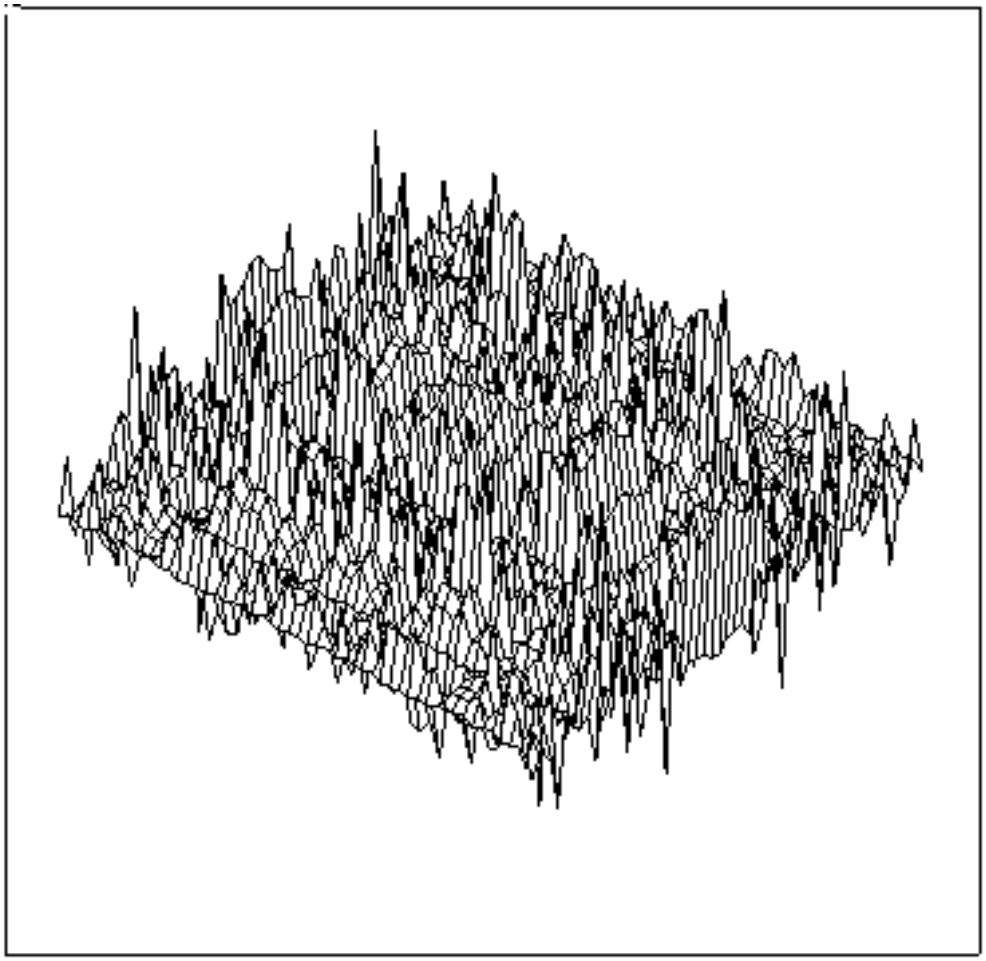


Figure 3

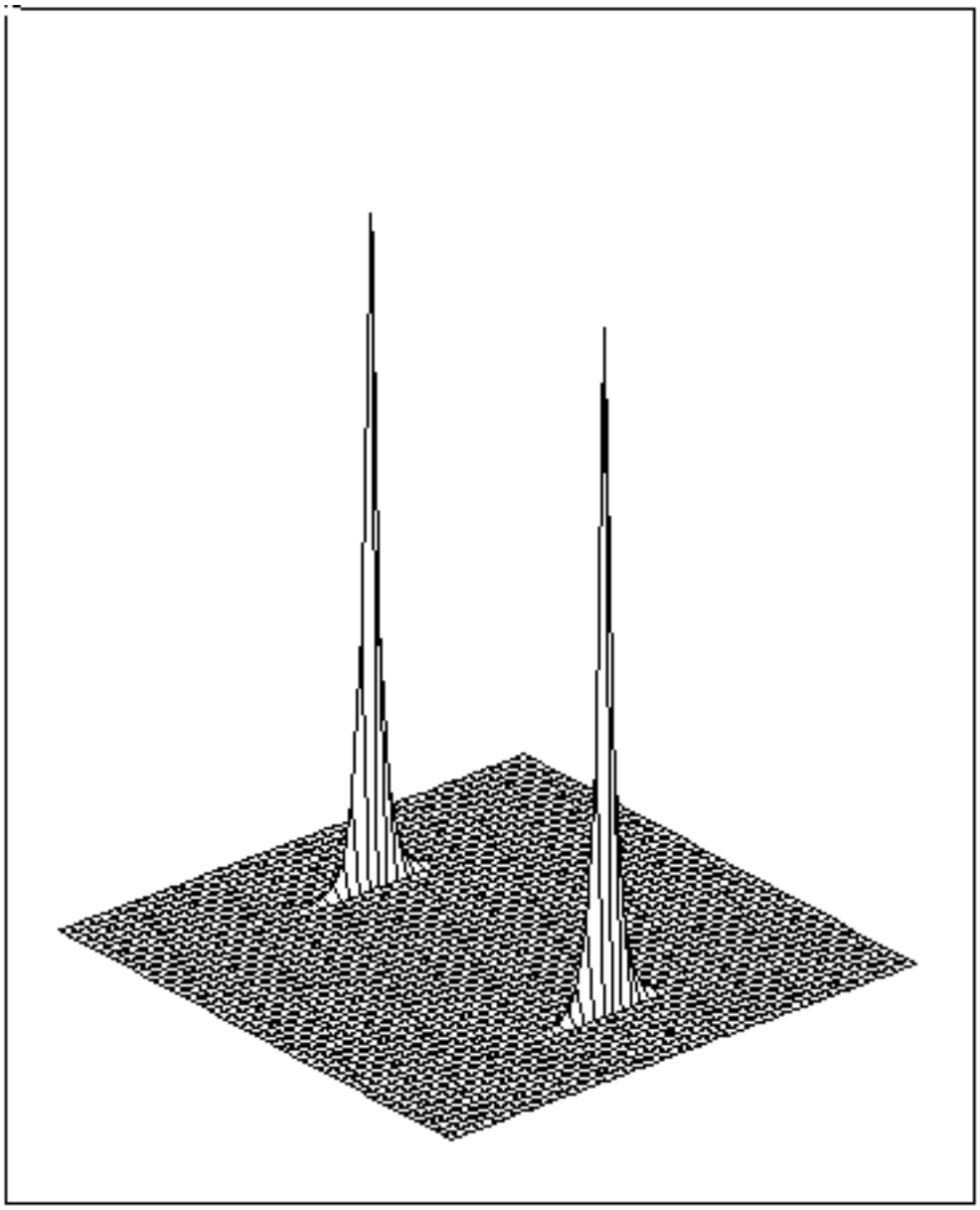


Figure 4a

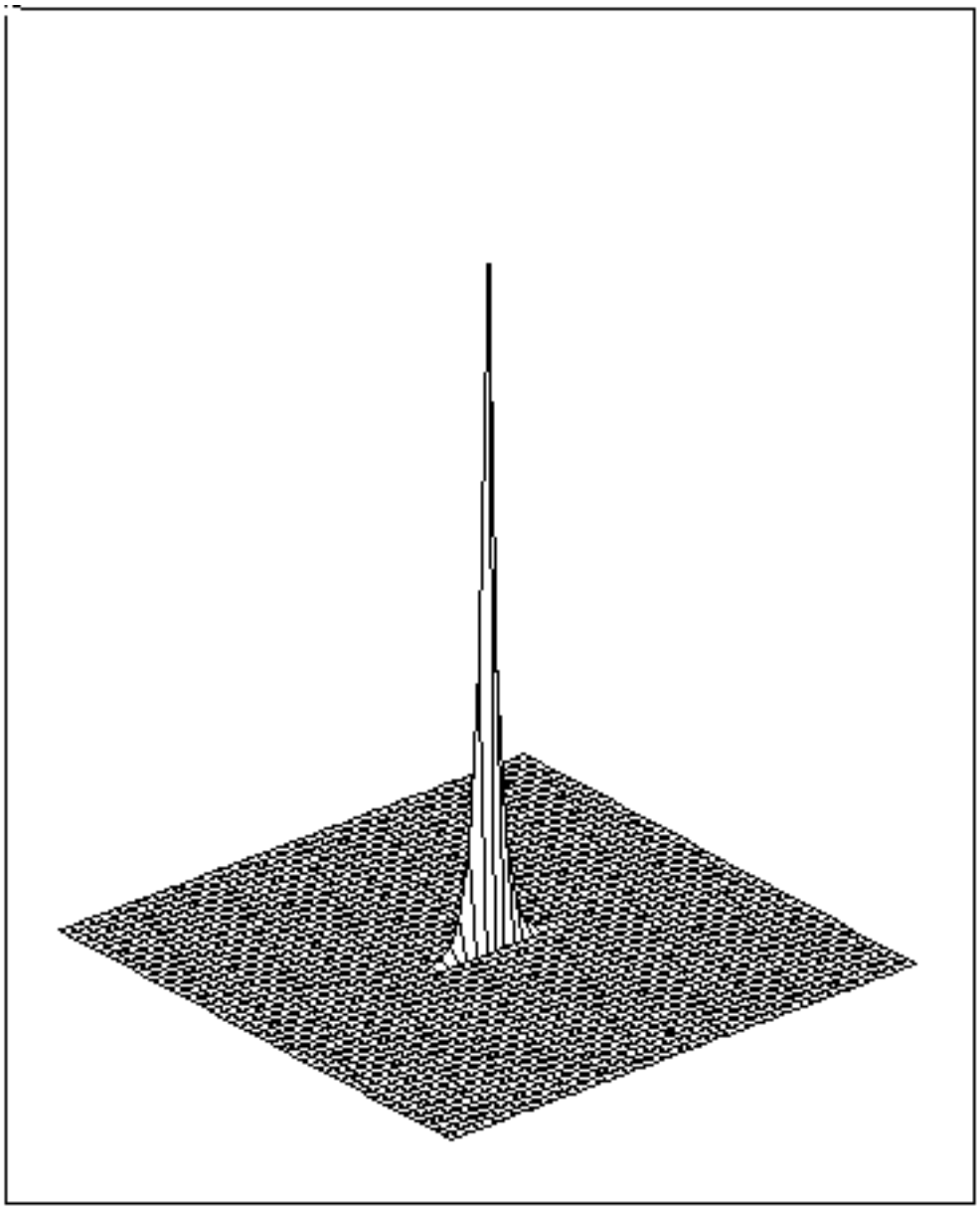


Figure 4b

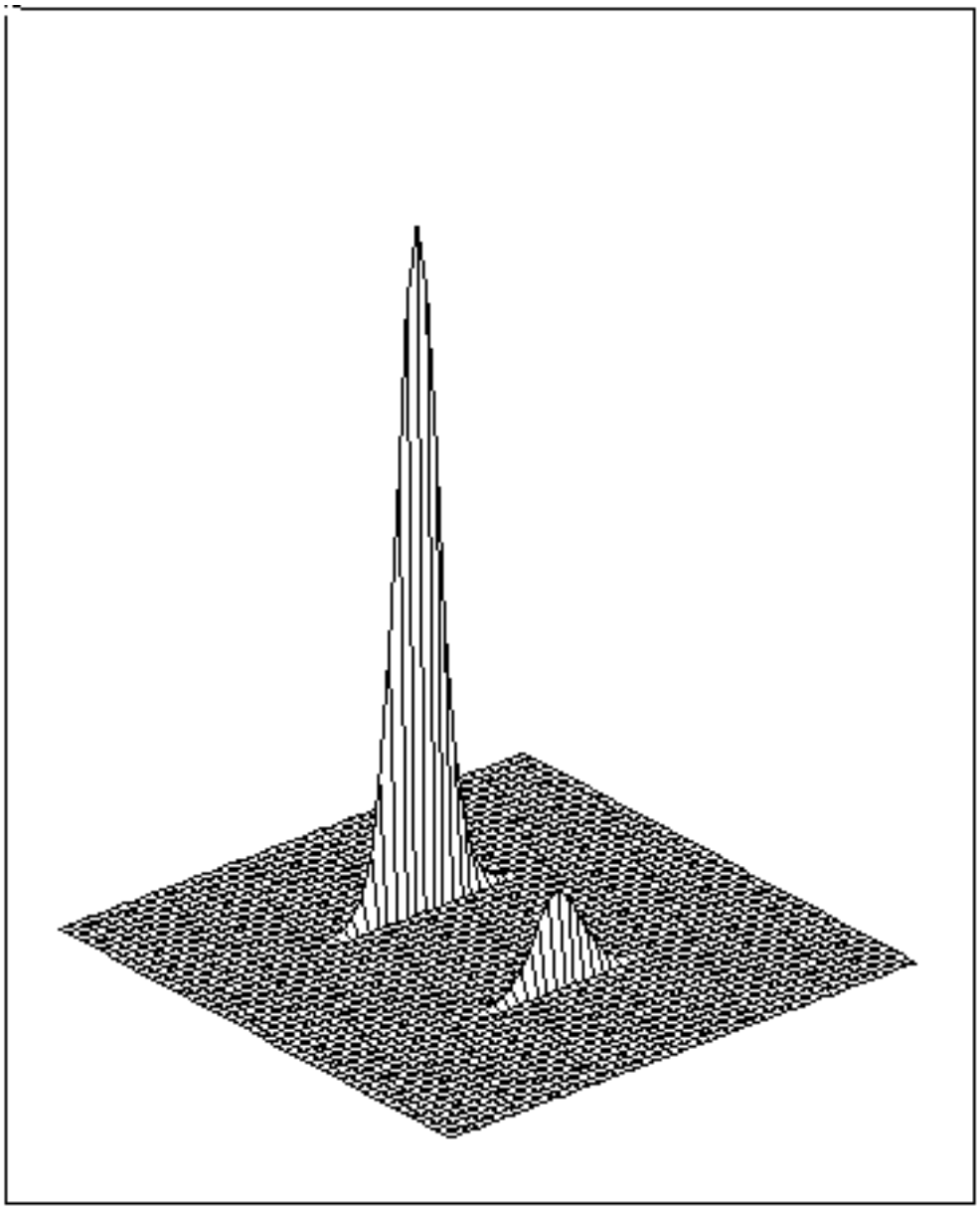


Figure 4c

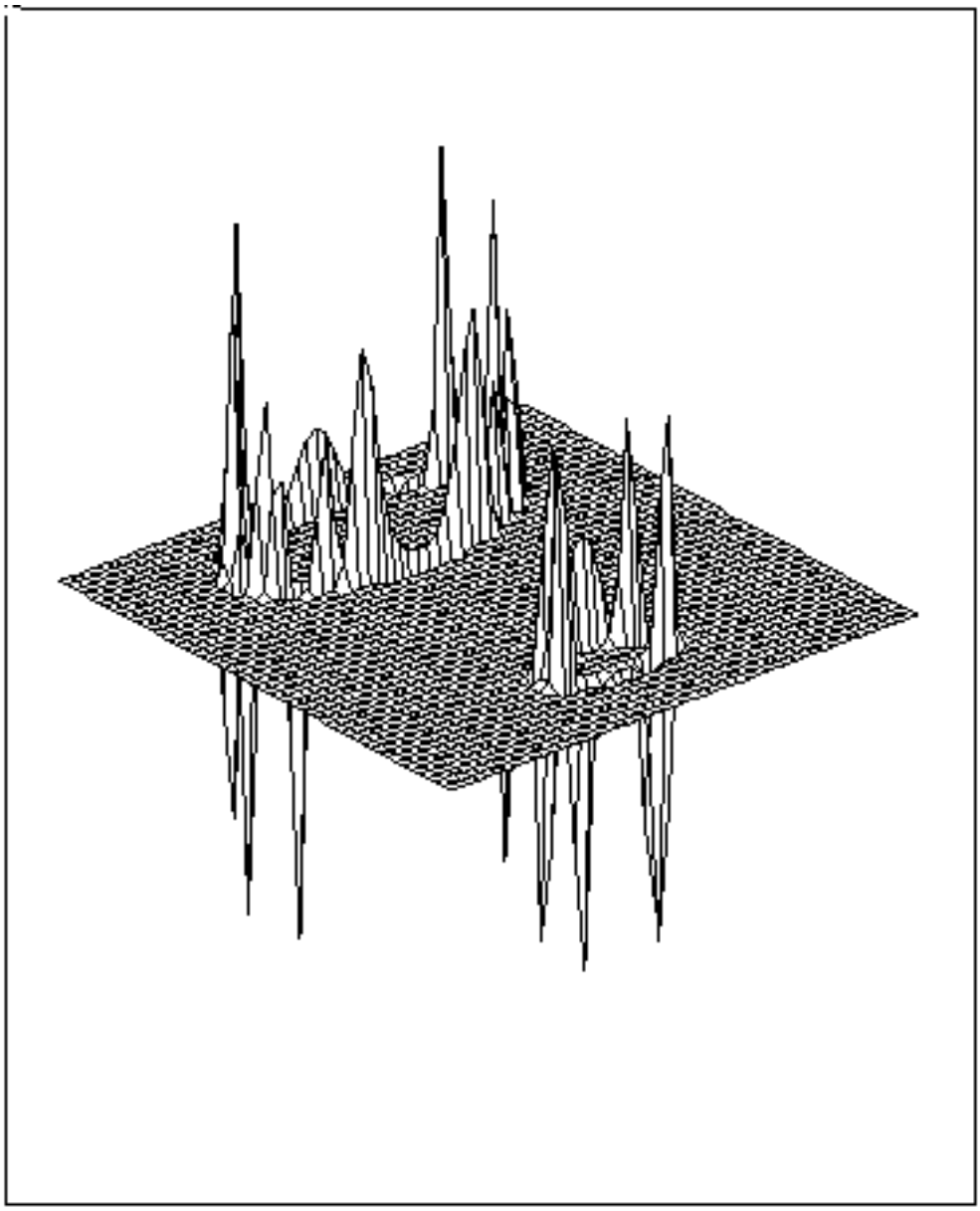


Figure 5a

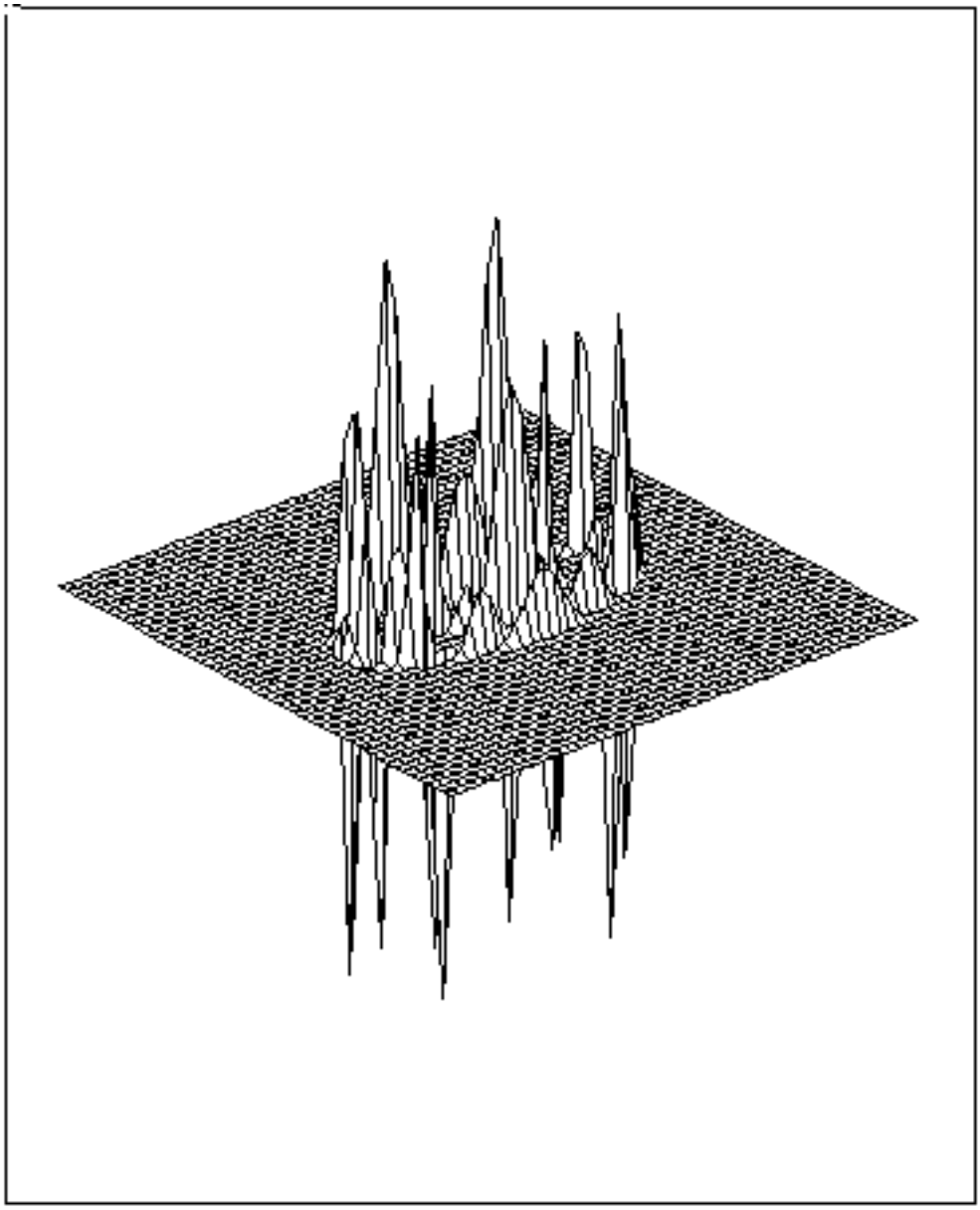


Figure 5b

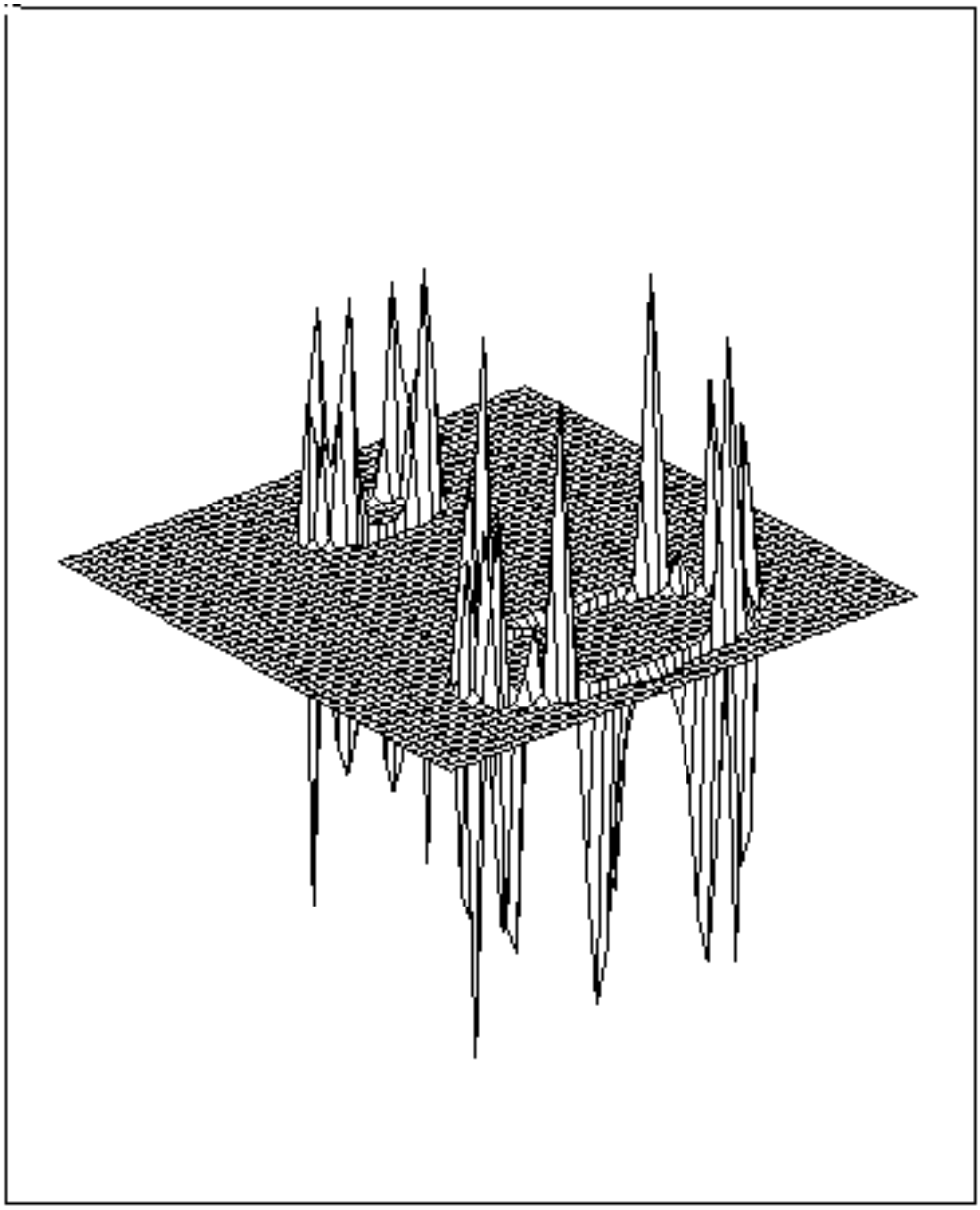


Figure 5c

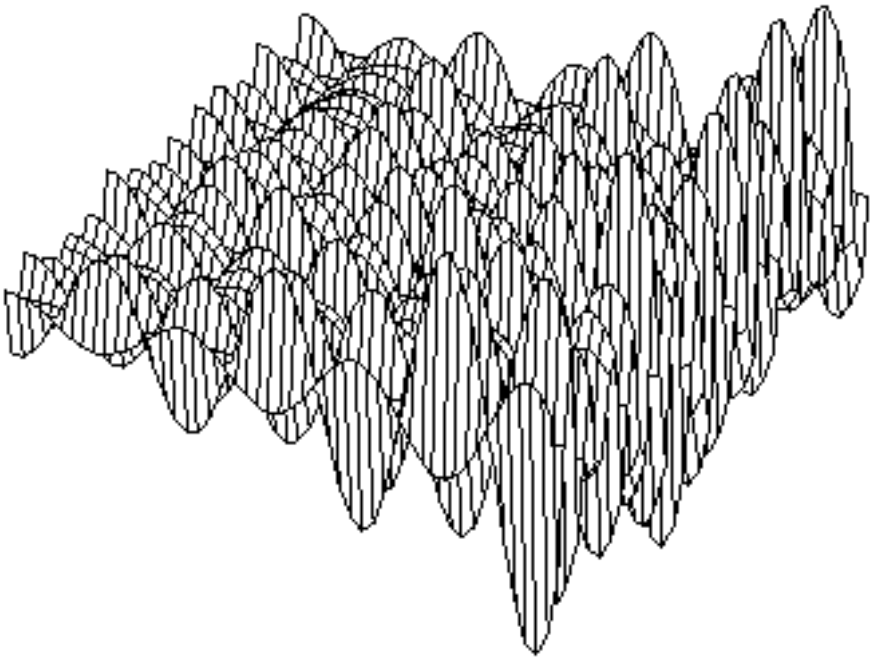


Figure 6

EJTP **3, No. 10** (2006) 89–107 *Electronic Journal of Theoretical Physics*

The Quantum Character of Physical Fields. Foundations of Field Theories

L.I. Petrova *

Faculty of Computational Mathematics and Cybernetics,
2nd Educational Building, Vorobjovy Gory,
Moscow State University,
Moscow 119899, Russia

Received 10 March 2006, Published 28 May 2006

Abstract: The existing field theories are based on the properties of closed exterior forms, which are invariant ones and correspond to conservation laws for physical fields. Hence, to understand the foundations of field theories and their unity, one has to know how such closed exterior forms are obtained. In the present paper it is shown that closed exterior forms corresponding to field theories are obtained from the equations modelling conservation (balance) laws for material media. It has been developed the evolutionary method that enables one to describe the process of obtaining closed exterior forms. The process of obtaining closed exterior forms discloses the mechanism of evolutionary processes in material media and shows that material media generate, discretely, the physical structures, from which the physical fields are formed. This justifies the quantum character of field theories. On the other hand, this process demonstrates the connection between field theories and the equations for material media and points to the fact that the foundations of field theories must be conditioned by the properties of material media. It is shown that the external and internal symmetries of field theories are conditioned by the degrees of freedom of material media. The classification parameter of physical fields and interactions, that is, the parameter of the unified field theory, is connected with the number of noncommutative balance conservation laws for material media.

Keywords: Foundations of Field Theories, Skew-Symmetric, Differential Forms
PACS (2006): 11.10.Cd, 11.10.-z, 11.10.Jj, 11.30.-j, 03.65.Ta

Introduction

Originally, beginning from the 17th century, the physics based on the differential equations, which describe physical processes. However, from the 20th century, the prob-

* ptr@cs.msu.su

lem of invariant (independent of the choice of the coordinate system) description of physical phenomena arose. As the result, the formalisms based on the tensor, group, variational methods, on the theories of symmetries, transformations and so on with the basic requirement of invariance were developed in physics.

This gave rise to building the field theories that enable one to describe physical fields and their interactions. Such theories were based on the postulates turned out practically not to be connected with the equations of mathematical physics, which describe physical processes. Just the absence of such a connection produced the emergence of the problems in field theories that are connected with investigation of general foundations of existing field theories, their unity, and constructing the general field theory.

In present paper it is shown that the connection of field theories with the equations describing physical processes in material media must lie at the basis of the general field theory.

The investigation of the foundations of field theories has been carried out using skew-symmetric differential forms. Skew-symmetric differential forms, which deal with differentials and differential expressions, can describe a conjugacy of various operators and objects. This is of principal importance for mathematical physics and field theories since the conjugated objects are invariants.

The properties of existing field theories are those of the closed exterior skew-symmetric differential forms [1,2], which are conjugated objects and correspond to conservation laws for physical fields. The properties of closed exterior forms explicitly or implicitly manifest themselves essentially in all formalisms of field theory, such as the Hamilton formalism, tensor approaches, group methods, quantum mechanics equations, the Yang-Mills theory and others. The gauge transformations (unitary, gradient and so on), the gauge symmetries and the identical relations of field theories are transformations, symmetries and relations of the theory of closed exterior forms.

Such connection between field theories and the theory of closed exterior forms enables one to understand the properties of field theories, which are common for all existing field theories.

However, this does not solve the basic problems of field theories. To understand the general foundations of field theories and their unity, one must know how the closed exterior forms connected with field theories are obtained.

It is known that the closed exterior forms are obtained from differential equations provided the requirements of integrability of these equations [1].

In the present paper it is shown, firstly, that the equations describing (balance) conservation laws for *material media* serve as the differential equations from which the closed exterior forms related to field theories and corresponding conservation laws for *physical fields* are obtained. And, secondly, it is developed the method, which is evolutionary one hence this method allows to find not only closed exterior forms, but also to describe the process of obtaining closed exterior forms.

The process of obtaining closed exterior forms, on one side, demonstrates the connection between field theories and the equations for material media, and, on other side,

discloses the mechanism of evolutionary processes in material media and shows that material media generate physical fields. This underlines the fact that the foundations of field theories, namely, the theories describing physical fields, must be conditioned by the properties of material media.

It has been possible to carry out the investigation of general foundations of field theories due to the skew-symmetric differential forms, which, unlike to the exterior forms, are defined on deforming manifolds and hence they possess the evolutionary properties. The mathematical apparatus of such forms includes some nontraditional elements such as nonidentical relations and degenerate transformation, and this enables one to describe the evolutionary processes, discrete transitions, quantum jumps, and generation of various structures.

In the first section the general properties of field theories are investigated with the help of closed exterior forms. In the next section the analysis of the equations of the balance conservation laws for material media, which describe the state of material system and the mechanism of generating physical structures forming physical fields. In the last section the general foundations of field theories obtained from the analysis of the equations for material media are discussed.

1. Connection Of Field Theories with the Theory of Closed Exterior Forms

Closed Exterior Forms and Conservation Laws

From the closure condition of the exterior form θ^p (p-form):

$$d\theta^p = 0 \tag{1.1}$$

one can see that the closed exterior form θ^p is a conserved quantity. This means that this can correspond to a conservation law, namely, to some conservative quantity. If the form is closed only on pseudostructure, i.e. this form is a closed inexact one, the closure conditions are written as

$$d_\pi \theta^p = 0 \tag{1.2}$$

$$d_\pi {}^*\theta^p = 0 \tag{1.3}$$

where ${}^*\theta^p$ is the dual form.

Condition (1.3), i.e. the closure condition for dual form, specifies the pseudostructure π. {Cohomology, sections of cotangent bundles, the eikonal surfaces, potential surfaces, pseudo-Riemannian and pseudo-Euclidean spaces, and others are examples of the pseudostructures and manifolds that are made up by pseudostructures.}

From conditions (1.2) and (1.3) one can see the following. The dual form (pseudostructure) and closed inexact form (conservative quantity) made up a conjugated conservative object that can also correspond to some conservation law. The conservation laws, to which physical fields are subject, are just such conservation laws.

The conservative object made up by the closed inexact exterior form and corresponding dual form is a differential-geometrical structure. (Such differential-geometrical structures are examples of G-structures.) The physical structures, which made up physical fields, and corresponding conservation laws are such differential-geometrical structures,

Properties of Closed Exterior Differential Forms

Invariance. Qauge transformations. It is known that the closed exact form is the differential of the form of lower degree:

$$\theta^p = d\theta^{p-1} \tag{1.4}$$

Closed inexact form is also a differential, and yet not total but interior on pseudostructure

$$\theta^p_\pi = d_\pi\theta^{p-1} \tag{1.5}$$

Since the closed form is a differential (a total one if the form is exact, or an interior one on the pseudostructure if the form is inexact), it is obvious that the closed form turns out to be invariant under all transformations that conserve the differential. The unitary transformations, the tangent, the canonical, the gradient transformations and so on are examples of such transformations. *These are gauge transformations of field theories.*

With the invariance of closed forms it is connected the covariance of relevant dual forms.

Conjugacy. Duality. Symmetries. The closure of exterior differential forms is the result of conjugating the elements of exterior or dual forms. The closure property of the exterior form means that any objects, namely, the elements of exterior form, the components of elements, the elements of the form differential, the exterior and dual forms, the forms of sequential degrees and others, turn out to be conjugated.

With the conjugacy it is connected the duality.

The example of a duality having physical sense: the closed exterior form is a conservative quantity corresponding to conservation law, and the closed form (as the differential) can correspond to a certain potential force.

The conjugacy is possible if there is one or another type of symmetry.

The gauge symmetries, which are interior symmetries of field theory and with which gauge transformations are connected, are symmetries of closed exterior differential forms. Symmetries of closed dual forms are exterior symmetries of the equations of field theory.

Identical relations of exterior forms. Since the conjugacy is a certain connection between two operators or mathematical objects, it is evident that, to express the conjugacy mathematically, it can be used relations. These are identical relations.

The identical relations express the fact that each closed exterior form is the differential of some exterior form (with the degree less by one). In general form such an identical relation can be written as

$$d_\pi\varphi = \theta^p_\pi \tag{1.6}$$

In this relation the form in the right-hand side has to be a *closed* one.

Identical relations of exterior differential forms are a mathematical expression of various types of conjugacy that leads to closed exterior forms.

Such relations like the Poincare invariant, vector and tensor identical relations, the Cauchi-Riemann conditions, canonical relations, the integral relations by Stokes or Gauss-Ostrogradskii, the thermodynamic relations, the eikonal relations, and so on are examples of identical relations of closed exterior forms that have either the form of relation (1.6) or its differential or integral analogs.

The Analysis of the Properties of Field Theories Using Closed Exterior Forms

One can see that the properties of closed exterior differential forms correspond to the properties of field theories. Hence, the mathematical principles of the theory of closed exterior differential forms made up the basis of field theories that is common for all existing field theories. (It should be emphasized that the field theories are connected with the properties of *inexact* closed exterior forms.)

The connection between field theories and the theory of closed exterior forms is primary explained by the fact that the closure conditions of exterior and dual forms correspond to conservation laws to which physical fields are subject. It is known that the conservation laws for physical fields are those that state an existence of conservative physical quantities or objects. The physical structures, which made up physical fields, and corresponding conservation laws are differential-geometrical structures formed by closed exterior forms and dual ones. [Below, using the evolutionary forms it will be shown that such physical structures arise in material media discretely.]

The properties of closed exterior and dual forms, namely, invariance, covariance, conjugacy, and duality, lie at the basis of the group, structural and other invariant methods of field theories.

The nondegenerate transformations of field theory are transformations of closed exterior forms. These are gauge transformations for spinor, scalar, vector, and tensor fields, which are transformations of closed (0-form), (1-form), (2-form) and (3-form) respectively.

The gauge, i.e. internal, symmetries of the field theory equations (corresponding to the gauge transformations) are those of closed exterior forms. The external symmetries of the equations of field theory are symmetries of closed dual forms.

The basis of field theory operators is connected with the nondegenerate transformations of exterior differential forms. If, in addition to the exterior differential, we introduce the following operators: (1) δ for transformations that convert the form of $(p+1)$ degree into the form of p degree, (2) δ' for cotangent transformations, (3) Δ for the $d\delta - \delta d$ transformation, (4) Δ' for the $d\delta' - \delta' d$ transformation, one can write down the operators in the field theory equations in terms of these operators that act on the exterior differential forms. The operator δ corresponds to Green's operator, δ' does to the canonical transformation operator, Δ does to the d'Alembert operator in 4-dimensional space, and

Δ' corresponds to the Laplace operator.

It can be shown that the equations of existing field theories are those obtained on the basis of the properties of the exterior form theory. The Hamilton formalism is based on the properties of closed exterior form of the first degree and corresponding dual form. The closed exterior differential form $ds = -Hdt+p_j dq_j$ (the Poincare invariant) corresponds to the field equation related to the Hamilton system. The Schrődinger equation in quantum mechanics is an analog to field equation, where the conjugated coordinates are changed by operators. It is evident that the closed exterior form of zero degree (and dual form) correspond to quantum mechanics. Dirac's *bra-* and *cket-* vectors constitute a closed exterior form of zero degree [3]. The properties of closed exterior form of the second degree (and dual form) lie at the basis of the electromagnetic field equations. The Maxwell equations may be written as [4] $d\theta^2 = 0$, $d^*\theta^2 = 0$, where $\theta^2 = \frac{1}{2}F_{\mu\nu}dx^\mu dx^\nu$ (here $F_{\mu\nu}$ is the strength tensor). Closed exterior and dual forms of the third degree correspond to the gravitational field. (However, to the physical field of given type it can be assigned closed forms of less degree. In particular, to the Einstein equation [5] for gravitational field it is assigned the first degree closed form, although it was pointed out that the type of a field with the third degree closed form corresponds to the gravitational field.)

One can recognize that the gauge transformations as well as the symmetries and equations of field theories are connected with closed exterior forms of given degree. This enables one to introduce a classification of physical fields and interactions according to the degree of closed exterior form. (If denote the degree of corresponding closed exterior forms by k, the case $k = 0$ will correspond to strong interaction, $k = 1$ will correspond to weak interaction, $k = 2$ will correspond to electromagnetic interaction, and $k = 3$ will correspond to gravitational interaction.) This shows that there exists a commonness between field theories describing physical fields of different types. The degree of closed exterior forms is a parameter that integrates fields theories into unified field theory.

Thus one can see that existing invariant field theories are based on the properties of closed exterior forms. And such a connection also discloses the problems of existing invariant field theories.

There are no answer to the question of how closed inexact exterior forms, which correspond to physical structures and reflect the properties of conservation laws and on which properties field theories are based, are obtained.

Below we will show that the answer to these question may be obtained from the analysis of differential equations describing the conservation laws for material media. These are just the equations from which the closed exterior forms whose properties correspond to field theories are obtained.

The evolutionary method of investigating these equations applied in the present paper enables one to understand how physical fields are formed and what must lie at the basis of the general field theory. [The method that enables one to find the closed exterior forms (the invariants) had been proposed by Cartan [1]. The differential equations are imposed by the requirement of obeying the closure condition of exterior form made up by the derivatives of these equations (it is added the requirement that the external form differential vanishes), and next one finds the conditions

that these requirements are satisfied (the integrability conditions). This method enables one *to find* the closed exterior forms (the invariants) that can possess the equation under consideration. However, for the evolutionary equations of mathematical physics describing physical processes it is important not only *to find* closed forms, but it is also important to know how these forms *are obtained*, in other words, it is important to know how the closure conditions of exterior forms are realized evolutionary. For this a principally new *evolutionary* method is necessary.]

2. The Equations of Balance Conservation Laws for Material System: The Evolutionary Processes in Material Media. Origination of Physical Structures

The conservation laws for material media (material media will be considered as material systems) are the balance conservation laws for energy, linear momentum, angular momentum, and mass. They are described by differential equations [6-8]. [Material system is a variety of elements that have internal structure and interact to one another. Thermodynamic and gas dynamical systems, systems of charged particles, cosmic systems, systems of elementary particles and others are examples of material systems. Examples of elements that constitute the material system are electrons, protons, neutrons, atoms, fluid particles, cosmic objects and others. The conservation laws for material systems are balance ones. These are conservation laws that establish a balance between the variation of physical quantity of material system and the corresponding external action.]

Nonconjugacy of the Balance Conservation Law Equations: Noncommutativity of the Balance Conservation Laws

The conservation laws for material systems have a peculiarity, namely, they are noncommutative ones. To this it points out the analysis of the equations of the balance conservation laws. (Just the noncommutativity of the balance conservation laws is a moving force of the evolutionary processes in material media that lead to generation of physical fields.)

It turns out that, even without a knowledge of the concrete form of the equations for balance conservation laws, with the help of skew-symmetric differential forms one can see their specific features. To carry out such an investigation, in addition to exterior skew-symmetric forms the skew-symmetric differential forms, which possesses the evolutionary properties (and for this reason the author named those as "evolutionary" ones), will be used. These are skew-symmetric differential forms, which, unlike to exterior forms, are defined on deforming manifolds. Such skew-symmetric differential forms have a specific feature, namely, they cannot be closed. The differential of such form does not vanish. This differential includes the metric form differential of deforming manifold, which is obtained due to differentiating the basis and is nonzero. The evolutionary form commutator, in addition to the commutator made up by the derivatives of the coefficients of the form itself, includes (in contrast to the commutator of the exterior form) the metric form

commutator being nonzero.

(The role of evolutionary forms in mathematical physics and field theory is due to the fact that they, as well as exterior forms, correspond to conservation laws. However, these conservation laws are those not for physical fields but for material media.)

We will analyze the equations that describe the balance conservation laws for energy and linear momentum.

If firstly to write down these equations in the inertial reference system and next pass to the accompanying reference system (this system is connected with the manifold built by the trajectories of the material system elements), in the accompanying reference system the energy equation will be written in the form

$$\frac{\partial \psi}{\partial \xi^1} = A_1 \tag{2.1}$$

Here ψ is the functional specifying the state of material system (the action functional, entropy or wave function can be regarded as examples of such a functional), ξ^1 is the coordinate along the trajectory, A_1 is the quantity that depends on specific features of material system and on external (with respect to local domain made up by the element and its neighborhood) energy actions onto the system.

In a similar manner, in the accompanying reference system the equation for linear momentum appears to be reduced to the equation of the form

$$\frac{\partial \psi}{\partial \xi^\nu} = A_\nu, \quad \nu = 2, ... \tag{2.2}$$

where ξ^ν are the coordinates in the direction normal to the trajectory, A_ν are the quantities that depend on the specific features of material system and on external force actions.

Eqs. (2.1) and (2.2) can be convoluted into the relation

$$d\psi = A_\mu \, d\xi^\mu, \quad (\mu = 1, \nu) \tag{2.3}$$

where $d\psi$ is the differential expression $d\psi = (\partial\psi/\partial\xi^\mu)d\xi^\mu$.

Relation (2.3) can be written as

$$d\psi = \omega \tag{2.4}$$

here $\omega = A_\mu \, d\xi^\mu$ is the skew-symmetric differential form of the first degree.

Since the balance conservation laws are evolutionary ones, the relation obtained is also an evolutionary relation.

Relation (2.4) was obtained from the equation of the balance conservation laws for energy and linear momentum. In this relation the form ω is that of the first degree. If the equations of the balance conservation laws for angular momentum be added to the equations for energy and linear momentum, this form in the evolutionary relation will be a form of the second degree. And in combination with the equation of the balance conservation law for mass this form will be a form of degree 3.

Thus, in general case the evolutionary relation can be written as

$$d\psi = \omega^p \tag{2.5}$$

where the form degree p takes the values $p = 0, 1, 2, 3$. The evolutionary relation for $p = 0$ is similar to that in the differential forms, and it was obtained from the interaction of time and energy of material system.

It could be noted that the degree p is connected with the number of interacting conservation laws that is equal to $(p + 1)$.

Relation obtained from the equation of the balance conservation laws has a specific feature, namely, this relation turns out to be nonidentical.

To justify this we shall analyze relation (2.4). This relation proves to be nonidentical since the left-hand side of the relation is a differential, which is a closed form, but the right-hand side of the relation involves the differential form ω, which is unclosed evolutionary form. The metric form commutator of the manifold, on which the form ω is defined, is nonzero since this manifold is an accompanying, deforming, manifold. The commutator made up by the derivatives of coefficients A_μ the form ω itself is also nonzero, since the coefficients A_μ are of different nature, that is, some coefficients have been obtained from the energy equation and depend on the energetic actions, whereas the others have been obtained from the equation for linear momentum and depend on the force actions.

In a similar manner one can prove the nonidentity of relation (2.5).

The nonidentity of evolutionary relation means that the balance conservation law equations are inconsistent (nonconjugated). This reflects the properties of the balance conservation laws that have a governing importance for the evolutionary processes in material media, namely, their *noncommutativity*. [The nonidentity of evolutionary relation points to the fact that *on the initial manifold* the equations of the balance conservation laws are nonintegrable ones: the derivatives of these equations do not make up the differential, that is, a closed form which can be directly integrated. This is explained by the fact that these equations, like any equations describing physical processes, include nonpotential terms.]

Physical Meaning of the Equations of Balance Conservation Laws: Description of the State of Material System

Nonequilibrium state of material system. The evolutionary relation obtained from the equations of balance conservation laws discloses a physical meaning of these equations – these equations describe the state of material system.

It is evident that if the balance conservation laws be commutative, the evolutionary relation would be identical and from that it would be possible to get the differential $d\psi$, this would indicate that the material system is in the equilibrium state.

However, as it has been shown, in real processes the balance conservation laws are noncommutative. The evolutionary relation is not identical and from this relation one cannot get the differential $d\psi$. This means that the system state is nonequilibrium. That

is, due to noncommutativity of the balance conservation laws the material system state turns out to be nonequilibrium. It is evident that the internal force producing such nonequilibrium state is described by the evolutionary form commutator. Everything that gives the contribution to the commutator of the form ω^p leads to emergence of internal force. (Internal force is a force that acts inside the local domain of material system, i.e. a domain made up by the element and its neighborhood.)

Nonidentical evolutionary relation also describes how the state of material system varies. This turns out to be possible due to the fact that the evolutionary nonidentical relation is a selfvarying one. This relation includes two objects one of which appears to be unmeasurable. The variation of any object of the relation in some process leads to variation of another object and, in turn, the variation of the latter leads to variation of the former. Since one of the objects is an unmeasurable quantity, the other cannot be compared with the first one, and hence, the process of mutual variation cannot stop. This process is governed by the evolutionary form commutator, that is, by interaction between the commutator made up by derivatives of the form itself and by metric form commutator of deforming manifold made up by the trajectories of material system. (This is an exchange between quantities of different nature, between physical quantities and space-time characteristics.)

[In essence, the evolutionary equation is a correlative relation. When changing the terms of this relation cannot become equal to one another, but in this case they correlate to one another. The terms of the evolutionary form commutator in nonidentical relation also correlate to one another.]

Selfvariation of nonidentical evolutionary relation points to the fact that the nonequilibrium state of material system turns out to be selfvarying. State of material system changes but holds nonequilibrium during this process.

Transition of material system from nonequilibrium state to the locally-equilibrium state. Origination of physical structure. The significance of the evolutionary relation selfvariation consists in the fact that in such a process it can be realized conditions under which the inexact, closed *on pseudostructure*, exterior form is obtained from the evolutionary form. This transition is possible only as the degenerate transformation, namely, a transformation that does not conserve the differential. The conditions of degenerate transformation are those that determine the direction on which interior (only along a given direction) differential of the evolutionary form vanishes. These are the conditions that defines the pseudostructure, i.e. the closure conditions of dual form, and leads to realization of the exterior form closed on pseudostructure.

As it has been already mentioned, the differential of the evolutionary form ω^p involved into nonidentical relation (2.5) is nonzero. That is,

$$d\omega^p \neq 0 \tag{2.6}$$

If the conditions of degenerate transformation are realized, it will take place the transition

$d\omega^p \neq 0 \rightarrow$ (degenerate transformation) $\rightarrow d_\pi\omega^p = 0$, $d_\pi{}^*\omega^p = 0$

The relations obtained

$$d_\pi\omega^p = 0, d_\pi{}^*\omega^p = 0 \tag{2.7}$$

are the closure conditions for exterior inexact form and dual form. This means that it is realized the exterior form closed on pseudostructure.

In this case on the pseudostructure π evolutionary relation (2.5) converts into the relation

$$d_\pi \psi = \omega_\pi^p \tag{2.8}$$

which proves to be an identical relation. Since the form ω_π^p is a closed one, on the pseudostructure this form turns out to be a differential. There are differentials in the left-hand and right-hand sides of this relation. This means that the relation obtained is an identical one.

Here it should be emphasized that under degenerate transformation the evolutionary form remains to be unclosed and the evolutionary relation itself remains to be nonidentical one. (The evolutionary form differential vanishes only on pseudostructure: the differential, which equals zero, is an interior one, the total differential of the evolutionary form is nonzero.)

The transition from nonidentical relation (2.5) obtained from the balance conservation laws to identical relation (2.8) means the following. Firstly, the emergence of the closed (on pseudostructure) inexact exterior form (relation (2.7) and right-hand side of relation (2.8)) points to origination of physical structure. And, secondly, the existence of the state differential (left-hand side of relation (2.8)) points to the transition of material system from nonequilibrium state to the locally-equilibrium state. *(But in this case the total state of the material system turns out to be nonequilibrium.)*

Identical relation (2.8) points to the fact that the origination of physical structures is connected with the transition of material system to the locally-equilibrium state.

The origination of physical structures in material system manifests itself as an emergence of certain observable formations, which develop spontaneously. Such formations and their manifestations are fluctuations, turbulent pulsations, waves, vortices, creating massless particles, and others. (One can see that the processes described also explain such phenomena as turbulence, radiation and others.)

Conditions of degenerate transformation: degrees of freedom of material system. The conditions of degenerate transformation that lead to emergence of closed inexact exterior form are connected with any symmetries. Since these conditions are closure conditions of dual (metric) form, they can be caused by symmetries of coefficients of the metric form commutator (for example, these can be symmetrical connectednesses).

Under describing material system the symmetries are conditioned by degrees of freedom of material system. The translational degrees of freedom, internal degrees of freedom of the system elements, and so on can be examples of such degrees of freedom.

The conditions of degenerate transformation (vanishing the dual form commutator) define the pseudostructure. These conditions specify the derivative of implicit function, which defines the direction of pseudostructure. The speeds of various waves are examples of such derivatives: the speed of light, the speed of sound and of electromagnetic waves, the speed of creating particles and so on. It can be shown that the equations for surfaces

of potential (of simple layer, double layer), integral surfaces, equations for one, two, ... eikonals, of the characteristic and of the characteristic surfaces, the residue equations and so on serve as the equations for pseudostructures.

To the degenerate transformation it must correspond a vanishing of some functional expressions, such as Jacobians, determinants, the Poisson brackets, residues and others. Vanishing of these functional expressions is the closure condition for dual form.

And it should be emphasized once more that *the degenerate transformation is realized as a transition from the accompanying noninertial coordinate system to the locally inertial system*. The evolutionary form is defined in the noninertial frame of reference (deforming manifold). But the closed exterior form created is obtained with respect to the locally-inertial frame of reference (pseudostructure).

Characteristics and Classification of Physical Structures. Forming Pseudometric and Metric Spaces

Characteristics of physical structures. The physical structure is a differential-geometrical structure made up by the dual form and closed inexact form. This is a pseudostructure (dual form) with conservative quantity (closed inexact form). The conservative quantities describe certain charges.

Since the physical structures are generated by material media by means of the balance conservation laws, their characteristics are connected with the characteristics of material systems and with the characteristics of evolutionary forms obtained from the equations of balance conservation laws.

It was already mentioned that the pseudostructure is obtained from the condition of degenerate transformation, which is connected with the degrees of freedom of material system.

The total differential of evolutionary form, which holds to be nonzero, defines two another characteristics of physical structure.

The first term of the evolutionary form differential, more exactly, its commutator, determines the value of discrete change (the quantum), which the quantity conserved on the pseudostructure undergoes during transition from one pseudostructure to another. The second term of the evolutionary form commutator determines the bending of pseudostructure. The bending specifies the characteristics that fixes the character of the manifold deformation, which took place before physical structure emerged. (Spin is an example of such a characteristics).

The closed exterior forms obtained correspond to the state differential for material system. The differentials of entropy, action, potential and others are examples of such differentials.

As it was already mentioned, in material system the created physical structure is revealed as an observable formation. It is evident that the characteristics of the formation (intensity, vorticity, absolute and relative speeds of propagation of the formation), as well as those of created physical structure, are determined by the evolutionary form and its

commutator and by the material system characteristics.

Classification of physical structures. The connection of the physical structures with the skew-symmetric differential forms allows to introduce a classification of these structures in dependence on parameters that specify the skew-symmetric differential forms and enter into nonidentical and identical relation. To determine these parameters one has to consider the problem of integration of the nonidentical evolutionary relation.

Under degenerate transformation from the nonidentical evolutionary relation one obtains a relation being identical on pseudostructure. Since the right-hand side of such a relation can be expressed in terms of differential (as well as the left-hand side), one obtains a relation that can be integrated, and as a result he obtains a relation with the differential forms of less by one degree.

The relation obtained after integration proves to be nonidentical as well.

By sequential integrating the nonidentical relation of degree p (in the case of realization of corresponding degenerate transformations and forming the identical relation), one can get a closed (on the pseudostructure) exterior form of degree k, where k ranges from p to 0.

In this case one can see that after such integration the closed (on the pseudostructure) exterior forms, which depend on two parameters, are obtained. These parameters are the degree of evolutionary form p in the evolutionary relation and the degree of created closed forms k.

In addition to these parameters, another parameter appears, namely, the dimension of space.

What is implied by the concept "space"?

In the process of deriving the evolutionary relation two frames of reference were used and, correspondingly, two spatial objects. The first frame of reference is an inertial one, which is connected with the space where material system is situated and is not directly connected with material system. This is an inertial space, it is a metric space. (This space is also formed by the material systems.) The second frame of reference is a proper one, it is connected with the accompanying manifold, which is not a metric manifold.

While generating closed forms of sequential degrees $k = p$, $k = p - 1$, ..., $k = 0$ the pseudostructures of dimensions $(n + 1 - k)$: 1, ..., $n + 1$ are obtained, where n is the dimension of inertial space. As a result of transition to the exact closed form of zero degree *the metric* structure of the dimension $n + 1$ is obtained.

The parameters of physical structures generated by the evolutionary relation depend on the degree of differential forms p and k and on the dimension of original inertial space n.

With introducing the classification by numbers p, k and n one can understand the internal connection between various physical fields. Since physical fields are the carriers of interactions, such classification enables one to see the connection between interactions.

Such a classification may be presented in the form of the table given below. This table corresponds to elementary particles.

[It should be emphasized the following. Here the concept of "interaction" is used in a twofold

meaning: an interaction of the balance conservation laws that relates to material systems, and the physical concept of "interaction" that relates to physical fields and reflects the interactions of physical structures, namely, it is connected with exact conservation laws].

TABLE

interaction	$k\backslash p,n$	**0**	**1**	**2**	**3**
gravitation	**3**				**graviton** ⇑ electron proton neutron photon
electro-magnetic	**2**			**photon2** ⇑ electron proton neutrino	**photon3**
weak	**1**		**neutrino1** ⇑ electron quanta	**neutrino2**	**neutrino3**
strong	**0**	**quanta0** ⇑ quarks?	**quanta1**	**quanta2**	**quanta3**
particles material nucleons?	exact forms	**electron**	**proton**	**neutron**	**deuteron?**
N		1 time	2 time+ 1 coord.	3 time+ 2 coord.	4 time+ 3 coord.

In the Table the names of the particles created are given. Numbers placed near particle names correspond to the space dimension. Under the names of particles the sources of interactions are presented. In the next to the last row we present particles with mass (the elements of material system) formed by interactions (the exact forms of zero degree obtained by sequential integrating the evolutionary relations with the evolutionary forms of degree p corresponding to these particles). In the bottom row the dimension of the *metric* structure created is presented.

From the Table one can see the correspondence between the degree k of the closed forms realized and the type of interactions. Thus, $k = 0$ corresponds to strong interaction, $k = 1$ corresponds to weak interaction, $k = 2$ corresponds to electromagnetic interaction, and $k = 3$ corresponds to gravitational interaction.

The degree k of the closed forms realized and the number p connected with the number of interacting balance conservation laws determine the type of interactions and the type of particles created. The properties of particles are governed by the space dimension. The last property is connected with the fact that closed forms of equal degrees k, but obtained from the evolutionary relations acting in spaces of different dimensions n, are distinctive because they are defined on pseudostructures of different dimensions (the dimension of pseudostructure $(n+1-k)$ depends on the dimension of initial space n). For this reason the realized physical structures with closed forms of degrees k are distinctive in their properties.

The parameters p, k, n can range from 0 to 3. They determine some completed cycle. The cycle involves four levels, to each of which are assign their own values of p $(p = 0, 1, 2, 3)$ and space dimension n.

In the Table one cycle of forming physical structures is presented. Each material system has his own completed cycle. This distinguishes one material system from another system. One completed cycle can serve as the beginning of another cycle (the structures formed in the preceding cycle serve as the sources of interactions for beginning a new cycle). This may mean that one material system (medium) proves to be imbedded into the other material system (medium). The sequential cycles reflect the properties of sequentially imbedded material systems. And yet a given level has specific properties that are inherent characteristics of the same level in another cycles. This can be seen, for example, from comparison of the cycle described and the cycle in which to the exact forms there correspond conductors, semiconductors, dielectrics, and neutral elements. The properties of elements of the third level, namely, of neutrons in one cycle and of dielectrics in another, are identical to the properties of so called "magnetic monopole" [9,10].

Forming pseudometric and metric spaces The mechanism of creating the pseudostructures lies at the basis of forming the pseudometric surfaces and their transition into metric spaces.

It was shown above that the evolutionary relation of degree p can generate (in the presence of degenerate transformations) closed forms of the degrees $p, p-1, .., 0$. While generating closed forms of sequential degrees $k = p, k = p-1, .., k = 0$ the pseudostructures of dimensions $(n+1-k)$: $1, ..., n+1$ are obtained. As a result of transition to the exact closed form of zero degree the metric structure of the dimension $n+1$ is obtained. Under the effect of external actions (and in the presence of degrees of freedom) the material system can convert the initial inertial space of the dimension n into the space of the dimension $n+1$.

Here the following should be pointed out. Physical structures are generated by local domains of material system. These are elementary physical structures. By combining

with one another they can form large-scale structures making up pseudomanifolds and physical fields.

Sections of the cotangent bundles (Yang-Mills fields), cohomologies by de Rham, singular cohomologies, pseudo-Riemannian and pseudo-Euclidean spaces, and others are examples of pseudostructures and spaces that are formed by pseudostructures. Euclidean and Riemannian spaces are examples of metric manifolds that are obtained when going to the exact forms.

What can be said about the pseudo-Riemannian manifold and Riemannian space?

The distinctive property of the Riemannian manifold is an availability of the curvature. This means that the metric form commutator of the third degree is nonzero. Hence, the commutator of the evolutionary form of third degree ($p = 3$), which involves into the proper metric form commutator, is not equal to zero. That is, the evolutionary form that enters into the evolutionary relation is unclosed, and the relation is nonidentical one.

When realizing pseudostructures of the dimensions $1, 2, 3, 4$ and obtaining the closed inexact forms of the degrees $k = 3, k = 2, k = 1, k = 0$ the pseudo-Riemannian space is formed, and the transition to the exact form of zero degree corresponds to the transition to the Riemannian space.

It is well known that while obtaining the Einstein equations it was assumed that there are satisfied the conditions [4,8]: 1) the Bianchi identity is satisfied, 2) the coefficients of connectedness are symmetric, 3) the condition that the coefficients of connectedness are the Christoffel symbols, and 4) an existence of the transformation under which the coefficients of connectedness vanish. These conditions are the conditions of realization of degenerate transformations for nonidentical relations obtained from the evolutionary nonidentical relation with evolutionary form of the degree $p = 3$ and after going to the identical relations. In this case to the Einstein equation the identical relations with forms of the first degree are assigned.

From the description of evolutionary processes in material media one can see that physical fields are generated by material media. (And thus the causality of physical processes and phenomena is explained.)

Here it should be emphasized that the conservation laws for material media, i.e. the balance conservation laws for energy, linear momentum, angular momentum, and mass, which are noncommutative ones, play a controlling role in these processes. This is precisely the noncommutativity of the balance conservation laws produced by external actions onto material system, which is a moving force of evolutionary processes leading to origination of physical structures (to which exact conservation laws are assigned). [Noncommutativity of balance conservation laws for material media and their controlling role in evolutionary processes accompanied by emerging physical structures practically have not been taken into account in the explicit form anywhere. The mathematical apparatus of evolutionary differential forms enables one to take into account and to describe these points.]

3. On the Foundations of General Field Theory

The results of the analysis of the equations of conservation laws for material media shows the connection of physical fields with material media.

This points to the fact that the fields theories that describe physical fields must be connected with the equations that describe material systems.

Such a connection, which is common to all field theories, discloses the general foundations of field theories, their quantum character, justifies the unity of field theories and can serve as an approach to general field theory.

The theories of exterior and evolutionary skew-symmetric differential forms, which reflect the properties of conservation laws for physical fields and material media, allow to disclose and justify the general principles of field theories. In this case the properties of closed exterior forms demonstrate these principles, and the theory of evolutionary forms justifies this. Below we present certain of concepts that lie at the basis of field theories. (The results obtained with using the evolutionary forms are italicized).

1. Physical fields are formatted by physical structures that are described by closed inexact exterior and dual forms. *Physical structures are generated by material media. Characteristics of physical structures relate to the characteristics of material systems.*

2. *The conservation laws for physical fields, on which the field theories are based, are connected with the conservation laws for material media (with the balance conservation laws for energy, linear momentum, angular momentum, and mass and the analog of such law for the time).*

3. Internal and external symmetries of field theories are those of closed exterior and dual forms. *They are conditioned by the degrees of freedom of material media.*

4. *The origination of physical structures, from which physical fields are made up, proceeds* **discretely** *under realization of the degrees of freedom of material systems.* This explains the quantum character of field theories.

5. The gauge transformations of field theories are transformations of closed exterior forms. *They are connected with the degenerate transformations of the equations of conservation laws for material media.*

6. *The constants of field theory must be connected with the characteristics of material systems.*

7. The classification parameter of physical fields and interactions, that is, the parameter of the unified field theory, is the degree of closed exterior forms corresponding to conservation laws for physical fields. *This parameter is connected with the number of the equations of interacting noncommutative balance conservation laws. This connection justifies the parameter of the united field theory.*

The results obtained show that when building the general field theory it is necessary to take into account the connection of existing field theories (which are based on the conservation laws for physical fields) with the equations of noncommutative conservation laws for material media (the balance conservation laws for energy, linear momentum, angular momentum and mass and the analog of such laws for the time, which takes into

account the noncommutativity of the time and the energy of material system).

References

[1] Cartan E., Les Systemes Differentials Exterieus ef Leurs Application Geometriques. -Paris, Hermann, 1945.

[2] Schutz B. F., Geometrical Methods of Mathematical Physics. Cambridge University Press, Cambridge, 1982.

[3] Dirac P. A. M., The Principles of Quantum Mechanics. Clarendon Press, Oxford, UK, 1958.

[4] Wheeler J. A., Neutrino, Gravitation and Geometry. Bologna, 1960.

[5] Tonnelat M.-A., Les principles de la theorie electromagnetique et la relativite. Masson, Paris, 1959.

[6] Clark J. F., Machesney M., The Dynamics of Real Gases. Butterworths, London, 1964.

[7] Dafermos C. M. In "Nonlinear waves". Cornell University Press, Ithaca-London, 1974.

[8] Tolman R. C., Relativity, Thermodynamics, and Cosmology. Clarendon Press, Oxford, UK, 1969.

[9] Dirac P. A. M., Proc. Roy. Soc., **A133**, 60 (1931).

[10] Dirac P. A. M., Phys. Rev., **74**, 817 (1948).

[11] Einstein A. The Meaning of Relativity. Princeton, 1953.

EJTP **3, No. 10** (2006) 109–130 *Electronic Journal of Theoretical Physics*

Relativistic Causality and Quasi–Orthomodular Algebras

R. Nobili *

Dipartimento di Fisica "G. Galilei"
Università di Padova
Via Marzolo 8, 35131 Padova — ITALY

Received 30 April 2006, Published 28 May 2006

Abstract: The concept of *fractionability* or *decomposability* in parts of a physical system has its mathematical counterpart in the lattice–theoretic concept of *orthomodularity*. Systems with a finite number of degrees of freedom can be decomposed in different ways, corresponding to different groupings of the degrees of freedom. The orthomodular structure of these simple systems is trivially manifest. The problem then arises as to whether the same property is shared by physical systems with an infinite number of degrees of freedom, in particular by the quantum relativistic ones. The latter case was approached several years ago by Haag and Schroer (1962; Haag, 1992) who started from noting that the causally complete sets of Minkowski spacetime form an orthomodular lattice and posed the question of whether the subalgebras of local observables, with topological supports on such subsets, form themselves a corresponding orthomodular lattice. Were it so, the way would be paved to interpreting spacetime as an intrinsic property of a local quantum field algebra. Surprisingly enough, however, the hoped property does not hold for local algebras of free fields with superselection rules. The possibility seems to be instead open if the local currents that govern the superselection rules are driven by gauge fields. Thus, in the framework of local quantum physics, the request for algebraic orthomodularity seems to imply physical interactions! Despite its charm, however, such a request appears plagued by ambiguities and criticities that make of it an ill–posed problem. The proposers themselves, indeed, concluded that the orthomodular correspondence hypothesis is too strong for having a chance of being practicable. Thus, neither the idea was taken seriously by the proposers nor further investigated by others up to a reasonable degree of clarification. This paper is an attempt to re–formulate and well–pose the problem. It will be shown that the idea is viable provided that the algebra of local observables: (1) is considered all over the whole range of its irreducible representations; (2) is widened with the addition of the elements of a suitable intertwining group of automorphisms; (3) the orthomodular correspondence requirement is modified to an extent sufficient to impart a natural topological structure to the *intertwined algebra of observables* so obtained. A novel scenario then emerges in which local quantum physics appears to provide a general framework for non–perturbative quantum field dynamics.

Keywords: Quantum Field Theory, Orthomodular Algebra, Algebraic Topology, Topological Lattices, Group Theory
PACS (2006): 03.70.+k, 04.62.+v, 04.60.Nc, 05.50,+q, 02.20.-a, 02.10.-v

* rnobili@padova.infn.it

Introduction

A physical system will be called finite or infinite according as the number N of its degrees of freedom is finite or infinite. It is well–known from both classic and quantum mechanics that finite systems can be described in many equivalent ways by suitable rearrangement of their degrees of freedom. In classic physics, this is carried out by canonical transformations that preserve the symplectic structure of phase space. In quantum physics, the corresponding transformations are implemented by unitary automorphisms of the algebra of observables.

The possibility of rearranging and grouping the degrees of freedoms in different ways reflects the fact that the symplectic space of a finite systems can, in general, be decomposed into smaller symplectic spaces and composed of symplectic spaces to form larger symplectic spaces. This property, which is not as trivial as the decomposition and composition of the subsets of a set, characterizes the structure of a finite symplectic space as an orthomodular lattice of subspaces (App.1). Correspondingly, the algebra of conjugate observables of a finite quantum system (with $N > 1$) is characterized as an orthomodular lattice of subalgebras.

The problem now arises quite naturally as to whether the orthomodular property holds also for infinite systems, in particular for the quantum relativistic ones, as an euristic principle of the preservation of formal properties would suggest. From here on, however, only systems with a non–compact continuum of degrees of freedom will be considered.

Unfortunately, in the framework of quantum relativistic theories, both the concept of degree of freedom and that of canonical conjugate quantities become elusive. The first difficulty arises because a continuum of physical quantities is not a simple set but a topological space. The second one because the canonical commutation relations between conjugate quantities are destroyed by physical interactions. We must therefore expect that in this more delicate context the orthomodularity condition needs to be formulated quite differently from the finite case.

Rearranging a continuum of degrees of freedom is not a simple matter of grouping. Transferring degrees of freedom from a region of a continuum to another is not as simple as picking up handfuls of degrees of freedom from one side to put them into the other side. We must instead detach topological sets from one side and merge them into the other, and this cannot be done without creating and dissolving topological boundaries. To put it differently, managing a continuum of degrees of freedom is tantamount to managing topological boundaries.

The topological boundary problem was posed since the very beginning of quantum mechanics. The representation of the quantum mechanical world as a relationship between an observing system and an observed system implies interfacing the two parts across a common boundary, whose location, however, is quite arbitrary (von Neumann, 1932, 1955). In the framework of general quantum physics, decomposing a system in two parts has its algebraic counterpart in factorizing the operator algebra of the physical

system into two mutually commuting subalgebras. In the framework of local quantum physics, this process is complicated by the need of partitioning conservative quantities between complementary regions of spacetime. Thus, from a theoretical standpoint, drawing a boundary is ultimately related to the problem of representing the infinite ways of partitioning conservative quantities among regions separated by topological boundaries. Moreover, the reconstruction of the observer–observed system as a whole requires that the entire algebraic structure is recovered, by some algebraic procedure, probably in a framework of thermodynamic limits, from the whole collection of representations corresponding to the different partitions. As will be seen, this is precisely the central point of the topological boundary problem in local quantum physics.

An interesting approach towards the orthomodularization of local quantum physics was proposed by Haag and Schoer (Haag and Schroer, 1962; Haag, 1992), basing on the facts that lattices formed by *causally complete sets*, or *causal completions*, of Minkowski spacetime (App.2) are orthomodular. They considered the possibility of establishing a correspondence between lattices of causally complete sets and subalgebras of local observables with topological supports on such sets. Remarkably enough, the orthomodular property holds also if the spacetime is equipped with a pseudo–Riemannian metric (Cegla and Jadczyk, 1977; Casini, 2002).

For reasons that will be clear in the next, however, it is preferable to restrict our consideration to the orthomodular sublattices of causal completions formed by *causal shadows* of spacelike sets of events (App.2). The main question is then of whether, in the framework of local quantum physics, suitable subalgebras of observables localized in causal shadows can form an orthomodular lattice with respect to the operations of decomposing an algebra into subalgebras and combining subalgebras to generate larger subalgebras.

The correspondence *causal–shadows* $\leftrightarrow$ *subalgebras* would indeed bear an interesting physical meaning. The observables localized in the causal shadow $\langle S\rangle$ of a region S of a spacelike surface Σ are physically protected from the influence of all the events occurring in the orthocomplement $\langle S'\rangle$ of $\langle S\rangle$, i.e. the causal shadow of the region S' external to S on Σ. Here, the observers with their measurement devices are supposed to reside after preparing a physical state in S. The extension of this correspondence over finer and finer causal–shadow decompositions has its algebraic counterpart in the infinite decomposability of observable quantities, ultimately in the infinite fractionability of matter. This is a property that all good quantum field theories are supposed to possess.

Were the correspondence possible, the way would be paved also for regarding spacetime as an intrinsic property of the algebraic structure of local quantum physics. This perspective is very appealing as it is precisely the principle of general relativity in its most abstract form (Rovelli, 2004).

Here we touch the heart of the problem. How can an algebraic lattice be organized to an extent sufficient to incorporate a topological structure? What are the algebraic analogs of closed sets, open sets, boundaries and all the operations that we are supposed to be able to do in general topology?

To proceed along the path here indicated, the consolidated repertoire of algebraic concepts that form the basis of the current approaches to local quantum physics must be suitably reorganized (Doplicher *et al.*, 1969a, 1969b, 1971, 1974). This attempt will be carried out without any pretensions of completeness and rigor in Sec.s 1, 2, 3 and 4.

In Sec.1, the general concepts that seem to be necessary to well–pose the problem will be introduced. In Sec.2, the notion of orthomodular algebra will be presented and the idea of Haag–Schroer briefly illustrated in order to show that its effective implementation requires a substantial widening of the current vistas on the algebras of observables. The concepts of quasi–orthocomplemented and quasi–orthomodular lattices will be introduced in Sec.3 for the purpose of evidencing certain natural topological properties of von Neumann algebras. The physical meaning and the perspectives of the approach will be finally discussed in Sec.4.

1. Looking for Well–Posing the Problem

Since an orthomodular lattice is *orthocomplemented* (App.1), it is worthwhile describing how orthocomplementation can be implemented in an algebra of bounded operators (the reason why we need to ground our analysis on bounded operators is that sums and products of unbounded operators do not exist in general). To avoid terminological misunderstanding, we introduce a few basic concepts on the algebras of quantum mechanical systems.

1.1 C*–algebras

A selfadjoint algebra of bounded linear operators of a Hilbert space $\mathcal{H}$, *including the unit operator*[1] and closed in the topology of the norm (uniform topology), will be simply called a *C*–algebra*.

The basic properties of a C*–algebra are that the eigenvalue spectrum of any element and the inverse of any element with non zero eigenvalues are unambiguously defined. The norm $|X|$ of an element X of the C*–algebra is the least upper bound of the absolute eigenvalues of X. Closure in the norm topology means that, if a sequence of elements $X_1, X_2, \ldots$ exists such that $|X_1 - X|, |X_2 - X|, \ldots$ converges to zero, then X belongs to the algebra.

Since these properties can be formulated in purely algebraic terms, i.e. independently of the fact that the elements of the algebra operate on the vectors of $\mathcal{H}$, the same notions can be ascribed to the abstract counterpart of the algebra (Bratteli and Robinson, 1987). In the following, however, C*–algebras will be understood sometimes as abstract algebras and some other times as representations of abstract algebras. This ambiguity, a little bit confusing though, has the advantage of simplifying greatly the language.

Let $\mathcal{M}$ be the C*–algebra formed by all bounded operators of $\mathcal{H}$. Let $\mathcal{I}$ be the trivial subalgebra of $\mathcal{M}$ formed by the multiples of the identity I. Any self–adjoint subset of

[1] In other contexts the inclusion of the unit is not required.

$\mathcal{M}$ including the unit and closed in the norm topology forms a C*–algebra $\mathcal{A}$. From a lattice–theoretic point of view, $\mathcal{A}$ is an element of a partially ordered set in which $\mathcal{M}$ is the supremum and $\mathcal{I}$ is the infimum, i.e. $\mathcal{I} \leq \mathcal{A} \leq \mathcal{M}$.

The lattice structure of $\mathcal{M}$ is immediately established as soon as we realize that for any two C*–algebras $\mathcal{A}, \mathcal{B} \leq \mathcal{M}$, both the *join* $\mathcal{A}\widetilde{\vee}\mathcal{B}$, i.e. the smallest C*–algebra that contains $\mathcal{A}$ and $\mathcal{B}$, and the *meet* $\mathcal{A}\widetilde{\wedge}\mathcal{B}$, i.e. the greatest C*–algebra including all the operators common to $\mathcal{A}$ and $\mathcal{B}$, exist in $\mathcal{M}$. Here the symbols $\widetilde{\vee}, \widetilde{\wedge}$ are used to indicate that algebraic closure is achieved in the uniform topology. The property $\mathcal{A}\widetilde{\wedge}\mathcal{B} = \mathcal{A} \cap \mathcal{B}$ holds, i.e. $\widetilde{\wedge}$ is equivalent to the set–theoretic intersection of $\mathcal{A}$ and $\mathcal{B}$.

Let $\mathcal{A}'$ be the C*–algebra formed by all bounded operators that commute with all the elements of $\mathcal{A}$, i.e. the *commutant* of $\mathcal{A}$. The relationships $\mathcal{A}\widetilde{\vee}\mathcal{A}' \leq \mathcal{M}$, $\mathcal{A}\widetilde{\wedge}\mathcal{A}' \geq \mathcal{I}$ can be easily proved.

1.2 Von Neumann algebras

A C*–algebra closed in the *weak topology* (Haag, 1992) is called a *von Neumann algebra*. The simplest way to formulate the criterion of weak closeness is the following: An operator sequence $X_1, X_2, \ldots$ converges weakly to X if the sequences $|\langle\alpha, (X_1 - X)\alpha\rangle|, |\langle\alpha, (X_2 - X)\alpha\rangle|, \ldots$ converge to zero for all vectors $|\alpha\rangle \in \mathcal{H}$. Weak closure means that X belongs to the algebra whenever a sequence $X_1, X_2, \ldots$ weakly converging to X exists. Closure in the uniform topology implies weak closure, but the converse is not true.

Note that the identity $4\langle\alpha, A\beta\rangle = \sum_{n=0}^{3} i^{-n}\langle\alpha + i^n\beta, A(\alpha + i^n\beta)\rangle$ assures that the criterion stated above is equivalent to requiring convergence to zero for the sequence $|\langle\alpha, (X_i - X)\beta\rangle|$, $i = 1, 2, \ldots$, for any pair $|\alpha\rangle, |\beta\rangle \in \mathcal{H}$.

From here on, the von Neumann algebra obtained by weak closure of a C*–algebra $\mathcal{A}$ will be denoted by the barred symbol $\bar{\mathcal{A}}$. For terminology simplicity, von Neumann algebras will be called *closed algebras* and the weak closure operation will be called *algebraic closure*.

The lattice structure of closed algebras can be established by defining the weakly closed join $\bar{\mathcal{A}}_1 \vee \bar{\mathcal{A}}_2$ of any two closed algebras $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$ as the smallest closed algebra that contains $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$, and, for the sake of completeness, the weakly closed meet $\bar{\mathcal{A}}_1 \wedge \bar{\mathcal{A}}_2$ as the greatest closed algebra that includes $\bar{\mathcal{A}}_1$ and $\bar{\mathcal{A}}_2$. The following relationships

$$\bar{\mathcal{A}}_1\widetilde{\vee}\bar{\mathcal{A}}_2 \leq \bar{\mathcal{A}}_1 \vee \bar{\mathcal{A}}_2\,, \quad \bar{\mathcal{A}}_1\widetilde{\wedge}\bar{\mathcal{A}}_2 \equiv \bar{\mathcal{A}}_1 \wedge \bar{\mathcal{A}}_2 \equiv \bar{\mathcal{A}}_1 \cap \bar{\mathcal{A}}_2 \tag{1}$$

mark then the difference between the uniform and the weak closure.

The most relevant fact regarding closed algebras is that also the concept of weak closure is purely algebraic, as the following theorem due to von Neumann states:

Proposition 1..1 (Bicommutant theorem)**.** Let $\mathcal{A}$ be a C*–algebra (with unit), $\mathcal{A}'$ its commutant and $\mathcal{A}'' \equiv (\mathcal{A}')'$ its *bicommutant*. $\mathcal{A}$ is closed in the weak topology if and only if $\mathcal{A} = \mathcal{A}''$ *(Bratteli and Robinson, 1987).*

Since $\mathcal{A}''' \equiv \mathcal{A}'$, the equality $\mathcal{A}' = \bar{\mathcal{A}}'$ follows. Thus, algebraic closure can be equivalently achieved by including into the algebra all bounded operators that can be obtained either spatially, by the weak closure of Hilbert space representations, or algebraically, i.e. abstractly, by bicommutation.

The algebraic closure possesses remarkable properties, the most important of which are the *duality* relationships:

$$(\bar{\mathcal{A}}_1 \wedge \bar{\mathcal{A}}_2)' = \bar{\mathcal{A}}_1' \vee \bar{\mathcal{A}}_2',\ (\bar{\mathcal{A}}_1 \vee \bar{\mathcal{A}}_2)' = \bar{\mathcal{A}}_1' \wedge \bar{\mathcal{A}}_2',\ \mathcal{M}' = \mathcal{I},\ \mathcal{I}' = \mathcal{M}. \tag{2}$$

Let $\mathcal{I}$ be the trivial subalgebra of $\mathcal{M}$ formed by the multiples of the identity I. Then $\bar{\mathcal{A}} \geq \mathcal{I}$, $\bar{\mathcal{A}}' \geq \mathcal{I}$ and $\mathcal{Z} = \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}' \geq \mathcal{I}$ is in general a non trivial algebra. Since $\mathcal{Z}$ is formed by all the elements of $\mathcal{M}$ that commute with all the elements of both $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$, it is an Abelian algebra. It is called the *center* of $\bar{\mathcal{A}}$ (and of $\bar{\mathcal{A}}'$).

1.3 Orthocomplemented and orthomodular algebras

Let us specify the conditions for a lattice of closed algebras to be orthocomplemented. Let $\mathcal{Z}$ be the center of a closed algebra $\bar{\mathcal{A}}$. If $\mathcal{Z} > \mathcal{I}$ (properly), then $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$, as Hilbert space representations, are reducible. In this case, the eigenvalues of a suitable set of independent elements of $\mathcal{Z}$ different from I can be used to label all possible representations. It is then clear (App.1) that in order for the lattice to be orthocomplemented the equality $\bar{\mathcal{A}} \wedge \bar{\mathcal{A}}' = \mathcal{I}$, and consequently its dual $\bar{\mathcal{A}} \vee \bar{\mathcal{A}}' = \mathcal{M}$, must be added to Eq.s (2), i.e. the representation must be *irreducible*, in which case $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$ are called *factors*.

Now assume by hypothesis that a formal correspondence between an orthomodular lattice $\mathcal{R}$ of causal shadows and an orthomodular lattice of C*–subalgebras of a C*–algebra exists. It is then clear that if the elements $\langle S \rangle \in \mathcal{R}$ are one–to–one with certain closed subalgebras $\bar{\mathcal{A}}_S$, then the lattice formed by these subalgebras is orthomodular by isomorphism. In particular, all $\bar{\mathcal{A}}_S$ are orthocomplemented, i.e. $\bar{\mathcal{A}}_S \wedge \bar{\mathcal{A}}_S' = \mathcal{I}$, implying that all $\bar{\mathcal{A}}_S$ are factors. The orthomodularity property is then assured provided that whenever $\bar{\mathcal{A}}_{S_1} \leq \bar{\mathcal{A}}_{S_2}$ the equality

$$(\bar{\mathcal{A}}_{S_1}' \wedge \bar{\mathcal{A}}_{S_2}) \vee \bar{\mathcal{A}}_{S_1} = \bar{\mathcal{A}}_{S_2} \tag{3}$$

is satisfied, i.e. $\bar{\mathcal{A}}_{S_1}' \wedge \bar{\mathcal{A}}_{S_2}$ is the orthocomplement of $\bar{\mathcal{A}}_{S_1}$ relative to $\bar{\mathcal{A}}_{S_2}$.

These properties are certainly satisfied for a finite system, as in this case the orthomodular property is tantamount to the decomposability of the Hilbert space into a direct product of as many Hilbert spaces as there are degrees of freedom. This is precisely the case we are not interested in.

2. Untenability of the Orthomodular Correspondence

We are now in a position that allows us to realize how and why the orthomodular correspondence hypothesis fails in the framework of local quantum physics. The first trouble comes from quantum free fields.

Consider a quantum theory of local free fields endowed with electrical charge density $\rho(x,t)$ and assume that a certain causal shadow $\langle S\rangle$ of cross–section S corresponds to the subalgebra $\bar{\mathcal{A}}_S$. Integrating $\rho(x,t)$ over S we obtain the observable Q_S that belongs to $\bar{\mathcal{A}}_S$ by construction, but which must be assigned to the orthocomplement $\bar{\mathcal{A}}'_S = \bar{\mathcal{A}}_{S'}$ as it commutes with all the observables of $\bar{\mathcal{A}}_S$. The orthomodular correspondence assumption is thus patently violated.

Haag and Schroer noted, however, that this difficulty can be circumvented if the current is the source of an electromagnetic field, as in this case Q_S can be expressed by Gauss' theorem as a the flux of the electric field across the boundary of S. If S is open, then its boundary is found in S'. Correspondingly, Q_S is found in $\bar{\mathcal{A}}'_S$. The orthomodular requirement is then saved.

Similar considerations hold if S is a ring–shaped spacelike set. The operator that represents a constant electrical current I_S circulating in S commutes with the subalgebra of the observables localized in $\bar{\mathcal{A}}_S$. Stoke's theorem then assures that I_S is proportional to the circulation of the magnetic field around the surface of the ring. If the ring is an open set, I_S can be ascribed to $\bar{\mathcal{A}}'_S$.

These examples seem to indicate that the attempt to establish the orthomodular correspondence forces the observables into a status of physical interaction.

Unfortunately the idea is untenable. If the algebraic lattice is orthomodular, not only $\bar{\mathcal{A}}$ but also any one of its subalgebras $\bar{\mathcal{A}}_S$ is a factor, i.e. $\bar{\mathcal{A}}_S$ is an irreducible representation. This implies that only one electric charge sector must be ascribed to each region of a spacelike surface.

Actually, the irreducibility requirement trivializes the whole question as, if $\bar{\mathcal{A}}_S$ is an irreducible representation, Q_S is a multiple of the identity, i.e. it belongs to both $\bar{\mathcal{A}}_S$ and $\bar{\mathcal{A}}'_S$. The mystery of Q_S ubiquity is thus explained without invoking the intervention of gauge fields. To put it differently, if we want that more eigenvalues of Q_S be associated with S, so as to avoid the trivialization, then $\bar{\mathcal{A}}_S$ must be a reducible representation, hence the lattice cannot be orthomodular.

2.1 Quasi–orthomodularity

It is then definitely clear that a way out from this impasse can be found only if we modify the way of implementing the orthomodular property in systems with a non–compact continuum of degrees of freedom. The problem seems strictly related to the topological character of the continuum. Indeed, passing from the discrete to the continuum, discreteness is replaced by suitable separability properties and sums are replaced by integrals. Thus, in the framework of local quantum physics, operators and subalgebras, rather than being indexed by the points of a set, are expected to be functions of measurable subsets of a topological space.

As noted App.1, the semi–complete distributive lattice formed by the open or the closed sets of a topology is not orthocomplemented simply because the orthocomplement of a closed set is open and that of an open set is closed. Precisely this asymmetry

prevents the lattice from being orthomodular. Clearly, the problem does not exist for discrete topologies, since in this case all topological subsets are both open and closed. The basic difference between the degrees of freedom of a finite system and those of an infinite system is precisely this.

Can we now somehow reconcile the structure of a topological space with the orthocomplementation property? A very similar question can be posed in the causal–shadow context. Can we somehow reconcile spacetime topology with the orthomodular properties of a causal–shadow lattice?

As pointed out in App.2, the orthomodular structure of a causal–shadow lattice $\mathcal{R}$ is a direct inheritance of the orthomodular structure of the Boolean lattice formed by their set–theoretic cross–sections. Most properties of the two structures are therefore closely related. For instance, since cross–sections are locally measurable regions of a spacelike surface, and since Boolean lattices are measurable provided that are discrete–complete (Grätzer, 1978), $\mathcal{R}$ too must be discrete–complete. In this view, however, the topological properties of the spacelike surfaces are neglected and it would be certainly preferable if $\mathcal{R}$ inherited also these. Unfortunately, this is incompatible with the orthomodular requirement. It is then clear that the orthomodularization problem must be primarily solved in the topological context.

For the topologies of measurable sets there is a way to recover, at least formally, orthocomplementarity, and orthomodularity with it: ignoring boundaries. Since the boundaries of measurable sets have measure zero, we can neglect these and regard the closed and the open sets as equivalent, *modulo boundaries*. All closed set of measure zero are then collected in the equivalence class of the lattice null element. By means of this peeling procedure, the fundamental asymmetry of the topological lattice is removed. This does not mean, however, that zero–measure sets, in particular topological boundaries, can be ignored at other levels of the analysis.

If we apply this procedure to a topological lattice of causal shadows, we obtain an orthomodular lattice whose elements are the elements of the topological lattice *modulo cross–section boundaries*. In this way, both the orthomodular structure and the good properties of the underlying topology are preserved.

From here on, orthocomplemented and orthomodular lattices, whose elements are equivalent "modulo something", will be respectively called *quasi–orthocomplemented* and *quasi–orthomodular*. Note that, despite their names, they actually are fully orthocomplemented and orthomodular. The prefix *quasi* is here intended as a way of saying that the halo of an underlying topology is maintained on the background.

In the following sections we will study whether an analogous procedure can be applied to impart a quasi orthomodular structure to an algebra of local quantum physics operators.

2.2 Topological lattices

Preliminary to the problem, a brief glossary of topological notions from a lattice–theoretic standpoint is here presented.

General topologies are distributive lattices generated by a join–complete but generally meet–incomplete family $\{S_\alpha\}$ of distinguished sets S_α, called *open*. Here α is an index running over some set. A theorem by Birkhoff (1933) and Stones (1936) assures that S_α can be interpreted as a lattice of subsets of a set T of points (Grätzer, 1978). This makes any S_α inherit the set–theoretic orthocomplementation $S_\alpha \to S'_\alpha$, with the well–known properties $S''_\alpha = S_\alpha$, $S_\alpha \cap S'_\alpha = 0$, $S_\alpha \cup S'_\alpha = T$.

Since orthocomplementation exchanges joins and meets, the orthocomplements $\{S'_\alpha\}$ of $\{S_\alpha\}$ form a meet–complete but generally join–incomplete family of distinguished sets, which are called *closed*. In the following, open sets will be denoted by S_α and closed sets by $\bar{S}_\alpha$. Redudantly though, we will write $\{\bar{S}'_\alpha\}$ instead of $\{S'_\alpha\}$ to make it explicit that we are dealing with closed sets.

The smallest closed set $\bar{S}$ that contains an open set S is called the *closure* of S. The greatest open set S contained in a closed set $\bar{S}$ is called the *interior* of $\bar{S}$. Thus, given any S_α of the open–set lattice we can form both its orthocomplement $\bar{S}'_\alpha$ and its closure $\bar{S}_\alpha$. Coherently with our notations, the interior of $\bar{S}'_\alpha$ will be denoted by S'_α.

The set $\bar{B}_\alpha = \bar{S}'_\alpha \cap \bar{S}_\alpha$, which is manifestly closed, will be called the *boundary* of S_α. Clearly enough, it is also the boundary of $\bar{S}_\alpha$, S'_α and $\bar{S}'_\alpha$. It is also clear that a general topology can be equivalently based on the lattice properties of its closed sets. Correspondingly, the lattice–theoretic structure of a topological space is characterized by the properties

$$\bar{S}_\alpha \cap \bar{S}'_\alpha = \bar{B}_\alpha \,, \tag{4}$$

$$\bar{S}_\alpha \cup \bar{S}'_\alpha = T \,. \tag{5}$$

From $\bar{S}_\alpha \subset \bar{S}_\beta$, the following equalities also follow

$$(\bar{S}'_\alpha \cap \bar{S}_\beta) \cup \bar{S}_\alpha = \bar{S}_\beta \,, \quad (\bar{S}'_\alpha \cap \bar{S}_\beta) \cap \bar{S}_\alpha \equiv \bar{B}_\alpha \cap \bar{S}_\beta = \bar{B}_{\alpha,\beta} \subset \bar{S}_\beta \,.$$

Interpreting $(S'_\alpha)_\beta = \bar{S}'_\alpha \cap \bar{S}_\beta$ as the quasi–orthocomplement of $\bar{S}_\alpha$ relative to $\bar{S}_\beta$ modulo the boundary $\bar{B}_{\alpha,\beta}$, we see that the closed sets of a topological space form a quasi–orthomodular lattice. As it is well–known from general topology, these simple properties are sufficient to define the concepts of *compactness* and *connectedness* (Simmons, 1963).

Let us enrich our vocabulary with a few auxiliary concepts. Any two closed sets will be called *adjacent* if their intersection is a non–empty common subset of their boundaries. Forming the join of adjacent sets makes their common pieces of boundary dissolve, in the sense that these pieces cannot be recovered anymore from the join. A set of adjacent sets will be called a *tiling* if their join covers the whole space T.

Assume as topological space that formed by the closed sets of a spacelike surface. As explained in App.2, the lattice formed by the causal–shadows of such closed sets inherit the quasi–orthomodular structure of the spacelike surface topology. Consequently,

it makes sense to speak of compactness, connectedness, adjacency and tiling of causal shadows. Quite differently from the cross–section topology case, however, the merging of pieces of boundaries following the joining of adjacent sets is inherited by the causal–shadows as the disappearance of the merged pieces and the formation of closed sets of events that are substantially greater than their mere set–theoretic unions. We can characterize this fact saying that boundary merging causes causal–shadow expansion.

It is then clear that, if a correspondence between causal shadows and subalgebras of operators is somehow possible, something equivalent is expected to happen on the algebraic side.

2.3 Pseudo–topological properties of closed algebras

Interpreting bicommutation as an analog of the topological closure, closed algebras appear to share some formal properties with the closed sets of a topology. As in the topological case, infinite meets of closed algebras are closed algebras. This happens because meets are equivalent to set theoretic intersections. Finite joins of closed algebras, although they are not set theoretic, they are nevertheless closed by definition. Since infinite joins can be formally defined as the algebras generated by an infinite number of closed subalgebras, they too make sense. But in general their closure is not assured as infinite products of operators are usually plagued by the phenomenon of disjoint representations. The formal analogy can be pushed even further basing on the following definitions

Definition 2..1 (Boundaries and interiors of closed algebras)**.** Let $\bar{\mathcal{A}}$ be a closed algebra. Define $\bar{\mathcal{B}} = \mathcal{Z} - \mathcal{I}$, i.e. the C*–algebra formed by the elements of the center $\mathcal{Z} = \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}'$ modulo multiples of the identity $\mathcal{I}$, as the *boundary* of $\bar{\mathcal{A}}$ (and of $\bar{\mathcal{A}}'$). Define $\mathcal{A} = \bar{\mathcal{A}} - \bar{\mathcal{B}}$, i.e. the C*–algebra obtained by removing from $\bar{\mathcal{A}}$ all the elements belonging to its boundary $\bar{\mathcal{B}}$, as the *interior* of $\bar{\mathcal{A}}$. The algebra $\mathcal{A}$ will be called *open*.

We omit proving that the lattice of open algebras is meet–incomplete but join–complete. $\mathcal{I}$ and $\mathcal{M}$ can be considered both open and closed. Since $\mathcal{Z}$ is closed and $\mathcal{I}$ is both open and closed, $\bar{\mathcal{B}}$ is closed. Since $\mathcal{A}' = \bar{\mathcal{A}}'$, then $\mathcal{A}'' = \bar{\mathcal{A}}$ holds, i.e. $\mathcal{Z}$, hence $\bar{\mathcal{B}}$, can be immediately recovered by bicommutation. Moreover, $\mathcal{A} \wedge \bar{\mathcal{A}} = \mathcal{I}$ and $\mathcal{A}' \wedge \bar{\mathcal{A}}' = \mathcal{I}$, i.e. both $\mathcal{A}$ and $\mathcal{A}'$ are C*–algebras. So, $\mathcal{I} \leq \bar{\mathcal{A}}, \mathcal{A}, \bar{\mathcal{A}}', \mathcal{A}' \leq \mathcal{M}$ is a lattice of closed and open C*–algebras with $\mathcal{I}$ as infimum and $\mathcal{M}$ as supremum. In order for the topological analogy to be meaningful, however, also the analog of Eq.(5) must be algebraically implemented in some way. Here we meet the crucial point.

Assume that $\mathcal{M}$, as the algebra of all bounded operators in $\mathcal{H}$, contains all the irreducible representation of its subalgebras. If $\bar{\mathcal{A}}$ is a subalgebra of $\mathcal{M}$ with a non trivial boundary $\bar{\mathcal{B}}$, then the operators of $\bar{\mathcal{B}}$ label the different irreducible representations $\pi_\alpha(\bar{\mathcal{A}})$ of $\bar{\mathcal{A}}$ within $\mathcal{M}$ (here α represents the label provided by $\bar{\mathcal{B}}$). Thus, the entire system of reducible representations $\Pi(\bar{\mathcal{A}})$ of $\bar{\mathcal{A}}$ and $\Pi(\bar{\mathcal{A}}')$ of $\bar{\mathcal{A}}'$ can be written as a direct sum of

the form

$$\Pi(\bar{\mathcal{A}}) = \bigoplus_{\alpha} \pi_\alpha(\bar{\mathcal{A}})\,, \qquad \Pi(\bar{\mathcal{A}}') = \bigoplus_{\alpha} \pi_\alpha(\bar{\mathcal{A}}')\,, \tag{6}$$

where $\bigoplus_\alpha$ is a summation symbol standing, in general, for Stiltjes–Lebegues integration. Since each $\pi_\alpha(\bar{\mathcal{A}})$ separately considered is a factor, the following relationships hold

$$\pi_\alpha(\bar{\mathcal{A}}) \wedge \pi_\alpha(\bar{\mathcal{A}}') = \mathcal{I}_\alpha\,, \qquad \pi_\alpha(\bar{\mathcal{A}}) \vee \pi_\alpha(\bar{\mathcal{A}}') = \pi_\alpha(\mathcal{M})\,,$$

where $\mathcal{I}_\alpha$ are multiples of the identity in the Hilbert spaces $\mathcal{H}_\alpha$ of the representations $\pi_\alpha(\bar{\mathcal{A}})$ and $\pi_\alpha(\mathcal{M})$ are inequivalent representations of $\mathcal{M}$. With a suitable normalization, we can then write

$$\Pi(\bar{\mathcal{A}}) \wedge \Pi(\bar{\mathcal{A}}') = \Pi(\bar{\mathcal{B}}) = \bigoplus_{\alpha} \mathcal{P}_\alpha\,, \tag{7}$$

where $\mathcal{P}_\alpha$ are the projectors of $\mathcal{H}$ on $\mathcal{H}_\alpha$, and

$$\Pi(\bar{\mathcal{A}}) \vee \Pi(\bar{\mathcal{A}}') = \bigoplus_{\alpha} \pi_\alpha(\mathcal{M}) \leq \mathcal{M}\,. \tag{8}$$

Equation (7) can be readily recognized as the analog of Eq. (4), with $\bar{\mathcal{B}}$, the algebra (without unit) of projectors $\mathcal{P}_\alpha$, playing the role of the topological boundary common to $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$.

Equation (8), however, is not the analog of Eq.(5). Thus, to complete the analogy we should find an extension $\sqcup$ of the weakly closed join $\vee$ so that the equation $\Pi(\bar{\mathcal{A}}) \sqcup \Pi(\bar{\mathcal{A}}') = \mathcal{M}$ holds instead of Eq.(8). We will focus on this problem in the next section.

2.4 Hyperfactors and intertwined joins

From here on, the direct sums $\Pi(\bar{\mathcal{A}}), \Pi(\bar{\mathcal{A}}')$ will be called *hyperfactors*. For the sake of simplicity, however, they will be denoted by $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}'$. Accordingly, the meet $\Pi(\bar{\mathcal{B}})$ of the two hyperfactors will be simply denoted by $\bar{\mathcal{B}}$. This is consistent with identifying abstract algebras with their widest disjoint representations.

As discussed in the previous section, the problem of establishing a correspondence between topological properties and algebraic properties leads us to the following dilemma: Either $\bar{\mathcal{A}}$ is a factor of $\mathcal{M}$, in which case the analog $\bar{\mathcal{A}} \vee \bar{\mathcal{A}}' = \mathcal{M}$ of Eq.5 holds but the boundary becomes trivial, or the boundary is not trivial, i.e. $\bar{\mathcal{A}} \wedge \bar{\mathcal{A}}' - \mathcal{I} = \bar{\mathcal{B}} > \mathcal{O}$, where $\mathcal{O}$ is the null algebra, in which case the analogy with Eq.5 fails, as the equality $\bar{\mathcal{A}} \vee \bar{\mathcal{A}}' = \bigoplus_\alpha \pi_\alpha(\mathcal{M}) < \mathcal{M}$ instead holds. Since the correspondence works perfectly for boundaries, we need only to focus on the problem of reversing the decomposition $\mathcal{M} \to \bar{\mathcal{A}}, \bar{\mathcal{A}}'$.

Let $\bar{\mathcal{A}}$ be a hyperfactor of $\mathcal{M}$ and $\mathcal{U}_{\mathcal{A}}$ the group of all unitary operators $U \in \mathcal{M}$ such that

$$U\bar{\mathcal{A}}U^\dagger = \bar{\mathcal{A}}\,, \qquad U\bar{\mathcal{A}}'U^\dagger = \bar{\mathcal{A}}'\,.$$

Since $\mathcal{U}_{\mathcal{A}}$ provides a group of automorphisms for both $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$, we simply write

$$\mathcal{U}_{\mathcal{A}}\bar{\mathcal{A}}\mathcal{U}_{\mathcal{A}}^{\dagger} = \bar{\mathcal{A}}\,, \qquad \mathcal{U}_{\mathcal{A}}\bar{\mathcal{A}}'\mathcal{U}_{\mathcal{A}}^{\dagger} = \bar{\mathcal{A}}'\,.$$

The equality $\mathcal{U}_{\mathcal{A}}\bar{\mathcal{B}}\mathcal{U}_{\mathcal{A}}^{\dagger} = \bar{\mathcal{B}}$ is easily proved, meaning that $\mathcal{U}_{\mathcal{A}}$ provides also a group of automorphisms for $\bar{\mathcal{B}}$. In general for $B \in \bar{\mathcal{B}}$ and $U \in \mathcal{U}_{\mathcal{A}}$ it is $UBU^{\dagger} = \hat{B} \in \bar{\mathcal{B}}$ but in general $\hat{B} \neq B$.

Now consider the subgroup $\mathcal{V}_{\mathcal{A}} \subset \mathcal{U}_{\mathcal{A}}$ formed by all the unitaries of $\bar{\mathcal{A}}$ and the subgroup $\mathcal{V}'_{\mathcal{A}} \subset \mathcal{U}_{\mathcal{A}}$ formed by all the unitaries of $\bar{\mathcal{A}}'$. Clearly, for all $V \in \mathcal{V}_{\mathcal{A}}$, $V' \in \mathcal{V}'_{\mathcal{A}}$ and $B \in \bar{\mathcal{B}}$ we have $VBV^{\dagger} = B$, $V'BV'^{\dagger} = B$. In other terms, the automorphisms generated by all the elements VV' of the direct product $\mathcal{V}_{\mathcal{A}} \times \mathcal{V}'_{\mathcal{A}}$, which is an invariant group of $\mathcal{U}_{\mathcal{A}}$, leaves the elements of $\bar{\mathcal{B}}$ unchanged.

It is then evident that the automorphism group that acts non trivially on $\bar{\mathcal{B}}$ is the factor group $\mathcal{F}_{\mathcal{A}} = \mathcal{U}_{\mathcal{A}}/(\mathcal{V}_{\mathcal{A}} \times \mathcal{V}'_{\mathcal{A}})$, i.e. the equivalence class of $\mathcal{U}_{\mathcal{A}}$ modulo $\mathcal{V}_{\mathcal{A}} \times \mathcal{V}'_{\mathcal{A}}$. This means that there are in $\mathcal{M}$ many equivalent groups that provide equivalent automorphisms for $\bar{\mathcal{A}}$, $\bar{\mathcal{A}}'$ and $\bar{\mathcal{B}}$.

Definition 2..2 (Intertwining group)**.** Let $\hat{\mathcal{U}}_{\mathcal{A}}$ a unitary group of the type just described, i.e. a representative of the equivalence class $\mathcal{F}_{\mathcal{A}}$, and assume that $\bar{\mathcal{A}} \vee \bar{\mathcal{A}}' \vee \hat{\mathcal{U}}_{\mathcal{A}} = \mathcal{M}$. Since $\mathcal{V}_{\mathcal{A}}$ and $\mathcal{V}'_{\mathcal{A}}$ are already elements of $\bar{\mathcal{A}}$ $\bar{\mathcal{A}}'$, the result does not depend on the particular choice of the representative. $\hat{\mathcal{U}}_{\mathcal{A}}$ will be called an *intertwining group* of $\bar{\mathcal{A}}$.

Thus, the existence of an intertwining group depends only on the structure of the algebraic automorphisms provided by $\mathcal{F}_{\mathcal{A}}$.

Here is the definition of extended join that we need to complete the formal analog with topologies.

Definition 2..3 (Intertwined join and meet)**.** Let $\bar{\mathcal{A}}$ any hyperfactor of $\mathcal{M}$ and assume that an intertwining group $\hat{\mathcal{U}}_{\mathcal{A}}$ exists for $\bar{\mathcal{A}}$, we define the operation

$$\bar{\mathcal{A}} \sqcup \bar{\mathcal{A}}' \equiv \bar{\mathcal{A}} \vee \bar{\mathcal{A}}' \vee \hat{\mathcal{U}}_{\mathcal{A}} = \mathcal{M}$$

the *intertwined join* of $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$. For the sake of completeness we define also, improperly though, their *intertwined meet* as the common boundary of $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$:

$$\bar{\mathcal{A}} \sqcap \bar{\mathcal{A}}' \equiv \bar{\mathcal{A}} \wedge \bar{\mathcal{A}}' - \mathcal{I} = \bar{\mathcal{B}}\,.$$

From here on, to simplify the language, whenever an algebra $\bar{\mathcal{A}}$ will be called or understood as a hyperfactor, the existence of an intertwining group $\hat{\mathcal{U}}_{\mathcal{A}}$ and of the intertwined join $\bar{\mathcal{A}} \sqcup \bar{\mathcal{A}}' = \mathcal{M}$ will be assumed.

Note that the intertwined join expands the algebra generated by $\bar{\mathcal{A}}$ and $\bar{\mathcal{A}}'$ while dissolving the common boundary $\bar{\mathcal{B}}$. This provides the algebraic analog of the expansion of two causal shadows $\langle S\rangle$, $\langle S'\rangle$ paired to the dissolution of their common boundary $\langle S\rangle \cap \langle S'\rangle$.

Studying under what conditions intertwined joins exist requires an algebraic analysis that we cannot carry out in this paper. Their existence for suitable algebras is plausible, however, otherwise no implementation of local quantum physics would be possible. In a certain sense, the construction here proposed is a generalization of the quantum coordinatization procedure for elementary quantum systems suggested by Hermann Weyl many years ago, and, ultimately, it might even be very similar, if not equivalent, to an algebra of fields. Here, however, we will not try to investigate any further this problem, but only to spend some more words in order to make it explicit the physical meaning of this approach.

The degrees of freedom of a classic system are primarily described by the coordinates of a configuration space. What makes of a physical quantity a coordinate is not the fact of being a simple set of numerical values, rather of being a set of numerical values ordered by a translation group. In elementary quantum physics, the eigenvalues of an observable Q representing positions on a real axis are translated by the Abelian group generated by the Heisenberg's conjugate operator P.

In *The theory of groups and quantum mechanics* (1931), Weyl attributed particular importance to the fact that Q and P can be though of as the generators of two continuous Abelian groups of unitary operators $U(x) = \exp(iPx/\hbar)$ and $V = \exp(iQy/\hbar)$, each one of which transforms isomorphically the Abelian algebra generated by the other:

$$U(a)QU^{\dagger}(a) = Q + a\,, \qquad V(b)PV^{\dagger}(b) = P - b\,.$$

The Hilbert space implementation of these transforms is usually described as a projective representation of the corresponding classical group of canonical transformations.

Generalizing this fact, Weyl proposed that the coordinatization of quantum mechanical systems without classic analogs be accomplished by the inclusion of projective representations of discrete Abelian groups, which are equivalent to discrete Abelian groups equipped with Abelian groups of isomorphisms. He showed that spin and fermion algebras fall within this class.

Dealing with systems formed by identical particles, following Bose, Fermi statistics or some parastatistics, the Abelian group of isomorphisms must be widened so as to account for the permutation invariance of the degrees of freedom. Note that, as accounted for in (Doplicher and Roberts, 1972), parastatistics provide an alternative schema for introducing non–Abelian gauge fields. Thus, in general, the coordinatization of a quantum system is provided by Abelian algebras equipped with automorphisms groups of a more general type. In other terms, the coordinatization is ruled by a group generated by an Abelian group $\mathcal{V}$ and a generally non–Abelian conjugated group $\mathcal{U}$ which transforms $\mathcal{V}$ automorphically. Let us call it *Weyl's coordinatization group*. To evidence how this view is related to our main subject we introduce the following notion:

Definition 2..4 (Conjugate group)**.** Let $\hat{\mathcal{V}}_{\mathcal{B}}$ the group formed by all the unitaries of an algebraic boundary $\bar{\mathcal{B}}$. Clearly $\hat{\mathcal{V}}_{\mathcal{B}}$ is a subset of both $\mathcal{V}_{\mathcal{A}}$ and $\mathcal{V}'_{\mathcal{A}}$, then of $\mathcal{V}_{\mathcal{A}} \times \mathcal{V}'_{\mathcal{A}}$. Since $\hat{\mathcal{U}}^{\dagger}_{\mathcal{A}}\mathcal{V}_{\mathcal{B}}\hat{\mathcal{U}}_{\mathcal{A}} = \mathcal{V}_{\mathcal{B}}$, the equality $\mathcal{V}_{\mathcal{B}}\hat{\mathcal{U}}_{\mathcal{A}}\mathcal{V}^{\dagger}_{\mathcal{B}} = \hat{\mathcal{U}}_{\mathcal{A}}$ also holds. $\mathcal{V}_{\mathcal{B}}$ will be called the *conjugate group* of $\hat{\mathcal{U}}_{\mathcal{A}}$.

We cannot avoid noting that $\bar{\mathcal{V}}_{\mathcal{B}}$ and $\hat{\mathcal{U}}_{\mathcal{A}}$, as defined in the context of hyperfactors, agree with the definition of a Weyl's coordinatization group. This suggests that the boundaries of all hyperfactors of an algebra of local observables, together with their respective intertwining groups, provide the coordinatization of local quantum physics by gauge fields. This view agrees with the role played by the intertwiners according to Doplicher *et al.* (1969a, 1969b, 1971, 1974).

3. Quasi–Orthomodular Algebras

The meaning of quasi–orthocomplementation for the hyperfactors of an algebra $\mathcal{M}$ is provided by the following definition.

Definition 3..1 (Quasi–orthocomplementation)**.** Assume first that $\mathcal{M}$ is an irreducible representation of a closed algebra. We will say that $\mathcal{M}$ is *quasi–orthocomplemented* if and only if every closed subalgebra $\bar{\mathcal{A}} \leq \mathcal{M}$ is a hyperfactor. Then assume that $\mathcal{M}$ is an abstract closed algebra. We say that $\mathcal{M}$ is *quasi–orthocomplemented* if every representation of $\mathcal{M}$ is quasi–orthocomplemented.

The collection of the hyperfactors $\bar{\mathcal{A}}$ in all representations of $\mathcal{M}$ will be still called hyperfactor. Thus $\bar{\mathcal{A}}$ must be understood as a hyperfactor in one sense or in the other according as $\mathcal{M}$ is understood as an irreducible representation or as an abstract algebra. In both cases, the orthocomplementation property is expressed by the condition that the relationships

$$\bar{\mathcal{A}} \sqcap \bar{\mathcal{A}}' = \bar{\mathcal{B}}\,, \qquad \bar{\mathcal{A}} \sqcup \bar{\mathcal{A}}' = \mathcal{M} \tag{9}$$

hold for any closed subalgebra $\bar{\mathcal{A}} \leq \mathcal{M}$. The meet symbol $\sqcap$ is here introduced with the meaning of $\wedge$ modulo $\mathcal{I}$. The formal correspondence with Eq.s (4), (5) is thus accomplished.

The quasi–orthomodular property can now be introduced as *relative quasi–orthocomplementation*:

Definition 3..2 (Quasi–orthomodularity)**.** Let $\bar{\mathcal{A}}_{\alpha;\beta} \leq \bar{\mathcal{A}}_{\beta} \leq \mathcal{M}$ and denote by $\bar{\mathcal{A}}'_{\alpha;\beta}$ the quasi–orthocomplement of $\bar{\mathcal{A}}_{\alpha;\beta}$ relative to $\bar{\mathcal{A}}_{\beta}$, i.e. $\bar{\mathcal{A}}'_{\alpha;\beta} \equiv (\bar{\mathcal{A}}_{\alpha;\beta})' \sqcap \bar{\mathcal{A}}_{\beta}$. Let $\bar{\mathcal{B}}_{\alpha;\beta}$ be the boundary of $\bar{\mathcal{A}}_{\alpha;\beta}$. Its portion in $\bar{\mathcal{A}}_{\beta}$, i.e. $\bar{\mathcal{B}}_{\alpha,\beta} = \bar{\mathcal{B}}_{\alpha;\beta} \sqcap \bar{\mathcal{A}}_{\beta}$, will be called the *interface* between $\bar{\mathcal{A}}_{\alpha;\beta}$ and $\bar{\mathcal{A}}'_{\alpha;\beta}$. The lattice formed by the closed subalgebras of $\mathcal{M}$, whether in the sense of an irreducible representation or of an abstract algebra, is called *quasi–orthomodular* provided that the relationships

$$\bar{\mathcal{A}}_{\alpha;\beta} \sqcap \bar{\mathcal{A}}'_{\alpha;\beta} = \bar{\mathcal{B}}_{\alpha,\beta}\,, \qquad \bar{\mathcal{A}}_{\alpha;\beta} \sqcup \bar{\mathcal{A}}'_{\alpha;\beta} = \mathcal{A}_{\beta} \tag{10}$$

hold for all subalgebras $\bar{\mathcal{A}}_{\beta;\alpha} \leq \bar{\mathcal{A}}_{\beta} \leq \mathcal{M}$. Thus all $\bar{\mathcal{A}}_{\alpha;\beta}$ are hyperfactors of $\bar{\mathcal{A}}_{\beta}$.

We are now in a position to draw a correspondence between quasi–orthomodular lattices of causal shadows and quasi–orthomodular lattices of hyperfactors. This is immediately accomplished by making any causal shadow $\langle S \rangle$, modulo its topological boundary

B_S, one–to–one with the closed subalgebra $\bar{\mathcal{A}}_S$, modulo $\bar{\mathcal{B}}_S = \bar{\mathcal{A}}_S \sqcap \bar{\mathcal{A}}'_S$. This map can then be extended to the underlying topology by ascribing $\bar{\mathcal{B}}_S$ to B_S.

The identifications can be further extended by coherence. Let $\{B_S\}$ be the set of boundaries of a closed–set tiling of a spacetime surface Σ. Since the algebraic boundaries $\bar{\mathcal{B}}_S$ corresponding to different B_S commute with each other, the set $\{\bar{\mathcal{B}}_S\}$ has the structure of a Boolean lattice of Abelian subalgebras without unit. It will be called the *boundary lattice*. If B_{S_1} and B_{S_2} are disjoint, i.e. $B_{S_1} \wedge B_{S_2} = 0$, where 0 is the null set, then $\bar{\mathcal{B}}_{S_1} \sqcap \bar{\mathcal{B}}_{S_2} = \mathcal{O}$ (the null algebra).

If B_{S_1} and B_{S_2} are not disjoint, let $\langle S_A \rangle$ and $\langle S_B \rangle$ be any two adjacent causal shadows of the tiling, then $B_{AB} = B_{S_A} \wedge B_{S_B}$. In this case, the *interface* between S_A and S_B, can be identified with the support of the Abelian subalgebra $\bar{\mathcal{B}}_{AB} = \bar{\mathcal{B}}_A \sqcap \bar{\mathcal{B}}_B$. Since $B_{AB} = \overline{S}_A \wedge \overline{S}_B$ corresponds to $\bar{\mathcal{B}}_{AB} = \bar{\mathcal{A}}_{S_A} \sqcap \bar{\mathcal{A}}_{S_B}$, $\bar{\mathcal{B}}_{AB}$ can be defined the *algebraic interface* of $\bar{\mathcal{A}}_{S_1}$ and $\bar{\mathcal{A}}_{S_2}$. The set $\{\bar{\mathcal{B}}_{AB}\}$ forms a Boolean lattice of Abelian subalgebras without unit that contains the boundary lattice. It will be called the *interface lattice*.

The main advantage of these identifications lies in the fact that, so doing, the algebras of observables completed by the intertwining operators inherit quite naturally all the topological properties of causal shadows. Thus, the notions of compactness and connectness for both measurable spacelike regions and boundaries can be transferred to the algebraic side. For instance, we can single out from the topological lattice a closed compact and simply connected set endowed with a simply connected boundary and transfer it with all its general topological properties to the algebraic side.

Due to the quasi–orthomodularity property, each causal shadow can be decomposed in many ways into a Boolean lattice of smaller causal shadows. The cross–sections of the causal shadows in each of these decomposition form a lattice of measurable topological sets on some spacelike surface. On the algebraic side, each closed algebra, whether factor or hyperfactor, can be decomposed in many ways into a Boolean lattice of closed subalgebras modulo boundaries, i.e sub–hyperfactors. The Boolean character of the decomposition is strictly related to the commutativity of the subalgebras.

We meet here the relevant property of our construction. Causal shadows do not form a topological space, but their cross–sections do. Correspondingly, quasi–orthomodular subalgebras do not have the formal properties of topological sets, but the subalgebras of any one of its quasi–orthomodular decomposition do. Indeed, the two lattices, that of cross–sections and that of the algebraic decomposition, correspond to each other.

3.1 Intertwined algebra of local observables

From what has been said so far, it is definitely clear that the idea of representing local quantum physics as a mere algebra of observables conflicts with the quasi–orthomodular requirement, and is therefore untenable. The point is that a quasi–orthomodular algebra cannot be devised as an irreducible or reducible algebra of observables, but as a reducible algebra of observables embedded in larger algebra by a net of intertwining operators. From here on, this extended structure will be called *intertwined algebra of observables*.

The need for an extension of the algebra of observables, however, was already implicit in the analysis carried out by Doplicher, Haag and Roberts (DHR analysis; Doplicher *et al.*, 1979–1974), in which the role of the intertwining operators was extensively described and fully clarified. Since intertwining operators represent in some way actions of unobservable fields, intertwined algebras of observables may appear a sort of quantum field theories. In an intertwined algebra, however, the subalgebra of local observables is a distinguished structure which forms the backbone of a local quantum physics representation. What marks the difference between the two structures, however, is the strength of the quasi–orthomodular requirement, which is likely to restrict considerably the spectrum of intertwined algebra representations. Whether these constraints are so strong to make any concrete representation impossible is not known, although, of course, we hope that this is not the case.

4. A Novel View

Once established the correspondence in this more general sense, Haag and Schroer's proposal, as well as the DHR analysis mentioned above, appears in a new light. Each causal shadow $\langle S \rangle$ of a compact and connected region S of a spacelike surface Σ can be interpreted as the integrity domain of an algebra closed by an Abelian boundary. The latter can be interpreted as the algebraic structure that provides the labels of all possible configurations of conservative quantities contained in S, i.e. of all possible states of the matter contained in S. Correspondingly, the content of matter of the original domain is fractioned according to all possible partitions of the conservative quantities there contained.

The structure appears even richer if we consider that boundary algebras provide more than conservative quantity labels. Indeed, in the framework of Haag–Schroer's interpretation, the Abelian algebra $\bar{\mathcal{B}}_{AB}$ with support on the topological interface B_{AB} of any two adjacent causal shadows $\langle S_A \rangle$ and $\langle S_B \rangle$ must be thought of as formed by the fluxes across B_{AB} of the gauge fields associated with the conservative quantities contained in S_A and S_B. So, if the intertwined algebras of observables is localized within the causal shadows, those of the gauge fields must be localized on the interfaces of adjacent causal–shadows.

In order for all these identifications to be physically meaningful, however, we need a new basic assumption. If matter homogeneity is postulated, the labels of all algebraic boundaries must run over a common set of possible eigenvalues. In order for this property to hold, the algebraic boundaries $\bar{\mathcal{B}}_A$, $\bar{\mathcal{B}}_B$, respectively corresponding to any two homeomorphic causal–shadow boundaries $B_A, B_B \subset \Sigma$, must be intertwined either by a unitary automorphism or, more in general, by an isometric automorphism (isometries preserve eigenvalues as well). Correspondingly, the interface Abelian algebras themselves must be related by homeomorphisms. In conclusion, the whole algebraic lattice must be coherently intertwined not only *vertically*, i.e. all over the irreducible representations of a same hyperfactor, as described in Sec.2.4, but also *horizontally*, i.e. through irreducible

representations of different hyperfactors, by suitable intertwining relationships.

APPENDIX 1

A few basic concepts on *lattices*, in particular the *orthocomplemented*, the *Boolean*, the *orthomodular* and the *modular* ones, are here flashed (Birkhoff, 1933; Stone, 1936; Birkhoff and von Neumann, 1936; Piron, 1964; Grätzer, 1978). The theory of lattices forms one of the highest levels of mathematical abstraction, being overcome only by the theory of categories. Its relevance resides in the MacNeille completion theorem: *every partial ordering can be uniquely embedded into a complete lattice up to isomorphisms* (MacNeille, 1937).

Lattices in general. A lattice $\mathcal{L}$ is a set of objects $A, B, C, \ldots$ of comparable magnitude or size, called the elements of $\mathcal{L}$, equipped with a *partial ordering* $\leq$, a *meet* $\vee$ and a *join* $\wedge$. $A \leq B$ (resp. $B \geq A$) means that A is not greater than B (resp. not smaller than B). $A \leq B$ and $B \geq A$ together means $A = B$. $A < B$ (resp. $B > A$) means $A \leq B$ but $A \neq B$. Sometimes the symbol $\leq$ is better interpreted as *being a part of* or *being included in*. Some other times, e.g. in logics, $\leq$ is better interpreted as *implies* and denoted by $\rightarrow$. In all cases it will be used with such particular meanings in place of the usual symbols.

The *join* of the elements $A, B, \ldots, Z \in \mathcal{L}$ is defined as their *least upper bound* (with respect to the partial ordering) and denoted by $A \vee B \vee \cdots \vee Z$. The *meet* of the same elements is defined as their *greatest lower bound* and denoted by $A \wedge B \wedge \cdots \wedge Z$. Joins and meets of any finite set of elements are supposed to exist in $\mathcal{L}$. In other terms, $\mathcal{L}$ is closed with respect to finite joins and meets. Often, also the greatest lower bound O (*infimum*) and the the least upper bound I (*supremum*) of all the elements of $\mathcal{L}$ are supposed to exist. In any case, assuming that O and I exist does not impose any additional constraint to a lattice.

A lattice closed with respect to infinite (countable and uncountable) joins and meets is called *complete*. Such is, for instance, the lattice formed by all the subsets of a set. A lattice closed with respect to infinite joins (meets) but only finite meets (joins) is called *join–complete* (*meet–complete*). Both are called *semi–complete*. The open (closed) sets of a topology form a join–complete (meet–complete) lattice. A lattice closed only with respect to discrete joins and meets will be called *discrete–complete*.

A and B are called *disjoint* if $A \wedge B = O$. If A and B are disjoint and $A \vee B = I$, the two element are called *complements* of each other. An element of a lattice may possess more than one complement. For instance, let $\mathcal{L}$ be a lattice formed by the linear subspaces of a vector plane, then O is the common origin of all vectors and I is the plane. All vectors which are not parallel to a vector A are complements of A.

Sublattices. Given a lattice and two elements $A, B \in \mathcal{L}$, with $A \leq B$, the set of elements $\{A \leq X \leq B, X \in \mathcal{L}\}$ form a *sublattice*, which has A as infimum and B as supremum.

Chains. A *chain* $\cdots < A < B < C < \cdots$ is a totally ordered subset of a lattice.

The elements of a chain can be indexed by ordinal numbers. Thus a chain may be finite, infinite, continuous open, closed, possess a minimum, a maximum etc as is the case for ordinal numbers. A chain C is *maximal* if there is no chain C' containing C as a proper subset. Using the axiom of choice it can be proved that any partial or total ordering contains a maximal chain (Hausdorff maximality theorem).

Orthocomplemented lattices. $\mathcal{L}$ is called *orthocomplemented* if for every $A \in \mathcal{L}$ there is an $A' \in \mathcal{L}$ such that $A \wedge A' = O$, $A \vee A' = I$ (*complementation property*), $A'' = A$ (*involution property*) and $(A \wedge B)' = A' \vee B'$ (*duality property*). A' is called the *orthocomplement* of A. It can be easily proved that A' is unique and that $A \leq B$ implies $A' \geq B'$. The latter property can be postulated in place of the duality property. Typical examples of orthocomplements are the complement of a set in the usual sense and the subspace orthogonal to a closed subspace of a Hilbert space.

Distributive lattices. A lattice is called *distributive* if $A \vee (B \wedge C) = (A \vee B) \wedge (A \vee C)$ or, equivalently by duality, if $A \wedge (B \vee C) = (A \wedge B) \vee (A \wedge C)$, holds for all the elements of $\mathcal{L}$. It has been proved (Birkoff, 1933; Stones, 1936) that any distributive lattice can be embedded in the lattice formed by the subsets of a set. Typical examples are the lattices formed by topological sets. In this case, however, the lattice is join–complete if its elements are the open sets of the topology, or it is meet–complete if its elements are the closed sets.

Boolean lattices. A lattice that is both orthocomplemented and distributive is called *Boolean*. A classic theorem by Stones (1936) states that any Boolean lattice is isomorphic to a lattice formed by selected subsets of a set. Typical examples are the lattice of all the subsets of a set, the lattice of subalgebras of an Abelian algebra, fields of measurable sets. In the latter case, however, only joins and meets of countable sets of elements are supposed to exist. This depends on the fact that sums make sense only over countable sets of summands.

Orthomodular lattices. The concept of *orthomodularity* is equivalent to that of *relative orthocomplementarity*. The importance of orthomodular lattices in physics was recognized for the first time by Constantin Piron (1964), who discovered their fundamental role in the partial ordering of quantum logical propositions. Consider a system S that, in some general sense, can be decomposed into disjoint parts $A, B, \ldots$, which in turn can be decomposed into smaller disjoint parts and so on, up to possible terminations at some atomic parcels, or forever. The concept of *relative orthocomplementation* corresponds to the fact that for each $A \leq B$ the element $A'_B \equiv A' \wedge B$ is the orthocomplement of A within the sublattice of $\mathcal{L}$ that has 0 as infimum and B as supremum, i.e. $A'_B \wedge A = 0$ and $A'_B \vee A = B$. In other terms, the class of orthomodular lattices are characterized by the property $(A' \wedge B) \vee A = B$ whenever $A \leq B$. Another characteristic property of orthomodular lattices is that all the elements of a chain generate a Boolean lattice, which can be supported on sets. Taking the infimum and the supremum of the lattice as chain extremals, we obtain many different orthomodular decompositions of the lattice. The subsets of a set and the closed subspaces of a Hilbert space are typical examples of orthomodular lattices. The decomposition of a Hilbert space into a refinable system of

orthogonal subspaces is strictly related to the resolution of the identity into a refinable commutative system of projectors.

Modular lattices. A lattice is called *modular* provided that whenever $A \leq B$ holds, the equality $A \vee (B \wedge C) = B \wedge (A \vee C)$ also holds. It can be proved that a lattice is modular if and only if a measure $D(A) > D(O) = 0$ exists, with O the lattice infimum and $A > O$, such that $D(A \vee B) + D(A \wedge B) = D(A) + D(B)$. $D(A)$ is called the *dimension* of A. This property is strictly related to the following theorem: in a modular lattice any two maximal chains $A < \cdots X \cdots < B$, $A < \cdots Y \cdots < B$ with same extremals A and B are isomorphic. Modular lattices are complemented but, in general, not orthocomplemented. Finite orthomodular lattices, however, are modular. The lattice formed by the subspaces of a projective geometry is modular. The invariant subgroups of a group form a modular lattice. The equivalence of the invariant–subgroup decomposition series (Jordan–Hölder theorem) is a consequence of the modular structure.

APPENDIX 2

Causal completions. *Causally complete sets*, or *causal completions*, are defined as follows: Let A_S be any subset of events of Minkowski spacetime M. Let A' be the set of all events that are spacelike to A_S and call it the *causal complement* of A_S. Let A be the set of all events that are spacelike to A' and call it the *causal completion* of A_S. Since $A'' = A$ and $A''' = A'$, A and A' are uniquely determined by A_S. A set will be called *causally complete* if it is the causal completion of a set A_S. For any two causal completions A and B, define $A \wedge B$ as the greatest lower bound of A and B, i.e. the largest causal completion contained in A and B. It coincides with the set–theoretic intersection of A and B. Define $A \vee B$ as the least upper bound of A and B, i.e., the smaller causal completion containing A and B. In general, it is greater than the set–theoretic union of A and B. Since $(A \vee B)' = A' \wedge B'$, the lattice of is orthocomplemented. Also $A \wedge B = 0$, the null set, $A \vee B = M \equiv 1$ hold. Remarkably, the lattice of causal completions is even orthomodular (Haag, 1992), meaning that the relative orthocomplement $A'_B \equiv A' \wedge B$ does exist for any $A \leq B$ and $A'_B \vee A = B$.

Handling with the whole lattice of causal completions is a hard task. Causal completions comprise for instance sets of both zero and non–zero measure, sets of points of light cones, isolated points, etc. The point is that with a pseudo–Euclidean metric, and more in general with a pseudo–Riemannian metric, it is difficult to define unambiguously the concept of closeness of two points and consequently, for instance, the boundaries of the causal completions. We are therefore motivated to look for a better approach.

The lattice of causal shadows. The task simplifies considerably if we limit our considerations to a much simpler orthomodular sublattice of the lattice just described. Taking advantage of the existence of spacelike surfaces, consider the lattice formed by the causal completions of the subsets of a spacelike surface Σ. The causal completion of any $S \subset \Sigma$ is its *causal shadow*, i.e. the set $\langle S \rangle$ of all points that cannot be reached by light rays passing through the points of the set–theoretic complement $S' \subset \Sigma$. The

relationships $\langle S\rangle \wedge \langle S'\rangle = 0$ and $\langle S\rangle \vee \langle S'\rangle = M$ are then evident.

The lattices formed in this way are manifestly orthomodular as they inherit directly the orthomodular structure of the subsets of Σ. Since Σ can be spanned by measurable sets, we can restrict our consideration to causal shadows of measurable cross–sections. Thus, on account of the lattice–theoretic properties of measurable sets (Grätzer, 1978), we can characterize the lattice of causal shadows as discrete–complete.

Causal shadows inherit quite naturally also the topological structure of Σ. In fact, a causal completion can be regarded as open or closed according as S is an open or closed set. Hence, both the closure $\overline{\langle S\rangle}$ and the boundary $B_S = \overline{\langle S\rangle} \wedge \overline{\langle S'\rangle}$ of $\langle S\rangle$ can be unambiguously defined. Note that B_S coincides with the boundary of S in Σ.

It is also manifest that the same closed causal completion can be generated by two closed spacelike sets S_1 and S_2 belonging to different surfaces Σ_1 and Σ_2, i.e. $\langle S_1\rangle = \langle S_2\rangle$, provided that S_1 and S_2 share the same boundary. Actually, it can be equivalently generated by an infinity of spacelike surfaces. From here on, any one S_α of such generating surfaces will be called a *cross–section* of $\langle S\rangle$.

It is worth mentioning that also the concepts of *compactness* and *connectness* can be transferred from the topology of spacelike sets to that of causal completions.

We find here a critical point. The lattice formed by the open sets of a topology is not orthocomplemented. Consequently, we cannot equip the lattice of causal shadows with the topology inherited from Σ without losing the orthomodular property. If we do this, we still obtain a lattice, but the orthocomplementation does not exist in it.

The expansion property. An obvious but important property of the causal–shadow lattice is that the partition of Minkowski spacetime into a causal–shadow pair $\langle S\rangle$, $\langle S'\rangle$ is uniquely determined by the boundary B_S. Indeed, $\langle S\rangle$ and $\langle S'\rangle$, as sets of events, include all and only those events that are not within the (double) light cones originating from the points of B_S. Some implications of this statement are worth noticing. Assume that S is a closed sphere in some spacelike surface of the Minkowski spacetime. Its causal shadow $\langle S\rangle$ is the double–cone (spherical diamond) external to the light cones originated from the points of S'. Let O be the center of S, then the causal shadow $\langle S - O\rangle$ differs from $\langle S\rangle$ by much more than one point. It is indeed the *crown* obtained by removing from $\langle S\rangle$ all the point belonging to the double light–cone diverging from O.

The phenomenon just described belongs to a class of facts that can be characterized by the concepts of *disjointness* and *interactivity*. Two causal shadows are called disjoint if their cross–sections are disjoint. Their join is simply the set theoretic union the two causal shadows. This property survives any change of shape or position of the two cross sections on a given spacelike surface provided that these keep disjoint. Something dramatic happens, however, if the two causal shadows come in touch. In this case, with the disappearing of a piece of boundary a wider set of point, which were previously excluded because they were in the light cones of the piece of boundary, are recruited by the join.

Acknoledgements

The Author is very grateful to Prof. Pier Alberto Marchetti of the Department of Physics of Padova University for his valuable comments and precise pieces of criticism, which have been of great help for improving the conceptual coherence of the approach here presented, and to Prof. Ignazio Licata of the IxtuCyber for Complex Systems and Editor of Majorana Legacy, for his flattering interest on the subject and his tenacious and invaluable encouragement.

References

[1] Birkhoff, G. (1933) On the combination of subalgebras. *Proc. of the Cambridge Philos. Society*. 29:442–464.

[2] Birkhoff, G. and von Neumann, J. (1936) The Logic of Quantum Mechanics. *Annals of Mathematics*, 27:823–843.

[3] Bratteli, O, and Robinson, D.W. (1987) *Operator algebras and quantum statistical mechanics I. C^*– and W^*–algebras, symmetry groups, decomposition of states*. II Ed. Springer–Verlag, New York.

[4] Casini, H. (2002) The logic of casually closed space–time subsets. *Class. Quant. Grav.* 19:6389–6404.

[5] Cegla, W. and Jadczyk, A.Z. (1977) Orthomodularity of causal logics *Comm. Math. Phys.* 57:213.

[6] Doplicher, S., Haag R., Roberts, J.E (1969) Fields, Observables and Gauge Transformations I. *Commun. math. Phys.* 13:1–23.

[7] Doplicher, S., Haag R., Roberts, J.E (1969) Fields, Observables and Gauge Transformations I. *Commun. math. Phys.* 15:173–200.

[8] Doplicher, S., Haag R., Roberts, J.E (1971) Local Observables and Particle Statistics I. *Commun. math. Phys.* 23:199–230.

[9] Doplicher, S. and Roberts, J.E (1972) Fields, Statistics and Non–Abelian Gauge groups. *Commun. math. Phys.* 28:331–348.

[10] Doplicher, S., Haag R., Roberts, J.E (1974) Local Observables and Particle Statistics I. *Commun. math. Phys.* 35:49–85.

[11] Grätzer, G. (1978) *General Lattice Theory*. Birkhäuser Verlag, Basel.

[12] Haag, R. and Schroer, B. (1962) Postulates of quantum field theory. *J. Math. Phys.* 3:248.

[13] Haag, R. (1992) *Local Quantum Physics*. Springer–Verlag Berlin Heidelberg.

[14] Kelley, J.L. (1961) *General Topology*. Springer–Verlag.

[15] MacNeille, H.M. (1936) Partially ordered sets. *Trans. Amer. Math. Soc.* 42:416–460.

[16] Murray, F.J. and von Neumann, J. (1936) On Rings of Operators I, in *von Neumann, Collected Works*, 1956.

[17] von Neumann, J. (1932) *Matematische Grundlagen der Quantenmechanik*, Springer–Verlag, Heidelberg; (1955) *Mathematical Foundations of Quantum Mechanics*, *Princeton Univ. Press.*

[18] von Neumann, J. (1939a) On infinite direct products, *Compositio Mathematica*, 6:1–77, Noordhoff-Gröningen.

[19] von Neumann, J. (1939b–1949) On Rings of Operators II, III, IV, V, in *Collected Works*, 1956;

[20] Piron, C. (1964) Axiomatique Quantique, *Helvetica Physica Acta*, 37:439–468.

[21] Rovelli, C. (2004) *Quantum Gravity*. Cambridge Univ. Press.

[22] Simmons,G.F. (1963) *Introduction to Topology and Modern Analysis* McGraw–Hill. New York.

[23] Stone, M.H. (1936) The theory of representations for Boolean algebras. *Trans. Amer. Math. Soc.* 122:379–398.

[24] Weyl, H. (1931) *The theory of groups and quantum mechanics*. Dover.

EJTP **3, No. 10** (2006) 131–142 *Electronic Journal of Theoretical Physics*

Lorentz Invariant Majorana Formulation of the Field Equations and Dirac-like Equation for the Free Photon

Tomislav Ivezić *

Ruder Bošković Institute, P.O.B. 180, 10002 Zagreb, Croatia

Received 3 May 2006, Published 28 May 2006

Abstract: In this paper we present a new geometric formulation (Clifford algebra formalism) of the field equations, which is independent of the reference frame and of the chosen system of coordinates in it. This formulation deals with the complex 1-vector $\Psi = E - icB$ (i is the unit imaginary), which is four-dimensional (4D) geometric generalization of Majorana's complex 3D quantity $\mathbf{\Psi} = \mathbf{E} - ic\mathbf{B}$. When the sources are absent the field equations with the complex Ψ become Dirac-like relativistic wave equations for the free photon. In the frame of "fiducial" observers (the observers who measure fields are at rest) and in the standard basis the component form of the field equations with 4D Ψ reproduces the component form of Majorana-Maxwell equations with 3D field $\mathbf{\Psi}$. The important differences between the approach with the 4D Ψ and that one with the 3D $\mathbf{\Psi}$ are discussed.

Keywords: Lorentz invariant field equations, Majorana-Maxwell equations
PACS (2006): 03.30.+p, 03.50.De

1. Introduction

In Majorana formulation of electrodynamics the Maxwell equations are written in terms of complex combination of the three-dimensional (3D) vectors of the electric and the magnetic fields $\mathbf{E}$ and $\mathbf{B}$ respectively, $\mathbf{\Psi} = \mathbf{E} - ic\mathbf{B}$, see [1,2]. (The vectors in the 3D space will be designated in bold-face.) In terms of $\mathbf{\Psi}$ the Maxwell equations in vacuum can be cast in a Dirac-like form using the correspondence principle $W \to i\hbar\partial/\partial t$, $\mathbf{p} \to -i\hbar \nabla$. In that case $\mathbf{\Psi}^* \cdot \mathbf{\Psi} = \mathbf{E}^2 + c^2\mathbf{B}^2$ is proportional to the probability density function for a photon. An important advantage of Majorana formulation of electrodynamics is that it does not make use of the intermediate electromagnetic potentials but deals with

* ivezic@irb.hr

observable quantities, the electric and the magnetic fields.

Covariant Majorana formulation is developed in [2]. There the covariant form of the complex field $\Psi_\mu = E_\mu - iB_\mu$ is introduced. The covariant Maxwell equation with Ψ_μ are written only for the free fields, i.e., when $j^\beta = 0$. It is worth noting that E_μ, B_μ and Ψ_μ are components that are determined in the specific system of coordinates, which we call Einstein's system of coordinates. In Einstein's system of coordinates the standard, i.e., Einstein's synchronization [3] of distant clocks and Cartesian space coordinates x^i are used in the chosen inertial frame. We also point out that in [2] E^μ and B^μ are treated as the "auxiliary fields," while the 3D vectors $\mathbf{E}$ and $\mathbf{B}$ are considered as the physical fields.

Further generalization of Majorana formulation is presented in [4]. There a geometric approach to special relativity is developed, which deals with tensors as 4D geometric quantities. We note that such geometric approach with tensors as geometric quantities is considered not only in [4] but in [5, 6] as well, while a similar treatment in which 4D geometric quantities are Clifford multivectors is presented in [7-10]. The approach to special relativity with 4D geometric quantities is called the invariant special relativity. In the the invariant special relativity one considers that the 4D geometric quantities are well-defined both theoretically and *experimentally* in the 4D spacetime, and not, as usual, the 3D quantities. All physical quantities are defined without reference frames, i.e., as absolute quantities (AQs) or, when some basis has been introduced, they are represented as 4D coordinate-based geometric quantities (CBGQs) comprising both components and a *basis*. It is shown in the mentioned references that such geometric approach is in a complete agreement with the principle of relativity and, what is the most important, with experiments, see [5] (tensor formalism) and [8-10] (geometric algebra formalism). In [4] Sec. 6.3 the invariant Majorana electromagnetic field Ψ^a is defined as $\Psi^a = E^a - icB^a$, where E^a, B^a and Ψ^a are the 4D AQs with definite physical meaning and not the "auxiliary fields". In the same section the field equation with Ψ^a is presented, which for $j^\beta = 0$ is reduced to the Dirac-like relativistic wave equation for the free photon.

In this paper we shall explore a similar Lorentz invariant Majorana formulation in which physical quantities will be represented by Clifford multivectors. To simplify the derivation of all important relations we shall employ recently developed axiomatic geometric formulation of electromagnetism [10] in which the primary quantitity for the whole electromagnetism is the electromagnetic field F (bivector). New Lorentz invariant Majorana form of the field equations and Dirac-like equations for the free photon are reported. The similarities and the differences between our Lorentz invariant field equations with the 4D Ψ and Majorana-Maxwell equations with the 3D $\mathbf{\Psi}$ are discussed.

2. A brief summary of geometric algebra

In this paper the investigation with 4D geometric quantities will be done in the geometric algebra formalism, see, e.g., [11] and [12]. First we provide a brief summary of Clifford algebra with multivectors. Clifford vectors are written in lower case (a) and general multivectors (Clifford aggregate) in upper case (A). The space of multivectors is graded and multivectors containing elements of a single grade, r, are termed homogeneous and

often written A_r. The geometric (Clifford) product is written by simply juxtaposing multivectors AB. A basic operation on multivectors is the degree projection $\langle A\rangle_r$ which selects from the multivector A its $r-$ vector part (0 = scalar, 1 = vector, 2 = bivector, ...). We write the scalar (grade-0) part simply as $\langle A\rangle$. The geometric product of a grade-r multivector A_r with a grade-s multivector B_s decomposes into $A_rB_s = \langle AB\rangle_{r+s} + \langle AB\rangle_{r+s-2} ... + \langle AB\rangle_{|r-s|}$. The inner and outer (or exterior) products are the lowest-grade and the highest-grade terms respectively of the above series $A_r \cdot B_s \equiv \langle AB\rangle_{|r-s|}$, and $A_r \wedge B_s \equiv \langle AB\rangle_{r+s}$. For vectors a and b we have $ab = a\cdot b + a\wedge b$, where $a\cdot b \equiv (1/2)(ab+ba)$, and $a\wedge b \equiv (1/2)(ab-ba)$. In the case considered in this paper Clifford algebra is developed over the field of the complex numbers. Complex reversion is the operation which takes the complex conjugate of the scalar (complex) coefficient of each of the 16 elements in the algebra and reverses the order of multiplication of vectors in each multivector, $\overline{AB} = \overline{BA}$, where, e.g., the complex reversion of A is denoted by an overbar $\overline{A}$.

Any multivector A is a geometric 4D quantity defined without reference frame, i.e., an AQ. When some basis has been introduced A can be written as a CBGQ comprising both components and a basis. Usually [11, 12] one introduces the standard basis. The generators of the spacetime algebra are taken to be four basis vectors $\{\gamma_\mu\}, \mu = 0, ...3$ (the standard basis) satisfying $\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = diag(+---)$. This basis is a right-handed orthonormal frame of vectors in the Minkowski spacetime M^4 with γ_0 in the forward light cone. The γ_k $(k = 1, 2, 3)$ are spacelike vectors. The basis vectors γ_μ generate by multiplication a complete basis for the spacetime algebra: $1, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu\gamma_5, \gamma_5$ (16 independent elements). γ_5 is the pseudoscalar for the frame $\{\gamma_\mu\}$.

Observe that the standard basis corresponds, in fact, to Einstein's system of coordinates. However different systems of coordinates are allowed in an inertial frame and they are all equivalent in the description of physical phenomena. For example, in [4], two very different, but physically completely equivalent, systems of coordinates, Einstein's system of coordinates and the system of coordinates with a nonstandard synchronization, the everyday (radio) ("r") synchronization, are exposed and exploited throughout the paper. In order to treat different systems of coordinates on an equal footing we have developed such form of the Lorentz transformations (LT) which is independent of the chosen system of coordinates, including different synchronizations, [4] (tensor formalism) and [7] (Clifford algebra formalism). Furthermore in [4] we have presented the transformation matrix that connects Einstein's system of coordinates with another system of coordinates in the same reference frame. For the sake of brevity and of clearness of the whole exposition, we shall write CBGQs only in the standard basis $\{\gamma_\mu\}$, but remembering that the approach with 4D geometric quantities holds for any choice of basis. All equations written with 4D AQs and 4D CBGQs will be manifestly Lorentz invariant equations.

As mentioned above a Clifford multivector A, an AQ, can be written as a CBGQ, thus with components and a basis. Any CBGQ is an invariant quantity under the passive Lorentz transformations; both the components and the basis vectors are transformed but the whole 4D geometric quantity remains unchanged, e.g., the position 1-vector x can be decomposed in the S and S' (relatively moving) frames and in the standard

basis $\{\gamma_\mu\}$ and some non-standard basis $\{r_\mu\}$ as $x = x^\mu\gamma_\mu = x'^\mu\gamma'_\mu = ... = x'^\mu_r r'_\mu$. The primed quantities are the Lorentz transforms of the unprimed ones. The invariance of some 4D CBGQ under the passive Lorentz transformations reflects the fact that such mathematical, invariant, geometric 4D quantity represents the same physical quantity for relatively moving observers.

3. The relations that connect F with E, B and with Ψ, $\overline{\Psi}$

In contrast to the usual Clifford algebra approaches [11, 12], and all other previous approaches, we have shown in [10] that the bivector field F, and not the 3D vectors **E** and **B** or the electromagnetic potentials, can be considered as the primary physical quantity for the whole electromagnetism. From the known F one can find different 4D quantities that represent the 4D electric and magnetic fields; they are considered in [8] and [9]. One of these representations, which is examined in [7-9], uses the decomposition of F into 1-vectors E and B

$$\begin{aligned} &F = (1/c)E \wedge v + (B \wedge v)I, \\ &E = (1/c)F \cdot v, \quad B = (1/c^2)(F \wedge v)I;\ E \cdot v = B \cdot v = 0, \end{aligned} \tag{1}$$

where I is the unit pseudoscalar. (I is defined algebraically without introducing any reference frame, as in [13] Sec. 1.2.) The velocity v can be interpreted as the velocity (1-vector) of a family of observers who measures E and B fields. That velocity v and all other quantities entering into (1) are defined without reference frames, i.e., they are AQs.

It is proved in [8, 9] (Clifford algebra formalism) and [6] (tensor formalism) that the observers in relative motion see the same field, e.g., the E field in the S frame is the same as in the relatively moving S' frame; $E^\mu\gamma_\mu = E'^\mu\gamma'_\mu$, where all primed quantities are the Lorentz transforms of the unprimed ones. The LT transform the components E^μ from the S frame again to the components E'^μ from the S' frame, in the same way as for any other 1-vector. For example, the transformations for the components of the 1-vector E are

$$E'^0 = \gamma(E^0 - \beta E^1), E'^1 = \gamma(E^1 - \beta E^0), E'^2 = E^2, E'^3 = E^3, \tag{2}$$

and the same for the transformations of the components of the 1-vector B. Thus the Lorentz transformed E'^μ are not expressed by the mixture of components E^μ and B^μ of the electric and magnetic fields respectively from the S frame. This is in sharp contrast to all previous formulations of electromagnetism, starting with Einstein's work [3], in which the components E'_i of the 3D $\mathbf{E}'$ are expressed by the mixture of components of E_i and B_i from the S frame. For example, the transformations for the components of the 3D **E** are

$$E'_x = E_x,\ E'_y = \gamma(E_y - \beta c B_z),\ E'_z = \gamma(E_z + \beta c B_y), \tag{3}$$

and similarly for the components of the 3D **B**, see, e.g., [14] Sec. 11.10. In all textbooks and papers treating relativistic electrodynamics these usual transformations of the components of the 3D **E** (3) and **B** (e.g., [14] Sec. 11.10) are considered to be the LT, but

the fundamental results obtained in [6] and [8, 9] exactly show that they drastically differ from the LT of the 4D quantities that represent the electric and the magnetic fields.

Next we introduce the complex fields, the 4D AQ Ψ and its complex reversion $\overline{\Psi}$. They are defined in terms of 1-vectors of the electric and magnetic fields E and B as

$$\begin{aligned} &\Psi = E - icB, \quad \overline{\Psi} = E + icB, \\ &E = (1/2)(\Psi + \overline{\Psi}), \quad B = (i/2c)(\Psi - \overline{\Psi});\ v \cdot \Psi = v \cdot \overline{\Psi} = 0. \end{aligned} \tag{4}$$

The complex Ψ and $\overline{\Psi}$ are homogeneous, grade-1, multivectors. The meanings of v and I are the same as in (1).

Using (1) we find that the F formulation and the complex Ψ formulation are connected by the relations

$$\begin{aligned} &F = (1/2c)\{(\Psi + \overline{\Psi}) \wedge v + i[(\Psi - \overline{\Psi}) \wedge v]I\}. \\ &\Psi = (1/c)F \cdot v + (i/c)I(F \wedge v). \end{aligned} \tag{5}$$

We note that one can construct the formulation of electrodynamics with the complex 1-vectors Ψ and $\overline{\Psi}$ as 4D AQs, i.e., Lorentz invariant Majorana formulation of electrodynamics using the relations (5) and the work [10]. Such formulation is perfectly suited for the transition to the quantum electrodynamics.

4. Lorentz invariant Majorana form of the field equation and Dirac-like equation for the free photon

As already mentioned we shall use the F formulation [10] to find the field equation for Ψ. In the F formulation [10] the electromagnetic field is represented by a bivector-valued function $F = F(x)$ on the spacetime. The source of the field is the electromagnetic current j which is a 1-vector field and the gradient operator ∂ is also 1-vector. A single field equation for F was first given by M. Riesz [15] as

$$\partial F = j/\varepsilon_0 c, \quad \partial \cdot F + \partial \wedge F = j/\varepsilon_0 c. \tag{6}$$

The trivector part is identically zero in the absence of magnetic charge.

Using (5) and (6) we write the field equation in terms of the complex 1-vector Ψ as

$$\partial \cdot (\Psi \wedge v) + i\,[\partial \wedge (\Psi \wedge v)]\, I = j/\varepsilon_0. \tag{7}$$

This form of the field equation (in which $\overline{\Psi}$ does not appear) is achieved separating vector and trivector parts and then combining them to eliminate $\overline{\Psi}$. The equation (7) is the most general basic equation for the Lorentz invariant Majorana formulation of electrodynamics.

From this field equation with AQs one can get more familiar field equation with CBGQs that are written in the standard basis $\{\gamma_\mu\}$. Thus instead of (7) we have

$$\partial_\alpha[(\delta^{\alpha\beta}{}_{\mu\nu} - i\varepsilon^{\alpha\beta}{}_{\mu\nu})\Psi^\mu v^\nu]\gamma_\beta = (j^\beta/\varepsilon_0)\gamma_\beta, \tag{8}$$

where $\delta^{\alpha\beta}{}_{\mu\nu} = \delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}$. The equation (8) can be also written as

$$\partial_\alpha[(\Gamma^\alpha)^\beta{}_\mu \Psi^\mu]\gamma_\beta = (j^\beta/\varepsilon_0)\gamma_\beta,$$
$$(\Gamma^\alpha)^\beta{}_\mu = \delta^{\alpha\beta}{}_{\nu\rho} v^\rho g^\nu{}_\mu + i\varepsilon^{\alpha\beta}{}_{\nu\mu} v^\nu. \tag{9}$$

We note that the same equation as (9) is obtained in the tensor formalism in [4]. Observe that our $(\Gamma^\alpha)^\beta{}_\mu$ differ from the expression for the corresponding quantity $(\gamma^\alpha)^\beta{}_\mu$, Eq. (30) in [2].

In the case when the sources are absent, $j = 0$, and when it is assumed that the velocity 1-vector v is independent of x, then the field equation with the 4D Ψ as AQ, (7), becomes

$$v(\partial \cdot \Psi) - (v \cdot \partial)\Psi + i\,[v \wedge (\partial \wedge \Psi)]\, I = 0. \tag{10}$$

Then we introduce a generalization of the correspondence principle that deals with 4D AQs

$$i\hbar\partial \to p. \tag{11}$$

Inserting (11) into (10) we reveal Dirac-like relativistic wave equation for the free photon, which is written with AQs

$$v(p \cdot \Psi) - (v \cdot p)\Psi + i\,[v \wedge (p \wedge \Psi)]\, I = 0. \tag{12}$$

If we write Eq. (10) with CBGQs in the standard basis $\{\gamma_\mu\}$ then we get an equation that is very similar to (9)

$$[(\Gamma^\alpha)^\beta{}_\mu(\partial_\alpha \Psi^\mu)]\gamma_\beta = 0. \tag{13}$$

Remember that v in (10) and (13) is independent of x, whereas $(\Gamma^\alpha)^\beta{}_\mu$ is the same as in (9). When the generalized correspondence principle (11) is written with CBGQs in the $\{\gamma_\mu\}$ basis it takes the form

$$\gamma^\alpha i\hbar\partial_\alpha \to \gamma^\alpha p_\alpha. \tag{14}$$

Inserting (14) into Eq. (13) we find the following equation

$$[(\Gamma^\alpha)^\beta{}_\mu(p_\alpha \Psi^\mu)]\gamma_\beta = 0. \tag{15}$$

The equation (15) is Dirac-like relativistic wave equation for the free photon, but now written with CBGQs in the $\{\gamma_\mu\}$ basis.

It is clear from the form of the equations (8), (9) (with some general v^μ) and (13), (15) (with v^μ independent of x) that they are invariant under the passive LT, since every 4D CBGQ is invariant under the passive LT. The field equations with primed quantities, thus in a relatively moving S' frame, are exactly equal to the corresponding equations in S, which are given by the above mentioned relations. Thus these equations are not only covariant but also the Lorentz invariant field equations. The principle of relativity is automatically included in such formulation.

In addition let us briefly examine how one can get the field equations in the formulation with 1-vectors of the electric field E and the magnetic field B from the corresponding

Eq (7), when AQs are used, and (8) or (9), when CBGQs in the $\{\gamma_\mu\}$ basis are used. These equations are already obtained and discussed in detail in [9] and previously in [7]. Substituting the decomposition of Ψ into E and B from (4) into (7) one gets two equations with real multivectors

$$\begin{aligned} &\partial \cdot (E \wedge v) + [\partial \wedge (cB \wedge v)]\, I = j/\varepsilon_0. \\ &\partial \cdot (cB \wedge v) - [\partial \wedge (E \wedge v)]\, I = 0. \end{aligned} \tag{16}$$

The equations (16) are the same as Eq. (39) in [9]. Similarly, starting with (8) we find

$$\begin{aligned} &\partial_\alpha(\delta^{\alpha\beta}{}_{\mu\nu} E^\mu v^\nu + \varepsilon^{\alpha\beta\mu\nu} v_\mu c B_\nu)\gamma_\beta = (j^\beta/\varepsilon_0)\gamma_\beta, \\ &\partial_\alpha(\delta^{\alpha\beta}{}_{\mu\nu} v^\mu c B^\nu + \varepsilon^{\alpha\beta\mu\nu} v_\mu E_\nu)\gamma_\beta = 0, \end{aligned} \tag{17}$$

which is the same as Eq. (40) in [9]. Of course, these equations, (16) and (17), are also Lorentz invariant field equations but with 1-vectors E and B.

5. Comparison with Majorana-Maxwell equations with the 3D Ψ

Let us now see how our results can be reduced to Majorana-Maxwell equations with the 3D $\mathbf{\Psi}$. In the presence of sources these equations are

$$div\mathbf{\Psi} = \rho/\varepsilon_0, \quad irot\mathbf{\Psi} = \mathbf{j}/\varepsilon_0 c + (1/c)\partial\mathbf{\Psi}/\partial t, \tag{18}$$

see, e.g., Eqs. (2) in [1]. When the sources are absent, $\rho = 0$, $\mathbf{j} = \mathbf{0}$, and when the correspondence principle $W \to i\hbar\partial/\partial t$, $\mathbf{p} \to -i\hbar\nabla$ is used in (18), then Eq. (18) with the 3D $\mathbf{\Psi}$ leads to the transversality condition and to Majorana-Maxwell equation in a Dirac-like form

$$\mathbf{p} \cdot \mathbf{\Psi} = 0, \quad W\mathbf{\Psi} + i\mathbf{p} \times \mathbf{\Psi} = \mathbf{0}, \tag{19}$$

see Eqs. (43) and (44) in [2].

As seen from (5) (or (4) and (1)) the complex 1-vectors Ψ and $\overline{\Psi}$ (or 1-vectors E and B) are not uniquely determined by F, but their explicit values depend also on v. Let us choose the frame in which the observers who measure Ψ and $\overline{\Psi}$, i.e., E and B, are at rest. For them $v = c\gamma_0$. This frame will be called the frame of "fiducial" observers (for that name see [16]), or the γ_0 - frame. All quantities in that frame will be denoted by the subscript f, e.g., Ψ_f, E_f and B_f. Furthermore, the standard basis $\{\gamma_\mu\}$ will be chosen in the γ_0 - frame. Then in that frame the velocity $v = c\gamma_0$ has the components $v^\alpha = (c, 0, 0, 0)$ and Ψ and $\overline{\Psi}$ (E and B) become Ψ_f and $\overline{\Psi}_f$ (E_f and B_f) and they do not have temporal components, $\Psi^0_f = \overline{\Psi}^0_f = 0$, $E^0_f = B^0_f = 0$. In the γ_0 - frame Eq. (8) becomes

$$(\partial_i \Psi^i_f - j^0/\varepsilon_0 c)\gamma_0 + (i\varepsilon^{ki}{}_{0j}\partial_k \Psi^j_f - \partial_0 \Psi^i_f - j^i/\varepsilon_0 c)\gamma_i = 0. \tag{20}$$

All terms in (20) are CBGQs that are written in the $\{\gamma_\mu\}$ basis. The equation (20) cannot be further simplified as a geometric equation. However if one compares the components

from Eq. (20) and the components from Majorana-Maxwell equations (18) then it is seen that they are the same. Hence it is the component form of Eq. (20) (the "fiducial" frame and the standard basis $\{\gamma_\mu\}$) which agrees with the component form of Majorana-Maxwell equations with the 3D $\mathbf{\Psi}$ (18).

In the case when $j^\mu = 0$, and with the replacement (14), Eq. (20) can be written as

$$p_i \Psi^i_f \gamma_0 + (-p_0 \Psi^i_f + i\varepsilon^{ki}{}_{0j} p_k \Psi^j_f)\gamma_i = 0. \tag{21}$$

The same result (21) follows from Eq. (15) when it is considered in the γ_0 - frame in which the $\{\gamma_\mu\}$ basis is chosen. The whole equation (21) is written with CBGQs in the standard basis $\{\gamma_\mu\}$ and cannot be further simplified as a geometric equation. Comparing that equation with Majorana-Maxwell equations (19) we again see that only component forms of both equations can be compared. From the first term (with γ_0) in Eq. (21) we find the component form of the transversality condition written with 4D p and Ψ_f, $p_i \Psi^i_f = 0$ (remember that in the γ_0 - frame $\Psi^0_f = 0$), which agrees with the component form of the transversality condition with the 3D $\mathbf{p}$ and $\mathbf{\Psi}$ from Eq. (19), $p_i \Psi^i = 0$. The second term (with γ_i) in Eq. (21) yields the component form of Dirac-like equation for the free photon that is written with 4D p and Ψ_f. It agrees with the component form of the corresponding equation (19) with the 3D $\mathbf{p}$ and $\mathbf{\Psi}$.

Similarly, in the frame of "fiducial" observers and in the $\{\gamma_\mu\}$ basis, we can derive the component form of the usual Maxwell equations with the 3D $\mathbf{E}$ and $\mathbf{B}$ from Eq. (17). This is discussed in detail in [9].

However, it is worth noting that there are very important differences between our Eqs. (20) and (21), or, better to say, our Eqs. (13) and (15), and Majorana-Maxwell equations (18) and (19). Our equations (20), (21), (13) and (15) are written with 4D CBGQs and the components are multiplied by the unit 1-vectors γ_μ, whereas Majorana-Maxwell equations (18) and (19) are written with 3D vectors and the components are multiplied by the unit 3D vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. Only in the frame of "fiducial" observers and in the $\{\gamma_\mu\}$ basis the temporal component of the complex 1-vector Ψ is zero, but in all other relatively moving inertial frames this component is different from zero. Furthermore in any frame other than the γ_0 - frame the "fiducial" observers are moving and the velocity v has the spatial components as well.

The complex 1-vector Ψ transforms under the LT as every 1-vector transforms, e.g., the components transform as in Eq. (2), whereas the unit 1-vectors γ_μ transform by the inverse LT. This gives that the whole Ψ is unchanged, i.e., it holds that $\Psi^\mu \gamma_\mu = \Psi'^\mu \gamma'_\mu$ as for any other 4D CBGQ. On the other hand there is no transformation which transforms the unit 3D vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ into the unit 3D vectors $\mathbf{i}'$, $\mathbf{j}'$, $\mathbf{k}'$. Hence it is not true that, e.g., the 3D vector $\mathbf{E}'{=}E'_1\mathbf{i}' + E'_2\mathbf{j}' + E'_3\mathbf{k}'$ is obtained by the LT from the 3D vector $\mathbf{E} = E_1\mathbf{i} + E_2\mathbf{j} + E_3\mathbf{k}$. Namely the components E_i of the 3D $\mathbf{E}$ are transformed by the usual transformations (3), which differ from the LT (2), and, as said above, there is no transformation for the unit 3D vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$. The same holds for the transformations of the 3D $\mathbf{B}$ and consequently for the transformations of the 3D $\mathbf{\Psi}$. This means that the correspondence of the 4D picture with complex 1-vector Ψ and the 3D picture with

Majorana 3D complex vector $\mathbf{\Psi}$ exists only in the frame of "fiducial" observers and in the $\{\gamma_\mu\}$ basis and not in any other relatively moving inertial frame, or in some nonstandard basis. Moreover, that correspondence in the γ_0 - frame and in the $\{\gamma_\mu\}$ basis refers only to the component forms of the corresponding equations. Our equations with 4D geometric quantities are the same in all relatively moving inertial frames, i.e., they are Lorentz invariant equations, whereas it is not true for Majorana-Maxwell equations with the 3D $\mathbf{\Psi}$.

Similarly it is proved in [9] that, contrary to the generally accepted opinion, Maxwell equations with the 3D $\mathbf{E}$ and $\mathbf{B}$ are not covariant under the LT. The field equations for the electric and magnetic fields that are Lorentz invariant are, e.g., the equations with 1-vectors E and B, Eqs. (17).

The situation with the physical importance of the 4D fields Ψ and $\overline{\Psi}$ and the corresponding 3D fields $\mathbf{\Psi}$ and $\mathbf{\Psi}^*$ is the same as it is the situation with the physical importance of the 4D fields E and B and the corresponding 3D fields $\mathbf{E}$ and $\mathbf{B}$. The comparison with experiments, the motional electromotive force in [8], the Faraday disk in [9] and the Trouton-Noble experiment in [10], strongly support our conclusions that the 4D fields E and B are not the "auxiliary fields," as explicitly considered in [2] and tacitly assumed in all previous works, but that an independent physical reality must be attributed to such 4D fields E and B (or even better to the electromagnetic field F, [10]) and not to the corresponding 3D fields $\mathbf{E}$ and $\mathbf{B}$. More generally, it is shown in [5] that there is a true agreement, which is independent of the chosen reference frame and the coordinate system in it, between the theory that deals with 4D geometric quantities and the well-known experiments which test special relativity, the "muon" experiment, the Michelson-Morley type experiments, the Kennedy-Thorndike type experiments and the Ives-Stilwell type experiments. It is also discovered in [5] that, contrary to the common opinion, there is no such agreement between Einstein's formulation of special relativity and the mentioned experiments.

6. Conclusions

The consideration presented in this paper reveals that in the 4D spacetime the complex fields, the 4D Ψ and its complex reversion $\overline{\Psi}$, are physically important and well-defined quantities that correctly transform under the LT, whereas it is not the case with the 3D complex field $\mathbf{\Psi}$ and its complex conjugate field $\mathbf{\Psi}^*$. In the 4D spacetime Majorana-Maxwell equations with the 3D $\mathbf{\Psi}$, (18) and (19), have to be replaced with our Lorentz invariant field equations with the 4D Ψ.

For $j \neq 0$ we have presented new Lorentz invariant field equation (7) in which only the 4D AQs are used. Eqs. (8) and (9) are the corresponding field equations with 4D CBGQs written in the standard basis $\{\gamma_\mu\}$. For $j = 0$ we have field equations for the 4D Ψ, (10) and (12) with 4D AQs, and (13) and (15) with 4D CBGQs.

A new generalization of the correspondence principle is introduced by Eq. (11), where the AQs are used, or by Eq. (14) with CBGQs.

The equations (12) (with AQs) and (15) (with CBGQs) are new forms for Dirac-like relativistic wave equations for the free photon, which are not yet reported in the literature, as I am aware. They will be the starting point for the construction of the observer independent stress-energy vector $T(n)$ (1-vector) and all other quantities that are derived from $T(n)$, as are the energy density U (scalar, i.e., grade-0 multivector), the Poynting vector S (1-vector), etc. All these quantities will be expressed by means of the 4D Ψ and $\overline{\Psi}$ in a complete analogy with the construction of these quantities in the axiomatic F formulation [10].

References

[1] R. Mignani, E. Recami and M. Baldo, "About a Dirac-like equation for the photon according to Ettore Majorana", Lett. Nuovo Cimento **11**, 568 (1974).

[2] S. Esposito, "Covariant Majorana formulation of electrodynamics", Found. Phys. **28**, 231 (1998).

[3] A. Einstein, "On the electrodynamics of moving bodies", Ann. Phys. **17**, 891 (1905), tr. by W. Perrett and G.B. Jeffery, in The Principle of Relativity (Dover, New York, 1952)..

[4] T. Ivezić, ""True transformations relativity" and electrodynamics", Found. Phys. **31**, 1139 (2001).

[5] T. Ivezić, "An invariant formulation of special relativity, or the "true transformations relativity," and comparison with experiments", Found. Phys. Lett. **15**, 27 (2002); physics/0103026, (with 'radio' synchronization as well).

[6] T. Ivezić, "The proof that the standard transformations of E and B are not the Lorentz transformations" Found. Phys. **33**, 1339 (2003).

[7] T. Ivezić, "Invariant relativistic electrodynamics. Clifford algebra approach", hep-th/0207250.

[8] T. Ivezić, "The difference between the standard and the Lorentz transformations of the electric and magnetic fiels. Application to motional EMF", Found. Phys. Lett. **18**, 301 (2005).

[9] T. Ivezić, "The proof that Maxwell's equations with the 3D E and B are not covariant upon the Lorentz transformations but upon the standard transformations. The new Lorentz-invariant field equations", Found. Phys. **35,** 1585 (2005).

[10] T. Ivezić, "Axiomatic geometric formulation of electromagnetism with only one axiom: the field equation for the bivector field F with an explanation of the Trouton-Noble experiment" Found. Phys. Lett. **18**, 401 (2005).

[11] D. Hestenes, Space-Time Algebra (Gordon and Breach, New York, 1966); Space-Time Calculus*;* available at: http://modelingnts.la. asu.edu/evolution. html; New Foundations for Classical Mechanics (Kluwer, Dordrecht, 1999) 2nd. edn.; Am. J Phys. **71**, 691 (2003).

[12] C. Doran, and A. Lasenby, Geometric algebra for physicists (Cambridge University, Cambridge, 2003).

[13] D. Hestenes and G. Sobczyk, Clifford Algebra to Geometric Calculus (Reidel, Dordrecht, 1984).

[14] J.D. Jackson, Classical Electrodynamics (Wiley, New York, 1977) 2nd edn.

[15] M. Riesz, Clifford Numbers and Spinors, Lecture Series No. 38, The Institute for Fluid Dynamics and Applied Mathematics, University of Maryland (1958).

[16] H.N. Núñez Yépez, A.L. Salas Brito, and C.A. Vargas, "Electric and magnetic four-vectors in classical electrodynamics", Revista Mexicana de Física **34**, 636 (1988).

EJTP **3, No. 10** (2006) 143–156 *Electronic Journal of Theoretical Physics*

"Anticoherent" Spin States via the Majorana Representation

Jason Zimba *

Departments of Physics and Mathematics,
Bennington College, Bennington,
Vermont 05201 USA

Received 20 April 2006, Published 28 May 2006

Abstract: In this article we define and exhibit "anticoherent" spin states, which are in a sense "the opposite" of the familiar coherent spin states. Since the familiar coherent states are as "classical" as spin states can be, the anticoherent states may turn out to be better candidates for applications involving non-classical behaviors such as quantum entanglement. Thanks to the Majorana representation of spinors as $2s$-tuples of points on the Riemann sphere, classes of anticoherent states are easy to find; the development of such examples also leads us into some curious geometry involving the perfect solids.

If we create a universe, let it not be abstract or vague but rather let it concretely represent recognizable things. M.C. Escher[1]

Keywords: Majorana Representation of Spinors, Spin States, Anticoherent, Riemann Sphere
PACS (2006): 02.40.Ky, 03.75.Mm, 03.65.Fd, 03.50.De

1. Introduction

In 1992, I first learned about the Majorana[2] representation of spinors by reading the fascinating paper of Roger Penrose, "On Bell nonlocality without probabilities: Some curious geometry."[3] I was astounded to learn that a complex spinor could be fully visualized in a way that preserved rotational structure. In the context of spin-3/2 states, it was as if I were being offered a glimpse of four-dimensional complex space. And poised in that veritable hall of mirrors was a beautiful set of 40 complex rays that Penrose had defined, based on the geometry of the dodecahedron.

At that time I was a research student in Professor Penrose's group, and over the course

* jzimba@bennington.edu

of my two years at Oxford, I would come to admire his almost uncanny insight into the geometry of four dimensions. Undoubtedly seeing a hint of some additional structure in the crystalline set he had devised, he asked me to search it out. The results proved interesting and led to my M.Sc. thesis,[4] a combined project in mathematics and the foundations of quantum mechanics, and to a 1993 paper that continued the geometric theme: "On Bell nonlocality without probabilities: More curious geometry."[5] A few years later, Massad and Aravind[6] presented an elementary account of this work, translating the Majorana approach into standard methods more familiar to physicists—a worthwhile program, but, as the authors pointed out, one that sacrificed some of the simplicity and insight of the Majorana methods. In the meantime, a parallel track of investigations into Bell's theorem without probabilities and Kochen-Specker type noncontextuality proofs had begun in the 1980's, with contributions by Greenberger, Horne, Zeilinger, Peres and others; for a quick survey of this thread, see the first section of the 2003 article by Peres,[7] and references therein. Some additional useful references are given in the endnotes.[9, 8]

It is a privilege to contribute to the present collection of articles in celebration of the centenary of the birth of Ettore Majorana, whose fascinating method for visualizing spinors has proved so fruitful in the past and will doubtless lead to more insights in the future. And in honor also of Roger Penrose's role in repopularizing the Majorana representation, I shall take a little time in what follows to explore just a bit more of the curious geometry which the Majorana representation reveals.

2. "Anticoherent" Spin States

In the usual way of speaking, a coherent spin state $|\hat{\mathbf{n}}\rangle$ is a spin state that corresponds as nearly as possible to a classical spin vector pointing in a given direction $\hat{\mathbf{n}}$. Coherent spin states are a conceptual attractor for physicists in part because they mimic classical spin angular momentum as much as possible. As useful as this makes the coherent states, in theory and in practice, it is also fair to say that today we care just as much, if not more, about ways in which quantum phenomena do *not* mimic classical phenomena—the distinct capabilities of quantum computers being just one example. From this perspective, a question arises as to whether there might in fact be certain limits to the usefulness of coherent states as a resource. As a case in point, Markham and Vedral have recently noted that coherent spin states are in a sense resistant to entanglement formation.[10]

It might then be worthwhile to ask the following question: What kind of spin state might serve as the *opposite* of a coherent state? If states of this kind could be thought of as being as "non-classical" as possible, then this might eventually make them distinctly useful conceptually or even practically.

Probably there are a number of reasonable ways to formalize the idea of a state that is "the opposite" of a coherent state. Perhaps the most natural approach is to consider a state $|\psi\rangle$ whose polarization vector vanishes, $\mathbf{p} \equiv \langle\psi|\mathbf{S}|\psi\rangle = \mathbf{0}$. Such a state "points nowhere," in the mean, and this is certainly one way to serve as the opposite of a state that points, as much as possible, somewhere. For example, the state $|s = 1, m_{\hat{\mathbf{z}}} = 0\rangle \equiv |0\rangle$

has vanishing polarization. So by this criterion at least, $|0\rangle$ would qualify as the opposite of coherent.

But one might also wish to make a stricter demand. For let us imagine that one experimenter measures $\hat{\mathbf{n}}_1 \cdot \mathbf{S}$ many times on a system prepared in state $|0\rangle$, while another experimenter measures $\hat{\mathbf{n}}_2 \cdot \mathbf{S}$ many times on the same kind of system prepared in the same state $|0\rangle$. The two experimenters will obtain the same mean value for their results. But, in general, the *variances* of their results will differ. At the level of the variances in their data, the signature will remain of the direction along which the spin has been measured. To this extent, the state $|0\rangle$ itself still contains a certain amount of directional information.

Erasing this directional signature would require finding a state $|\psi\rangle$ for which the variance in measurements of spin in any one direction is the same as the variance in measurements of spin in any other direction—in other words, a state $|\psi\rangle$ for which the function

$$\Delta S^2_{\hat{\mathbf{n}}} = \frac{\langle\psi|(\hat{\mathbf{n}} \cdot \mathbf{S})^2|\psi\rangle}{\langle\psi|\psi\rangle} - \frac{\langle\psi|\hat{\mathbf{n}} \cdot \mathbf{S}|\psi\rangle^2}{\langle\psi|\psi\rangle^2} \tag{1}$$

is uniform over the unit sphere.

Generally, the variance function for a state is nonuniform on the unit sphere; Fig. 1 shows several examples. But if the variance function $\Delta S^2_{\hat{\mathbf{n}}}$ is uniform over the sphere in the state $|\psi\rangle$, then we shall call $|\psi\rangle$ a *uniform state*.

If $|\psi\rangle$ is a uniform state with vanishing polarization vector, then we shall call $|\psi\rangle$ an *anticoherent state*. Anticoherent states will serve as our beginning notion of "the opposite" of coherent states.

A. Criteria for Uniformity and Anticoherence

In this section, we make a few observations on anticoherent states and establish some necessary and sufficient conditions for anticoherence and uniformity.

When the polarization vector vanishes, $\Delta S^2_{\hat{\mathbf{n}}}$ reduces to $\langle(\hat{\mathbf{n}} \cdot \mathbf{S})^2\rangle$. Denoting the uniform value of $\Delta S^2_{\hat{\mathbf{n}}}$ for an anticoherent state by c, we have

$$\begin{aligned} c + c + c &= \Delta S_x^2 + \Delta S_y^2 + \Delta S_z^2 && (2)\\ &= \langle S_x^2\rangle + \langle S_y^2\rangle + \langle S_z^2\rangle && (3)\\ &= s(s+1) && (4) \end{aligned}$$

so that any anticoherent state must have $\Delta S^2_{\hat{\mathbf{n}}} = s(s+1)/3$, independent of $\hat{\mathbf{n}}$.

In addition, if $|\psi\rangle$ is anticoherent, then we must have $\langle S_i S_j\rangle = 0$ for perpendicular axes i and j. This can be seen from the identity

$$S_i S_j + S_j S_i = (\tfrac{1}{\sqrt{2}} S_i + \tfrac{1}{\sqrt{2}} S_j)^2 - (\tfrac{1}{\sqrt{2}} S_i - \tfrac{1}{\sqrt{2}} S_j)^2 \,. \tag{5}$$

The right-hand side is $S_u^2 - S_v^2$, where the axes u and v are 45-degree rotations of axes i and j. So if we take expectation values of both sides in an anticoherent state $|\psi\rangle$, then

the right-hand side vanishes. Hence $\langle S_i S_j\rangle + \langle S_j S_i\rangle = 0$. But also $\langle S_i S_j\rangle - \langle S_j S_i\rangle = i\hbar\langle S_k\rangle = 0$, since $|\psi\rangle$ is polarization-free. Hence $\langle S_i S_j\rangle = 0$.

Conversely, if a state satisfies $\langle S_x\rangle = \langle S_y\rangle = \langle S_z\rangle = \langle S_x S_y\rangle = \langle S_y S_z\rangle = \langle S_z S_x\rangle = 0$ and $\langle S_x^2\rangle = \langle S_y^2\rangle = \langle S_z^2\rangle$, then it must be anticoherent. This is easily seen by expanding $\Delta S^2_{\hat{\mathbf{n}}} = \langle(\sin\theta\cos\phi\ S_x + \sin\theta\sin\phi\ S_y + \cos\theta\ S_z)^2\rangle$.

A distinct criterion for uniformity that is sometimes useful can be given by considering $|\psi\rangle$ fixed and interpreting $\Delta S^2_{\hat{\mathbf{n}}}$ as a vector in the Hilbert space $L^2[\mathcal{S}^2]$ of square-integrable functions on the sphere $\mathcal{S}^2$, with inner product denoted
$\langle\!\langle Q|R\rangle\!\rangle \equiv \int_0^{2\pi}\!\int_0^{\pi} Q^*(\theta,\phi)R(\theta,\phi)\sin\theta\, d\theta d\phi$. By the Cauchy-Schwarz inequality, uniformity is equivalent to
$|\langle\!\langle Y_0^0|\Delta S^2_{\hat{\mathbf{n}}}\rangle\!\rangle|^2 = \langle\!\langle \Delta S^2_{\hat{\mathbf{n}}}|\Delta S^2_{\hat{\mathbf{n}}}\rangle\!\rangle\langle\!\langle Y_0^0|Y_0^0\rangle\!\rangle$, where $Y_0^0(\theta,\phi) = \frac{1}{\sqrt{4\pi}}$ is the normalized $\ell = 0$, $m = 0$ spherical harmonic. Explicitly then, the necessary and sufficient uniformity condition is[11]

$$\left[\int_0^{2\pi}\!\int_0^{\pi} \frac{1}{\sqrt{4\pi}}\Delta S^2_{\hat{\mathbf{n}}}\,\sin\theta\, d\theta d\phi\right]^2 = \int_0^{2\pi}\!\int_0^{\pi}\left[\Delta S^2_{\hat{\mathbf{n}}}\right]^2\sin\theta\, d\theta d\phi\,. \tag{6}$$

B. The Majorana Representation as a Strategy for Finding Anticoherent States

Even for relatively low spin quantum numbers, the anticoherence conditions for a general state $|\psi\rangle$ with given symbolic components will involve several complex variables and numerous terms, making it taxing to generate examples of anticoherent states by attacking the algebraic conditions head-on. But with the Majorana representation at our disposal, examples of anticoherent states can easily be found.

The Majorana representation of a $2s+1$-component spinor of the form $c_0|+s\rangle + \cdots + c_{2s}|-s\rangle$ as a $2s$-tuple of points on the sphere $\mathcal{S}^2$ is summarized by the following chain of bijective correspondences:

$$c_0|+s\rangle + \cdots + c_{2s}|-s\rangle \leftrightarrow \text{roots of } M(z) \ \leftrightarrow\ \text{points in } \mathcal{S}^2\,, \tag{7}$$

where the Majorana polynomial $M(z)$ is formed from the spinor components according to

$$M(z) = \sum_{k=0}^{2s} c_k \binom{2s}{k}^{\frac{1}{2}} z^{2s-k}\,, \tag{8}$$

and where the mapping between points (X,Y,Z) of $\mathcal{S}^2$ and the roots of $M(z)$ in $\mathbf{C}\cup\{\infty\}$ is by stereographic projection, $(X,Y,Z) \leftrightarrow (X+iY)/(1-Z)$.

The Majorana representation of a coherent state consists of a single point with multiplicity $2s$. At the opposite extreme (in some intuitive sense), we can imagine states whose Majorana representations are spread "nicely" over the sphere. Nicest of all are what we shall call *perfect states*: states whose Majorana representations comprise the vertices of one of the five perfect solids (tetrahedron, octahedron, cube, icosahedron, and

dodecahedron). These states exist for $s = 2, 3, 4, 6$, and 10. Constructing these states via the Majorana representation offers the possibility of generating concrete examples of anticoherent states without having to solve the anticoherence conditions directly. To this construction we now turn.

3. Perfect States

A. Concrete Examples of the Perfect States

Given the vertices of a perfect solid, the Majorana correspondence determines a perfect state, of which there are five specimens, up to rotational equivalence. It is straightforward to construct concrete examples of perfect states, which we denote by $|\psi_{\rm tet}\rangle$, $|\psi_{\rm oct}\rangle$, $|\psi_{\rm cube}\rangle$, $|\psi_{\rm icos}\rangle$, and $|\psi_{\rm dodec}\rangle$. The results of these calculations are:

$$|s=2,\psi_{\rm tet}\rangle = \frac{1}{\sqrt{3}}\left(\sqrt{2}|+1\rangle - |-2\rangle\right) \tag{9a}$$

$$|s=3,\psi_{\rm oct}\rangle = \frac{1}{\sqrt{2}}\left(|+2\rangle - |-2\rangle\right) \tag{9b}$$

$$|s=4,\psi_{\rm cube}\rangle = \frac{1}{2\sqrt{6}}\left(\sqrt{5}\,|+4\rangle + \sqrt{14}\,|0\rangle + \sqrt{5}\,|-4\rangle\right) \tag{9c}$$

$$|s=6,\psi_{\rm icos}\rangle = \frac{1}{5}\left(-\sqrt{7}\,|+5\rangle + \sqrt{11}\,|0\rangle + \sqrt{7}\,|-5\rangle\right) \tag{9d}$$

$$|s=10,\psi_{\rm dodec}\rangle = a\,|+10\rangle + b\,|+5\rangle + |0\rangle - b\,|-5\rangle + a\,|-10\rangle\,. \tag{9e}$$

In $|\psi_{\rm dodec}\rangle$, if we take $a = \sqrt{11/38 + 7b^2/38}$ then we ensure anticoherence, and if we take $b \approx 1.593$ (hence $a \approx 0.870$), then we obtain a dodecahedral Majorana representation.

The vertices of the perfect solids corresponding to the above states are, respectively, $\{(0,0,1),\ (\frac{2\sqrt{2}}{3},0,-\frac{1}{3}),\ (-\frac{\sqrt{2}}{3},\sqrt{\frac{2}{3}},-\frac{1}{3}),\ (-\frac{\sqrt{2}}{3},-\sqrt{\frac{2}{3}},-\frac{1}{3})\}$; $\{(0,0,1),\ (1,0,0),\ (0,1,0),\ (-1,0,0),\ (0,-1,0),\ (0,0,-1)\}$; $\{(\pm1,\pm1,\pm1)/\sqrt{3}\}$; and, in the last two cases, the vertices defined for the icosahedron and dodecahdron in the standard Mathematica package Geometry`Polytopes`, normalized to the unit sphere.

To expound briefly on one of these examples, consider the state $|\psi_{\rm dodec}\rangle$. The roots of the Majorana polynomial for this state are found from

$$az^{20} + 4b\sqrt{969}\,z^{15} + 2\sqrt{46189}\,z^{10} - 4b\sqrt{969}\,z^5 + a = 0\,. \tag{10}$$

Eq. (10) is quartic in the variable $z^5 \equiv w$. If we imagine placing a dodecahedron flat on a table, then the twenty vertices form four tiers with five vertices in each tier, the five vertices of a given tier all lying in the same horizontal plane. Tier k corresponds to the $k^{\rm th}$ root of the quartic, $w_k = |w_k|e^{i\phi_k}$; and the five vertices in tier k correspond to the five fifth-roots of w_k, $|w_k|^{1/5}e^{i(\phi_k+2\pi n)/5}$ $(n = 0, 1, 2, 3, 4)$. Via stereographic projection, the modulus $|w_k|^{1/5}$ determines the height of the $k^{\rm th}$ tier, and the complex phases $(\phi_k+2\pi n)/5$ serve to arrange the five vertices of the tier with equal spacing around a circle of latitude. With only four tiers of vertices to map, only a quartic polynomial is required, which is why a "sparse" state in the pattern of $|\psi_{\rm dodec}\rangle$ suffices.

B. A Tetrahedral Basis in $\mathbf{C}^5$

Five tetrahedra can be oriented in space in an interlocking fashion so that their twenty vertices together comprise the vertices of a dodecahedron. (See Fig. 2.) This configuration offers a highly symmetrical way of inscribing five tetrahedra in the sphere. Now, as the tetrahedral perfect states happen to "live" in a space of five dimensions, a question naturally presents itself: Is it possible that a dodecahedral collection of tetrahedra inscribed in the unit sphere generates an orthonormal basis in $\mathbf{C}^5$? The answer (surprisingly it seems) is yes.

Specifically, suppose we choose a particular tetrahedron from among the ten contained in a given dodecahedron inscribed in the unit sphere, and that from this first tetrahedron we generate a series of four more tetrahedra by rotating the original tetrahedron in successive angular steps of $2\pi/5$ about an axis passing perpendicularly through the centers of two opposite pentagonal faces of the dodecahedron. The five tetrahedra that result from this process will between them include all 20 vertices of the dodecahedron. Denoting the five tetrahedral states corresponding to these five tetrahedra by $\{|\tau_1\rangle, |\tau_2\rangle, |\tau_3\rangle, |\tau_4\rangle, |\tau_5\rangle\}$, we find that $\langle\tau_i|\tau_j\rangle = \delta_{ij}$; the successive rotations in 3-space act in $\mathbf{C}^5$ to rotate the original tetrahedral state $|\tau_1\rangle$ into a succession of mutually orthogonal states. (To make an imprecise analogy, it is somewhat as if the vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ were to be rotated in 3-space by an angle of $2\pi/3$ about an axis spanned by $(1, 1, 1)$, with resulting actions $\hat{\mathbf{x}} \to \hat{\mathbf{y}} \to \hat{\mathbf{z}} \to \hat{\mathbf{x}}$.)

At this point, I do not have an insightful argument to explain why the specified rotations of the dodecahedron should act in $\mathbf{C}^5$ to rotate the original tetrahedral state into a succession of mutually orthogonal states—just a symbolic Mathematica calculation of the inner products $\langle\tau_i|\tau_j\rangle$. So at present, the existence of the "tetrahedral basis" is something of a Platonic mystery.

Given a basis $\{|\tau_k\rangle\}$ of tetrahedral states corresponding to a set of five interlocking tetrahedra, there also exists another basis $\{|\overline{\tau}_k\rangle\}$ that is related to the original one by reflecting each of the five given tetrahedra through the center of the sphere. (See Fig. 3.) The orthogonality relations between these two bases are summarized by $\langle\tau_i|\overline{\tau}_j\rangle = c_{ij}(1 - \delta_{ij})$, where the c_{ij} are nonzero complex numbers. In other words, each vector in one of the bases is orthogonal to its corresponding vector in the "inverted" basis, but to no others. (Two orthonormal bases in $\mathbf{C}^n$ may have this sort of relationship to one another provided $n \geq 4$.)

C. Anticoherence of the Perfect States

With perfect states in hand, we can determine whether any or all of them are anticoherent. Starting with the tetrahedral state $|\psi_{\text{tet}}\rangle$, one easily finds $\mathbf{p} = \mathbf{0}$ and $\Delta S^2_{\hat{\mathbf{n}}} = 2$, independent of $\hat{\mathbf{n}}$; the tetrahedral state is indeed anticoherent. See Fig. 4. Likewise, all of the perfect states listed above can be shown to be anticoherent. While these particular states were only representatives of equivalence classes under rotations, the property of

anticoherence is itself invariant under rotations, so it suffices to make the calculations with representatives. Thus, all perfect states are anticoherent.[12]

However, all anticoherent states are not perfect. To see this, consider the following family of states:

$$|s \geq 3, \psi_{\rm ac}\rangle = e^{i\alpha}\sqrt{\frac{s+1}{6s}}|+s\rangle + \sqrt{\frac{2s-1}{3s}}|0\rangle + e^{i\alpha}\sqrt{\frac{s+1}{6s}}|-s\rangle\,. \tag{11}$$

It is straightforward to show that for integral spin $s \geq 3$, $|\psi_{\rm ac}\rangle$ is anticoherent for any real α, with $\mathbf{p} = \mathbf{0}$ and variance $\Delta S^2_{\hat{\mathbf{n}}} = s(s+1)/3$ independent of $\hat{\mathbf{n}}$.

Up to an overall scalar multiple, the Majorana polynomial of $|\psi_{\rm ac}\rangle$ has the form

$$z^{2s} + e^{-i\alpha}\binom{2s}{s}^{\frac{1}{2}}\sqrt{\frac{4s-2}{2s+1}}\;z^s + 1\,, \tag{12}$$

which is quadratic in z^s. Thus, the Majorana representation of $|\psi_{\rm ac}\rangle$ consists of two tiers of points on the sphere, each tier consisting of s points spaced equally around a circle of latitude. The cubic perfect state $|\psi_{\rm cube}\rangle$ given in Eq. (9c) is an example of this general family, with $s = 4$ and $\alpha = 0$. For $s > 4$, the Majorana representation of $|\psi_{\rm ac}\rangle$ will not form a perfect solid. See Fig. 5.

4. Higher Orders of Anticoherence

In the foregoing sections, we exhibited anticoherent states for all integral spin quantum numbers $s \geq 2$. For $s = 1/2$, $s = 1$, and $s = 3/2$, it is possible to show that no anticoherent states exist.[13] In general, one expects anticoherent states to exist for all $s \geq 2$, based on the following count of degrees of freedom.

A spin-s state is represented by $2s$ points on the unit sphere, one of which may be rotated to become the North Pole. For $s \geq 1$, this leaves $2s - 1$ other points on the unit sphere, and thus $4s - 2$ real degrees of freedom—except that one of the points may always be rotated to azimuthal position $\phi = 0$. Thus, whenever a problem is rotationally invariant, there are $4s - 3$ real degrees of freedom for a spin-s state with $s \geq 1$.

A constraint equation on the complex spinor components will reduce the number of degrees of freedom by two, except when the constraint equation fails to have a nontrivial imaginary part. Thus, the requirement that $\mathbf{p} = \mathbf{0}$ will in general reduce the number of real degrees of freedom by three, leaving us with a manifold of $4s - 6$ real degrees of freedom. And finally, the uniformity condition Eq. (6) reduces the number of degrees of freedom by one more, leaving $4s - 7$ degrees of freedom. For finitely many isolated solutions we should have $4s - 7 = 0$, or $s = 1.75$. So the count of degrees of freedom is consistent with the fact that no anticoherent states exist for $s \leq 1.5$, and it suggests that for $s \geq 2.0$, there will be enough freedom in the choice of a state to satisfy the requirements of anticoherence.[16]

In fact, by advancing to high enough spin quantum numbers, it should be possible to find states $|\psi\rangle$ in which ever-higher moments of the probability distribution $\{|\langle s, m_{\hat{\mathbf{n}}}|\psi\rangle|^2\}$ are independent of $\hat{\mathbf{n}}$. Thus, one expects that greater and greater anticoherence is possible for higher and higher values of spin. The salient problem would be in finding concrete examples of states with the desired order of anticoherence. See Table 1.

Order	Condition	Least s for existence	Example
0	none	$\frac{1}{2}$	any $s = \frac{1}{2}$
1	$\langle \hat{\mathbf{n}} \cdot \mathbf{S} \rangle \neq f(\hat{\mathbf{n}})$	1	$\|s = 1, m_{\hat{\mathbf{z}}} = 0\rangle$
2	$\langle (\hat{\mathbf{n}} \cdot \mathbf{S})^2 \rangle \neq f(\hat{\mathbf{n}})$	2	$\|\psi_{\rm tet}\rangle$
3	$\langle (\hat{\mathbf{n}} \cdot \mathbf{S})^3 \rangle \neq f(\hat{\mathbf{n}})$	2, $\frac{5}{2}$, or 3	$\|\psi_{\rm oct}\rangle$
4	$\langle (\hat{\mathbf{n}} \cdot \mathbf{S})^4 \rangle \neq f(\hat{\mathbf{n}})$	?	?
⋮	⋮	⋮	⋮

Table 1 Ascending orders of anticoherence for pure states. Anticoherence of order q means that all of the conditions up to and including condition q apply. The condition for $q = 1$, $\langle \hat{\mathbf{n}} \cdot \mathbf{S} \rangle \neq f(\hat{\mathbf{n}})$, is equivalent to the vanishing of the polarization vector.

5. Anticoherent Mixtures

In this section we present just a few observations concerning anticoherent mixtures. We say that a mixed state ρ is anticoherent to order q if $\mathrm{Tr}\big[\rho(\hat{\mathbf{n}} \cdot \mathbf{S})^\ell\big]$ is independent of $\hat{\mathbf{n}}$ for $\ell \leq q$. Mixed states incorporate more degrees of freedom than do pure states, so one expects anticoherent mixtures to be easier to find in lower-dimensional spaces. Indeed, for any s, the reduced density operator for a fully entangled spin, $\rho = \frac{1}{2s+1} I$, is anticoherent to all orders $q > 0$. (But for $s = 1/2$, $\rho = \frac{1}{2} I$ is the only mixed state that is anticoherent to any order.)

If a mixed state is diagonal in an anticoherent basis, then the mixture itself is anticoherent. Specifically, if $\{|\alpha_k\rangle\}$ is an orthonormal basis of pure states, all of which are anticoherent to order q, then for any numbers $r_1, \ldots, r_{2s+1}$ with $0 \leq r_k \leq 1$ and $\sum_k r_k = 1$, it is easy to see that the mixed state $\rho = \sum_k r_k |\alpha_k\rangle\langle\alpha_k|$ is itself anticoherent to order q. Thus, for $s = 2$, the tetrahedral basis may be used to construct mixtures anticoherent to second order: $\rho = \sum_{k=1}^{5} r_k |\tau_k\rangle\langle\tau_k|$.

The converse is not true, in the sense that we can have a nondegenerate mixed state that is anticoherent to some order, even when the basis diagonalizing the state is not anticoherent to any order. For example, with $s = 3/2$, consider the mixed state diagonal in the standard basis $\{|m_{\hat{\mathbf{z}}}\rangle\}$ with eigenvalues $\frac{1}{18}$, $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{1}{9}$. This mixture is easily shown to be anticoherent to order $q = 1$, even though none of its eigenvectors has this property.

6. Conclusion

By way of conclusion, we point to some further questions worth investigating with regard to anticoherent states.

First of all, returning to the "non-classical" perspective that led us to consider anticoherent states in the first place, it would be worthwhile to examine the question of whether states of high anticoherence manage to avoid the resistance to entanglement formation noted in the case of coherent states by Markham and Vedral.[10]

Second, a question may be posed as to whether anticoherent states have any interesting dynamical features. For example, when a system in an initially anticoherent state $|\psi_0\rangle$ is placed in a uniform, static magnetic field $B_0\hat{\mathbf{n}}$, the observable degree of development away from the initial state as a function of time, $|\langle\psi_0|\psi_t\rangle|^2$, is independent of the field direction to lowest nonvanishing order in time. (This is easy to see by expanding the unitary evolution $e^{-iHt/\hbar}$ in power series.) And if the initial state has a greater degree of anticoherence, then the $\hat{\mathbf{n}}$-independence of $|\langle\psi_0|\psi_t\rangle|^2$ extends to higher order in time. This dynamical feature of anticoherent states may make it worthwhile to examine whether anticoherent states may be exploited in some fashion—theoretically, computationally, or practically—in the context of slowly-varying or imperfectly-known external fields.

Acknowledgments

This work was supported by Bennington College

References

[1] M.C. Escher, quoted in *M.C. Escher: 29 Master Prints* (Harry N. Abrams, 1983), p. 56.

[2] E. Majorana, "Atomi orientati in campo magnetico variabile," *Nuovo Cimento* **9**, 43-50 (1932).

[3] R. Penrose, "On Bell non-locality without probabilities: Some curious geometry." This paper existed in some form as early as 1992, but it seems not to have been published until some years later, in J. Ellis and D. Amati (Eds.), *Quantum Reflections* (Cambridge University Press, 2000).

[4] J. Zimba, "Finitary proofs of contextuality and non-locality using Majorana representations of spin-3/2 states," Oxford University M.Sc. thesis (Queen's College, 1993).

[5] J. Zimba and R. Penrose, "On Bell non-locality without probabilities: More curious geometry," *Stud. Hist. Phil. Sci.* **24**(5), 697-720 (1993).

[6] J.E. Massad and P.K. Aravind, "The Penrose dodecahedron revisited," *Am. J. Phys.* **67**(7), 631-638 (1999).

[7] A. Peres, "What's wrong with these observables?" *Found. Phys.*, **33**(10), 1543-1547 (2003).

[8] H. Brown, "Bell's other theorem and its connection with nonlocality. Part I," in A. van der Merwe, F. Selleri and G. Tarozzi (Eds.), *Bell's Theorem and the Foundations of Modern Physics* (World Scientific Publishing Company, Singapore, 1992), 104-116.

[9] M. Pavicic, "The KS Theorem," a helpful powerpoint presentation to be found on the web: www.aloj.us.es/adanoptico/Lecture/KS18%20reduced.ppt.

[10] D. Markham and V. Vedral, "Classicality of spin-coherent states via entanglement and distinguishability," *Phys. Rev. A* **67**, 042113 (2003).

[11] One can also define a uniformity function U on spin states via

$$U(\psi) = \frac{|\langle\langle Y_0^0 | \Delta S^2_{\hat{\mathbf{n}}} \rangle\rangle|}{\sqrt{\langle\langle \Delta S^2_{\hat{\mathbf{n}}} | \Delta S^2_{\hat{\mathbf{n}}} \rangle\rangle}}\,. \tag{13}$$

In this characterization, the uniformity of any state lies between 0 and 1, and the uniform states are those for which $U(\psi) = 1$.

[12] If we place $r > 1$ repeated points at each vertex of a perfect solid, then we determine a "degenerate" perfect state. For example, a doubly degenerate octahedral state is given by

$$|s = 6, \mathrm{oct}, r = 2\rangle = \frac{1}{4\sqrt{2}} \left(\sqrt{14}\,|+4\rangle - 2\,|0\rangle + \sqrt{14}\,|-4\rangle \right) . \tag{14}$$

This state is anticoherent to order 3, not higher. A triply degenerate tetrahedral state is given by

$$|s = 6, \mathrm{tet}, r = 3\rangle = \frac{1}{3\sqrt{183}} \left(8\sqrt{14}\,|+3\rangle - 4\sqrt{15}\,|0\rangle + 3\sqrt{14}\,|-4\rangle - \sqrt{385}\,|-6\rangle \right) . \tag{15}$$

This state is anticoherent to order 2, not higher. The orthogonality relations in $\mathbf{C}^{13}$ between $|s = 6, \mathrm{tet}, r = 3\rangle$, $|s = 6, \mathrm{oct}, r = 2\rangle$, and $|s = 6, \mathrm{icos}, r = 1\rangle$ are an interesting question.

[13] For $s = 3/2$ in particular, one can show that although there is, up to rotational equivalence, a single uniform state, nevertheless, this state is not anticoherent: the polarization vector in this state does not vanish. Alternatively, one can show that if a normalized spin-3/2 state is polarization-free, then the left-hand side of Eq. (6) is necessarily equal to $25\pi/4$, independent of the state, whereas the right-hand side is necessarily equal to $141\pi/20$, also independent of the state. Hence, uniformity is impossible for a polarization-free spin-3/2 state, and anticoherent spin-3/2 states do not exist.

[14] C. Séquin, *Tangle of Five Tetrahedra*. Computer rendering. http://http.cs.berkeley.edu/∼sequin/SFF/spec.tangle5tetra.html.

[15] M.C. Escher, *Double Planetoid*, 1949. Woodcut. In *M.C. Escher: 29 Master Prints* (Harry N. Abrams, 1983), p. 56.

[16] The analysis of degrees of freedom suggests that in the case $s = 2$ we should have a continuous one-dimensional manifold of rotationally inequivalent anticoherent states, with the tetrahedral states representing a point on a continuum.

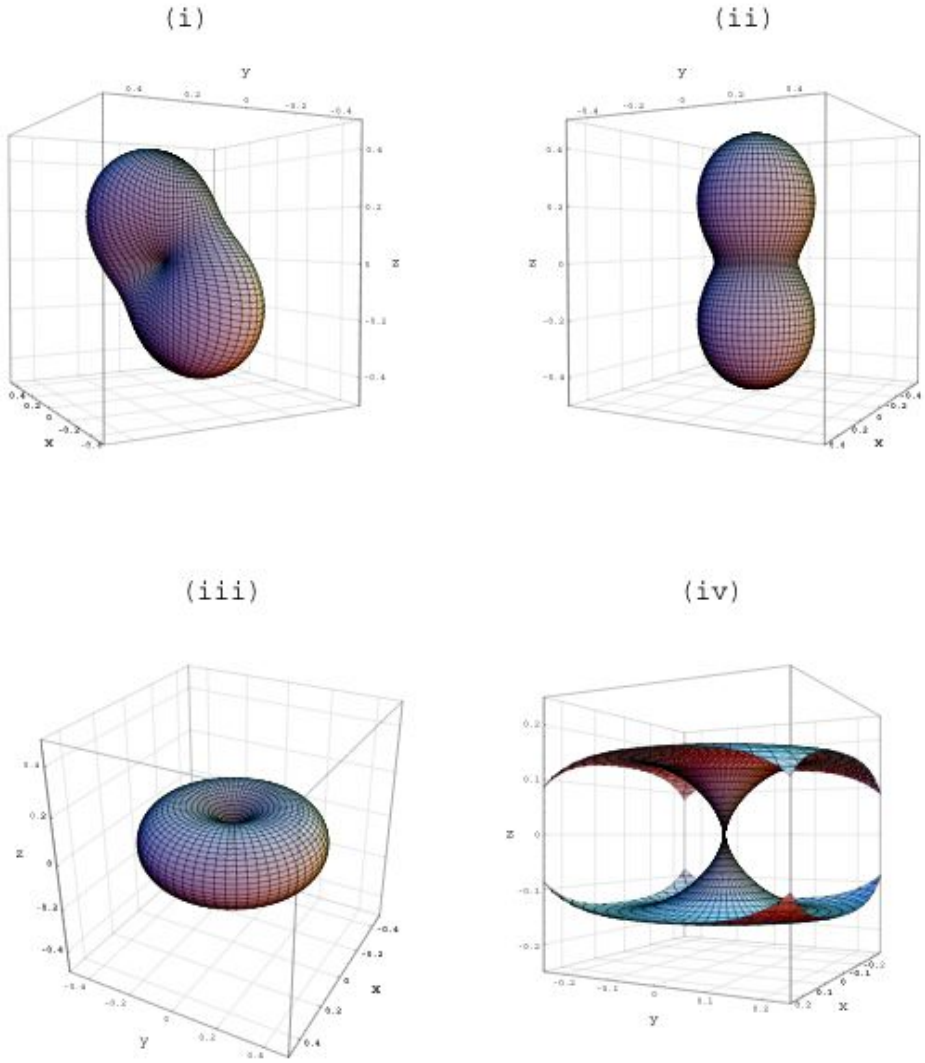


Fig. 1 Spherical parametric plots of the $L^2[\mathcal{S}^2]$-normalized variance $\Delta S^2_{\hat{\mathbf{n}}}/||\Delta S^2_{\hat{\mathbf{n}}}||$ for (i) a randomly chosen $s = 2$ state; (ii) the state $\sqrt{\frac{1}{2}}(|s = \frac{3}{2}, m_{\hat{\mathbf{z}}} = +\frac{3}{2}\rangle + |s = \frac{3}{2}, m_{\hat{\mathbf{z}}} = -\frac{3}{2}\rangle)$, with polarization vector $\mathbf{p} = \mathbf{0}$ and Majorana representation $\{(\frac{1}{2}, \frac{\sqrt{3}}{2}, 0), (\frac{1}{2}, -\frac{\sqrt{3}}{2}, 0), (0, 0, -1)\}$; (iii) the coherent state $|s = 2, m_{\hat{\mathbf{z}}} = +2\rangle$, with polarization vector $\mathbf{p} = 2\hat{\mathbf{z}}$ and Majorana representation (SP, SP, SP, SP) (where "SP" = South Pole); and (iv) a cutaway view of plot (iii).

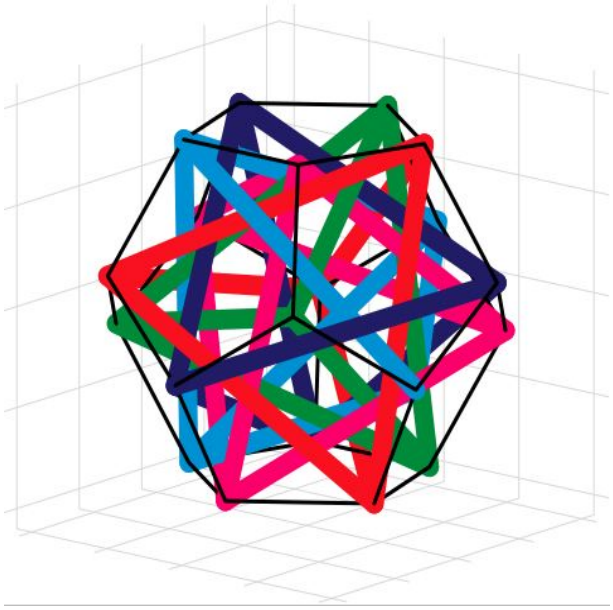


Fig. 2 Five interlocking tetrahedra whose vertices collectively form a dodecahedron. Vertices of tetrahedra of the same color comprise the Majorana representation of a tetrahedral state $|\tau_k\rangle$. Tetrahedral states corresponding to tetrahedra of different colors are orthogonal. (For a more accomplished rendering, see the work by artist/computer scientist Carlos Séquin.[14])

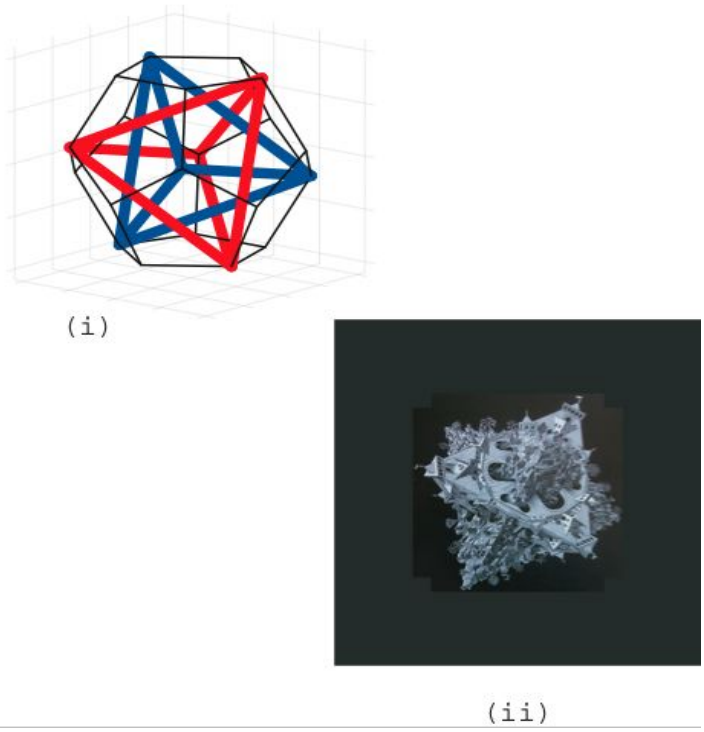


Fig. 3 (i) Reflections of the original five tetrahedra in the center of the sphere produce a new set of tetrahedra, corresponding to a new orthogonal basis $\{|\overline{\tau}_k\rangle\}$. (ii) M.C. Escher, *Double Planetoid*, 1949.[15]

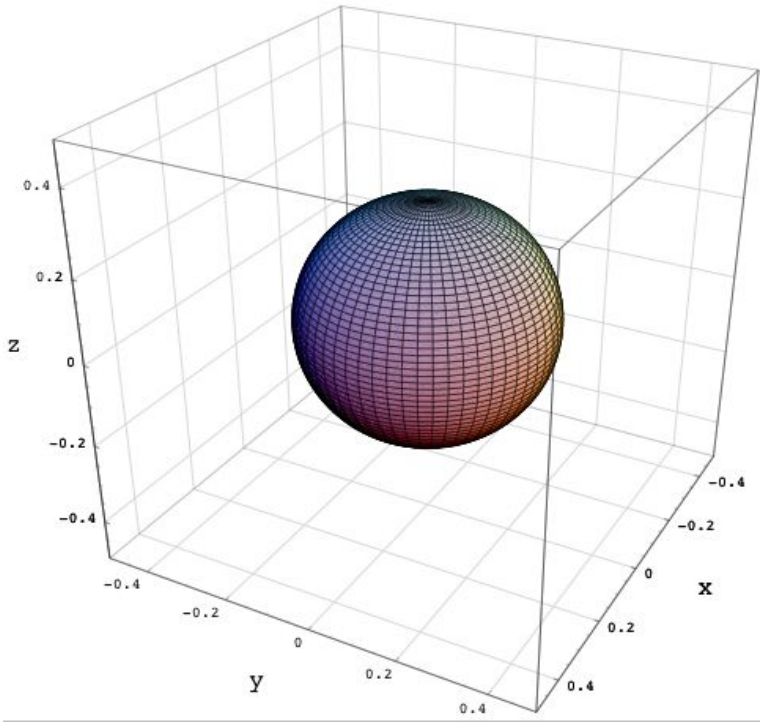


Fig. 4 A plot of the $L^2[\mathcal{S}^2]$-normalized variance $\Delta S^2_{\hat{\mathbf{n}}}/||\Delta S^2_{\hat{\mathbf{n}}}||$ for the tetrahedral state $|\psi_{\rm tet}\rangle$ given in Eq. (9a). Compare Fig. 1.

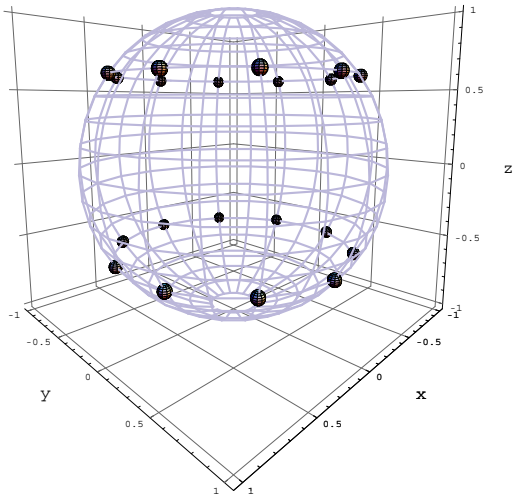


Fig. 5 The Majorana representation of the anticoherent state in Eq. (11), for $s = 10$. The unit sphere is shown in wire mesh. Points in the Majorana representation are shown as small spheres.

EJTP **3, No. 10** (2006) 157–190 *Electronic Journal of Theoretical Physics*

Stretching the Electron as Far as it Will Go

Gordon W. Semenoff[1] * and Pasquale Sodano [2] †

[1] *Department of Physics and Astronomy*
University of British Columbia, Vancouver, British Columbia, Canada V6T 1Z1

[2] *Progetto Lagrange, Fondazione C.R.T. e Fondazione I.S.I.*
c/o Dipartimento di Fisica- Politecnico di Torino,
Corso Duca degli Abruzzi 24 Torino-Italy

Received 8 May 2006, Published 28 May 2006

Abstract: Effects associated with the existence of isolated zero modes of Majorana fermions are discussed. It is argued that the quantization of this system necessarily contains highly extended quantum states and that populating and depopulating such states by interacting with the quantum system leads to long-ranged teleportation-like processes. Also leads to spontaneous violation of fermion parity symmetry. A quasi-realistic model consisting of a quantum wire embedded in a p-wave superconductor is discussed as an explicit example of a physical system with an isolated Majorana zero mode.

Keywords: Majorana Fermions, Teleportation, quantum Wire P-Wave superconductor, Soliton
PACS (2006): 71.10.Pm, 05.30.Fk, 05.45.Yv, 78.67.Lt,85.35.Be, 74.20.-z

1. Introduction

It is a great pleasure to dedicate this article to the 100'th anniversary of the birth of Ettore Majorana. As a testimony to his lasting influence on science, we shall describe how one of his great insights, used in a modern context, can be related to a particular macroscopic quantum phenomenon.

The idea is related to the observation by Majorana that a relativistic fermion such as the electron can be meaningfully decomposed into more basic degrees of freedom, essentially by taking the real and imaginary parts of its wave-function [1]. In relativistic field theory, what one obtains are called Majorana fermions, which have become the basic building blocks of supersymmetric field theories and supply a scenario whereby the

* This work is supported in part by NSERC of Canada.
† Dipartimento di Fisica e Sezione INFN, Universit degli Studi di Perugia, Via A. Pascoli, 06123, Perugia

neutrinos which are observed in nature can have mass. We shall use this idea in a different context. In quantum condensed matter, the appearance of an emergent Majorana fermion would provide an excitation of a system that has minimal degrees of freedom. The wave-function of the single-particle state would obey a Majorana condition, which would forbid quantum fluctuations of its phase. The utility of this fact has already been recognized in the context of quantum computing [2]-[4]. In the present manuscript, we will elaborate on our previous observation [5] that in some cases this can provide isolated states with wave-functions which are peaked at multiple, well separated locations. In a controlled setting, this can be used to create a condensed matter realization of the Einstein-Podolsky-Rosen effect and even a version of teleportation by long-ranged tunnelling.

Majorana's original motivation for inventing the Majorana fermion was to avoid the negative energy states that relativistic particles invariably seem to possess by identifying the negative and positive energy states of a relativistic wave equation as manifestations of the same quantum excitation.

In second quantization, the positive energy state can be occupied by a particle. Filling a positive energy state creates an excited state of the system with positive energy. On the other hand, a negative energy state should be regarded as typically being already filled by a particle. An excitation of the system is then found by emptying the negative energy state, or creating a hole. The system is put in a higher energy state by removing a negative energy particle, equivalently, creating a positive energy hole.

Majorana's idea can be implemented when there is a particle-hole symmetry. Then, for a given particle state, there exists a hole state with the same energy and with a wave-function that is related to the particle wave-function by a simple transformation. Then, by making the appropriate identification, one could indeed identify these as one and the same quantum state. Of course, the resulting system has half as many degrees of freedom.[1]

To illustrate the idea, let us recall the conventional second quantization of complex fermions, which could be either relativistic or non-relativistic. We begin with the assumption that in some approximation it makes sense to discuss a single non-interacting particle whose wave-function obeys the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\Psi(\vec{x},t) = H_0\Psi(\vec{x},t) \tag{1}$$

where H_0 is the single-particle Hamiltonian operator. Generally, as in the case of the Dirac equation, the Hamiltonian H_0 could be a matrix, as well as a differential operator, and $\Psi(\vec{x},t)$ a column vector whose indices we shall suppress.[2] The second-quantized field

[1] For a comprehensive account of issues to do with positive and negative energy modes of relativistic bosons and fermions, see the series of papers [6]-[11].

[2] An example is the Dirac Hamiltonian in 3+1-dimensions

$$H_0 = i\vec{\alpha}\cdot\vec{\nabla} + \beta m$$

where $\vec{\alpha}$ and β are a set of four Hermitian, anti-commuting 4×4 Dirac matrices. There exists a matrix Γ with the property $\Gamma\vec{\alpha}\,\Gamma = \vec{\alpha}^*$ and $\Gamma\beta\Gamma = -\beta^*$, so that, $\Gamma H_0\Gamma = -H_0^*$ and $\Gamma\psi_E^* = \psi_{-E}$. This is a

operator also typically obeys this wave equation plus the equal-time anti-commutation relation

$$\left\{\Psi(\vec{x},t),\Psi^\dagger(\vec{y},t)\right\}=\delta(\vec{x}-\vec{y}) \tag{2}$$

It is this anti-commutator which defines $\Psi(x,t)$ as an operator. It can further be used to derive the wave equation (1) from the second quantized Hamiltonian,

$$H=\int dx:\Psi^\dagger(x,t)H_0\Psi(x,t): \tag{3}$$

using the Hamilton equation of motion

$$i\hbar\frac{\partial}{\partial t}\Psi(x,t)=[\Psi(x,t),H]$$

We shall assume that H_0 is a Hermitian operator which has eigenfunctions and a spectrum of real eigenvalues

$$H_0\psi_E(x)=E\psi_E(x)$$

The energy E can be both positive and negative, in fact for the relativistic electron, if (1) were the Dirac equation, there are necessarily negative eigenvalues and the spectrum is unbounded below. The eigenfunctions obey the orthogonality and completeness relations

$$\int d\vec{x}\psi_E^\dagger(\vec{x})\psi_{E'}(\vec{x})=\delta_{EE'}\quad,\quad\sum_E\psi_E(\vec{x})\psi_E^\dagger(\vec{y})=\delta(\vec{x}-\vec{y}) \tag{4}$$

The delta function and summation in these formulae should be understood in a generalized sense where they are a Kronecker delta and a sum for discrete components of the spectrum and a Dirac delta function and integral for continuum spectrum.

In this system, one then forms the second quantized field operator by superposing the wave-functions with creation and annihilation operators,

$$\Psi(x,t)=\sum_{E>0}\psi_E(x)e^{-iEt/\hbar}a_E+\sum_{E<0}\psi_E(x)e^{-iEt/\hbar}b_{-E}^\dagger$$

Here, a_E is the annihilation operator for a particle with energy E and $b_{-E}^\dagger$ is the creation operator for a hole with energy $-E$. When they obey the algebra with non-vanishing anti-commutators

$$\left\{a_E,a_{E'}^\dagger\right\}=\delta_{EE'}\quad,\quad\left\{b_{-E},b_{-E'}^\dagger\right\}=\delta_{EE'}$$

the $\Psi(\vec{x},t)$ obeys the anticommutator (2). The completeness condition in Eq. (4) is essential for establishing this.

---

one-to-one mapping of positive to negative energy states. Explicitly, if the matrices are represented by $\vec{\alpha}=\begin{pmatrix}\vec{\sigma} & 0\\ 0 & -\vec{\sigma}\end{pmatrix}$ with $\vec{\sigma}$ the Pauli matrices and $\beta=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, then we can form the matrix $\Gamma=\begin{pmatrix}0 & -i\sigma^2\\ i\sigma^2 & 0\end{pmatrix}$. Note that, in this case $\Gamma=\Gamma^*$ and $\Gamma^2=1$. A Majorana fermion obeys the reality condition $\Psi(\vec{x},t)=\Gamma\Psi^*(\vec{x},t)$.

The ground state of the system, $|0>$, is the state where all positive energy levels are empty and where all negative energy levels are filled, or alternatively all hole states are empty. In the second quantized language, it is annihilated by the annihilation operators,

$$a_E|0>=0=b_{-E}|0>$$

Excited states are created by operating on $|0>$ with $a_E^\dagger$ and $b_{-E}^\dagger$. The excitations created by $a_E^\dagger$ are particles, those created by $b_{-E}^\dagger$ are anti-particles, or holes. A typical state is

$$a_{E_1}^\dagger \dots a_{E_m}^\dagger b_{E_1}^\dagger \dots b_{E_n}^\dagger |0>$$

and such states form a basis for the Fock space of the second quantized theory.

One can formulate Majorana fermions for a system of this kind if there exists a particle-hole symmetry, or, in the relativistic context, a charge conjugation symmetry. For example, consider the situation where a matrix Γ exists such that, for eigenstates of H_0,

$$\psi_{-E}(x) = \Gamma\psi_E^*(x) \tag{5}$$

(This implies that $\Gamma^*\Gamma = 1 = \Gamma\Gamma^*$.) Then, the particles and holes have identical spectra. A Majorana fermion is formed by treating the particle and hole with the same energy as a single excitation. The second quantized field operator is

$$\Phi(x,t) = \sum_{E>0}\left(\psi_E(x)e^{-iEt/\hbar}a_E + \Gamma\psi_E^*(x)e^{iEt/\hbar}a_E^\dagger\right)$$

This fermion does not have both particles and anti-particles. The ground state $|0>$ is annihilated by a_E

$$a_E|0>=0 \quad \forall a_E$$

and $a_E^\dagger$ creates particles, so that the excited states of the system are

$$a_{E_1}^\dagger a_{E_2}^\dagger ... a_{E_k}^\dagger |0>$$

The field operator is (pseudo-)real in the sense that it obeys

$$\Phi(x,t) = \Gamma\Phi^*(x,t) \tag{6}$$

It obeys the anti-commutation relation

$$\left\{\Phi(\vec{x},t), \Phi^\dagger(\vec{y},t)\right\} = \delta(\vec{x}-\vec{y}) \tag{7}$$

To be concrete, in a system of complex fermions where the Hamiltonian such that the spectrum has the particle-hole symmetry (5), we could decompose the complex fermion into two Majorana fermions by taking the real and imaginary parts,

$$\Phi_1(x,t) = \frac{1}{\sqrt{2}}\left(\Psi(x,t) + \Gamma\Psi^*(x,t)\right)$$

$$\Phi_2(x,t) = \frac{1}{\sqrt{2}i}\left(\Psi(x,t) - \Gamma\Psi^*(x,t)\right)$$

Then each of $\Phi_1(x,t)$ and $\Phi_2(x,t)$ are a Majorana fermion.

In spite of the beautiful simplicity of this idea, Majorana fermions are not easy to come by in nature. One could, for example, decompose the relativistic electron, whose wave equation does have a charge-conjugation symmetry, into its real and imaginary parts. However, the interaction of the electron with photons is not diagonal in this decomposition. The real and imaginary components would be rapidly re-mixed by electromagnetic interactions, they cannot be stationary states of the full Hamiltonian of quantum electrodynamics.

One place where we might have better luck is to look for emergent Majorana fermions in quantum condensed matter systems. For example, in a superconductor, the electromagnetic interactions are effectively screened. Indeed, the Bogoliubov quasi-electrons in a superconductor behave like neutral particles. However, even there, in an ordinary s-wave superconductor, the anti-particle of a quasi-electron is another quasi-electron with opposite spin. Indeed, the quasi-electron operator in an s-wave superconductor is the two-component object

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\downarrow(x) \end{pmatrix}$$

where $(\uparrow,\downarrow)$ denotes spin up and down. It obeys the charge conjugation condition

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\downarrow(x) \end{pmatrix}^* = \begin{pmatrix} \psi_\downarrow(x) \\ \psi^*_\uparrow(x) \end{pmatrix}$$

which is not an analog of the Majorana condition in eqn. (6), since it entails both conjugation and a flip of the spin.

In order to find a medium where the quasi-electron is a Majorana fermion, we need to consider a superconductor where the condensate has Cooper pairs with the same spin, so that the quasi-electron has the form

$$\begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\uparrow(x) \end{pmatrix}$$

Then, quasi-electron is pseudo-real, complex conjugation of its wave-function is equivalent to multiplying by the matrix $\Gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\downarrow(x) \end{pmatrix}^* = \begin{pmatrix} \psi_\uparrow(x) \\ \psi^*_\uparrow(x) \end{pmatrix}$$

This gives a physical realization of a Majorana fermion. An example of such a superconductor is one with a p-wave condensate, such as Strontium Ruthenate [12]. There, the condensate has the form $\langle\psi_\uparrow(x)\vec{x}\times\vec{\nabla}\psi_\uparrow(x)\rangle$ (and can in principle have an admixture of spin down as well). Thus, we see that, in such a material, the quasi-electron is a two-component object obeying a Majorana condition. We will make use of this example later in this Paper.

Our particular interest in the following will be in situations where the fermion spectrum has mid-gap, or zero energy states. These are well known to lead to interesting phenomena. Already for complex electrons, mid-gap states give rise to fractional quantum numbers [13, 14]. With Majorana fermions, they are known to lead to peculiar representations of the anti-commutator algebra which can violate basic symmetries [15, 16]. Some interesting effects in the context of zero modes on cosmic strings have also been examined [17]-[20].

To illustrate, let us consider the second quantization of a complex fermion whose spectrum has a zero mode,

$$H_0\psi_0(x) = 0$$

The conjugation symmetry implies that

$$\psi_0(x) = \Gamma\psi_0^*(x)$$

If the fermion is complex (not Majorana), the second quantized field has a term with the zero mode wave-function and an operator, the first term in the following expansion:

$$\Psi(x,t) = \psi_0(x)\alpha + \sum_{E>0}\psi_E(x)e^{-iEt/\hbar}a_E + \sum_{E<0}\psi_E(x)e^{-iEt/\hbar}b^\dagger_{-E} \quad .$$

Here, α obeys the algebra

$$\left\{\alpha,\alpha^\dagger\right\} = 1 \tag{8}$$

and it anti-commutes with all of the other creation and annihilation operators. The existence of this zero mode leads to a degeneracy of the fermion spectrum. The vacuum state is annihilated by all of the annihilation operators a_E and b_E. However, it must also carry a representation of the algebra (8). The minimal representation is two-dimensional. There are two vacuum states, $(|\uparrow>, |\downarrow>)$, which obey

$$a_E|\uparrow> = 0 = a_E|\downarrow> \quad , \quad b_E|\uparrow> = 0 = b_E|\downarrow>$$

and

$$\alpha^\dagger|\downarrow> = |\uparrow> \quad , \quad \alpha^\dagger|\uparrow> = 0$$

$$\alpha|\downarrow> = 0 \quad , \quad \alpha|\uparrow> = |\downarrow>$$

The entire spectrum has a 2-fold degeneracy, with two towers of excited states,

$$a^\dagger_{E_1}...a^\dagger_{E_m}b^\dagger_{E_1}...b^\dagger_{E_n}|\uparrow>$$

and

$$a^\dagger_{E_1}...a^\dagger_{E_m}b^\dagger_{E_1}...b^\dagger_{E_n}|\downarrow>$$

having the identical energies $\sum_i E_i$.

This quantization of the zero mode α was argued by Jackiw and Rebbi [13] to lead to states with fractional fermion number. Indeed, the suitably normal ordered second quantized number operator

$$Q = \int d\vec{x}\frac{1}{2}\left[\psi^\dagger(x,t),\psi(\vec{x},t)\right] = \sum_{E>0}\left(a^\dagger_E a_E - b^\dagger_{-E}b_{-E}\right) + \alpha^\dagger\alpha - \frac{1}{2} \tag{9}$$

has fractional eigenvalues, for example

$$Q|\uparrow>=+\frac{1}{2}|\uparrow> \quad , \quad Q|\downarrow>=-\frac{1}{2}|\downarrow>$$

In actuality, the charge operator is defined only up to an overall additive constant. However, there does exist a symmetry of the theory, gotten at the second quantized level by replacing $\Psi(x,t)$ by $\Gamma\Psi^*(x,t)$. This transformation interchanges particles and anti-particles, and is a symmetry of the suitably normal ordered second quantized Hamiltonian. It should also flip the sign of Q. It implies that, if there is an eigenstate of Q in the system with eigenvalue q,

$$Q|q>=q|q>$$

then there must exist another eigenstate $|-q>$ in the spectrum of Q with eigenvalue $-q$:

$$Q|-q>=-q|-q>$$

In addition, it is easy to argue that the eigenvalues of Q are space by integers, i.e. if q_1 and q_2 are any two eigenvalues, then $q_1 - q_2$ =integer. This is essentially because the raising and lowering operators for Q are $\Psi^\dagger$ and Ψ, respectively and they raise and lower in units of integers. In particular, this implies that

$$q-(-q) = 2q = \text{integer}$$

Thus, the only possibilities are that the entire spectrum of states have integer eigenvalues of Q, q =integer, or the entire spectrum of states have half-odd-integer eigenvalues $q = \frac{1}{2}$-odd integer. It is easy to see that the operator Q as written in (9) indeed flips sign if we interchange $a_E \leftrightarrow b_E$ and $\alpha \leftrightarrow \alpha^\dagger$ and the offset of -1/2 that appears explicitly there is essential for this transformation to work. This leads to the conclusion that, with a single fermion zero mode, the fermion number charge is quantized in half-odd-integer units.

Now, consider what happens for a Majorana fermion with a single zero mode. [3] In this case, a charge analogous to Q is not defined, so the issue of fractional charge is not

[3] We will later construct an explicit example where this precisely this situation occurs.

relevant. But the quantization of the system is still interesting. The second quantized operator is

$$\Phi(x,t) = \psi_0(x)\alpha + \sum_{E>0} \psi_E(x) e^{-iEt/\hbar} a_E + \sum_{E<0} \psi_E(x) e^{-iEt/\hbar} a^\dagger_{-E}$$

This fermion contains half of the degrees of freedom of the previous complex one. Here, the b_E are absent and the zero mode operator is real, $\alpha = \alpha^\dagger$.

The creation and annihilation operator algebra is now

$$\left\{ a_E, a^\dagger_{E'} \right\} = \delta_{EE'}$$

as before, and

$$\alpha^2 = 1/2 \quad , \quad \{\alpha, a_E\} = 0 = \left\{ \alpha, a^\dagger_E \right\} \tag{10}$$

A minimal representation can be constructed by defining a vacuum state where

$$a_E|0>=0 \quad \text{for all } E>0$$

Then, we can represent the zero mode by the operator

$$\alpha = \frac{1}{\sqrt{2}} (-1)^{\sum_{E>0} a^\dagger_E a_E} \tag{11}$$

Indeed

$$\alpha = \alpha^\dagger$$

and, since

$$\sum_E a^\dagger_E a_E |0>=0$$

we have

$$\alpha|0>=\frac{1}{\sqrt{2}}|0>$$

The Klein operator, $(-1)^{\sum_{E>0} a^\dagger_E a_E}$, anti-commutes with a_E and $a^\dagger_E$. A basis for the Hilbert space consists of the vacuum and excited states which are obtained from the vacuum by operating creation operators

$$a^\dagger_{E_1} a^\dagger_{E_2} ... a^\dagger_{E_k} |0>$$

These are eigenstates of $\sum_{E>0} a^\dagger_E a_E$ with integer eigenvalues. Thus, in this basis, $\alpha^2 = 1/2$ when operating on each basis vector, and thus the identity operator on the whole space. The operator in (11) thus satisfies the algebra (10).

Another, inequivalent representation can be obtained by starting with

$$\tilde{\alpha} = -\frac{1}{\sqrt{2}} (-1)^{\sum_{E>0} a^\dagger_E a_E} \tag{12}$$

and a similar construction leads to a Hilbert space whose states are orthogonal the one found above. We emphasize here that there are two inequivalent representations of the

anti-commutator algebra, one where the zero mode operator is represented by α in eq. (11) and one where it is represented by $\tilde{\alpha}$ in eq. (12). Both of these give an irreducible representation and the two representations are not related to each other by an internal automorphism.

We observe that these minimal representations of the anti-commutator algebra have the property that they break a symmetry of the fermion theory under $\Phi(\vec{x},t) \to -\Phi(\vec{x},t)$, which we shall call "fermion parity". Fermion parity is a symmetry of the linear wave equation even when $\Phi(\vec{x},t)$ is a Majorana fermion. At the quantum level, fermion parity symmetry leads to a conservation law for the number of fermions modulo 2. By this conservation law, any physical process must entail creation or destruction of an even number of fermions. For example, if a quantum state is initially prepared with an even number of fermions, after any physical process, the number should remain even. In operator language, there should exist an operator $(-1)^F$ which anti-commutes with $\Phi(\vec{x},t)$,

$$(-1)^F \Phi(\vec{x},t) + \Phi(\vec{x},t)(-1)^F = 0$$

and which therefore commutes with the full second quantized Hamiltonian,

$$(-1)^F H = H(-1)^F$$

where

$$H = \int d\vec{x} \frac{1}{2} : \Phi^\dagger(\vec{x},t) H_0 \Phi(\vec{x},t) :$$

However, we see that in the minimal representations of the anti-commutation algebras (10) discussed above, in the first representation (11),

$$< 0|\Phi(x,t)|0 >= +\frac{1}{\sqrt{2}}\psi_0(x)$$

and in the second representation (12)

$$< 0|\Phi(x,t)|0 >= -\frac{1}{\sqrt{2}}\psi_0(x)$$

In both of these representations, neither the vacuum state, nor any of the excited states can be eigenstates of fermion parity, the operator $(-1)^F$. Thus fermion parity symmetry is broken by the minimal quantization of this model.

Fermion parity is a sacred symmetry of physics in four dimensional space-time [21]. All fundamental fermions in nature have half-odd-integer spin. A flip in sign of all fermion operators can then be realized as a rotation by an angle 2π. Nature should be symmetric under a rotation by 2π. This means that, if we superpose a state with even fermion number and a state with odd fermion number,

$$c_1|\text{even} > +c_2|\text{odd} >$$

no experiment should be devisable, even in principle, to measure the relative sign of c_1 and c_2. In the four dimensional world, unless rotation invariance is broken at a the level

of fundamental physics, we should always be free to insist that $(-1)^F$ is a good symmetry and that we can take all physical states as eigenstates. Of course, this applies in four space-time dimensions. The emergent Majorana fermions that we want to consider here are embedded in four space-time dimensions. We therefore feel free to insist on fermion parity.

This brings up a contradiction with the previous discussion, where we found that fermion parity is necessarily broken by the quantization of the zero mode Majorana fermion system. The only way to restore the symmetry is to use a reducible representation of the anti-commutator algebra. The minimal modification of the representation is equivalent to the introduction of another degree of freedom – and subsequent use of irreducible representations. The new degree of freedom acts like a hidden variable. In the anti-commutator algebra it would be another anti-commuting variable β which has identical properties to α,

$$\beta^2 = 1/2$$

and anti-commutes with all other variables. Then the algebra of α and β would have a two dimensional representation which we could find by considering the fermionic oscillators

$$a = \frac{1}{\sqrt{2}}\left(\alpha + i\beta\right) \quad , \quad a^\dagger = \frac{1}{\sqrt{2}}\left(\alpha - i\beta\right) \tag{13}$$

$$\alpha = \frac{1}{\sqrt{2}}\left(a + a^\dagger\right) \quad , \quad \beta = \frac{1}{\sqrt{2}i}\left(a - a^\dagger\right) \tag{14}$$

which obey

$$a^2 = 0 \quad , \quad a^{\dagger 2} = 0 \quad , \quad \left\{a, a^\dagger\right\} = 1$$

We could then find a vacuum state which is annihilated by a, and another state which is created from the vacuum by $a^\dagger$,

$$a|- >= 0 \quad , \quad a^\dagger|- >= |+ >$$

$$a|+ >= |- > \quad , \quad a^\dagger|+ >= 0$$

so that both are eigenstates of $(-1)^F$ and fermion parity is restored. Later we will see that the hidden variable β can have a physical interpretation.

2. Degeneracy, Tunnelling and Teleportation

In this paper, the most speculative use of Majorana fermions that we shall find is for a kind of teleportation by quantum tunnelling. In the context in which quantum tunnelling is normally studied, a classical object can exist in allowed regions. There exist other forbidden regions where it is not allowed to be. Then, quantum tunnelling makes use of the fact that, when the particle is quantum mechanical, its wave-function does not necessarily go to zero in a classically forbidden region, but decays exponentially. That means that it could, in principle, have support on the other side of such a region and

there is some small probability that an object will be found on the other side. This is called tunnelling.

One might try to make use of quantum tunnelling to transport an object through a classically forbidden region. Unfortunately, the exponential decay of the wave-function across any classical barrier of appreciable size renders it too small to be of any practical use in this regard. A more sophisticated approach would be to create a scenario where the wave-function has peaks of appreciable size at spatially separated locations, perhaps with a forbidden region in between. This too will fail, but for a more sophisticated reason which, since it is related to our later use of Majorana fermions, we will outline. Consider, for example the double well potential depicted in Fig. 1. If the locations of the minima

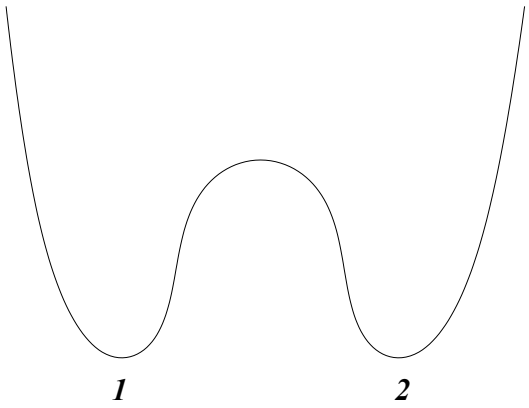


Fig. 1 A double-well potential.

are well separated and the barrier in between them is large, semi-classical reasoning can be applied to this system. Then, the ground state of a particle in this potential should indeed have a peak near each of the minima, and should be approximately symmetric under interchanging the locations of the minima. The typical profile of such a wave-function is drawn in Fig. 2.

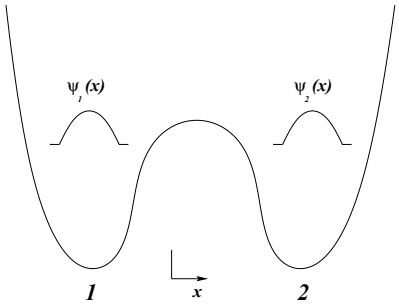


Fig. 2 The ground state of a particle in a double well has two peaks, localized at 1 and 2.

Now, we ask the question. Is this state of use for tunnelling? If this were the energy landscape in which a quantum mechanical particle lived, could we, for example, populate this ground state by interacting with the system in the vicinity of minimum 1 and then depopulate the state by interacting with the system near the other minimum - 2, effectively teleporting the particle from location 1 to location 2?

The answer to this question is 'no'. The reason for this answer is degeneracy, or approximate degeneracy of the quantum state that we are considering. In such a system, when our classical reasoning is good, there must always be a second state, perhaps at slightly higher energy but approximately degenerate with the ground state, whose wave-function is approximately an anti-symmetric function of the positions of the minima. Its typical profile is depicted in Fig. 3. The ground state wave-function has the form $\psi_0(x) = \psi_1(x) + \psi_2(x)$ where $\psi_1(x)$ is localized near minimum 1 and $\psi_2(x)$ is localized near minimum 2. The anti-symmetric state would have the form $\psi_a(x) = \psi_1(x) - \psi_2(x)$.

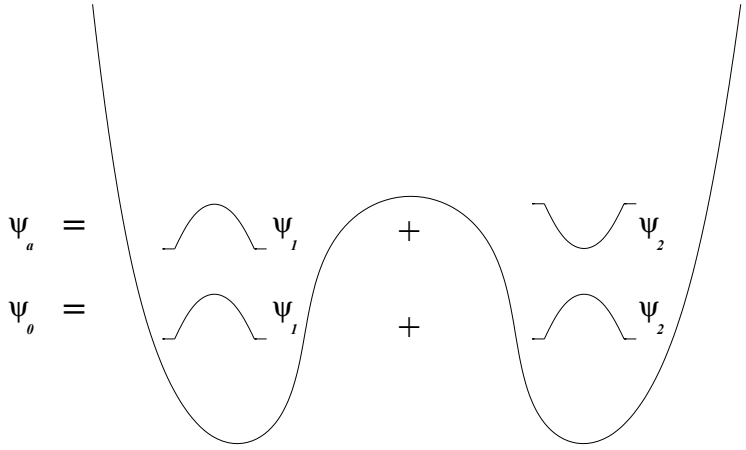


Fig. 3 The almost degenerate state ψ_a also has two peaks but with differing signs.

Now, when we interact with the system near minimum 1, while we overlap the ground state wave-function, $\psi_0(x)$, we also overlap $\psi_a(x)$ by the same amount. Of course, the state that we actually populate is a linear combination of the two,

$$\frac{1}{\sqrt{2}}\left(\psi_0(x) + \psi_a(x)\right) = \sqrt{2}\psi_1(x)$$

whose wave-function is entirely localized at the position of the first minimum. The particle initially has zero probability of appearing near the second minimum. Our attempt at teleportation by tunnelling has been foiled by degeneracy.

Anytime the Schrödinger equation can be analyzed semi-classically in this way, it seems to have a built in protection against the long-ranged behavior that we are looking for.

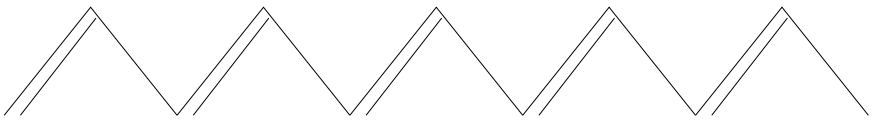
In this argument, because it is a superposition of two stationary states with slightly differing energies, $\psi_1(x)$ is not a stationary state. It should have a small time dependence which eventually mixes it with $\psi_2(x)$. But this time dependence mixes it slowly, in fact its origin is just the conventional tunnelling amplitude for the particle to move from location 1 to location 2 through the barrier in between.

What we need to find is a quantum system where a quantum state which is well isolated from other states in the spectrum can have peaks at different locations. From the argument above, it will be difficult to find states of this kind which obey the regular Schrödinger equation. Where we will look for such states is in quantum condensed matter

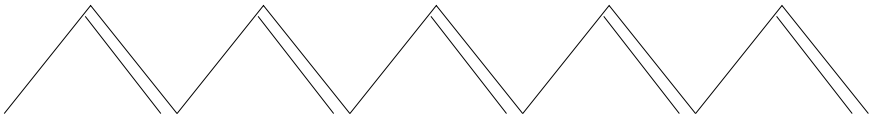
systems, where electrons, or more properly quasi-electrons, can satisfy equations that are very different from the Schrödinger equation. To motivate this, in the next Section, we review some of the pictorial arguments for the appearance of fractionally charged states in polyacetylene. Also, to set the stage for what comes next, we discuss what happens if the fermion spectrum of polyacetylene were Majorana, rather than complex fermions.

3. The Polyacetylene Story

Before we consider a more quantitative model which will illustrate our point, we pause to recall the example of the conducting polymer, polyacetylene. Polyacetylene is a hydrocarbon polymer where each Carbon atom bonds with a Hydrogen atom and as well forms two strong covalent bonds with neighboring Carbon atoms. The fourth valence electron is nominally a conduction electron. However, a Peirls instability localizes it into a charge density wave which is effectively a dimer. The result is a gap in the electron spectrum at the fermi surface and, without impurities or other structures, the material is an insulator. There are two degenerate ground states, depending on the direction chosen by the dimerization. We illustrate these as the A and B phases in the diagram in Fig. 4. In that figure, each line is a covalent bond, using two of the valence electrons of the



Phase A



Phase B

Fig. 4 The two degenerate ground states of polyacetylene.

Carbon atoms.

The conductivity of doped polyacetylene that is seen by experiments is thought to be mostly attributed to the transport of charged solitons along the polyacetylene molecules. A soliton in this system is a defect which interpolates between the two phases. We have depicted a soliton-anti-soliton pair in Fig. 5. Note that it can be obtained from one of the ground states by flipping the direction of the bonds that lie between the locations of the solitons. Also note that the energy of the system could be higher than that of the ground state, since the defects have non-minimal energy configurations, but the energy density should be concentrated in the vicinity of the solitons. Although it will not be an issue for us, since we are interested in other aspects of this system, the solitons turn out to be quite mobile. They also carry electric charge, and can thus account for the high

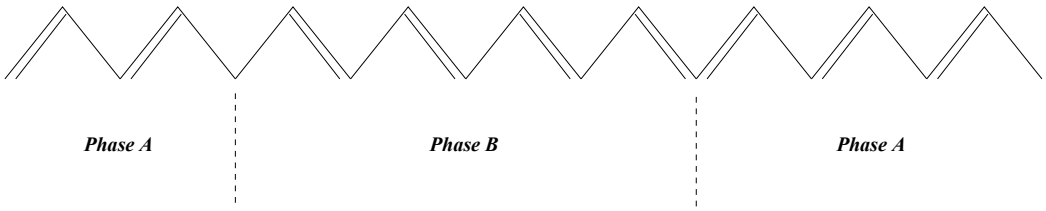


Fig. 5 Solitons form phase boundaries. The soliton anti-soliton pair can be created by flipping the direction of the bonds between them.

conductivity that is attainable in polyacetylene. The density of solitons can be controlled by doping. For some original literature on polyacetylene, see refs. [22]-[29].

There is a simple argument that shows that a soliton of polyacetylene has half of the quantum numbers of an electron [28]. In this argument, we will neglect the spin of the electron. Thus, for the purpose of our arguments, in figures 4 and 5, each bond stands for a single electron, rather than a spin up, spin down pair of electrons. Now, consider what happens when we add an electron to phase A, as in Fig. 6. By flipping the directions

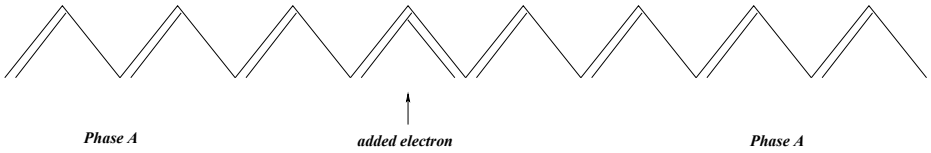


Fig. 6 Phase A with an additional electron.

of some bonds, we can see that we have created a soliton-anti-soliton pair, where each object seems to share half of the added electron. This state is depicted in Fig. 7.

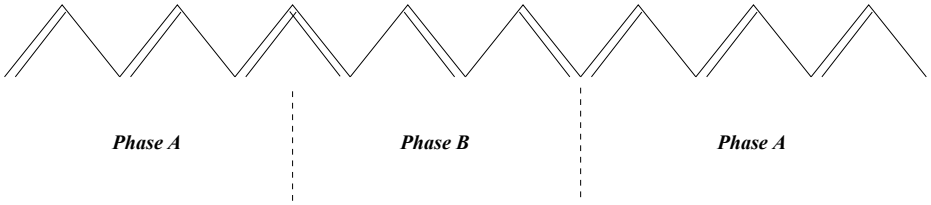


Fig. 7 Beginning with phase A and an additional electron, as shown in Fig. 6, we create a soliton-antisoliton pair which seems to share the electron.

This brings up the question, is the electron really 'split' between the two sites? Or does it exist in an entangled state of some sort which has some probability – $\frac{1}{2}$ – of the "whole" electron being located at either site. This question can be made more precise by asking about measurement of the electron charge, which is a conserved quantum number in this system. If, by further flipping bonds, we separate the solitons to a large distance, and then measure the electron charge in the vicinity of one of the solitons, is the result of the measurement -e/2? Or does this measurement manage to collapse the electron wave-function somehow so that the result is either 0 or -e? In the latter case, the average of many measurements might be -e/2, but any single measurement would either see a whole electron or no electron at all. The answer to this question was found long ago in ref. [30, 31]. The conclusion was that the measurement of the electron charge localized

near one of the solitons should yield -e/2. Put equivalently, the fractional charge of the soliton is a sharp quantum observable. How it manages to do this is interesting, and was discussed in ref. [31]. We shall review it here.[4] This issue has recently been reexamined [48] in conjunction with some ideas about entangled electron states in Helium bubbles [49].

The electron spectrum in polyacetylene has an electron-hole symmetry. We could have created a state with the same energy as the one depicted in Fig. 7 by removing, rather than adding an electron, to give a hole which is apparently split between the soliton and anti-soliton, as shown in Fig. 8.

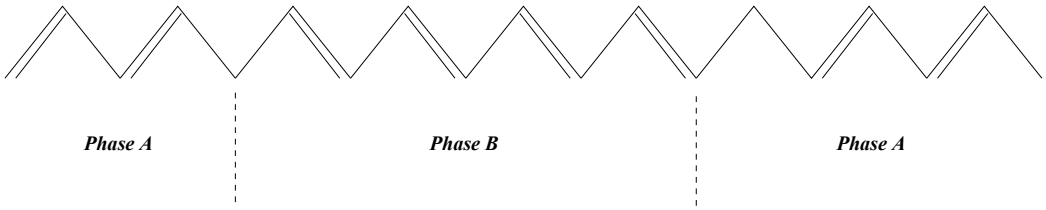


Fig. 8 A soliton-antisoliton pair with a deficit of one electron.

There are apparently four different states of the soliton-anti-soliton system. There are the two overall neutral states, one of which is depicted in Fig. 5 and the other obtained by flipping the intermediate bonds in the opposite direction. We could also go from one of these states to the other by transporting a whole electron from one soliton to the other. The other two states we can obtain by either adding or subtracting an electron from one of the ground states and are those that we have already discussed in Figs. 7 and 8. We can identify these charged soliton states in the low energy electron spectrum. In the single electron spectrum of polyacetylene with a soliton-anti-soliton pair, there are two near-mid-gap states which have small positive and negative energies. Thus the low energy electron spectrum has four states, a ground state, an electron state, a hole state and an electron-hole state.

By their quantum numbers, the electron and hole states can be identified with the configurations in Figs. 7 and 8, respectively. The ground state and the electron-hole state are neutral and must be formed from linear combinations of the two neutral states. Then, in the electron state, the electron wave-function indeed should have two peaks, as depicted in Fig. 9. Similarly the hole wave-function also should have two peaks, as is depicted in Fig. 10. Detailed analysis shows that one is an even and the other is an odd function of relative distance, as shown in the figures.

The electron wave-function has peaks at two locations. So, we could ask the question again: Can we use this system for teleportation? Could we populate the electron state by interacting with one of the solitons and subsequently extract the electron again, and thereby teleport it, by interacting with the other soliton? To understand the answer, which will be 'No!', it is necessary to realize that, once the electric charge is a sharp quantum observable, the electronic states of the solitons are disentangled by a local

[4] For some other literature on this and closely related issues, see refs. [32]-[47].

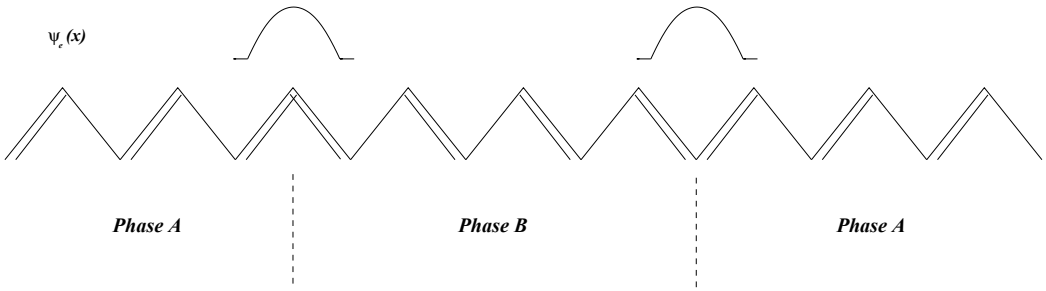


Fig. 9 The electron wave-function.

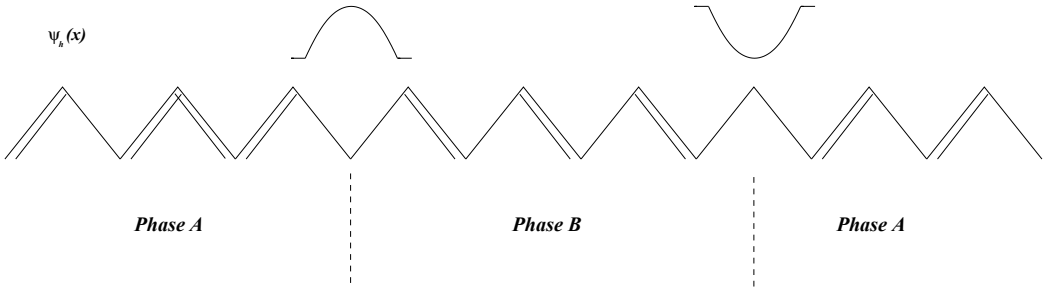


Fig. 10 The hole wave-function.

measurement of an observable such as the charge.

To see how this happens, let us consider a second quantization of this system. The electron operator has the form

$$\psi(x,t) = \psi_e(x)a + \psi_h(x)b^\dagger + ...$$

where we have identified an electron annihilation operator a for the positive energy state and a hole creation operator $b^\dagger$ for the negative energy state. We have neglected the time dependence (the energies of the two states are exponentially small in the soliton separation). We could as well write

$$\psi(x,t) = \psi_1(x)(a + b^\dagger) + \psi_2(x)(a - b^\dagger) + ... \tag{15}$$

where $\psi_1(x) = \frac{1}{\sqrt{2}}(\psi_e(x) + \psi_h(x))$ has support only in the region of the left-hand soliton and $\psi_2(x) = \frac{1}{\sqrt{2}}(\psi_e(x) - \psi_h(x))$ has support only near the right-hand soliton in Figs. 9 and 10.

Now, if we concentrate on the region near the left-hand soliton, $\psi(x,t)$ or $\psi^\dagger(x,t)$ will annihilate or create an excitation using the combination of operators

$$\alpha = \frac{1}{\sqrt{2}}\left(a + b^\dagger\right) \quad , \quad \alpha^\dagger = \frac{1}{\sqrt{2}}\left(a^\dagger + b\right)$$

Similarly, if we concentrate on the region around the right-hand soliton, excitations are created and annihilated using

$$\beta = \frac{1}{\sqrt{2}}\left(a - b^\dagger\right) \quad , \quad \beta^\dagger = \frac{1}{\sqrt{2}}\left(a^\dagger - b\right)$$

The set of operators $(\alpha, \alpha^\dagger, \beta, \beta^\dagger)$ are a Bogoliubov transformation of the creation and annihilation operators $(a, a^\dagger, b, b^\dagger)$. This transformation does not violate fermion number – it superposes operators with the same fermion number. Further, the excitations that the new operators create or annihilate are entirely localized on one or the other of the solitons.

Thus, again, we do not have a process whereby an electron or hole state which has two peaks can be populated by interacting with the system in the vicinity of one of the peaks. We have failed to find teleportation. Instead we have found fractional charge. The charge density integrated over the vicinity of one of the solitons turns out to be

$$Q = -e\left(\alpha^\dagger\alpha - 1/2\right) + \text{ charge of electrons } - \text{ charge of holes}$$

which indeed has half-odd-integer eigenvalues. This Bogoliubov transformation, as a mechanism for disentangling the charge quantum numbers of the solitons was originally found in ref. [31].

In eq. (15), we ignored the small time dependence of the near mid-gap states. At this point, the reader might wonder if the disentanglement of the soliton and anti-soliton charges that we find by the Bogoliubov transformation would not be undone by this time variation. Indeed, it would be, eventually. However the time scale is given by the inverse of the energy gap and is therefore exponentially large in the distance L between the soliton and anti-soliton, $T \sim m^{-1}e^{mL}$, where m is the energy gap. This is roughly the time for quantum mechanical tunnelling between the solitons assuming an energy barrier of height the energy gap extending over distance L. For macroscopic L this time T should be very large.

What has prevented teleportation in this second example is again a degeneracy, this time a slightly more subtle one since, even though the electron and hole state have identical energies, they have opposite signs of charge. Avoiding teleportation has led to fractional charge. It has done this by a hybridization, at the second quantized level, of the propensity of the electron field operator to create an electron and to annihilate a hole in a local state.

Now, imagine that, rather than complex electrons, polyacetylene had Majorana fermions which would be obtained by identifying the particle and hole states as the same excitations. (Here, we are ignoring the obvious disaster that this scenario would lead to in chemistry.) Then, in eqn. (15), we would have to identify $a = b$ and

$$\psi_{\text{Maj}}(x,t) = \psi_1(x)(a + a^\dagger) + \psi_2(x)(a - a^\dagger) + ... \tag{16}$$

Now, $a + a^\dagger$ cannot be an annihilation operator, in fact

$$(a + a^\dagger)^2 = 1$$

It is similar to the single zero mode operator "α" that we found for a Majorana fermion in the Eq. (14). In fact, the other combination $\frac{1}{\sqrt{2}i}(a - a^\dagger)$ now plays the role of "β", the "hidden variable". Its purpose in our previous discussion was to provide a quantization

which did not violate fermion parity. Here, this hidden variable is just the fermion zero mode living on the far-away anti-soliton. We could choose the fermion parity conserving quantization by using the states $(|->, |+>)$ defined by

$$a|->=0 \quad , \quad a^\dagger|->=|+>$$

$$a|+>=|-> \quad , \quad a^\dagger|+>=0$$

which can be eigenstates of $(-1)^F$. In these states, the expectation value of the fermion operator vanishes, for example $<0|\psi_{\rm Maj}(x,t)|0>=0$. However, the two solitons are invariably entangled. There is now no conserved fermion number that we can use to measure this entanglement, but there are other effects which we will discuss in later sections once we have made the present reasoning more solid by discussing it in the context of a field theoretical model and them formulated a more realistic model with emergent Majorana fermions.

4. Relativistic Majorana Fermions in a Soliton Background

Single-particle states that are in some sense isolated are well known to occur for Dirac equations, particularly when interacting with various topologically non-trivial background fields such as solitons, monopoles and instantons. The consequences of fermion zero modes such as chiral anomalies [50] and fractional fermion number [13], [14] are well known.

The polyacetylene example, in the context of discussions of fractional charge, that we used in the previous Section is a well-known example of this. In polyacetylene, the low energy electron spectrum can be approximately described by the Dirac equation [22, 51] and the solitons which we discussed using pictures have a mathematical description as soliton-like configurations of a scalar field which couples to the Dirac equation. In this Section, we will make the analysis of the previous Section more quantitative by considering the problem of a 1+1-dimensional relativistic Dirac equation coupled to a soliton background field and a soliton-anti-soliton pair.

Consider, for example, the simple one-dimensional model with Dirac equation

$$\left[i\gamma^\mu\partial_\mu+\phi(x)\right]\psi(x,t)=0 \tag{17}$$

The Dirac gamma-matrices obey the algebra

$$\{\gamma^\mu,\gamma^\mu\}=2g^{\mu\nu}$$

where $g^{\mu\nu}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$ is the (inverse of the) metric of two dimensional space-time.

This describes a fermion moving in one dimension and interacting with a scalar field $\phi(x)$ which we shall take to have a soliton-anti-soliton profile. For the purposes of this discussion, we take the ideal case of a step-function soliton located at position $x=0$ and

a step-function anti-soliton located at $x = L$,

$$\phi(x) = \begin{cases} \phi_0 & x < 0 \ , \ x > L \\ -\phi_0 & 0 < x < L \end{cases} \tag{18}$$

We will assume that the solitons are very massive, so they do not recoil when, for example, fermions scatter from them.

If we take

$$\psi(x,t) = \psi_E(x) e^{-iEt}$$

and choose an appropriate basis for the Dirac gamma-matrices, the Dirac equation becomes

$$i \begin{pmatrix} 0 & \frac{d}{dx} + \phi(x) \\ \frac{d}{dx} - \phi(x) & 0 \end{pmatrix} \begin{pmatrix} u_E(x) \\ v_E(x) \end{pmatrix} = E \begin{pmatrix} u_E(x) \\ v_E(x) \end{pmatrix} \tag{19}$$

This equation has a particle-hole symmetry

$$\psi_{-E}(x) = \psi_E^*(x)$$

It is easy to show that it has exactly two bound states. One is a state with small positive energy and the other is the associated hole state with a small negative energy. The wave-functions

$$E_+ \approx +\phi_0 e^{-\phi_0 L} \tag{20}$$

$$\psi_+(x) \approx \sqrt{\phi_0} \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\phi_0 x} + \mathcal{O}(e^{-\phi_0 L}) & x < 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\phi_0 x} + \begin{pmatrix} 0 \\ -i \end{pmatrix} e^{\phi_0 (x-L)} + \mathcal{O}(e^{-\phi_0 L}) & 0 < x < L \\ \begin{pmatrix} 0 \\ -i \end{pmatrix} e^{\phi_0 (L-x)} + \mathcal{O}(e^{-\phi_0 L}) & L < x \end{cases} \tag{21}$$

$$E_- \approx -\phi_0 e^{-\phi_0 L} = -E_+ \tag{22}$$

$$\psi_(x) - \approx \sqrt{\phi_0} \begin{cases} \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\phi_0 x} + \mathcal{O}(e^{-\phi_0 L}) & x < 0 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-\phi_0 x} + \begin{pmatrix} 0 \\ i \end{pmatrix} e^{\phi_0 (x-L)} + \mathcal{O}(e^{-\phi_0 L}) & 0 < x < L \\ \begin{pmatrix} 0 \\ i \end{pmatrix} e^{\phi_0 (L-x)} + \mathcal{O}(e^{-\phi_0 L}) & L < x \end{cases} \tag{23}$$

where, sufficient for our purposes, we give only the large L asymptotics – corrections to all quantities are of order $e^{-\phi_0 L}$. Note that $\psi_-(x)$ is indeed related to $\psi_+(x)$ by $\psi_-(x) = \psi_+^*(x)$.

These states have energy well separated from the rest of the spectrum, which is continuous and begins at $E = \pm\phi_0$. The energies are also exponentially close to zero as the separation L is large. Furthermore, each wave-function has two peaks, one near $x = 0$ and one near $x = L$. They have identical profile near $x = 0$ and they differ by a minus sign near $x = L$. This is the same feature of the electron and hole states that we claimed for the polyacetylene soliton-anti-soliton system in the previous Section.

The second quantized Dirac field now has the form

$$\psi(x,t) = \psi_+(x)e^{-iE_+t}a + \psi_+^*(x)e^{iE_+t}b^\dagger + \dots \tag{24}$$

When L is large, one can consider a second set of almost stationary states which are the superpositions

$$\psi_0(x) = \frac{1}{\sqrt{2}}\left(e^{iE_0t}\psi_+ + e^{-iE_0t}\psi_-\right) \tag{25}$$

$$\approx \sqrt{2\phi_0}\begin{cases} \begin{pmatrix}\cos E_0t\\ 0\end{pmatrix} e^{-\phi_0 x} + \mathcal{O}(e^{-\phi_0 L}) & x < 0 \\ \begin{pmatrix}\cos E_0t\\ 0\end{pmatrix} e^{-\phi_0 x} + \begin{pmatrix}0\\ \sin E_0t\end{pmatrix} e^{\phi_0(x-L)} + \mathcal{O}(e^{-\phi_0 L}) & 0 < x < L \\ \begin{pmatrix}0\\ \sin E_0t\end{pmatrix} e^{\phi_0(L-x)} + \mathcal{O}(e^{-\phi_0 L}) & L < x \end{cases} \tag{26}$$

which has most of its support near $x = 0$ and

$$\psi_L(x) = \frac{1}{\sqrt{2}i}\left(e^{iE_0t}\psi_+ - e^{-iE_0t}\psi_-\right) \tag{27}$$

$$\approx \sqrt{2\phi_0}\begin{cases} \begin{pmatrix}\sin E_0t\\ 0\end{pmatrix} e^{-\phi_0 x} + \mathcal{O}(e^{-\phi_0 L}) & x < 0 \\ \begin{pmatrix}\sin E_0t\\ 0\end{pmatrix} e^{-\phi_0 x} + \begin{pmatrix}0\\ -\cos E_0t\end{pmatrix} e^{\phi_0(x-L)} + \mathcal{O}(e^{-\phi_0 L}) & 0 < x < L \\ \begin{pmatrix}0\\ -\cos E_0t\end{pmatrix} e^{\phi_0(L-x)} + \mathcal{O}(e^{-\phi_0 L}) & L < x \end{cases} \tag{28}$$

which has most of its support near $x = L$.

In terms of these wave-functions, which are localized at the sites of the solitons,

$$\psi(x,t) = \psi_0(x,t)\frac{1}{\sqrt{2}}\left(a + b^\dagger\right) + \psi_L(x,t)\frac{1}{\sqrt{2}i}\left(-a + b^\dagger\right) + \dots \tag{29}$$

We could now consider the creation and annihilation operators

$$\alpha = \frac{1}{\sqrt{2}}\left(a + b^\dagger\right) \quad , \quad \alpha^\dagger = \frac{1}{\sqrt{2}}\left(a^\dagger + b\right)$$

$$\beta = \frac{1}{\sqrt{2}i}\left(a^\dagger - b\right) \quad , \quad \beta^\dagger = \frac{1}{\sqrt{2}i}\left(-a + b^\dagger\right)$$

By interacting with the system at $x = 0$, we could as well be dropping the fermion into the state ψ_0, which is localized there and which has exponentially vanishing probability of occurring at $x = L$ (until $\sin E_0 t$ becomes appreciable, which is just the usual estimate of tunnelling time through a barrier of height ϕ_0 and width L).

It might seem bizarre that, if we begin with the system in its ground state when L is small, then adiabatically increase L that we would not simply end up with the original ground state that has $\psi_-(x)$ populated, $\psi_+(x)$ empty. In fact, this is a possibility. However, as we have argued in the polyacetylene example in the previous Section, as $L \to \infty$, the result is an entangled state of (appropriately defined [30, 31]) fermion number. If we begin with the original ground state, measurement of the fermion number which is localized in the vicinity of one of the solitons will collapse the wave-function to one where the fermion, rather than occupying the negative energy state ψ_-, occupies either the state ψ_0 or the state ψ_L which are localized at $x = 0$ or $x = L$, respectively. As seen from the vicinity of each soliton, these are identical to the Jackiw-Rebbi states [13] of the fermion in a single soliton background, which have fermion number $\pm\frac{1}{2}$. These states are time-dependent, but again, just as in the polyacetylene example, the time scale for charge fluctuations is just the tunnelling time for a particle to go between the locations of the solitons.

What about teleportation? Now, our dumping a fermion into the bound state, if performed near $x = 0$ would populate the state $\psi_0(x)$, rather than $\psi_+(x)$, as all local operators would couple only to this state. It would have appreciable probability of appearing at $x = L$ only after a time over order $E_0^{-1} \sim \phi_0^{-1} e^{\phi_0 L}$.

The situation is somewhat different if we assume that the fermion is a Majorana fermion. The Hamiltonian of a Majorana fermion must have a symmetry which maps positive energy states onto negative energy states. In the case of (19), we have $\psi_{-E}(x) = \psi_E^*(x)$. Then, a fermion and an anti-fermion have the same spectrum, and we can identify them as the same particle.

Now, for the Majorana fermion, the pair of wave-functions $\psi_+(x)$ and $\psi_-(x)$ correspond to the same quantum state which can be either occupied or empty. (We can arbitrarily assign fermion parities $(-1)^F = -1$ for the unoccupied state and $(-1)^F = 1$ for the occupied state, although $+i$ and $-i$ might be more symmetric). In this case, the states ψ_0 and ψ_L are wave-functions for superpositions of the occupied and unoccupied states – they do not have definite fermion parity.

If we begin with the system where the quantum state is an eigenstate of fermion parity and we by some process dump a fermion into the bound state near $x = 0$, its wave-function automatically has a second peak at $x = L$ and it could in principle be extracted there. This defines what we mean by "teleportation".

If we concentrate on the region near $x = 0$ and we are unaware of the region near $x = L$, depending on the quantization, this teleportation will appear as either violation of conservation of fermion number mod 2 or the existence of a hidden variable in the local theory.

5. P-Wave Superconductor Model and Andreev States

Of course, the fermions in polyacetylene are not Majorana, they are electrons with complex wave-functions. The place to look for emergent Majorana fermions in nature is in superconductivity. Here we shall formulate a model whose basic excitations are Majorana fermions. We will do this by using contact with a p-wave superconductor to violate the conservation of total charge, leaving behind conservation of charge modulo 2. In such an environment, the real and imaginary parts of the electron can have different dynamics and the electron is essentially split into two Majorana fermions. They can further be coupled to soliton-like objects, in this case the boundaries of the space, in such a way that only one of the Majorana fermions has zero modes. Then, the scenario that we have been looking for, an isolated single-particle state, can be found.

In these materials, mid-gap bound states, called Andreev states, are a common occurrence. They typically live at surface of the superconductor [52]. In our case, these will be Majorana zero modes.

Majorana zero modes of the type that we are discussing are also known to be bound to vortices in p-wave superconductors where they have the remarkable effect of giving vortices non-Abelian fractional statistics [53],[54]. For concreteness we will consider a slightly simpler model one-dimensional model that was originally discussed by Kitaev [2] in the context of fermionic quantum computation.

We shall consider a quantum wire embedded in a bulk P-wave superconductor as is depicted in Fig. 11.

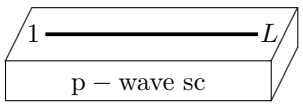


Fig. 11 A quantum wire embedded in a bulk P-wave superconductor.

We shall assume that the wire has a single channel. We shall also assume that the dynamics of electrons in the wire are adequately described by a one-dimensional tight-binding model. We will ignore the spin degree of freedom of the electron. The phenomenon that we will find is to a first approximation spin-independent.

We will assume that the coupling to the neighboring p-wave superconductor is weak and its net effect is to give electrons the possibility of entering and leaving the wire in

pairs by creating or destroying a p-wave cooper pair in the bulk. To describe the electrons, we will use the Hamiltonian

$$H = \sum_{n=1}^{L} \left(\frac{t}{2} a_{n+1}^{\dagger} a_n + \frac{t^*}{2} a_n^{\dagger} a_{n+1} + \frac{\Delta}{2} a_{n+1}^{\dagger} a_n^{\dagger} + \frac{\Delta^*}{2} a_n a_{n+1} + \mu a_n^{\dagger} a_n \right) \tag{30}$$

Sites on the quantum wire are labelled by $n = 1, 2, ..., L$. The operators a_n and $a_n^{\dagger}$ annihilate and create an electron at site n. They obey the anti-commutator algebra

$$\left\{ a_n, a_{n'}^{\dagger} \right\} = \delta_{nn'} \tag{31}$$

The first terms in the Hamiltonian, with coefficients t and t^* are the contribution to the energy of the hopping of electrons between neighboring sites. The second pair of terms, with Δ and Δ^*, arise from the presence of the super-conducting environment. They describe the amplitude for a pair of electrons to leave or enter the wire from the environment. It is assumed that they can do this as a Cooper pair when they are located on neighboring sites. This is effectively an assumption about the size and coherence of the cooper pairs in the superconductor. Even if it were not accurate, the smaller next-to-nearest neighbor, etc. terms that would arise could be taken into account and would not change our result significantly. The last term is the chemical potential, the energy of an electron sitting on a site of the wire. We shall assume a reasonable hierarchy of the parameters, that the amplitude for hopping along the wire is somewhat larger than hopping to and from the bulk, $|t| > |\Delta|$, and that the chemical potential is close enough to zero that the electron band has substantial filling, $|\mu| < |t|$.

5.1 Spectrum of Single-Particle States

Let us discuss the spectrum of the single-particle states in the many-body theory described by the Hamiltonian (30). If $t = |t|e^{i\phi}$ and $\Delta = |\Delta|e^{2i\theta}$, by redefining $a_n \to e^{i(\phi+\theta)} a_n$ for n odd and $a_n \to e^{i(\phi-\theta)} a_n$ for n even, we remove the complex phases of t and Δ, which we can henceforth assume to be positive real numbers. The equation of motion for the fermion wave-function is gotten by taking the commutator of its operator a_k with the Hamiltonian (30),

$$i\hbar \dot{a}_n = [a_n, H]$$

for which we get

$$i\hbar \frac{d}{dt} a_n = \frac{t}{2} \left(a_{n+1} + a_{n-1} \right) - \frac{\Delta}{2} \left(a_{n+1}^{\dagger} - a_{n-1}^{\dagger} \right) + \mu a_n \tag{32}$$

for the sites $n = 2, ..., L-1$.

Because we are using open boundary conditions – the chain simply ends at $n = 1$ and $n = L$, the equations for $\frac{d}{dt} a_1$ and $\frac{d}{dt} a_L$ differ from (32) by missing terms. When we solve (32) as a wave equation, it will be convenient to deal with this by extending the chain

by one site in each direction and then eliminating the extra sites by imposing Dirichlet boundary conditions,

$$a_0(t) = 0 \quad , \quad a_{L+1}(t) = 0$$

With these conditions, (32) describes the dynamics for all $n = 1, 2, ..., L$.

Now, it is most efficient to decompose the electron into real and imaginary parts, $a_n = b_n + ic_n$, and assemble them into a spinor

$$\psi_n = \begin{pmatrix} b_n \\ c_n \end{pmatrix} \tag{33}$$

Note that this spinor obeys the Majorana condition

$$\psi_n = \psi_n^* \tag{34}$$

The equation for the wave-function is

$$\begin{pmatrix} \mu & \hbar\frac{d}{dt} \\ -\hbar\frac{d}{dt} & \mu \end{pmatrix} \psi_n + \begin{pmatrix} \frac{1}{2}(t-\Delta) & 0 \\ 0 & \frac{1}{2}(t+\Delta) \end{pmatrix} \psi_{n+1} + \begin{pmatrix} \frac{1}{2}(t+\Delta) & 0 \\ 0 & \frac{1}{2}(t-\Delta) \end{pmatrix} \psi_{n-1} = 0 \tag{35}$$

In order to solve the equation, we will make the ansatz

$$\psi_n(t) = e^{-i\omega t/\hbar}\psi_n(\omega) \tag{36}$$

The Majorana condition for energy eigenstates is

$$\psi_n^*(\omega) = \psi_n(-\omega)$$

We will normalize the wave-functions with the condition

$$\sum_{n=1}^{L} |\psi_n(\omega)|^2 \;=\; 1 \;=\; \sum_{n=1}^{L} \left(|b_n(\omega)|^2 + |c_n(\omega)|^2\right)$$

Since the equation and boundary conditions are linear, we can further make the ansatz that the wave-functions are superpositions of plane waves,

$$\psi_n(\omega) = \zeta^n \begin{pmatrix} u(\zeta) \\ v(\zeta) \end{pmatrix} \tag{37}$$

Then, the difference equation (35) becomes

The equation for the wave-function is

$$\begin{pmatrix} \frac{1}{2}t(\zeta + 1/\zeta) - \frac{1}{2}\Delta(\zeta - 1/\zeta) + \mu & -i\omega \\ i\omega & \frac{1}{2}t(\zeta + 1/\zeta) - \frac{1}{2}\Delta(\zeta - 1/\zeta) + \mu \end{pmatrix} \begin{pmatrix} u(\zeta) \\ v(\zeta) \end{pmatrix} = 0 \tag{38}$$

which has a solution when the frequencies obey the dispersion relation

$$\omega^2 = \left[\frac{1}{2}t(\zeta + 1/\zeta) + \mu\right]^2 - \left[\frac{1}{2}\Delta(\zeta - 1/\zeta)\right]^2 \tag{39}$$

For a given real value of ω, there are generally four wave-vectors which satisfy this dispersion relation,

$$\zeta_\omega \ , \ 1/\zeta_\omega \ , \ \zeta_\omega^* \ , \ 1/\zeta_\omega^*$$

To find a solution of the wave equation, we must take superpositions of the four solutions of (38) with each of these four wave-vectors. Then we must adjust the four coefficients of the superposition in order to satisfy the four boundary conditions. (Remember that the boundary conditions are for spinors, so there are four boundary conditions in total.) Three of the boundary conditions can be solved by adjusting the coefficients in the superposition. The fourth superposition coefficient can eventually be determined up to phases by normalizing the wave-function. The fourth boundary condition, which has yet to be satisfied, then gives a condition that the wave-vector must obey. Plugging the resulting wave-vector back into the dispersion relation (39) then gives the allowed energy eigenvalue. This gives an algorithm for finding the energies, the allowed wave-vectors (which are $\frac{1}{i}\ln\zeta$ and are generally complex) and the wave-functions.

When L is large, the solutions are of two kinds. One are to a good approximation continuum states, where $\zeta = e^{ik}$ and the continuum spectrum is

$$\omega(k) = \pm\sqrt{\left[t\cos k + \mu\right]^2 + \Delta^2 \sin^2 k}$$

with $k \in (-\pi, \pi]$ (it is quantized approximately as $k = 2\pi \cdot \text{integer}/(L+1)$ which becomes a continuum when $L \to \infty$). This spectrum has an energy gap. The point of closest approach of the positive and negative energy bands occurs when $\cos k = -t\mu/(t^2 - \Delta^2)$ and the gap is $E_{\text{gap}} = 2\Delta\sqrt{\frac{t^2-\Delta^2-\mu^2}{t^2-\Delta^2}}$. We will assume that this gap is significant, so that the mid-gap states that we will discuss next are indeed well isolated.

The other modes in the spectrum are a pair of mid-gap states. When L is large, these states have energies that are exponentially small in L, one is positive, one is negative and they have equal magnitudes. In the following, we will solve for the spectrum of these mid-gap states in the approximation where effects that are exponentially small in L are neglected.

We begin with an un-normalized spinor

$$\zeta^n \begin{pmatrix} i\omega \\ \frac{1}{2}t(\zeta + 1/\zeta) - \frac{1}{2}\Delta(\zeta - 1/\zeta) + \mu \end{pmatrix} + A\zeta^{-n} \begin{pmatrix} i\omega \\ \frac{1}{2}t(\zeta + 1/\zeta) + \frac{1}{2}\Delta(\zeta - 1/\zeta) + \mu \end{pmatrix}$$
$$+B\zeta^{*n} \begin{pmatrix} i\omega \\ \frac{1}{2}t(\zeta^* + 1/\zeta^*) - \frac{1}{2}\Delta(\zeta^* - 1/\zeta^*) + \mu \end{pmatrix} + C\zeta^{*-n} \begin{pmatrix} i\omega \\ \frac{1}{2}t(\zeta^* + 1/\zeta^*) + \frac{1}{2}\Delta(\zeta^* - 1/\zeta^*) + \mu \end{pmatrix} \tag{40}$$

We will solve the boundary condition for the mid-gap state in the limit where L is large. There, we expect the solution to be very close to $\omega = 0$, for which we then need a wave-vector which solves $t(\zeta + 1/\zeta) + 2\mu = -\Delta(\zeta - 1/\zeta)$. Then, to a first approximation, the terms with A and C are absent from (40) and we must choose the $B = -(\zeta - 1/\zeta)/(\zeta^* - 1/\zeta^*)$ in order to satisfy the boundary condition at $n = 0$. Since

$$\zeta = -\frac{\mu}{2(t+\Delta)} + i\sqrt{\frac{t-\Delta}{t+\Delta}}\sqrt{1-\mu^2/4(t^2-\Delta^2)}$$

so that $\zeta\zeta^* = \frac{t-\Delta}{t+\Delta} < 1$, this gives a wave-function which is maximal at $n = 1$ and which decays exponentially as n increases. This would indeed be the solution for the mid-gap state on the half-line when $L \to \infty$. When L is finite, rather than infinite, in order to satisfy the boundary condition at $n = L+1$ we must include an amplitude for the growing solution. It can be obtained from the decaying one by simply replacing n by $L + 1 - n$ and multiplying the spinor by σ^2. Thus, to a good approximation the mid-gap solution is

$$\psi_n^+ = \sqrt{\frac{\Delta}{2t}\frac{t^2-\mu^2}{t^2-\Delta^2-\mu^2}}\left[\frac{\left(-\mu+i\sqrt{t^2-\Delta^2-\mu^2}\right)^n - \left(-\mu-i\sqrt{t^2-\Delta^2-\mu^2}\right)^n}{(t+\Delta)^n}\begin{pmatrix}0\\ i\end{pmatrix} + \right.$$
$$\left. + \frac{\left(-\mu+i\sqrt{t^2-\Delta^2-\mu^2}\right)^{L+1-n} - \left(-\mu-i\sqrt{t^2-\Delta^2-\mu^2}\right)^{L+1-n}}{(t+\Delta)^{L+1-n}}\right]\begin{pmatrix}1\\ 0\end{pmatrix}$$

This wave-function has infinitesimal positive energy. The wave-function with infinitesimal negative energy is given by

$$\psi_n^- = \sqrt{\frac{\Delta}{2t}\frac{t^2-\mu^2}{t^2-\Delta^2-\mu^2}}\left[\frac{\left(-\mu+i\sqrt{t^2-\Delta^2-\mu^2}\right)^n - \left(-\mu-i\sqrt{t^2-\Delta^2-\mu^2}\right)^n}{(t+\Delta)^n}\begin{pmatrix}0\\ i\end{pmatrix} - \right.$$
$$\left. - \frac{\left(-\mu+i\sqrt{t^2-\Delta^2-\mu^2}\right)^{L+1-n} - \left(-\mu-i\sqrt{t^2-\Delta^2-\mu^2}\right)^{L+1-n}}{(t+\Delta)^{L+1-n}}\right]\begin{pmatrix}1\\ 0\end{pmatrix}$$

We will abbreviate these by naming the function

$$\phi_n = i\sqrt{\frac{\Delta}{2t}\frac{t^2-\mu^2}{t^2-\Delta^2-\mu^2}}\frac{\left(-\mu+i\sqrt{t^2-\Delta^2-\mu^2}\right)^n - \left(-\mu-i\sqrt{t^2-\Delta^2-\mu^2}\right)^n}{(t+\Delta)^n} \tag{41}$$

where $\phi_n = \phi_n^*$ and we have normalized to

$$\sum_n |\phi_n|^2 = \frac{1}{2} \tag{42}$$

The function ϕ_n has maximum magnitude at $n = 1$ and it decays exponentially as n increases. We shall use the notation

$$\psi_n^+ = \phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \phi_{L+1-n} \begin{pmatrix} i \\ 0 \end{pmatrix} \tag{43}$$

$$\psi_n^- = \phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \phi_{L+1-n} \begin{pmatrix} i \\ 0 \end{pmatrix} \tag{44}$$

We have normalized the spinors so that

$$\sum_n \psi_n^{\pm\dagger} \psi_n^{\pm} = 1 \tag{45}$$

Note that, these wave-functions satisfy the Majorana condition $\psi_n^- = \psi_n^{+*}$. As expected, they have support near $n = 1$ and $n = L$ and are exponentially small in the interior of the quantum wire, far from the boundaries. Further, we have adjusted phases so that the wave-functions are identical in profile in the region near $n = 1$. Then, we expect that they differ in sign in the region near $n = L$ and we confirm from that above that this is so. Also, note that they are complex. To form the real, Majorana spinor, we must superpose them with a creation and annihilation operator,

$$\psi_n(t) = \psi_n^+ e^{-i\omega t} a + \psi_n^- e^{i\omega t} a^\dagger + \text{ non} - \text{zero energy states} \tag{46}$$

Here a and $a^\dagger$ are the annihilation and creation operators for the mid-gap state and ω is their exponentially small energy. Ignoring the energy, we can also write this operator as

$$\psi_n(t) = \phi_n \begin{pmatrix} 0 \\ 1 \end{pmatrix} (a + a^\dagger) + \phi_{L+1-n} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{i}(a - a^\dagger) + \ldots$$

The first term on the right-hand side has support near $n = 0$ and decays exponentially as n increases from 1. The second term has support near $n = L$ and decays exponentially as n decreases from L. They each multiply the operators $\alpha = \frac{1}{\sqrt{2}}\left((a + a^\dagger\right)$ and $\beta = \frac{1}{\sqrt{2}i}\left(a - a^\dagger\right)$, respectively. These are analogous to the operators which we introduced on Section 1. a and $a^\dagger$ must have the anti-commutator

$$\left\{a, a^\dagger\right\} = 1$$

which has a two-dimensional representation, the states $|- >$ and $|+ >$ of Section 1 which we copy here for the reader's convenience,

$$a|- >= 0 \quad , \quad a^\dagger|- >= |+ >$$

$$a|+ >= |- > \quad , \quad a^\dagger|+ >= 0$$

All other excited states of the system are created by operating creation operators for the other, non-zero energy excitations. Remember that it is the states $|+ >$ and $|- >$ which we expect to be eigenstates of fermion parity, $(-1)^F$.

5.2 Second Quantized Electron Operator

Now, we recall that the upper and lower components of the spinor $\psi_n(t)$ that we discussed in the previous subsection are simply the real and imaginary components of the electron field operator, which we can now reconstruct,

$$a_n(t) = \phi_{L+1-n} \frac{1}{i} \left(a - a^\dagger\right) + i\phi_n \left(a + a^\dagger\right) + \dots \tag{47}$$

This is now a complex operator, but its real and imaginary parts have support at opposite ends of the quantum wire. The part of the operator which has not been written, and is indicated by dots in (47), are superpositions of creation and annihilation operators for continuum states. All such states have energies above the gap and extended, plane-wave-like wave-functions. Note that now that the phase symmetry of the system has been broken by coupling to the superconductor, the real and imaginary parts of the electron operator will generally have different properties. This interesting fact will not concern us in the following and we will focus on the mid-gap, or zero mode part of the electron operator.

Note, now, if we operate with any local operator in the vicinity of $n = 1$, the electon operator acts as if it were composed of the combination of zero mode operators $(a + a^\dagger)$. As we have discussed before, this operator squares to a constant. There cannot be any states that it annihilates. Thus, operating it on any state of the system, in the region where the zero mode wavefunction has support, will have an effect. What it does is flip the state from $|- >$ to $|+ >$. Since it is a hermitian operator, it is possible to diagonalize it, the states $\frac{1}{\sqrt{2}}(|- > + |+ >)$ and $\frac{1}{\sqrt{2}}(|- > + |+ >)$ are its eigenvectors. However, these eigenvectors are not eigenstates of fermion parity.

6. Long Ranged Correlations of Electrons

What about teleportation? Let us imagine that we begin with the system in one of its ground states, say $|- >$ and inject an electron so that at time $T = 0$ it is resting at site #1. This means, we being with the state $a_1^\dagger|- >$, where, as we recall, $a_1^\dagger$ is the creation operator for an electron at site #1.

We then ask what is the quantum transition amplitude for the transition, after a time T has elapsed, of this state to one with the electron located at position $\#L$. The final quantum state is $a_L^\dagger|- >$. The amplitude is given by

$$\mathcal{A} =< -|\ a_L\ e^{iHT}\ a_1^\dagger\ |- >= |\phi_1^0|^2 + (T \text{ and } L\text{−dependent}) \tag{48}$$

The T- and L-dependent parts of this matrix element represent the usual propagation via excited quasi-electrons which must travel across the wire. The first term is non-zero and is T and L-independent. By 'teleportation', we are referring to this part of the amplitude. Here, we can evaluate the amplitude explicitly. It is

$$\mathcal{A}_{\text{Tel}} = \left(\frac{2\Delta}{t}\right)\left(\frac{t^2 - \Delta^2 - \mu^2}{(t + \Delta)^2}\right) \tag{49}$$

Which can be appreciable, in the 10-30 percent range, for a surprisingly wide choice of parameters.

However, the teleportation probability is the square of this amplitude, which is somewhat smaller. We could ask a more sophisticated question: What is the probability that the electron, once injected at $n = 1$ appears anywhere within the exponential range of the zero mode wave-function at $n = L$. This probability would be given by

$$\mathcal{P}_{\rm Tel} = \sum_n |\phi_n|^2 |\phi_1^0|^2 = \frac{1}{2}\left(\frac{2\Delta}{t}\right)\left(\frac{t^2 - \Delta^2 - \mu^2}{(t+\Delta)^2}\right) \tag{50}$$

This is what we shall call the "teleportation probability". Again, for a range of parameters t, Δ and μ, it can be appreciable.

7. Discussion

The apparently instantaneous propagation of an electron would seem to be a potential violation of Einstein causality, since in principle a message could be sent at a speed faster than that of light.

Let us review the nature of the system that we have constructed. Once the quantum wire - p-wave superconductor system is prepared, the extended Majorana state of the electron is already there, ready for use. The system has a two-fold degeneracy: at low energy, there are two states $|- >$ and $|+ >$. These are not normal quantum states in that they differ by a quantum number which we would like to preserve, fermion parity $(-1)^F$.

Thus, if we do not allow superpositions of these states, this is effectively a classical bit, like a classical switch that can either be OFF or ON, the wave-function can be in one state or the other.

The system moves from OFF to ON by absorbing or emitting an electron in a way that flips the vacuum from one state to the other. This should occur somewhere in the vicinity of the ends of the wire, where the zero mode wave-functions have support. It can move back from ON to OFF by the identical process, again absorbing or emitting a single electron.

This leads to the rather drastic conclusion that there could be super-luminal transfer of information in this system. One would need only to prepare the system in one of its ground states, with a sender sitting at 1 and a receiver sitting at L. Either ground state is sufficient and neither the sender nor the observer needs to know which it is. All the receiver has to do is wait for an electron to arrive. If it arrives with energy at or above the electron energy gap, he or she can conclude that it propagated normally and was sent at some time in the past. However, if it arrives at very low energy, he or she knows that it tunnelled and that it was sent by the sender at that instant. This is seemingly an instant transfer of information over a finite distance.

There is a obvious way out of this, but it means giving up the fermion parity symmetry that has until now been sacred. If we allow superpositions of the states $|- >$ and

$|+>$ which have even and odd fermion number, then the degenerate ground states are a quantum rather than classical two-level system, there are two states and any superpositions are allowed. Now, in this system, it is easy to prepare states where an electron can spontaneously appear or disappear. Take, for example an eigenstate of the operators that we called α and β. In their eigenstates, $\frac{1}{\sqrt{2}}(|- > \pm|+ >)$, the electron operator has an expectation value $< a_n(t) >= \pm\phi_n \pm \phi_{L+1-n}$. It would thus have an amplitude for simply vanishing or appearing spontaneously.

Then, when the observer at L detects the arrival of a low energy electron he or she cannot distinguish one which was sent from the other side of the wire from one which is spontaneously created. This restores Einstein causality at the expense of our having to admit states onto physics which are not eigenstates of fermion parity. There is the further question of whether such states are consistent with three dimensional physics.

Fermion number mod 2 is an important conservation law in three dimensional physics [55]. Even though the quantum wire that we have discussed is one-dimensional, it is embedded in three dimensional space and the electrons that we are discussing are spin-$\frac{1}{2}$ particles in three dimensional space. This means that their wave-functions individually change sign under a rotation by 2π. More importantly, a state with odd fermion number must change sign under a rotation by 2π whereas a state with even number should remain unchanged. A rotation by angle 2π should not affect physics. Thus, the relative sign of even and odd fermion number states should not have any physical consequences.

If we did allow a superposition of the two states, they would form a single qubit. We could parameterize the state-vector by a point on the Bloch sphere (θ, ϕ) where the state is

$$|\theta, \varphi > \;\; = \;\; \cos\frac{\theta}{2}|- > + e^{i\varphi}\sin\frac{\theta}{2}|+ > \tag{51}$$

Points on the two-dimensional unit sphere are specified by the unit vectors

$$\hat{n} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$$

and $0 \leq \theta \leq \pi$, $-\pi < \varphi \leq \pi$. However, as we have argued, the relative sign of the two states should not be an observable. Then the set of "physical states" of the qubit would be the Bloch sphere with a further identification

$$\varphi \sim \varphi + \pi \tag{52}$$

Of course, this identification is allowed only if there are no experiments, even in principle, which could measure the relative sign of the two states in the superposition. Normally, one could measure that sign by an interference experiment.

For example, we could attempt to observe the relative sign by examining interference between the electron which arrives by tunnelling and the one which arrives by conventional transport. However, the teleportation amplitude in the state ϕ

$$< \theta, \varphi|\; a_L\; e^{iHT}\; a_1^\dagger\; |\theta, \varphi >= \cos\theta[\text{teleportation}] + [\text{transport}] \tag{53}$$

The teleportation amplitude is diminished by a factor of $\cos\theta$ whereas the transport amplitude is unchanged. One can make the teleportation amplitude vanish by adjusting

$\theta = \pi/2$. However, the relative amplitude cannot be used to measure the relative sign of the two components of the wave-function.

There is an amplitude for an electron to vanish,

$$< \theta, \varphi| \; e^{iHT} \; a_1^\dagger \; |\theta, \varphi > \sim i \sin\theta \cos\varphi \;\; \cdot \phi_1 \tag{54}$$

and to appear spontaneously

$$< \theta, \varphi| \; a_L \; e^{iHT} \;\; |\theta, \varphi > \sim \sin\theta \sin\varphi \;\; \cdot \phi_1 \tag{55}$$

As we expect, the latter two amplitudes change sign when we put $\varphi \to \varphi + \pi$. Actual quantum observables are probabilities which are the modulus squares of amplitudes. They are also insensitive to the relative sign of the two parts of the wave-function.

The above probability amplitudes do not offer a way to distinguish the quantum states with φ and $\varphi + \pi$. At this point, we have not ruled out, but also we have not devised an experiment by which they could be distinguished. Indeed, if there is no such experiment, we are free to cut the Bloch sphere in half by the identification (52) and the ground states would form this peculiar qubit. Teleportation still happens, but so does the spontaneous disappearance or appearance of a single electron and the contradiction with Einstein locality is removed.

We cannot exclude the possibility that the effect that we have been discussing could be interfered with by the superconductor which the quantum wire is in contact with. Here, we have assumed that it acts as a simple bath which supplies and absorbs Cooper pairs but is otherwise innocuous. We cannot rule out that it also has exotic states that should be included in the picture.

Acknowledgements

G.S. thanks Duncan Haldane, Tony Leggett, Girard Milburn and Bill Unruh for discussions and acknowledges the hospitality of the I.H.E.S., Bures-sur-Yvette, the Werner Heisenberg Institute, Munich and the University of Perugia where parts of this work were completed. P.S. thanks M.Rasetti, P. Zanardi and V. Barone for discussions and acknowledges the Pacific Institute for Theoretical Physics, Vancouver and the Center for Theoretical Physics at MIT for hospitality. This work is supported in part by NSERC of Canada and by M.I.U.R. National Project "JOSNET" (grant n.2004027555).

References

[1] E. Majorana, Nuovo Cimento, Vol.5, 171 (1937).

[2] Alexei Kitaev, "Unpaired Majorana fermions in quantum wires" [arXiv:cond-mat/0010440]

[3] L. S. Levitov, T. P. Orlando, J. B. Majer and J. E. Mooij, "Quantum spin chains and Majorana states in arrays of coupled qubits", [arXiv:cond-mat/0108266]

[4] U. Dorner, P. Fedichev, D. Jaksch, K. Lewenstein and P. Zoller, "Entangling strings of neutral atoms in 1D atomic pipeline structures" [arXiv.org:quant-ph/0212039]

[5] Gordon W. Semenoff and P. Sodano, " Teleportation by a Majorana Medium" [arXiv:cond-mat/0601261].

[6] H. B. Nielsen and M. Ninomiya, "Dirac sea for bosons," arXiv:hep-th/9808108.

[7] Y. Habara, H. B. Nielsen and M. Ninomiya, "Boson sea versus Dirac sea: General formulation of boson sea through Int. J. Mod. Phys. A **19**, 5561 (2004) [arXiv:hep-th/0312302].

[8] H. B. Nielsen and M. Ninomiya, "Dirac sea for bosons. I: Formulation of negative energy sea for bosons," Prog. Theor. Phys. **113**, 603 (2005) [arXiv:hep-th/0410216].

[9] H. B. Nielsen and M. Ninomiya, "Dirac sea for bosons. II: Study of the naive vacuum theory for the Prog. Theor. Phys. **113**, 625 (2005) [arXiv:hep-th/0410218].

[10] Y. Habara, H. B. Nielsen and M. Ninomiya, "Negative energy (Dirac-like) sea for bosons and supersymmetry," arXiv:hep-th/0501095.

[11] Y. Habara, H. B. Nielsen and M. Ninomiya, "Hole theory of boson sea," arXiv:hep-th/0504173.

[12] Y.Maeno, T.M. Rice, M. Sigrist, Physics Today, **54**, 42 (2001).

[13] R. Jackiw and C. Rebbi, "Solitons With Fermion Number 1/2," Phys. Rev. D **13**, 3398 (1976).

[14] For a review, see A. J. Niemi and G. W. Semenoff, "Fermion Number Fractionization In Quantum Field Theory," Phys. Rept. **135**, 99 (1986) and for material closely related to the present paper, A. J. Niemi and G. W. Semenoff, "Spectral Asymmetry On An Open Space," Phys. Rev. D **30**, 809 (1984).

[15] A. Losev, M. A. Shifman and A. I. Vainshtein, "Single state supermultiplet in 1+1 dimensions," New J. Phys. **4**, 21 (2002) [arXiv:hep-th/0011027].

[16] A. Losev, M. A. Shifman and A. I. Vainshtein, "Counting supershort supermultiplets," Phys. Lett. B **522**, 327 (2001) [arXiv:hep-th/0108153].

[17] G. D. Starkman, D. Stojkovic and T. Vachaspati, "Neutrino zero modes on electroweak strings," Phys. Rev. D **63**, 085011 (2001) [arXiv:hep-ph/0007071].

[18] G. Starkman, D. Stojkovic and T. Vachaspati, "Zero modes of fermions with a general mass matrix," Phys. Rev. D **65**, 065003 (2002) [arXiv:hep-th/0103039].

[19] D. Stojkovic, "Fermionic zero modes on domain walls," Phys. Rev. D **63**, 025010 (2001) [arXiv:hep-ph/0007343].

[20] D. Stojkovic, "Neutrino zero modes and stability of electroweak strings," Int. J. Mod. Phys. A **16S1C**, 1034 (2001) [arXiv:hep-th/0103216].

[21] R. F. Streater and A. S. Wightman, "Pct, Spin And Statistics, And All That," Redwood City, USA: Addison-Wesley (1989) (Advanced book classics).

[22] W. P. Su, J. R. Schrieffer and A. J. Heeger, "Solitons In Polyacetylene," Phys. Rev. Lett. **42** (1979) 1698.

[23] M. J. Rice, Phys. Lett. A71, 152 (1979.

[24] S. A. Brazovskii, JETP Lett. 28, 656 (1978).

[25] B. Horovitz and J. A. Krumhansl, Solid State Commun. 26, 81 (1978),

[26] H. Shirakawa, T. Ito and S. Ikeda, Die Macromol. Chem. 197, 1565 (1978).

[27] H. Yakayama, Y. Lin-Liu and K. Maki, Phys. Rev. B21, 2388 (1980).

[28] R. Jackiw and J. R. Schrieffer, "Solitons With Fermion Number 1/2 In Condensed Matter And Relativistic Field Theories," Nucl. Phys. B **190**, 253 (1981).

[29] A. Heeger, Comments Solid State Phys. 10, 53 (1981).

[30] R. Rajaraman and J. S. Bell, "On Solitons With Half Integral Charge," Phys. Lett. B **116**, 151 (1982).

[31] R. Jackiw, A. K. Kerman, I. R. Klebanov and G. W. Semenoff, "Fluctuations Of Fractional Charge In Soliton Anti-Soliton Systems," Nucl. Phys. B **225**, 233 (1983).

[32] J. Ranft and A. Schiller, "Lattice Study Of Solitons With Half Integral Fermionic Charge In A (1+1)-Dimensional Field Theoretic Model," Phys. Lett. B **140**, 239 (1984).

[33] A. Z. Capri, R. Ferrari and L. E. Picasso, "Charge Of The Vacuum In (1+1)-Dimensional Fermion Theories With External Fields," Phys. Rev. D **30**, 2136 (1984).

[34] R. Blankenbecler and D. Boyanovsky, "Fractional Charge And Spectral Asymmetry In 1-D: A Closer Look," Phys. Rev. D **31**, 2089 (1985).

[35] L. Brown and M. Bernstein, "Fractional Charge In The Infinite Volume Limit," UW-ER-40048-36 P4

[36] H. Grosse, K. R. Ito and Y. M. Park, "Statistical Mechanics Of Polyacetylene (Ch)-X. 1," Nucl. Phys. B **270**, 379 (1986).

[37] A. S. Goldhaber, "Does Natural Duplexity Conceal Half Integral Fermion Number?," Phys. Rev. D **33**, 3697 (1986).

[38] M. B. Paranjape, "(1+1)-Dimension Axial Anomaly, Berry's Phase And Induced Charge, Statistics And Wess-Zumino Term," Phys. Rev. D **40**, 540 (1989).

[39] A. S. Goldhaber, "When Solitons Can't Halve Fermions," ITP-SB-89-61

[40] A. S. Goldhaber and S. A. Kivelson, "Charge!," Phys. Lett. B **255**, 445 (1991).

[41] A. S. Goldhaber, "Hairs on the unicorn: Fine structure of monopoles and other solitons," arXiv:hep-th/9712190.

[42] A. S. Goldhaber, "First are last fractional-charge solitons," arXiv:hep-th/9712192.

[43] G. V. Dunne and K. Rao, "Thermal fluctuations of induced fermion number," Phys. Rev. D **64**, 025003 (2001) [arXiv:hep-th/0103061].

[44] R. Rajaraman, "Fractional charge," arXiv:cond-mat/0103366.

[45] R. Jackiw and A. Shimony, "The depth and breadth of John Bell's physics," Phys. Perspect. **4**, 78 (2004) [arXiv:physics/0105046].

[46] G. V. Dunne, "Finite temperature induced fermion number," Nucl. Phys. Proc. Suppl. **108**, 155 (2002) [Nucl. Phys. Proc. Suppl. **109A**, 168 (2002 IMPAE,A17,890-897.2002)] [arXiv:hep-th/0112030].

[47] J. Javanainen and J. Ruostekoski, "Optical detection of fractional particle number in an atomic Fermi-Dirac gas," arXiv:cond-mat/0308113.

[48] R. Jackiw, C. Rebbi, J.R. Schrieffer, "Fractional Electrons in Liquid Helium?", arXiv:cond-mat/0012370.

[49] H. Maris, J. Low Temp. Phys. 120, 173 (2000); Physics Today 53, No. 11, 9 (2000); New Scientist, No. 2260, 25 (2000).

[50] For a review, see S. B. Treiman, E. Witten, R. Jackiw and B. Zumino, "Current Algebra And Anomalies," World Scientific, Singapore (1985).

[51] R. Jackiw and G. W. Semenoff, "Continuum Quantum Field Theory For A Linearly Conjugated Diatomic Polymer With Fermion Fractionization," Phys. Rev. Lett. **50**, 439 (1983).

[52] A.F. Andreev," The Thermal Conductivity of the Intermediate State in Superconductors", Sov.Phys. JETP **19**, 1228 (1964) .

[53] D. A. Ivanov, " Non Abelian Statistics of Half-Quantum Voirtices in p-Wave Superconductors", Phys.Rev.Lett. **86**, 268 (2001); A. Stern, F. von Oppen, E. Mariani, "Geometric phases and quantum entanglement as building blocks for nonabelian quasiparticle statistics", cond-mat/0310273.

[54] M. Stone and R. Roy, "Edge modes, edge currents and gauge invariance in $p_x + ip_y$ superfluids and superconductors", [arXiv:cond-mat/0308034].

[55] A. F. Andreev, "Manifestation of additional dimensions of space-time in mesoscopic systems," arXiv:hep-th/0302040.

EJTP **3, No. 10** (2006) 191–209 *Electronic Journal of Theoretical Physics*

Why do Majorana Neutrinos Run Faster than Dirac Neutrinos?

Zhi-zhong Xing * and He Zhang †

Institute of High Energy Physics, Chinese Academy of Sciences, P.O. Box 918, Beijing 100049, China

Received 15 February 2006, Published 28 May 2006

Abstract: The τ-lepton dominance in the one-loop renormalization-group equations (RGEs) of neutrinos sets a cute criterion to parametrize the 3×3 lepton flavor mixing matrix U: its elements U_{3i} (for $i = 1, 2, 3$) should be as simple as possible. Such a novel parametrization is different from the "standard" one used in the literature and can lead to greatly simplified RGEs for three mixing angles and the physical CP-violating phase(s), no matter whether neutrinos are Dirac or Majorana particles. We show that the RGEs of Dirac neutrinos are not identical with those of Majorana neutrinos even if two Majorana CP-violating phases vanish. As the latter can keep vanishing from the electroweak scale to the typical seesaw scale, it makes sense to explore the similarities and differences between the RGE running effects of Dirac and Majorana neutrinos. We conclude that Majorana neutrinos are in general expected to run faster (i.e., more significantly) than Dirac neutrinos.

Keywords: Majorana neutrinos, Dirac neutrinos, neutrino interactions
PACS (2006): 14.60.Pq, 13.15.+g, 25.30.Pt

1. Introduction

The fact that neutrinos have non-vanishing masses is a clean signal of new physics beyond the standard model (SM). To understand the small neutrino mass-squared differences and the large lepton flavor mixing angles observed in solar and atmospheric neutrino oscillation experiments [1, 2, 3, 4], many models based on either new flavor symmetries or some unspecified interactions have been proposed at some superhigh energy scales [5]. Their phenomenological consequences at low energy scales can be confronted with current experimental data, after radiative corrections to those neutrino mixing parameters are

* xingzz@ihep.ac.cn
† zhanghe@ihep.ac.cn

properly taken into account. Such radiative corrections can be very significant in some cases, for instance, when the masses of three light neutrinos are nearly degenerate or the value of $\tan\beta$ is very large in the minimal supersymmetric standard model (MSSM).

An elegant idea to explain the smallness of left-handed neutrino masses is to introduce very heavy right-handed neutrinos and lepton number violation into the SM or MSSM and to make use of the famous seesaw mechanism [6]. Below the seesaw scale, where heavy Majorana neutrinos become decoupled, the effective neutrino coupling matrix κ obeys the following one-loop renormalization-group equation (RGE) [7]:

$$16\pi^2 \frac{{\rm d}\kappa}{{\rm d}t} = \alpha_{\rm M}\kappa + C\left[\left(Y_l Y_l^\dagger\right)\kappa + \kappa\left(Y_l Y_l^\dagger\right)^T\right] \; , \tag{1}$$

where $t \equiv \ln(\mu/\Lambda_{\rm SS})$ with μ being an arbitrary renormalization scale between the electroweak scale $\Lambda_{\rm EW} \sim 10^2$ GeV and the typical seesaw scale $\Lambda_{\rm SS} \sim 10^{10\cdots 14}$ GeV, and Y_l is the charged-lepton Yukawa coupling matrix. In the SM, $C = -1.5$ and $\alpha_{\rm M} \approx -3g_2^2 + 6y_t^2 + \lambda$; and in the MSSM, $C = 1$ and $\alpha_{\rm M} \approx -1.2g_1^2 - 6g_2^2 + 6y_t^2$, where g_1 and g_2 denote the gauge couplings, y_t stands for the top-quark Yukawa coupling, and λ is the Higgs self-coupling in the SM.

There are also some good reasons to speculate that massive neutrinos might be the Dirac particles [8]. In this case, the Dirac neutrino Yukawa coupling matrix Y_ν must be extremely suppressed in magnitude, so as to reproduce the light neutrino masses of $\mathcal{O}(1)$ eV or smaller at the electroweak scale. Y_ν can run from a superhigh energy scale down to $\Lambda_{\rm EW}$ via the one-loop RGE

$$16\pi^2 \frac{{\rm d}\omega}{{\rm d}t} = 2\alpha_{\rm D}\omega + C\left[\left(Y_l Y_l^\dagger\right)\omega + \omega\left(Y_l Y_l^\dagger\right)\right] \; , \tag{2}$$

where $\omega \equiv Y_\nu Y_\nu^\dagger$, $\alpha_{\rm D} \approx -0.45g_1^2 - 2.25g_2^2 + 3y_t^2$ in the SM or $\alpha_{\rm D} \approx -0.6g_1^2 - 3g_2^2 + 3y_t^2$ in the MSSM [8]. In obtaining Eq. (2), we have safely neglected those tiny terms of $\mathcal{O}(\omega^2)$.

Eq. (1) or (2) allows us to derive the explicit RGEs for all neutrino mass and mixing parameters in the flavor basis where Y_l is diagonal and real (positive). In this basis, we have $\kappa = \mathcal{V}_{\rm M}\overline{\kappa}\mathcal{V}_{\rm M}^T$ with $\overline{\kappa} = {\rm Diag}\{\kappa_1, \kappa_2, \kappa_3\}$ for Majorana neutrinos; or $\omega = \mathcal{V}_{\rm D}\overline{\omega}\mathcal{V}_{\rm D}^\dagger$ with $\overline{\omega} = {\rm Diag}\{y_1^2, y_2^2, y_3^2\}$ for Dirac neutrinos. $\mathcal{V}_{\rm M}$ or $\mathcal{V}_{\rm D}$ is just the lepton flavor mixing matrix. At $\Lambda_{\rm EW}$, Majorana neutrino masses are given by $m_i = v^2\kappa_i$ (SM) or $m_i = v^2\kappa_i \sin^2\beta$ (MSSM), while Dirac neutrino masses are given by $m_i = vy_i$ (SM) or $m_i = vy_i \sin\beta$ (MSSM) with $v \approx 174$ GeV.

Note that $\mathcal{V}_{\rm M}$ (or $\mathcal{V}_{\rm D}$) can be parametrized in terms of three mixing angles and a few CP-violating phases. Their RGEs consist of the flavor-dependent contributions from $Y_l Y_l^\dagger$. Because of $y_e^2 \ll y_\mu^2 \ll y_\tau^2$, where y_e, y_μ and y_τ correspond to the electron, muon and tau Yukawa couplings, we only need to take account of the dominant τ-lepton contribution to those one-loop RGEs of neutrino mixing angles and CP-violating phases in an excellent approximation. A careful analysis shows that the τ-dominance is closely associated with the matrix elements $(\mathcal{V}_{\rm M})_{3i}$ or $(\mathcal{V}_{\rm D})_{3i}$ (for $i = 1, 2, 3$). This important observation implies that very concise RGEs can be obtained for those flavor mixing and CP-violating parameters, if $\mathcal{V}_{\rm M}$ (or $\mathcal{V}_{\rm D}$) is parametrized in such a way that its elements

$(\mathcal{V}_{\rm M})_{3i}$ (or $(\mathcal{V}_{\rm D})_{3i}$) are as simple as possible. One may then make use of this criterion to choose the most suitable parametrization of $\mathcal{V}_{\rm M}$ or $\mathcal{V}_{\rm D}$ in deriving the one-loop RGEs.

We find that the so-called "standard" parametrization (advocated by the Particle Data Group [9]), which has extensively been used in describing lepton flavor mixing, does not satisfy the above criterion. Instead, the parametrization recommended in Ref. [10] fulfills our present requirement

$$\begin{aligned} U &= \begin{pmatrix} c_l & s_l & 0 \\ -s_l & c_l & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\phi} & 0 & 0 \\ 0 & c & s \\ 0 & -s & c \end{pmatrix} \begin{pmatrix} c_\nu & -s_\nu & 0 \\ s_\nu & c_\nu & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} s_l s_\nu c + c_l c_\nu e^{-i\phi} & s_l c_\nu c - c_l s_\nu e^{-i\phi} & s_l s \\ c_l s_\nu c - s_l c_\nu e^{-i\phi} & c_l c_\nu c + s_l s_\nu e^{-i\phi} & c_l s \\ -s_\nu s & -c_\nu s & c \end{pmatrix} , \end{aligned} \tag{3}$$

where $c_l \equiv \cos\theta_l$, $s_l \equiv \sin\theta_l$, $c_\nu \equiv \cos\theta_\nu$, $s_\nu \equiv \sin\theta_\nu$, $c \equiv \cos\theta$ and $s \equiv \sin\theta$. In general, we have $\mathcal{V}_{\rm M} = Q_{\rm M} U P_{\rm M}$ for Majorana neutrinos or $\mathcal{V}_{\rm D} = Q_{\rm D} U P_{\rm D}$ for Dirac neutrinos, where $P_{\rm M}$ (or $P_{\rm D}$) and $Q_{\rm M}$ (or $Q_{\rm D}$) are two diagonal phase matrices. It is clear that U_{3i} (for $i = 1, 2, 3$) shown in Eq. (3) are simple enough to describe the τ-dominant terms in those one-loop RGEs of θ_l, θ_ν, θ and ϕ (as well as two Majorana phases of $\mathcal{V}_{\rm M}$ coming from $P_{\rm M}$). In the approximation that solar and atmospheric neutrino oscillations are nearly decoupled [11], three mixing angles of U can simply be related to those of solar, atmospheric and CHOOZ neutrino oscillations [1, 2, 3]: $\theta_{12} \approx \theta_\nu$, $\theta_{23} \approx \theta$ and $\theta_{13} \approx \theta_l \sin\theta$. Hence our parametrization is also a convenient option to describe current neutrino oscillation data.

One purpose of this paper is to show that Eq. (3) is actually a novel parametrization of τ-dominance in the one-loop RGEs of neutrino mixing angles and CP-violating phases. Compared with the "standard" parametrization used in the literature, Eq. (3) leads to greatly simplified results for relevant RGEs [12]. The latter can therefore allow us to understand the RGE running behaviors of lepton flavor mixing parameters in a much simpler and more transparent way, which is of course useful for model building at a superhigh energy scale to explore possible flavor symmetries or flavor dynamics responsible for the origin of neutrino masses and CP violation.

The other purpose of this paper is to explore the similarities and differences between the RGE running behaviors of Dirac and Majorana neutrinos in the especially interesting case that two Majorana CP-violating phases vanish [13]. We shall show that $\rho = \sigma = 0$ at a specific energy scale leads to $\dot{\rho} = \dot{\sigma} = 0$, implying that ρ and σ can keep vanishing at any energy scales between $\Lambda_{\rm EW}$ and $\Lambda_{\rm SS}$. In this case, only three mixing angles $(\theta_l, \theta_\nu, \theta)$ and the so-called Dirac CP-violating phase ϕ undergo the RGE evolution. Note that a kind of underlying flavor symmetry may actually forbid two Majorana phases to take non-zero values in a concrete neutrino model. It is therefore meaningful to ask whether the RGE running behaviors of Majorana neutrinos with $\rho = \sigma = 0$ are identical to those of Dirac neutrinos.

2. RG Equations for Majorana Neutrinos

The general strategy and tactics about how to derive the one-loop RGEs for Majorana neutrino mixing parameters have been outlined in Refs. [14, 15, 16, 17]. To be specific, we take $P_{\rm M} = {\rm Diag}\{e^{i\rho}, e^{i\sigma}, 1\}$ and $Q_{\rm M} = {\rm Diag}\left\{e^{i\phi_1}, e^{i\phi_2}, e^{i\phi_3}\right\}$. The phase parameters ρ and σ are physical and referred to as the Majorana phases. The phase parameters ϕ_i (for $i = 1, 2, 3$) are unphysical, but they have their own RGE evolution. Following the procedure described in Ref. [14] and taking the τ-dominance approximation, we obtain the RGEs of κ_i (for $i = 1, 2, 3$) from Eq. (1):

$$\dot{\kappa}_i \; = \; \frac{\kappa_i}{16\pi^2} \left(\alpha_{\rm M} + 2C y_\tau^2 \left|U_{3i}\right|^2\right) \; , \tag{4}$$

where $\dot{\kappa}_i \equiv {\rm d}\kappa_i/{\rm d}t$. In addition, the quantities ρ, σ, ϕ_i and U_{ij} (for $i, j = 1, 2, 3$) satisfy the following equations:

$$\begin{aligned}
\sum_{j=1}^{3} \left[U_{j1}^* \left(i\dot{U}_{j1} - U_{j1}\dot{\phi}_j\right)\right] &= \dot{\rho} \; , \\
\sum_{j=1}^{3} \left[U_{j2}^* \left(i\dot{U}_{j2} - U_{j2}\dot{\phi}_j\right)\right] &= \dot{\sigma} \; , \\
\sum_{j=1}^{3} \left[U_{j3}^* \left(i\dot{U}_{j3} - U_{j3}\dot{\phi}_j\right)\right] &= 0 \; ;
\end{aligned} \tag{5}$$

and

$$\begin{aligned}
\sum_{j=1}^{3} \left[U_{j1}^* \left(\dot{U}_{j2} + iU_{j2}\dot{\phi}_j\right)\right] &= -\frac{C y_\tau^2}{16\pi^2} \; e^{i(\rho-\sigma)} \left[\zeta_{12}^{-1} {\rm Re}\left(U_{31}^* U_{32} e^{i(\sigma-\rho)}\right) + i\zeta_{12} {\rm Im}\left(U_{31}^* U_{32} e^{i(\sigma-\rho)}\right)\right] \; , \\
\sum_{j=1}^{3} \left[U_{j1}^* \left(\dot{U}_{j3} + iU_{j3}\dot{\phi}_j\right)\right] &= -\frac{C y_\tau^2}{16\pi^2} \; e^{i\rho} \left[\zeta_{13}^{-1} {\rm Re}\left(U_{31}^* U_{33} e^{-i\rho}\right) + i\zeta_{13} {\rm Im}\left(U_{31}^* U_{33} e^{-i\rho}\right)\right] \; , \\
\sum_{j=1}^{3} \left[U_{j2}^* \left(\dot{U}_{j3} + iU_{j3}\dot{\phi}_j\right)\right] &= -\frac{C y_\tau^2}{16\pi^2} \; e^{i\sigma} \left[\zeta_{23}^{-1} {\rm Re}\left(U_{32}^* U_{33} e^{-i\sigma}\right) + i\zeta_{23} {\rm Im}\left(U_{32}^* U_{33} e^{-i\sigma}\right)\right] \; ,
\end{aligned} \tag{6}$$

where $\zeta_{ij} \equiv \left(\kappa_i - \kappa_j\right) / \left(\kappa_i + \kappa_j\right)$. Obviously, those y_τ^2-associated terms only consist of the matrix elements U_{3i} (for $i = 1, 2, 3$). If a parametrization of U assures U_{3i} to be as simple as possible, then the resultant RGEs of relevant neutrino mixing angles and CP-violating phases will be as concise as possible. One can see that the parametrization of U given in Eq. (3) just accords with such a criterion, while the "standard" parametrization advocated in Ref. [9] and used in many papers (e.g., Refs. [14, 15, 16, 17, 18]) does not satisfy this requirement.

Combining Eq. (3) with Eqs. (4), (5) and (6), we arrive at

$$\begin{aligned}
\dot{\kappa}_1 &= \frac{\kappa_1}{16\pi^2} \left(\alpha_{\rm M} + 2C y_\tau^2 s_\nu^2 s^2\right) \; , \\
\dot{\kappa}_2 &= \frac{\kappa_2}{16\pi^2} \left(\alpha_{\rm M} + 2C y_\tau^2 c_\nu^2 s^2\right) \; , \\
\dot{\kappa}_3 &= \frac{\kappa_3}{16\pi^2} \left(\alpha_{\rm M} + 2C y_\tau^2 c^2\right) \; ;
\end{aligned} \tag{7}$$

and

$$\dot{\theta}_l = \frac{Cy_\tau^2}{16\pi^2}\, c_\nu s_\nu c \left[\left(\zeta_{13}^{-1} c_\rho c_{(\rho-\phi)} + \zeta_{13} s_\rho s_{(\rho-\phi)}\right) - \left(\zeta_{23}^{-1} c_\sigma c_{(\sigma-\phi)} + \zeta_{23} s_\sigma s_{(\sigma-\phi)}\right)\right] ,$$
$$\dot{\theta}_\nu = \frac{Cy_\tau^2}{16\pi^2}\, c_\nu s_\nu \left[s^2 \left(\zeta_{12}^{-1} c_{(\sigma-\rho)}^2 + \zeta_{12} s_{(\sigma-\rho)}^2\right) + c^2 \left(\zeta_{13}^{-1} c_\rho^2 + \zeta_{13} s_\rho^2\right) - c^2 \left(\zeta_{23}^{-1} c_\sigma^2 + \zeta_{23} s_\sigma^2\right)\right] ,$$
$$\dot{\theta} = \frac{Cy_\tau^2}{16\pi^2}\, cs \left[s_\nu^2 \left(\zeta_{13}^{-1} c_\rho^2 + \zeta_{13} s_\rho^2\right) + c_\nu^2 \left(\zeta_{23}^{-1} c_\sigma^2 + \zeta_{23} s_\sigma^2\right)\right] ; \tag{8}$$

as well as

$$\dot{\phi} = \frac{Cy_\tau^2}{16\pi^2} \left[\left(c_l^2 - s_l^2\right) c_l^{-1} s_l^{-1} c_\nu s_\nu c \left(\zeta_{13}^{-1} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)}\right) \right.$$
$$\left. + \widehat{\zeta}_{12} s^2 c_{(\sigma-\rho)} s_{(\sigma-\rho)} + \widehat{\zeta}_{13} \left(s_\nu^2 - c_\nu^2 c^2\right) c_\rho s_\rho + \widehat{\zeta}_{23} \left(c_\nu^2 - s_\nu^2 c^2\right) c_\sigma s_\sigma \right] ,$$
$$\dot{\rho} = \frac{Cy_\tau^2}{16\pi^2} \left[\widehat{\zeta}_{12} c_\nu^2 s^2 c_{(\sigma-\rho)} s_{(\sigma-\rho)} + \widehat{\zeta}_{13} \left(s_\nu^2 s^2 - c^2\right) c_\rho s_\rho + \widehat{\zeta}_{23} c_\nu^2 s^2 c_\sigma s_\sigma \right] ,$$
$$\dot{\sigma} = \frac{Cy_\tau^2}{16\pi^2} \left[\widehat{\zeta}_{12} s_\nu^2 s^2 c_{(\sigma-\rho)} s_{(\sigma-\rho)} + \widehat{\zeta}_{13} s_\nu^2 s^2 c_\rho s_\rho + \widehat{\zeta}_{23} \left(c_\nu^2 s^2 - c^2\right) c_\sigma s_\sigma \right] , \tag{9}$$

where $\widehat{\zeta}_{ij} \equiv \zeta_{ij}^{-1} - \zeta_{ij} = 4\kappa_i \kappa_j / \left(\kappa_i^2 - \kappa_j^2\right)$, $c_a \equiv \cos a$ and $s_a \equiv \sin a$ (for $a = \rho$, σ, $\sigma - \rho$, $\rho - \phi$ or $\sigma - \phi$). Comparing the RGEs of three mixing angles and three CP-violating phases obtained in Eqs. (8) and (9) with their counterparts given in Refs. [14, 15, 16, 17, 18], which were derived by using the "standard" parametrization, we find that great simplification and conciseness have been achieved for our present analytical results.

As a by-product, the RGEs of three unphysical phases ϕ_i are listed below:

$$\dot{\phi}_1 = +\frac{Cy_\tau^2}{16\pi^2} \left[c_l s_l^{-1} c_\nu s_\nu c \left(\zeta_{13}^{-1} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)}\right) \right.$$
$$\left. + c^2 \left(\widehat{\zeta}_{13} s_\nu^2 c_\rho s_\rho + \widehat{\zeta}_{23} c_\nu^2 c_\sigma s_\sigma\right)\right] ,$$
$$\dot{\phi}_2 = -\frac{Cy_\tau^2}{16\pi^2} \left[c_l^{-1} s_l c_\nu s_\nu c \left(\zeta_{13}^{-1} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)}\right) \right.$$
$$\left. - c^2 \left(\widehat{\zeta}_{13} s_\nu^2 c_\rho s_\rho + \widehat{\zeta}_{23} c_\nu^2 c_\sigma s_\sigma\right)\right] ,$$
$$\dot{\phi}_3 = -\frac{Cy_\tau^2}{16\pi^2} \left[s^2 \left(\widehat{\zeta}_{13} s_\nu^2 c_\rho s_\rho + \widehat{\zeta}_{23} c_\nu^2 c_\sigma s_\sigma\right)\right] . \tag{10}$$

It is easy to check that the relationship $\dot{\phi} = \dot{\rho} + \dot{\sigma} + \dot{\phi}_1 + \dot{\phi}_2 + \dot{\phi}_3$ holds. That is why ϕ_i should not be ignored in deriving the RGEs of other physical parameters, although these three phases can finally be rotated away via rephasing the charged-lepton fields.

Some qualitative comments on the basic features of Eqs. (7)–(10) are in order.

(a) The RGE running behaviors of three neutrino masses m_i (or equivalently κ_i) are essentially identical and determined by $\alpha_{\rm M}$ [15], unless $\tan\beta$ is large enough in the MSSM to make the y_τ^2-associated term is competitive with the $\alpha_{\rm M}$ term. In our phase convention, $\dot{\kappa}_i$ or $\dot{m}_i$ (for $i = 1, 2, 3$) are independent of the CP-violating phase ϕ.

(b) Among three mixing angles, only the derivative of θ_ν contains a term proportional to ζ_{12}^{-1}. Note that $\zeta_{ij}^{-1} = -(m_i + m_j)^2 / \Delta m_{ji}^2$ with $\Delta m_{ji}^2 \equiv m_j^2 - m_i^2$ holds, and current

solar and atmospheric neutrino oscillation data yield $\Delta m^2_{21} \approx 8 \times 10^{-5}$ eV^2 and $|\Delta m^2_{32}| \approx |\Delta m^2_{31}| \approx 2.5 \times 10^{-3}$ eV^2 [11]. Thus θ_ν is in general more sensitive to radiative corrections than θ_l and θ. The RGE running of θ_ν can be suppressed through the fine-tuning of $(\sigma - \rho)$. The smallest mixing angle θ_l may get radiative corrections even if its initial value is zero, thus it can be radiatively generated from other mixing angles and CP-violating phases.

(c) The RGE running behavior of ϕ is quite different from those of ρ and σ, because it includes a peculiar term proportional to s_l^{-1}. This term, which dominates $\dot{\phi}$ when θ_l is sufficiently small, becomes divergent in the limit $\theta_l \to 0$. Indeed, ϕ is not well-defined if θ_l is exactly vanishing. But both θ_l and ϕ can be radiatively generated. We may require that $\dot{\phi}$ should remain finite when θ_l approaches zero, implying that the following necessary condition can be extracted from the expression of $\dot{\phi}$ in Eq. (9):

$$\zeta_{13}^{-1} c_\rho s_{(\rho-\phi)} - \zeta_{13} s_\rho c_{(\rho-\phi)} - \zeta_{23}^{-1} c_\sigma s_{(\sigma-\phi)} + \zeta_{23} s_\sigma c_{(\sigma-\phi)} \; = \; 0 \; . \tag{11}$$

It turns out that

$$\tan\phi = \frac{\widehat{\zeta}_{13} \sin 2\rho - \widehat{\zeta}_{23} \sin 2\sigma}{\left(\zeta_{13}^{-1} + \zeta_{13} + \widehat{\zeta}_{13} \cos 2\rho\right) - \left(\zeta_{23}^{-1} + \zeta_{23} + \widehat{\zeta}_{23} \cos 2\sigma\right)} \tag{12}$$

holds, a result similar to the one obtained in Eq. (25) of Ref. [15]. Note that the initial value of θ_l, if it is exactly zero or extremely small, may immediately drive ϕ to its *quasi-fixed point* (see Ref. [19] for a relevant study of the quasi-fixed point in the "standard" parametrization of lepton flavor mixing). In this interesting case, Eq. (12) can be used to understand the relationship between ϕ and two Majorana phases ρ and σ at the quasi-fixed point.

(d) On the other hand, the RGE running behaviors of ρ and σ are relatively mild in comparison with that of ϕ. A remarkable feature of $\dot{\rho}$ and $\dot{\sigma}$ is that they will vanish, if both ρ and σ are initially vanishing. This observation indicates that ρ and σ cannot simultaneously be generated from ϕ via the one-loop RGE evolution. In contrast, a different conclusion was drawn in Ref. [18], where the "standard" parametrization with a slightly changed phase convention was utilized.

(e) As for three unphysical phases, ϕ_2 and ϕ_3 only have relatively mild RGE running effects, while the running behavior of ϕ_1 may be violent for sufficiently small θ_l. A quasi-fixed point of ϕ_1 is also expected in the limit $\theta_l \to 0$ and under the circumstance given by Eq. (11) or (12).

3. RG Equations for Dirac Neutrinos

Now let us derive the one-loop RGEs for Dirac neutrino mixing parameters. To be specific, we take $P_{\rm D} = {\rm Diag}\left\{e^{i\varphi_1}, e^{i\varphi_2}, e^{i\varphi_3}\right\}$ and $Q_{\rm D} = {\rm Diag}\left\{e^{i\alpha}, e^{i\beta}, 1\right\}$. The phase matrix $P_{\rm D}$ can be cancelled in ω, thus it does not take part in the RGE evolution. The phase parameters α and β are also unphysical, but they have their own RGE running behaviors. Following the procedure described in Ref. [8] and taking the τ-dominance

approximation, we get the RGEs of y_i (for $i = 1, 2, 3$) from Eq. (2):

$$\dot{y}_i \; = \; \frac{y_i}{16\pi^2} \left(\alpha_{\rm D} + C y_\tau^2 \left| U_{3i} \right|^2 \right) \; , \tag{13}$$

where $\dot{y}_i \equiv {\rm d}y_i/{\rm d}t$. On the other hand, the quantities α, β and U_{ij} (for $i, j = 1, 2, 3$) satisfy the following equations:

$$\begin{aligned}
\sum_{j=1}^{3} \left(U_{j1}^* \dot{U}_{j2} \right) + i \left(\dot{\alpha} U_{11}^* U_{12} + \dot{\beta} U_{21}^* U_{22} \right) &= -\frac{C y_\tau^2}{16\pi^2} \; \xi_{12} U_{31}^* U_{32} \; , \\
\sum_{j=1}^{3} \left(U_{j1}^* \dot{U}_{j3} \right) + i \left(\dot{\alpha} U_{11}^* U_{13} + \dot{\beta} U_{21}^* U_{23} \right) &= -\frac{C y_\tau^2}{16\pi^2} \; \xi_{13} U_{31}^* U_{33} \; , \\
\sum_{j=1}^{3} \left(U_{j2}^* \dot{U}_{j3} \right) + i \left(\dot{\alpha} U_{12}^* U_{13} + \dot{\beta} U_{22}^* U_{23} \right) &= -\frac{C y_\tau^2}{16\pi^2} \; \xi_{23} U_{32}^* U_{33} \; ,
\end{aligned} \tag{14}$$

where $\xi_{ij} \equiv \left(y_i^2 + y_j^2 \right) / \left(y_i^2 - y_j^2 \right)$. Again, the y_τ^2-associated terms in Eqs. (13) and (14) only contain U_{3i} (for $i = 1, 2, 3$). These RGEs can therefore be specified in a relatively concise way, if the parametrization of U shown in Eq. (3) is taken into account.

Explicitly, the Yukawa coupling eigenvalues of three Dirac neutrinos obey the one-loop RGEs

$$\begin{aligned}
\dot{y}_1 &= \frac{y_1}{16\pi^2} \left(\alpha_{\rm D} + C y_\tau^2 s_\nu^2 s^2 \right) \; , \\
\dot{y}_2 &= \frac{y_2}{16\pi^2} \left(\alpha_{\rm D} + C y_\tau^2 c_\nu^2 s^2 \right) \; , \\
\dot{y}_3 &= \frac{y_3}{16\pi^2} \left(\alpha_{\rm D} + C y_\tau^2 c^2 \right) \; .
\end{aligned} \tag{15}$$

The RGEs of three neutrino mixing angles and one (physical) CP-violating phase are given by

$$\begin{aligned}
\dot{\theta}_l &= +\frac{C y_\tau^2}{16\pi^2} \; c_\nu s_\nu c c_\phi \left(\xi_{13} - \xi_{23} \right) \; , \\
\dot{\theta}_\nu &= +\frac{C y_\tau^2}{16\pi^2} \; c_\nu s_\nu \left[s^2 \xi_{12} + c^2 \left(\xi_{13} - \xi_{23} \right) \right] \; , \\
\dot{\theta} &= +\frac{C y_\tau^2}{16\pi^2} \; c s \left(s_\nu^2 \xi_{13} + c_\nu^2 \xi_{23} \right) \; , \\
\dot{\phi} &= -\frac{C y_\tau^2}{16\pi^2} \left(c_l^2 - s_l^2 \right) c_l^{-1} s_l^{-1} c_\nu s_\nu c s_\phi \left(\xi_{13} - \xi_{23} \right) \; ,
\end{aligned} \tag{16}$$

where $c_\phi \equiv \cos\phi$ and $s_\phi \equiv \sin\phi$. The RGEs of two unphysical phases α and β read

$$\begin{aligned}
\dot{\alpha} &= -\frac{C y_\tau^2}{16\pi^2} \; c_l s_l^{-1} c_\nu s_\nu c s_\phi \left(\xi_{13} - \xi_{23} \right) \; , \\
\dot{\beta} &= +\frac{C y_\tau^2}{16\pi^2} \; c_l^{-1} s_l c_\nu s_\nu c s_\phi \left(\xi_{13} - \xi_{23} \right) \; .
\end{aligned} \tag{17}$$

The relationship $\dot{\phi} = \dot{\alpha} + \dot{\beta}$ holds obviously, implying that α and β are not negligible in deriving the RGEs of other physical parameters. One can see that our analytical results are really concise, thanks to the novel parametrization of U that we have taken.

Some qualitative remarks on the main features of Eqs. (15), (16) and (17) are in order.

(1) Like the Majorana case, the RGE running behaviors of three Dirac neutrino masses m_i (or equivalently y_i) are nearly identical and determined by $\alpha_{\rm D}$ [8], unless $\tan\beta$ is sufficiently large in the MSSM. It is also worth mentioning that $\dot{y}_i$ or $\dot{m}_i$ (for $i = 1, 2, 3$) are independent of both the CP-violating phase ϕ and the smallest mixing angle θ_l in our parametrization.

(2) The derivative of θ_ν consists of a term proportional to $\xi_{12} = -(m_1^2 + m_2^2)/\Delta m_{21}^2$. Hence θ_ν is in general more sensitive to radiative corrections than θ_l and θ, whose derivatives are only dependent on $\xi_{13} = -(m_1^2+m_3^2)/\Delta m_{31}^2$ and $\xi_{23} = -(m_2^2+m_3^2)/\Delta m_{32}^2$. Given θ_ν and θ at a specific energy scale, the smallest mixing angle θ_l can be radiatively generated at another energy scale. In this case, however, it is impossible to simultaneously generate the CP-violating phase ϕ (see Ref. [8] for a similar conclusion in the "standard" parametrization of U). The reason is simply that ϕ can always be rotated away when θ_l is exactly vanishing, and the proportionality relationship between $\dot{\phi}$ and $\sin\phi$ forbids ϕ to be generated even when θ_l becomes non-vanishing.

(3) Different from the Majorana case, there is no non-trivial *quasi-fixed point* in the RGE evolution of ϕ for Dirac neutrinos. If $\dot{\phi}$ is required to keep finite when θ_l approaches zero, then ϕ itself must approach zero or π, as indicated by Eq. (16). On the other hand, $\dot{\theta}_l \propto \cos\phi$ implies that the RGE running of θ_l has a turning point characterized by $\phi = \pi/2$ (i.e., $\dot{\theta}_l$ flips its sign at this point). Hence two interesting conclusions analogous to those drawn in Ref. [8] can be achieved: first, θ_l can never cross zero if $\theta_l \neq 0$ and $\sin\phi \neq 0$ hold at a certain energy scale; second, CP will always be a good symmetry if $\theta_l = 0$ or $\sin\phi = 0$ holds at a certain energy scale.

(4) The RGE running behavior of α is quite similar to that of ϕ, because $\dot{\phi} = \dot{\alpha}\,(1 - \tan^2\theta_l)$ holds. In addition, $\dot{\beta} = -\dot{\alpha}\tan^2\theta_l$ holds, implying that β only gets some relatively mild RGE corrections.

Let us remark that the Jarlskog invariant of CP violation [20] takes the same form for Dirac and Majorana neutrinos: $\mathcal{J} = c_l s_l c_\nu s_\nu c s^2 s_\phi$. If neutrinos are Dirac particles, the one-loop RGE of $\mathcal{J}^{\rm D}$ can be expressed as

$$\dot{\mathcal{J}}^{\rm D} \; = \; \frac{C y_\tau^2}{16\pi^2}\, \mathcal{J}_{\rm D} \left[\left(c_\nu^2 - s_\nu^2\right) s^2 \xi_{12} + \left(c^2 - s_\nu^2 s^2\right) \xi_{13} + \left(c^2 - c_\nu^2 s^2\right) \xi_{23} \right] \; . \tag{18}$$

It becomes obvious that $\mathcal{J}^{\rm D} = 0$ will be a stable result independent of the renormalization scales, provided θ_l or $\sin\phi$ initially vanishes at a given scale. In comparison, we have

$$\begin{aligned} \dot{\mathcal{J}}^{\rm M} = \frac{C y_\tau^2}{16\pi^2} \left\{ \mathcal{J}_{\rm M} \left[\left(c_\nu^2 - s_\nu^2\right) s^2 \left(\zeta_{12}^{-1} c_{(\sigma-\rho)}^2 + \zeta_{12} s_{(\sigma-\rho)}^2\right) + \left(c^2 - s_\nu^2 s^2\right) \left(\zeta_{13}^{-1} c_\rho^2 + \zeta_{13} s_\rho^2\right) \right. \right. \\ \left. \left. + \left(c^2 - c_\nu^2 s^2\right) \left(\zeta_{23}^{-1} c_\sigma^2 + \zeta_{23} s_\sigma^2\right) \right] + c_\nu s_\nu c s^2 \left(C_{12} \widehat{\zeta}_{12} + C_{13} \widehat{\zeta}_{13} + C_{23} \widehat{\zeta}_{23} \right) \right\} \end{aligned} \tag{19}$$

for Majorana neutrinos, where

$$\begin{aligned} C_{12} &= c_l s_l s^2 c_\phi c_{(\sigma-\rho)} s_{(\sigma-\rho)} \; , \\ C_{13} &= \left[c_l s_l c_\phi \left(s_\nu^2 - c_\nu^2 c^2\right) + \left(c_l^2 - s_l^2\right) c_\nu s_\nu c \right] c_\rho s_\rho \; , \\ C_{23} &= \left[c_l s_l c_\phi \left(c_\nu^2 - s_\nu^2 c^2\right) - \left(c_l^2 - s_l^2\right) c_\nu s_\nu c \right] c_\sigma s_\sigma \; . \end{aligned} \tag{20}$$

One can see that $\mathcal{J}^{\rm M}$ can be radiatively generated from two non-trivial Majorana phases ρ and σ, even if it is initially vanishing at a specific scale. Taking $\rho = \sigma = 0$, we arrive at $C_{12} = C_{13} = C_{23} = 0$ as well as $\dot{\rho} = \dot{\sigma} = 0$. But it is impossible to obtain the equality $\dot{\mathcal{J}}^{\rm M}(\rho = \sigma = 0) = \dot{\mathcal{J}}^{\rm D}$, because $\zeta_{12}^{-1} = \xi_{12}$, $\zeta_{13}^{-1} = \xi_{13}$ and $\zeta_{23}^{-1} = \xi_{23}$ (or equivalently $m_1 m_2 = m_1 m_3 = m_2 m_3 = 0$) cannot simultaneously hold. This observation demonstrates again that the RGE running behavior of $\mathcal{J}^{\rm M}$ is essentially different from that of $\mathcal{J}^{\rm D}$.

4. Comparison Between Dirac and Majorana Neutrinos

The one-loop RGEs for three Yukawa coupling eigenvalues of Dirac neutrinos (y_i with $i = 1, 2, 3$) and their four flavor mixing parameters (θ_l, θ_ν, θ and ϕ) have been derived above. Here we replace y_i by m_i. The RGEs of three neutrino masses, three mixing angles and one CP-violating phase can then be written as

$$\begin{aligned}
\dot{m}_1 &= \frac{m_1}{16\pi^2}\left(\alpha_{\rm D} + C y_\tau^2 s_\nu^2 s^2\right) \; , \\
\dot{m}_2 &= \frac{m_2}{16\pi^2}\left(\alpha_{\rm D} + C y_\tau^2 c_\nu^2 s^2\right) \; , \\
\dot{m}_3 &= \frac{m_3}{16\pi^2}\left(\alpha_{\rm D} + C y_\tau^2 c^2\right) \; ;
\end{aligned} \tag{21}$$

and

$$\begin{aligned}
\dot{\theta}_l &= +\frac{C y_\tau^2}{8\pi^2}\, c_\nu s_\nu c c_\phi \frac{m_3^2\,(m_2^2 - m_1^2)}{(m_3^2 - m_1^2)\,(m_3^2 - m_2^2)} \; , \\
\dot{\theta}_\nu &= -\frac{C y_\tau^2}{16\pi^2}\, c_\nu s_\nu \left[s^2 \frac{m_2^2 + m_1^2}{m_2^2 - m_1^2} - c^2 \frac{2 m_3^2\,(m_2^2 - m_1^2)}{(m_3^2 - m_1^2)\,(m_3^2 - m_2^2)} \right] \; , \\
\dot{\theta} &= -\frac{C y_\tau^2}{16\pi^2}\, cs \left(s_\nu^2 \frac{m_3^2 + m_1^2}{m_3^2 - m_1^2} + c_\nu^2 \frac{m_3^2 + m_2^2}{m_3^2 - m_2^2} \right) \; , \\
\dot{\phi} &= -\frac{C y_\tau^2}{8\pi^2}\left(c_l^2 - s_l^2\right) c_l^{-1} s_l^{-1} c_\nu s_\nu c s_\phi \frac{m_3^2\,(m_2^2 - m_1^2)}{(m_3^2 - m_1^2)\,(m_3^2 - m_2^2)} \; .
\end{aligned} \tag{22}$$

Note that the neutrino mass-squared differences Δm_{31}^2 and Δm_{32}^2 are much larger in magnitude than Δm_{21}^2, as indicated by current experimental data. Typically, $\Delta m_{21}^2 \approx 8.0 \times 10^{-5}$ eV^2 and $|\Delta m_{31}^2| \approx |\Delta m_{32}^2| \approx 2.5 \times 10^{-3}$ eV^2 [11]. Among three neutrino mixing angles, the RGE running of θ_ν is expected to be most significant. The CP-violating phase ϕ may significantly evolve from one energy scale to another, if θ_l takes sufficiently small values. These qualitative features will become clearer in our subsequent numerical calculations.

The one-loop RGEs for three effective coupling eigenvalues of Majorana neutrinos (κ_i with $i = 1, 2, 3$) and their six flavor mixing parameters (θ_l, θ_ν, θ, ϕ, ρ and σ) have been presented in section II. Here we replace κ_i by m_i and take $\rho = \sigma = 0$ at either $\Lambda_{\rm EW}$ or $\Lambda_{\rm SS}$. As one can see from Eq. (9), $\rho = \sigma = 0$ leads to $\dot{\rho} = \dot{\sigma} = 0$. In other words, two Majorana phases ρ and σ keep vanishing at any energy scales between $\Lambda_{\rm EW}$ and $\Lambda_{\rm SS}$. One may safely simplify the RGEs of θ_l, θ_ν, θ and ϕ obtained in Eqs. (8) and (9) by setting

$\rho = \sigma = 0$, and then compare them with their Dirac counterparts on the same footing. In this case, we arrive at

$$\begin{aligned} \dot{m}_1 &= \frac{m_1}{16\pi^2}\left(\alpha_{\rm M} + 2Cy_\tau^2 s_\nu^2 s^2\right) \; , \\ \dot{m}_2 &= \frac{m_2}{16\pi^2}\left(\alpha_{\rm M} + 2Cy_\tau^2 c_\nu^2 s^2\right) \; , \\ \dot{m}_3 &= \frac{m_3}{16\pi^2}\left(\alpha_{\rm M} + 2Cy_\tau^2 c^2\right) \; ; \end{aligned} \tag{23}$$

and

$$\begin{aligned} \dot{\theta}_l &= +\frac{Cy_\tau^2}{8\pi^2}\, c_\nu s_\nu c c_\phi \frac{m_3(m_2-m_1)}{(m_3-m_1)\,(m_3-m_2)} \; , \\ \dot{\theta}_\nu &= -\frac{Cy_\tau^2}{16\pi^2}\, c_\nu s_\nu \left[s^2 \frac{m_2+m_1}{m_2-m_1} - c^2 \frac{2m_3\,(m_2-m_1)}{(m_3-m_1)\,(m_3-m_2)} \right] \; , \\ \dot{\theta} &= -\frac{Cy_\tau^2}{16\pi^2}\, cs \left(s_\nu^2 \frac{m_3+m_1}{m_3-m_1} + c_\nu^2 \frac{m_3+m_2}{m_3-m_2} \right) \; , \\ \dot{\phi} &= -\frac{Cy_\tau^2}{8\pi^2}\left(c_l^2 - s_l^2\right) c_l^{-1} s_l^{-1} c_\nu s_\nu c s_\phi \frac{m_3\,(m_2-m_1)}{(m_3-m_1)\,(m_3-m_2)} \; . \end{aligned} \tag{24}$$

As a consequence of $\Delta m^2_{21} \ll |\Delta m^2_{31}| \approx |\Delta m^2_{32}|$, the mixing angle θ_ν is most sensitive to radiative corrections. The RGE evolution of the CP-violating phase ϕ depends strongly on the smallness of θ_l, on the other hand. These qualitative features are essentially analogous to what we have pointed out for Dirac neutrinos.

It is interesting to note that Eq. (23) can actually be obtained from Eq. (21) with the replacements $\alpha_{\rm D} \Longrightarrow \alpha_{\rm M}$ and $C \Longrightarrow 2C$, while Eq. (24) can be achieved from Eq. (22) with the replacements $m_i^2 \Longrightarrow m_i$ (for $i = 1, 2, 3$). These similarities and differences imply that it is very non-trivial to distinguish between the RGE running behaviors of Dirac neutrinos and Majorana neutrinos with vanishing Majorana CP-violating phases.

Taking $\rho = \sigma = 0$ in the Majorana case, we obtain a simplified expression of $\dot{\mathcal{J}}^{\rm M}$,

$$\dot{\mathcal{J}}^{\rm M} \; = \; \frac{Cy_\tau^2}{16\pi^2}\, \mathcal{J}^{\rm M} \left[\left(c_\nu^2 - s_\nu^2\right) s^2 \frac{m_2+m_1}{m_2-m_1} + \left(c^2 - s_\nu^2 s^2\right) \frac{m_3+m_1}{m_3-m_1} + \left(c^2 - c_\nu^2 s^2\right) \frac{m_3+m_2}{m_3-m_2} \right] \; , \tag{25}$$

which is very analogous to $\dot{\mathcal{J}}^{\rm D}$ of Dirac neutrinos,

$$\dot{\mathcal{J}}^{\rm D} \; = \; \frac{Cy_\tau^2}{16\pi^2}\, \mathcal{J}^{\rm D} \left[\left(c_\nu^2 - s_\nu^2\right) s^2 \frac{m_2^2+m_1^2}{m_2^2-m_1^2} + \left(c^2 - s_\nu^2 s^2\right) \frac{m_3^2+m_1^2}{m_3^2-m_1^2} + \left(c^2 - c_\nu^2 s^2\right) \frac{m_3^2+m_2^2}{m_3^2-m_2^2} \right] \; . \tag{26}$$

It is obvious that Eq. (26) can be obtained from Eq. (25) with the replacements $m_i \Longrightarrow m_i^2$ (for $i = 1, 2, 3$). Note that $\dot{\mathcal{J}}^{\rm D} \propto \mathcal{J}^{\rm D}$ (or $\dot{\mathcal{J}}^{\rm M} \propto \mathcal{J}^{\rm M}$) holds. This result implies that the Jarlskog parameter will keep vanishing at any energy scales between $\Lambda_{\rm EW}$ and $\Lambda_{\rm SS}$, if it initially vanishes at either $\Lambda_{\rm EW}$ or $\Lambda_{\rm SS}$.

Because $\alpha_{\rm M} > \alpha_{\rm D}$ and $(m_j + m_i)/(m_j - m_i) > (m_j^2 + m_i^2)/(m_j^2 - m_i^2)$ hold (for $m_j > m_i$), the RGE running of each Majorana neutrino parameter is in general expected to be faster (i.e., more significant) than the RGE running of the corresponding Dirac neutrino parameter.

5. Numerical Results

In view of the fact that the absolute mass scale of three light neutrinos and the sign of Δm^2_{32} remain unknown at present, let us consider four typical patterns of the neutrino mass spectrum:

- Normal hierarchy (NH): $m_1 \ll m_2 \ll m_3$. For simplicity, we typically take $m_1 = 0$ at $\Lambda_{\rm EW}$ in our numerical calculations. Then $m_2 = \sqrt{\Delta m^2_{21}}$ and $m_3 = \sqrt{|\Delta m^2_{32}| + \Delta m^2_{21}}$ can be determined from current experimental data.

- Inverted hierarchy (IH): $m_3 \ll m_1 \lesssim m_2$. For simplicity, w typically take $m_3 = 0$ at $\Lambda_{\rm EW}$ in our numerical calculations. Then $m_2 = \sqrt{|\Delta m^2_{32}|}$ and $m_1 = \sqrt{|\Delta m^2_{32}| - \Delta m^2_{21}}$ can be determined from current experimental data.

- Near degeneracy (ND) with $\Delta m^2_{32} > 0$: $m_1 \lesssim m_2 \lesssim m_3$. For simplicity, we typically take $m_1 = 0.2$ eV at $\Lambda_{\rm EW}$ in our numerical calculations.

- Near degeneracy (ND) with $\Delta m^2_{32} < 0$: $m_3 \lesssim m_1 \lesssim m_2$. For simplicity, we typically take $m_1 = 0.2$ eV at $\Lambda_{\rm EW}$ in our numerical calculations.

In addition, we take $\Delta m^2_{21} = 8.0 \times 10^{-5}$ eV^2, $|\Delta m^2_{32}| = 2.5 \times 10^{-3}$ eV^2, $\theta_\nu = 33.8^\circ$, $\theta = 45^\circ$, $\theta_l = 0.5^\circ$ and $\phi = 90^\circ$ as typical inputs at $\Lambda_{\rm EW} \sim M_Z$ in our numerical calculations [13].

5.1 Neutrino Masses

In either the SM or the MSSM with small $\tan\beta$, the RGE running behaviors of three neutrino masses are dominated by $\alpha_{\rm D}$ or $\alpha_{\rm M}$. The y^2_τ-associated term of $\dot{m}^2_i$ in Eq. (21) or (23) becomes important only when $\tan\beta$ takes sufficiently large values in the MSSM [14, 15, 16, 17]. Note that $\alpha_{\rm M} = 2\alpha_{\rm D}$ holds in the MSSM, in which the running effects of m_i for Majorana neutrinos are twice as large as those for Dirac neutrinos.

The first plot in Fig. 1 illustrates the ratios $R_i \equiv m_i(\mu)/m_i(M_Z)$ changing with the energy scale μ in the SM for Dirac and Majorana neutrinos, where $m_1(M_Z) = 0.2$ eV and $M_H = 180$ GeV (the Higgs mass) have typically been input. Since the running of m_i is governed by $\alpha_{\rm D}$ or $\alpha_{\rm M}$, $R_1 \approx R_2 \approx R_3$ holds to a high degree of accuracy. Furthermore, the behaviors of R_i are actually independent of the initial value of m_1 and possible patterns of the neutrino mass spectrum. We observe that R_i in the Majorana case is always larger than R_i in the Dirac case, and their discrepancy can be as large as 0.7 at $\mu \sim \Lambda_{\rm SS} \sim 10^{14}$ GeV.

The relation $R_1 \approx R_2 \approx R_3$ is also a very good approximation in the MSSM with small $\tan\beta$, as shown by the second plot in Fig. 1, where $\tan\beta = 10$ has been input. It is clear that R_i in the Dirac case is numerically distinguishable from R_i in the Majorana case, in particular when the energy scale μ far exceeds M_Z.

If $\tan\beta$ is sufficiently large, the common scaling of three neutrino masses in the RGE evolution will fail [15]. The splitting of R_1, R_2 and R_3, which increases with the energy scale μ, is illustrated by the third plot in Fig. 1 with the input $\tan\beta = 50$. One can

see that R_i in the Dirac case is always smaller than R_i in the Majorana case, and their discrepancy is distinguishable at the scales $\mu \gg M_Z$.

5.2 Neutrino Mixing Parameters

Radiative corrections to three neutrino mixing angles, the Dirac CP-violating phase and the Jarlskog parameter are all controlled by the τ-lepton Yukawa coupling eigenvalue y_τ. Because of $y^2_\tau/(8\pi^2) \approx 1.3 \times 10^{-6}$ (SM) or $y^2_\tau/(8\pi^2) \approx 1.3 \times 10^{-6}\,(1 + \tan^2\beta)$ (MSSM) at M_Z, significant RGE running effects are expected to appear in the MSSM case when $\tan\beta$ is sufficiently large. To illustrate, here we simply concentrate on the MSSM with $\tan\beta = 50$ and consider four typical patterns of the neutrino mass spectrum in our subsequent discussions and calculations.

(1) In the NH case with $m_1 = 0$, the RGEs of θ_l, θ_ν, θ, ϕ and $\mathcal{J}$ can be simplified as

$$\begin{aligned}
\dot{\theta}_l &= +\frac{Cy^2_\tau}{16\pi^2}\, c_\nu s_\nu c c_\phi r \; , \qquad \dot{\theta}_\nu = -\frac{Cy^2_\tau}{16\pi^2}\, c_\nu s_\nu \left(s^2 - c^2 r\right) \; , \\
\dot{\theta} &= -\frac{Cy^2_\tau}{16\pi^2}\, cs\,(1 + r) \; , \qquad \dot{\phi} = -\frac{Cy^2_\tau}{16\pi^2} \left(c^2_l - s^2_l\right) c^{-1}_l s^{-1}_l c_\nu s_\nu c s_\phi r \; , \\
\dot{\mathcal{J}} &= \frac{Cy^2_\tau}{16\pi^2}\, \mathcal{J} \left[2\left(s^2_\nu s^2 - c^2\right) - \left(c^2 - c^2_\nu s^2\right) r\right] \; ,
\end{aligned} \tag{27}$$

in which $r = 2m^2_2/(m^2_3 - m^2_2)$ for Dirac neutrinos or $r = 2m_2/(m_3 - m_2)$ for Majorana neutrinos. Current experimental data yield $r_{\rm D} \approx 0.06$ and $r_{\rm M} \approx 0.4$. Both of them are too small to compensate for the strong suppression induced by y^2_τ in Eq. (27). Thus the RGE corrections to those flavor mixing and CP-violating parameters are not important in the NH case. Note, however, that the radiative correction to ϕ can be very significant when θ_l is extremely small or becomes vanishing. We find that ϕ quickly approaches its quasi-fixed point $\phi_{\rm QF} = 0$ or π in the $\theta_l \to 0$ limit, an interesting phenomenon which is remarkably different from the non-trivial quasi-fixed point of ϕ discovered in the general ($\rho \neq \sigma \neq 0$) case for Majorana neutrinos [19]. One can also see that both $\mathcal{J} = 0$ and $\dot{\mathcal{J}} = 0$ hold when θ_l vanishes; i.e., CP is a good symmetry in this limit.

(2) In the IH case with $m_3 = 0$, we arrive at

$$\begin{aligned}
\dot{\theta}_l &= \dot{\phi} = 0, \qquad \dot{\theta}_\nu = -\frac{Cy^2_\tau}{16\pi^2}\, c_\nu s_\nu s^2 r' \; , \qquad \dot{\theta} = +\frac{Cy^2_\tau}{16\pi^2}\, cs \; , \\
\dot{\mathcal{J}} &= \frac{Cy^2_\tau}{16\pi^2}\, \mathcal{J} \left[3c^2 - 1 + \left(s^2_\nu - c^2_\nu\right) s^2 r'\right] \; ,
\end{aligned} \tag{28}$$

where $r' = (m^2_2 + m^2_1)/(m^2_2 - m^2_1)$ for Dirac neutrinos or $r' = (m_2 + m_1)/(m_2 - m_1)$ for Majorana neutrinos. We observe that radiative corrections to θ_l and ϕ are vanishingly small, and the correction to θ is also insignificant. Nevertheless, the RGE running effects of θ_ν and $\mathcal{J}$ may get enhanced by r', whose typical value reads $r'_{\rm D} \approx 60$ or $r'_{\rm M} \approx 120$ at M_Z. Fig. 2 illustrates the evolution of θ_ν and $\mathcal{J}$ in the IH case. The discrepancy between Dirac and Majorana cases is obviously distinguishable for both parameters, when the energy scale is much higher than M_Z. In particular, $\mathcal{J}^{\rm D} \sim 2\mathcal{J}^{\rm M}$ holds at $\mu \sim \Lambda_{\rm SS}$,

because the corresponding value of θ_ν for Majorana neutrinos is roughly half of that for Dirac neutrinos.

(3) In the ND case with $\Delta m^2_{32} > 0$ and $m_1 = 0.2$ eV, the RGE corrections to those neutrino mixing parameters can significantly be enhanced by the ratios $(m_i^2 + m_j^2)/(m_i^2 - m_j^2)$ in Eqs. (22) and (26) for Dirac neutrinos, or by the ratios $(m_i + m_j)/(m_i - m_j)$ in Eqs. (24) and (9) for Majorana neutrinos. We illustrate the typical evolution behaviors of θ_l, θ_ν, θ and ϕ in Fig. 3. One can see that Majorana neutrinos undergo the RGE corrections more significantly than Dirac neutrinos. The discrepancy between these two cases is about 10° for either θ or ϕ at $\mu \gg M_Z$. It is therefore possible to distinguish the running of Majorana neutrinos from that of Dirac neutrinos. The difference between $\mathcal{J}^{\rm D}$ and $\mathcal{J}^{\rm M}$ is insignificant even at $\mu \sim \Lambda_{\rm SS}$, as shown in Fig. 4, partly because the increase (or decrease) of θ_l can somehow compensate for the decrease (or increase) of θ and ϕ in the Majorana (or Dirac) case.

(4) In the ND case with $\Delta m^2_{32} < 0$ and $m_1 = 0.2$ eV, we get similar enhancements in the RGEs of those neutrino mixing parameters induced by the ratios $(m_i^2 + m_j^2)/(m_i^2 - m_j^2)$ for Dirac neutrinos, or by $(m_i + m_j)/(m_i - m_j)$ for Majorana neutrinos. However, only the running of θ is sensitive to the sign flip of Δm^2_{32}, as one can see from Eqs. (22) and (24)–(26), in which $\dot\theta_\nu$ and $\dot{\mathcal{J}}$ are dominated by the term proportional to $(m_2^2 + m_1^2)/(m_2^2 - m_1^2)$ (Dirac) or $(m_2 + m_1)/(m_2 - m_1)$ (Majorana). Then the numerical results for θ_l, θ_ν, ϕ and $\mathcal{J}$ in the present case are very similar to those in the ND case with $\Delta m^2_{32} > 0$. For simplicity, we only illustrate the evolution of θ in Fig. 5 by taking $\Delta m^2_{32} < 0$. It is obvious that the running behavior of θ for either Dirac or Majorana neutrinos in Fig. 5 is essentially opposite (or complementary) to that in Fig. 3, just due to the sign flip of Δm^2_{32}.

6. Summery

We have pointed out that the τ-lepton dominance in the one-loop RG equations of relevant neutrino mixing quantities allows us to set a criterion for the choice of the most appropriate parametrization of the lepton flavor mixing matrix U: its elements U_{3i} (for $i = 1, 2, 3$) should be as simple as possible. Such a novel parametrization *does* exist, but it is quite different from the "standard" parametrization advocated by the Particle Data Group and used in the literature. We have shown that this parametrization can lead to greatly simplified RG equations for three mixing angles and the physical CP-violating phase(s), no matter whether neutrinos are Dirac or Majorana particles.

Another goal of this paper is to examine whether the RGE running behaviors of Majorana neutrinos are still different from those of Dirac neutrinos, if two Majorana CP-violating phases vanish at a given energy scale. For this purpose, it is essential to use the afore-mentioned parametrization of the 3×3 lepton flavor mixing matrix, such that its two Majorana phases keep vanishing in the RGE evolution from one scale to another. Taking $\rho = \sigma = 0$ at the electroweak scale, we have carefully compared the similarities and differences between the RGEs of θ_l, θ_ν, θ and ϕ for Majorana neutrinos and those

for Dirac neutrinos. Our numerical calculations show that it is possible to distinguish between these two cases in the MSSM with sizable $\tan\beta$, in particular when the masses of three neutrinos are nearly degenerate or have an inverted hierarchy. Furthermore, we conclude that Majorana neutrinos are in general expected to run faster (i.e., more significantly) than Dirac neutrinos from one energy scale to another.

Of course, the numerical examples presented in this work are mainly for the purpose of illustration. The point is that the nature of neutrinos determines their RGE running behaviors, and the latter may be crucial for building a realistic neutrino model. We expect that our analysis can not only complement those previous studies of radiative corrections to the physical parameters of Dirac and Majorana neutrinos, but also help us understand the dynamical role of Majorana phases in a more general picture of flavor physics.

This work is supported in part by the National Nature Science Foundation of China.

References

[1] SNO Collaboration, Q.R. Ahmad *et al.*, Phys. Rev. Lett. **89**, 011301 (2002).

[2] For a review, see: C.K. Jung *et al.*, Ann. Rev. Nucl. Part. Sci. **51**, 451 (2001).

[3] KamLAND Collaboration, K. Eguchi *et al.*, Phys. Rev. Lett. **90**, 021802 (2003); CHOOZ Collaboration, M. Apollonio *et al.*, Phys. Lett. B **420**, 397 (1998); Palo Verde Collaboration, F. Boehm *et al.*, Phys. Rev. Lett. **84**, 3764 (2000).

[4] K2K Collaboration, M.H. Ahn *et al.*, Phys. Rev. Lett. **90**, 041801 (2003).

[5] For recent reviews with extensive references, see: H. Fritzsch and Z.Z. Xing, Prog. Part. Nucl. Phys. **45**, 1 (2000); S.F. King, Rept. Prog. Phys. **67**, 107 (2004); G. Altarelli and F. Feruglio, New J. Phys. **6**, 106 (2004); R.N. Mohapatra *et al.*, hep-ph/0510213.

[6] P. Minkowski, Phys. Lett. B **67**, 421 (1977); T. Yanagida, in *Proceedings of the Workshop on Unified Theory and the Baryon Number of the Universe*, edited by O. Sawada and A. Sugamoto (KEK, Tsukuba, 1979), p. 95; M. Gell-Mann, P. Ramond, and R. Slansky, in *Supergravity*, edited by F. van Nieuwenhuizen and D. Freedman (North Holland, Amsterdam, 1979), p. 315; S.L. Glashow, in *Quarks and Leptons*, edited by M. Lévy *et al.* (Plenum, New York, 1980), p. 707; R.N. Mohapatra and G. Senjanovic, Phys. Rev. Lett. **44**, 912 (1980).

[7] P.H. Chankowski and Z. Pluciennik, Phys. Lett. B **316**, 312 (1993); K.S. Babu, C.N. Leung, and J. Pantaleone, Phys. Lett. B **319**, 191 (1993); S. Antusch, M. Drees, J. Kersten, M. Lindner, and M. Ratz, Phys. Lett. B **519**, 238 (2001); Phys. Lett. B **525**, 130 (2002).

[8] M. Lindner, M. Ratz, and M.A. Schmidt, hep-ph/0506280; and references therein.

[9] Particle Data Group, S. Eidelman *et al.*, Phys. Lett. B **592**, 1 (2004).

[10] H. Fritzsch and Z.Z. Xing, Phys. Lett. B **413**, 396 (1997); Phys. Rev. D **57**, 594 (1998).

[11] See, e.g., M. Maltoni, T. Schwetz, M.A. Tortola, and J.W.F. Valle, New. J. Phys. **6**, 122 (2004); A. Strumia and F. Vissani, Nucl. Phys. B **726**, 294 (2005).

[12] Z.Z. Xing, Phys. Lett. B **633**, 550 (2006).

[13] Z.Z. Xing and H. Zhang, hep-ph/0601106.

[14] J.A. Casas, J.R. Espinosa, A. Ibarra, and I. Navarro, Nucl. Phys. B **573**, 652 (2000).

[15] S. Antusch, J. Kersten, M. Lindner, and M. Ratz, Nucl. Phys. B **674**, 401 (2003).

[16] S. Antusch, J. Kersten, M. Lindner, M. Ratz, and M.A. Schmidt, JHEP **0503**, 024 (2005).

[17] J.W. Mei, Phys. Rev. D **71**, 073012 (2005).

[18] S. Luo, J. Mei, and Z.Z. Xing, Phys. Rev. D **72**, 053014 (2005).

[19] S. Luo and Z.Z. Xing, Phys. Lett. B **632**, 341 (2006).

[20] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985); Z.Z. Xing, Int. J. Mod. Phys. A **19**, 1 (2004).

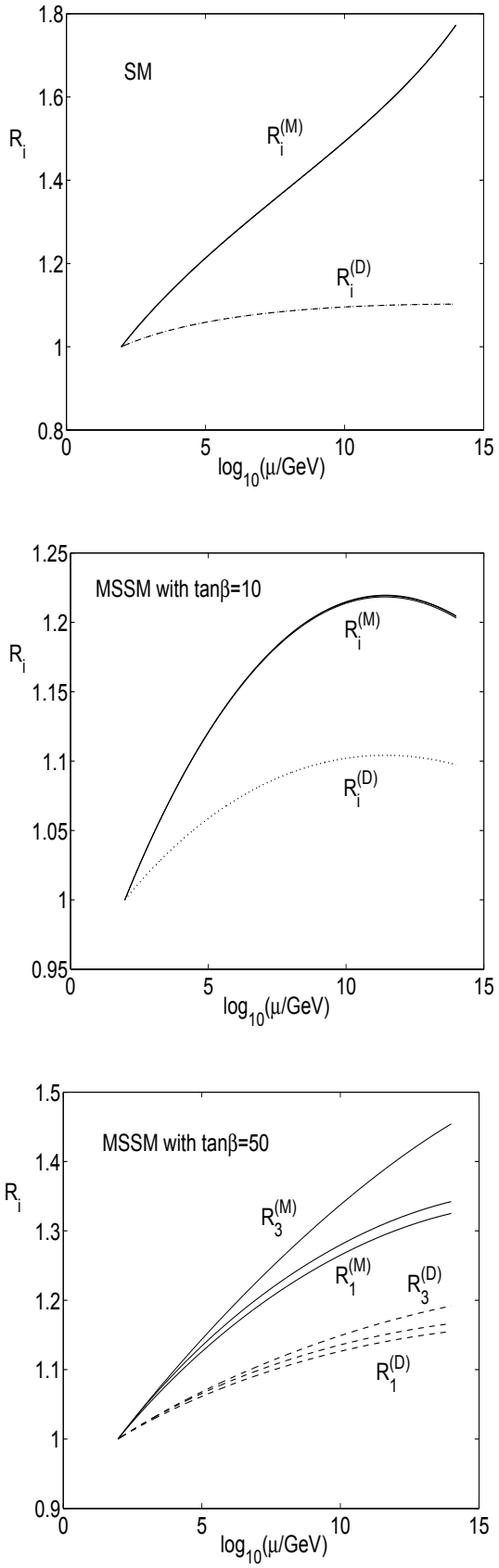


Fig. 1 The running neutrino mass ratios $R_i = m_i(\mu)/m_i(M_Z)$ (for $i = 1, 2, 3$), where the dashed and solid curves stand respectively for the Dirac and Majorana cases.

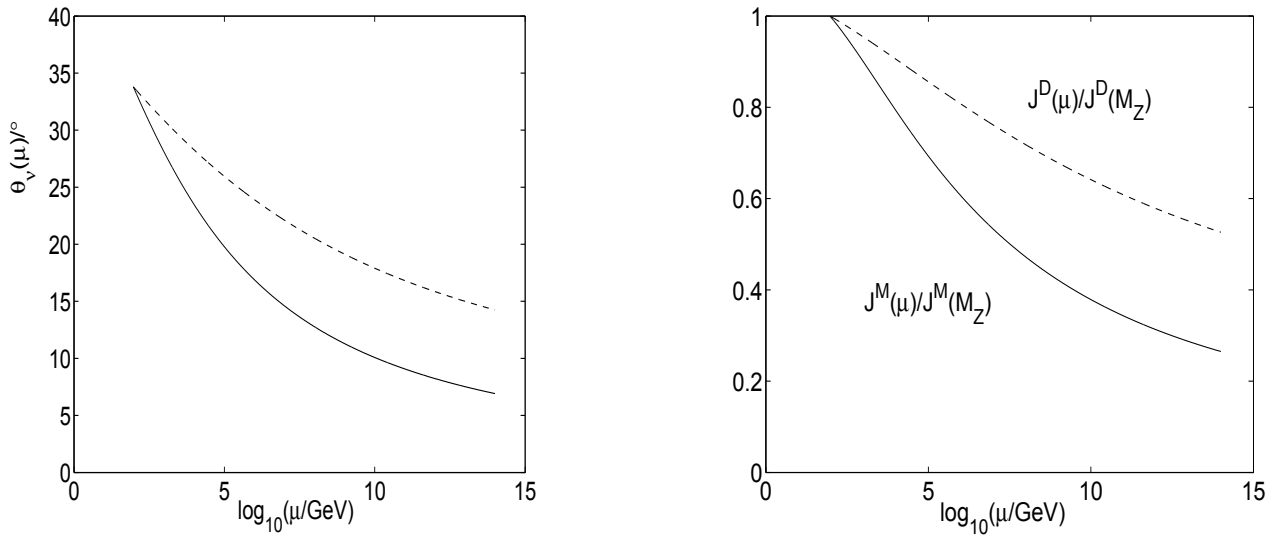


Fig. 2 The running behaviors of θ_ν and $\mathcal{J}$ in the IH case with $\tan\beta = 50$ and $m_3 = 0$ at M_Z within the MSSM, where the dashed and solid curves stand respectively for the Dirac and Majorana cases, and $\mathcal{J}^{\rm D}(M_Z) = \mathcal{J}^{\rm M}(M_Z) \approx 0.0014$.

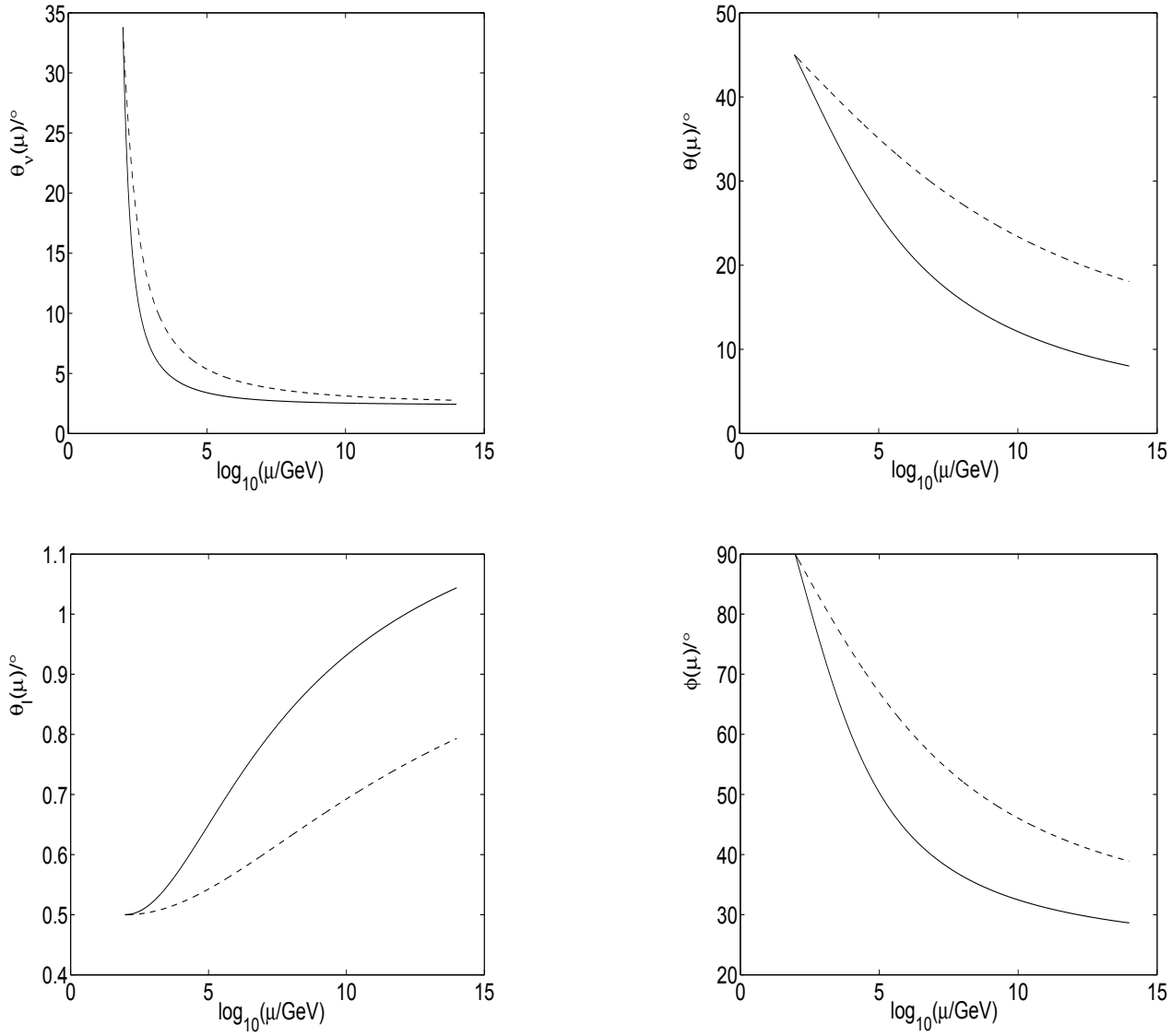


Fig. 3 The running behaviors of θ_l, θ_ν, θ and ϕ in the ND case with $\Delta m^2_{32} > 0$, $\tan\beta = 50$ and $m_1(M_Z) = 0.2$ eV within the MSSM, where the dashed and solid curves stand respectively for the Dirac and Majorana cases.

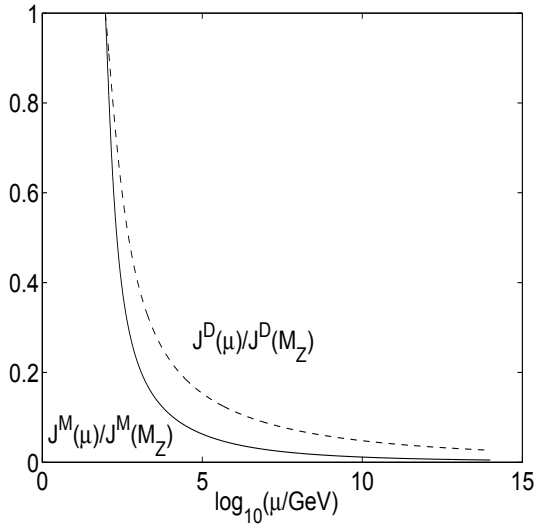


Fig. 4 The running behavior of $\mathcal{J}$ in the ND case with $\Delta m^2_{32} > 0$, $\tan\beta = 50$ and $m_1(M_Z) = 0.2$ eV within the MSSM, where the dashed and solid curves stand respectively for the Dirac and Majorana cases, and $\mathcal{J}^{\rm D}(M_Z) = \mathcal{J}^{\rm M}(M_Z) \approx 0.0014$.

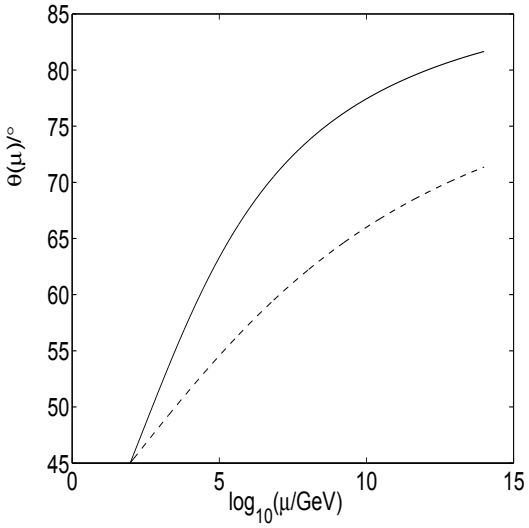


Fig. 5 The running behavior of θ in the ND case with $\Delta m^2_{32} < 0$, $\tan\beta = 50$ and $m_1(M_Z) = 0.2$ eV within the MSSM, where the dashed and solid curves stand respectively for the Dirac and Majorana cases.

EJTP **3, No. 10** (2006) 211–224 *Electronic Journal of Theoretical Physics*

Universe Without Singularities
A Group Approach to De Sitter Cosmology

Ignazio Licata *†

IxtuCyber for Complex Systems
via Favorita 9, 91025, Marsala (TP), Italy

Received 21 May 2006, Published 28 May 2006

Abstract: In the last years the traditional scenario of "Big Bang" has been deeply modified by the study of the quantum features of the Universe evolution, proposing again the problem of using "local" physical laws on cosmic scale, with particular regard to the cosmological constant role. The "group extention" method shows that the De Sitter group univocally generalizes the Poincarè group, formally justifies the cosmological constant use and suggests a new interpretation for Hartle-Hawking boundary conditions in Quantum Cosmology.

Keywords: Group Methods in Theoretical Physics, Projective Relativity, De Sitter Universe, Quantum Cosmology.
PACS (2006): 02.20.Bb, 02.20.Qs, 02.40.-k, 03.30.+p, 04.90.+e, 98.80.Qc

1. Introduction

There are strong theoretical coherence reasons which impose to critically reconsider the approach to cosmological problem on the whole. The Quantum Cosmology's main problem is to individuate the proper boundary conditions for the Universe's wave function in the Wheeler-DeWitt equation. These conditions have to be such to allow the confrontation between a probability distribution of states and the observed Universe. In particular, it is expected to select a path in the configuration space able to solve the still open problems of the Big-Bang traditional scenario: flat space, global homogeneity (horizon problem) and the "ruggedness" necessary to explain the tiny initial dishomogeneities which have led to the formation of the galactic structures.

The inflationary cosmology ideas has partly supplied with a solution to the standard

* Ignazio.licata@ejtp.info
† ignazio.licata@ixtucyber.org

model wants by introducing the symmetry breaking and phase transition notions which are at the core of Quantum Cosmology. The last one also finds its motivation in the necessity to provide with a satisfactory physical meaning to the initial singularity problem, unavoidable in GR under the condition of the Hawking-Penrose theorem (Hawking & Ellis, 1973).

The Hartle-Hawking "no-boundary" condition seems to provide a very powerful constraint for the Quantum Cosmology main requirements, but appears as an "ad hoc" solution which could be deduced by a fundamental approach. Particularly, the mix of topologies used to conciliate without boundary Universe symmetry with the Big-Bang evolutionary scenario is unsatisfactory.

We realize that most part of the Quantum Cosmology problems inherit the uncertainties of the Fridman model in GR, so they derive from the euristic use of the local laws on cosmic scale.

A possible way-out is the Fantappié-Arcidiacono group approach which allows to individuate a Universe model without recourse to arbitrary extrapolations of the symmetry groups valid in physics.

The group extension theory naturally finds again the Hartle-Hawking condition on the Universe wave function and allows to firmly founding theoretically the Quantum Cosmology. The price to pay is a subtle methodological question on using the GR in cosmology. In fact, in 1952 Fantappié pointed out that the problem of the use of *local laws* to define the cosmological boundary conditions is due to the fact that GR describes matter in terms of local curvature, but leaves the question of space-time global structure indeterminate. It happens because, differently from RR, GR has not be built on group base, which thing should be central in building any theory up, especially when it aims to express universally valid statements on physical world, the class of *the superb theories,* how Roger Penrose called them.

We are going to examine here the foundations of the group extension method (par. 2) and the relativity in the De Sitter Universe (par. 3, 4), we introduce the conditions to define matter-fields (par.5).In (par.6) we analyze the physical significance of the observers in an istantonic Universe at imaginary time, and in (par.7) investigate the physical meaning of the Hartle-Hawking condition in an hyper-spherical universe.

2. An Erlangen Program for Cosmology

In 1872 Felix Klein presented the so-called *Erlangen program* for geometry, centred upon the symmetry transformations group. From 1952, Fantappié, basing on a similar idea and in perfect consonance with Relativity spirit, proposed an *Erlangen program for physics*, where a Universe is univocally individuated by a symmetry group which let its physical laws invariant (Fantappié,1954, 1959). It has to be underlined that in the theory *Universe* means any physical system characterized by a symmetry group.

The space-time isotropy and homogeneity principle with respect to physical laws tells us that *the physical law concept itself is based upon symmetry.* So the essential idea

is to individuate physical laws starting from the transformations group which let them invariant. We observe here that there are infinite possible transformations group which individuate an isotropic and homogeneous space-time. In order to build the next improvements in physics using the *group extension* method, we can follow the path indicated by the two groups we know to be two valid description levels of the physical world: the Galilei group and the Lorentz-Poincarè one. It is useful to remember that the Galilei group is a particular case of the Lorentz one when $c \to \infty$,i.e. when it is not made use of the field notion and the interactions velocity is considered to be infinite. Staying within a quadrimensional space-time and consequently considering only groups at 10 parameters and continuous transformations, Fantappié showed that the Poincaré group can be considered a limit case of a broader group depending with continuity on c and another parameter r: the *Fantappié group*; moreover this group cannot be further extended under the condition to stay within a group at 10 parameters.

So we have the sequence:

$$G^{10}_{1+3} \to L^{10}_{1+3} \to F^{10}_{1+3}$$

Where G is the Galilei group, L the Lorentz one and F the Fantappié *final one*, from which with $R \to \infty$, we get the L group. It is shown that such sequence of universes is *univocal*.

The Lorentz group can be mathematically interpreted as the group of roto-translations such to let that particular *object* that is the Minkowski space-time invariant. Similarly, the Fantappié group is the one of the pentadimensional rotations of a new space-time: the hyper-spherical and at constant curvature De Sitter universe (maximally symmetric). We point out we have obtained the De Sitter model without referring to the gravitational interaction, differently from the GR where the De Sitter universe is one of the possible solutions of the Einstein equations with cosmological constant. From a formal viewpoint we make recourse to pentadimensional rotations because in the De Sitter universe there *appears* a new constant r, which can be interpreted as the Universe radius.

The group extension *mechanism* individuates an univocal sequence of symmetry groups; for each symmetry group we have a corresponding level of physical world description and a new universal constant, so providing the most general boundary conditions and constraining the form of the possibile physical laws. The Fantappié group fixes the c and $r$constants and defines a new relativity for the inertial observers in De Sitter Universe. In this sense, the *Theory of Universes*- based on group extension method- is actually a version of what is sought for in the Holographic Principle: the possibility to describe laws and boundaries in a compact and unitary way.

In 1956 G. Arcidiacono proposed to study the De Sitter S_4 *absolute* universe by means of the tangent relative spaces where observers *localize* and *describe* the physical events by using the Beltrami-Castelnuovo P^4 projective representation in the *Projective Special Relativity,* PSR (Arcidiacono,1956; 1976; 1984).

We note that we pass from hyper-spherical S_4 to its real representation as hyperboloid by means of an inverse Wick rotation, rotating $it \to \tau$ and associating the great circles on the hyper-sphere with a family of geodesics on the hyperboloid. In this way, we get a

realization of the Weyl Principle for defining a Universe model, because it fixes a set of privileged observers (Ellis & Williams, 1988). So, the choice of P^4Beltrami-Castelnuovo is equivalent to study a relativity in S^4.

3. The Fantappié Group Transformations

To study the De Sitter S^4 universe according to Beltrami-Castelnuovo representation we have to set the projectivities which let the Cayley-Klein interval invariant:

$$x^2 + y^2 + z^2 - c^2t^2 + r^2 = 0 \tag{3.1}$$

The (3.1) meets the time axis in the two $t = \pm t_0$ "singularities", where $t_0 = r/c$ is the time it takes light to run the Universe r radius. In this case the singularities' meaning is purely geometrical, not physical, and they represent the hyperboloid rims (3.1), since the De Sitter universe is lacking in "structural" singularities. The S^4 invariant transformations are the 5-dimensional space rotations which lead on the P^4 observer's space the projectivities that let the (3.1) unchanged.

Let's introduce the five homogeneous projective coordinates (Weierstrass condition):

$$\overline{x_a}\ \overline{x_a} = r^2, \quad \text{with} \quad a = 0, 1, 2, 3, 4 \tag{3.2}$$

The x_i space-time coordinates, with i = 1,2,3,4 are:

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict \tag{3.3}$$

The connection between the (3.2) and (3.3) is given by the relation:

$$x_i = r\overline{x_i}/\overline{x_0} \tag{3.4}$$

from which, owing to (3.2), we get the inverse relation:

$$\overline{x_0} = r/a, \quad \overline{x_i} = x_i/a, \tag{3.5}$$

where $a^2 = 1 + x_i x_i/r^2 = 1 + \alpha^2 - \gamma^2$, with $\vec{\alpha} = \vec{x}/r$ and $\gamma = t/t_0$.

The searched transformation between the two $O'$and O observers consequently has the form:

$$\overline{x}'_a = \alpha_{ab}\overline{x_b} \quad \text{with} \quad \alpha_{ab} \quad \text{orthogonal matrix} \tag{3.6}$$

Limiting ourselves, just for simplicity reasons, to the $\overline{x_0}, \overline{x_1}, \overline{x_4}$variables and following the standard method, also used in RR, we get 3 families of transformations:

A) the space translations along the x axis, given by the $(\bar{x}_0, \bar{x}_1)$ rotation:

$$\begin{aligned} \overline{x}'_1 &= \overline{x_1}\cos\vartheta + \overline{x_0}\sin\vartheta \\ \overline{x}'_0 &= -\overline{x_1}\sin\vartheta + \overline{x_0}\cos\vartheta \end{aligned} \tag{3.7}$$

$$\overline{x}_4^{'} = \overline{x_4}$$

Using the (3.4) and putting $tg\vartheta = {}^{T}\!/_{r} = \alpha$, we get the space-time transformations with T parameter:

$$x^{'} = \frac{x+T}{1-\alpha x/r}, \quad t^{'} = \frac{t\sqrt{1+\alpha^2}}{1-\alpha x/r} \tag{3.8}$$

The (3.8) for r indeterminate, i.e. $r \to \infty$, are reduced to the well-known space translations of the classical and relativistic cases, connected by the T parameter.

B) the T_0 parameter time translation, given by the $(\bar{x}_0, \bar{x}_4)$ rotation:

$$\begin{aligned} \overline{x}_4^{'} &= \overline{x_4}\cos\vartheta_0 + \overline{x_0}\sin\vartheta_0 \\ \overline{x}_0^{'} &= -\overline{x_4}\sin\vartheta_0 + \overline{x_0}\cos\vartheta_0 \\ \overline{x}_1^{'} &= \overline{x_1}. \end{aligned} \tag{3.9}$$

Putting $tg\vartheta_0 = iT_0/t_0 = i\gamma$ we obtain:

$$x^{'} = \frac{x\sqrt{1-\gamma^2}}{1+\gamma t/t_0}, \quad t^{'} = \frac{t+T_0}{1+\gamma t/t_0} \tag{3.10}$$

Also the (3.10), when $r \to \infty$ are reduced to the known cases of classical and relativistic physics.

C) the V parameter inertial transformations, given by the $(\bar{x}_1, \bar{x}_4)$ rotation:

$$\begin{aligned} \overline{x}_1^{'} &= \overline{x_1}\cos\varphi_0 + \overline{x_4}\sin\varphi_0 \\ \overline{x}_4^{'} &= -\overline{x_1}\sin\varphi_0 + \overline{x_4}\cos\varphi_0 \\ \overline{x}_0^{'} &= \overline{x_0}. \end{aligned} \tag{3.11}$$

Putting $tg\varphi = iV/c = i\beta$, here we find again the Lorentz transformations:

$$x^{'} = \frac{x+Vt}{\sqrt{1-\beta^2}}, \quad t^{'} = \frac{t+Vx/c^2}{\sqrt{1-\beta^2}} \tag{3.12}$$

The (A), (B) and (C) transformations form the Fantappié projective group which for two variables (x,t) and three parameters (T,T_0,V), with T translations and V velocity along x, can be written:

$$\begin{aligned} x^{'} &= \frac{ax + [\beta + (\alpha - \beta\gamma)\,\gamma]\,ct + bT}{b - (\alpha - \beta\gamma)\,ax/r + (\gamma - \alpha\beta)\,t/t_0}; \\ t^{'} &= \frac{a\beta x/c + [1 + (\alpha - \beta\gamma)\,\alpha]\,t + bT_0}{b - (\alpha - \beta\gamma)\,ax/r + (\gamma - \alpha\beta)\,t/t_0}, \end{aligned} \tag{3.13}$$

where we have put $a = \sqrt{1+\alpha^2-\gamma^2}$ and $b = \sqrt{1-\beta^2+(\alpha-\beta\gamma)^2}$, with $\alpha = x/r$, $\beta = V/c$ and $\gamma = t/t_0$.

For $r \to \infty$ we get $a = 1$ and $b = \sqrt{1-\beta^2}$, and from (3.13) we obtain the Poincaré group with three parameters (T, T_0,V).

The Fantappié group can be synthesized by a very clear geometrical viewpoint, saying that the De Sitter universe at $1/r^2$constant curvature shows an elliptic geometry in its hyper spatial global aspect (Gauss-Riemann) and an hyperbolic geometry in its space-time sections (Lobacevskij). Making the "natural" r unit of this two geometries tend towards infinity we obtain the parabolic geometry of Minkowski flat space.

4. The Projective Relativity in De Sitter Universe

The Projective Special Relativity (PSR) widens and contextualizes the relativistic results in De Sitter geometry.Just like in any physics there exists a well-defined connection between mechanics and geometry. Therefore the PSR makes use of the notion of observer's private space, redifining it on the basis of a constant curvature.

In PSR it is introduced a space temporal double scale which connects a (χ, τ) point of S^4 with a (x, t) one of P^4 by means the (3.1) projective invariant. Given a AB straight line and put as R and S the intersections with (1.3), the projective distance is given by the logarithm of the (ABRS) bi-ratio:

$$AB = (t_0/2)\log(ABRS) = (t_0/2)\log(AR \cdot BS)/(BR \cdot AS) \tag{4.1}$$

From the (4.1) we obtain:

$$\chi = r arctg\frac{x}{r} \quad \text{and} \quad \tau = \frac{t_0}{2}\log\frac{t_0 + t}{t_0 - t} \tag{4.2}$$

From the (4.2) second one, similar to the Milne's formula, we can see that the "formal" singularities are related to the projective description which depicts a universe with infinite space and finite time, whereas the De Sitter one is with finite space and infinite time. It is important to underline that such equivalence between an "evolutionary" model and a "stationary" one, differently from what is often stated, is purely geometrical and has nothing to do with the physical processes, but it deals with the cosmological observer definition.We will speak again about such fundamental point further.

The addition of durations' new law:

$$d = \frac{d_1 + d_2}{1 + d_1 d_2/t_0^2} \tag{4.3}$$

it is obtained by the (3.10) formulae and finds its physical meaning in the appearing of the new $t_0 = r/c$, interpretable as the "universe age" for any P^4observer family.

Let us consider a uniform motion with U velocity, given by $x^{'} = Ut^{'}$, by means of Fantappié transformations we have a uniform motion with W velocity given by:

$$W = \frac{(1+\alpha^2)U + (1-\gamma^2)V + \alpha\gamma c\,(1 - UV/c^2)}{a\,(1 + UV/c^2)} \tag{4.4}$$

For the visible universe of the O observer, inside the light-cone, it is valid the condition $\alpha = \pm\gamma$ and $a = 1$, and the (4.4) can be simplified as:

$$W = \frac{U + V \pm \alpha^2 c\,(1 + U/c)\,(1 - V/c)}{1 + UV/c^2} \tag{4.5}$$

For $V = c$ then $W = c$, according to RR, while for $U = c$ we have:

$$W = c \pm 2\alpha^2 c(1 - V/c)/(1 + V/c) \neq c \tag{4.6}$$

The (4.6) expresses the possibility of observing hyper-c velocity in PSR. The outcome is less strange than it can seem at first sight, because now the space-time of an observer is defined not only by the c constant but also by r, and the light-cone is at variable aperture. In straighter physical terms it means that when we observe a far universe region of the $t_0 = r/c$ order, the cosmic objects' velocity appears to be superior to c value, even if the region belongs to the light-cone of the observer's past. For b =0 we obtain the angular coefficients of the tangents to the (1.3) Cayley-Klein invariant starting from a P point of the Beltrami-Castelnuovo projection, which represent the two light-cone's straight lines. Differently from RR, here the light-cone's angle is not constant and depends on the P point according to the formula:

$$tg\vartheta = 2a/\left(\alpha^2 + \gamma^2\right) \tag{4.7}$$

From the (4.7) derives the C variation of the light velocity with time:

$$C = \frac{c}{\sqrt{1 - \gamma^2}}, \quad \text{with} \quad \gamma = \frac{t}{t_0} = \frac{ct}{r}, \tag{4.8}$$

from which follows that $C \to \infty$ in the two $\pm t_0$ singularities which fix the limit duration according to the addition of durations' new law (4.3).

Another remarkable consequence of the projective group is the expansion-collapse law, that is the connection between the two singularities. Differentiating the (3.10) and dividing them we obtain the velocities' variation law for a translation in time:

$$V^{'}\sqrt{1 - \gamma^2} = V\left(1 + \gamma t/t_0\right) - \gamma x/t_0 \tag{4.9}$$

For $\gamma = 1$ and $T_0 = t_0$ we have the law of projective expansion valid for $-t_0 < t < 0$:

$$V = \frac{x}{t + t_0}, \quad \text{or} \quad \text{also} \quad \beta = \frac{\alpha}{1 + \gamma} \tag{4.10}$$

If $\gamma = 0\,(t = 0)$, we can write

$$V = x/t_0 = Hx, \quad (\beta = \alpha) \tag{4.11}$$

where $H = c/r = 1/t_0$ is the well-known Hubble constant.

The analogous procedure will be followed for the law of projective collapse valid for $0 < t < t_0$, with $\gamma = -1$ and $T_0 = -t_0$:

$$V = \frac{x}{t - t_0}, \quad \text{or} \quad \beta = \frac{\alpha}{\gamma - 1} \tag{4.12}$$

We note that in singularities the expansion-collapse velocity becomes infinite. In PSR such process, differently from GR, is not connected to gravitation, but derives from Beltrami-Castelnuovo geometry.

From the Fantappié group it also follows a new formula for the Doppler effect:

$$\omega^{'} = \omega\sqrt{(1-\beta)/(1+\beta)+\alpha^2}, \tag{4.13}$$

where ω is the frequency. For $\beta = 1$, which is $V = c$, we get nothing but the traditional proportionality between distance and frequency, $\omega^{'} = \alpha\omega$. For V =0 there follows a Doppler effect depending on distance:

$$\omega^{'} = \omega\sqrt{1+\alpha^2} \tag{4.14}$$

The z red-shift is defined by $1 + z = \omega/\omega^{'}$ and the (13.4) becomes:

$$1/(1+z) = \sqrt{(1-\beta)/(1+\beta)+\alpha^2}, \tag{4.15}$$

which was historically introduced- in a 1930 Accademia dei Lincei famous memoir- by Castelnuovo to explain the "new" Hubble observations on galactic red-shift. If we are placed on the observer's light-cone where the (12.4) becomes $\beta = \alpha/(1-\alpha)$, the (4.15) will be:

$$1 + z = 1/(1-\alpha) \tag{4.16}$$

The red-shift tends towards infinity for $x = r$, and hyper-c velocities are possible if $z > 1$.

As everybody would naturally expect, modifying geometry implies, as well as in RR, a deep redefinition of mechanics. In PSR, the m mass of a body varies with velocity and distance according to:

$$m = m_0 a^2/b \tag{4.17}$$

From the (4.17) it follows that for $a = 0$, in singularities, the mass is null, while on the light-cone, for $b = 0$, $m \to \infty$. The mass of a body at rest varies with t according to:

$$m = m_0\left(1-\gamma^2\right), \tag{4.18}$$

from which we deduce that at the initial and final instant, $\gamma = \pm 1$, the mass vanishes.

Another greatly important outcome (Arcidiacono, 1977) is the relation between m mass and the J polar inertia momentum of a body:

$$J = mr^2 \tag{4.19}$$

A remarkable consequence is that the universe Mmass varies with t:

$$M(t) = M_0\left(1-\gamma^2\right) + \frac{J}{r^2}, \tag{4.20}$$

where M_0 is the mass for $t = 0$, and J the polar momentum with respect to the observer.

So the overall picture for an inertial observer in a De Sitter Universe is that of a universe coming into existence in a singularity at $-t_0$ time, expanding and collapsing at t_0 time and where c light velocity is only locally constant. In the initial and final instants the light velocity is infinite and the global mass is zero while in the expansion-collapse

time it varies according to (4.20). In the projective scenario the space flatness is linked to the observer geometry in a universe at constant curvature. All this is linked to the fact that in PSR the translations and rotations are indivisible. In the singularities there is no "breakdown" of the physical laws because the global space-time structure is univocally individuated by the group which is independent of the matter-energy distribution. In this case, the singularities in P^4 are – more properly- an horizon of events with a natural "cosmic censure" fixed by observers' geometry.

5. The Projective Gravitation

The connection between the metric approach to Einstein gravitation and Fantappié-Arcidiacono group one is the aim of Projective General Relativity(PGR), which describes a universe globally at constant curvature and locally at variable curvature. It can be done by following the Cartan idea, where any V^4 Riemann manifold is associated with an infinite family of Euclidean, pseudo-Euclidean, non-Euclidean spaces tangent to it in each of its P points. Those spaces' geometry is individuated by a holonomy group. The Cartan connection law links the tangent spaces so as to obtain both the V^4 local characteristics (curvature and torsion) and the global ones (holonomy group). The GR holonomy group is the one at four dimension rotations, i.e. the Lorentz group. So we get a general method which builds a bridgeway up between differential geometry and group theory (Pessa, 1973; Arcidiacono, 1986)

To make a PGR it is introduced the V^5 Riemann manifold which allows as holonomy group the De Sitter-Fantappié one, isomorphic to the S^5 five-dimensional rotations' group. The V^5 geometry is successively written in terms of Beltrami projective inducted metric for a anholomonous V^4 manifold at variable curvature. The Veblen projective connection:

$$\pi^A_{BC} = \left\{ {A \atop BC} \right\} = \frac{1}{2} g^{AS} \left(\bar{\partial}_C g_{BS} + \bar{\partial}_B g_{CS} - \bar{\partial}_S g_{BC} \right) \tag{5.1}$$

defines a projective translation law which let the field of the Q quadrics invariant in the tangent spaces, in each V^4 point, $Q = g_{AB} \bar{x}^A \bar{x}^B = 0$,where g_{AB} are the coefficients of the five-dimensional metric, the $\bar{x}^K$ are the homogeneous projective coordinates, and (ABC)=0,1,..,4.From the (5.1) we build the projective torsion-curvature tensor:

$$R^A_{BCD} = \bar{\partial}_C \pi^A_{BD} - \bar{\partial}_D \pi^A_{BC} + \pi^A_{SC} \pi^S_{BD} - \pi^A_{SD} \pi^S_{BC} \tag{5.2}$$

So the gravitation equations of Projective General Relativity are:

$$R_{AB} - \frac{1}{2} R g_{AB} = \chi T_{AB}, \tag{5.3}$$

with T_{AB}energy-momentum tensor, and χ Einstein gravitational constant. The (5.2) tensor is projectively flat, i.e. when it vanishes we get the De Sitter space at constant curvature. The deep link between rotations and translations in S^4 naturally leads the (5.3) to include the torsion, showing an interesting formal analogy with Einstein-Cartan-Sciama-Kibble spin-fluids theory. The construction is analogous to the GR one, but in lieu

of the relation between Riemann curvature and Minkowski s-t, we get here a curvature-torsion connected to the De Sitter-Fantappié holonomy group. It has to be noted that, in concordance with the equivalence principle, the PGR gives a metric description of the *local* gravity, valid for single(i.e., non cosmological) systems.

It is here proposed again the problem of the relations between local physics and its extension on cosmic scale. In fact, if we take the starting expression of standard cosmology based upon GR, i.e. *let us consider the whole matter of Universe*, and transfer it within the ambit of PGR, we can ask ourselves if the torsion role, associated to the rotation one, could get a feed-back on the background metric, modifying it deeply. Generally, the syntax of a purely group-based theory does not get the tools to give an answer, because it is independent from gravity and the hypotheses on T_{AB}. For example, Snyder (Snyder, 1947) showed that in a De Sitter space it is introduced an uncertainty relation linked to a curvature of the kind: $\Delta x^i \Delta x^k \approx 1/r^2$. Only a third quantization formalism, able to take into account the dynamical two-way inter-relations between local and global, will succeed in giving an answer.

The essential point we have to underline here is that the introduction of a cosmological constant, both as additional hypothesis on Einstein equations or via group, is a radical alternative to the "machian philosophy" of the GR.

So, for a Universe without metter-fields we assume the constant curvature as a sort of "pre-matter" which describes in topological terms the most general conditions for the quantum vacuum. Therefore the Einstein equations in the following form are valid:

$$G^{AB} = \Lambda g^{AB} \quad \text{and} \quad R^{AB} = (R/2 - \Lambda)\, g^{AB}, \tag{5.4}$$

with their essentially physical content, i.e. the deep connection among curvature, radius and matter-energy's density ρ_{vac} by means of the cosmological constant:

$$\rho_{vac} = \frac{c^2 \Lambda}{8\pi G} \tag{5.5}$$

6. De Sitter Observers, Singularities and Wick Rotations

From a quantum viewpoint the S^4 interesting aspect is that it is at imaginary cyclic time and without singularities. It means that it is impossible to define on De Sitter a global temporal coordinate. So it has an istanton feature, individuated by its Euler topological number which is 2 (Rajaraman,1982). This leads to a series of formal analogies both with black holes' quantum physics and the theoretical proposals for the "cure" for singularities.

Let us consider the De Sitter-Castelnuovo metric in real time:

$$ds^2 = -\left(1 - \frac{H^2}{c^2} r^2\right) dt^2 + \left(1 - \frac{H^2}{c^2} r^2\right)^{-1} dr^2 + r^2 d\Omega^2, \tag{6.1}$$

where $d\Omega^2 = d\vartheta^2 + \sin^2 \vartheta d\varphi^2$in polar coordinates.

As we have seen in PSR, the singularity in $r = c/H$ becomes an horizon of events for any observer when it passes to the Euclidean metric with $\tau \to -it$:

$$ds^2 = d\tau^2 + \frac{1}{H^2}\cos H\tau \left(dr^2 + \sin^2 r d\Omega^2\right), \tag{6.2}$$

with a close analogy with the Schwarzschild solution's case. The τperiod is $\beta = 2\pi/H$; for the observers in De Sitter it implies the possibility to define a temperature, an entropy and an area of the horizon, respectively given by:

$$T_b = \frac{H}{2\pi} = \beta^{-1}; \quad S = \frac{\pi}{H^2} = \frac{\beta^2}{4\pi}; \quad A = \frac{4\pi}{H^2} = \frac{\beta^2}{\pi} \tag{6.3}$$

From the (6.3) we get the following fundamental outcome:

$$S = \frac{1}{4}A, \tag{6.4}$$

which is the well-known expression of the t'Hooft-Susskind-Bekenstein Holographic Principle (Susskind, 1995). The (6.4) connects the non-existence of a global temporal coordinate with the information accessible to any observer in the De Sitter model. In this way we obtain a deep physical explanation for applying the Weyl Principle in the De Sitter Universe, and sum up that in cosmology, as well as in QM, a physical system cannot be fully specified without defining an observer. G. Arcidiacono stated that the hyper-spherical Universe is like *a book written with seven seals* (Apocalypse, 6-11), and consequently two operations are necessary to investigate its physics: 1) inverse Wick rotation and 2) Beltrami-Castelnuovo representation. That's the way we can completely define a relativity in De Sitter.

The association of imaginary time with temperature gets a remarkable physical significance which implies some considerations on the statistical partition function (Hawking, 1975). For our aims it will be sufficient to say that such temperature is linked to the (6.4) relation, i.e. to the information that an observer spent within his area of events. Which thing has patent implications from the dynamical viewpoint, because it is the same as to state that, as well as in Schwarzschild black hole' s case, the De Sitter space and the quantum field defined on it behave as if they were immersed in background fluctuations. The transition amplitude from a configuration of a ϕ generic field in $t_2 - t_1 = dt$ time will be given by the e^{-iHdt} matrix element which acts as a $U(1)$ group transformation of the $U(1)_{space} \Leftrightarrow U(1)_{time}$. It means that a transition amplitude on S^4 will appear to an observer as the $R(t)$scale factor's variation with Hvariation rate.

It makes possible to link the hyper-spherical description with the Big-Bang evolutionary scenario and to get rid of the thermodinamic ambiguities which characterize its "beginning" and "ending" notions. The last ones have to be re-interpretated as purely quantum dynamics of the matter-fields on the hyper-sphere free of singularities.

7. Physical Considerations for Further Developments

Such considerations suggest a research program we are going here to shortly delineate ; it furthermore develops the analogy between black holes, istantons and De Sitter

Universes (see – for example – Frolov, Markov, Mukhanov, 1989;Strominger, 1992). It is known that the Hartle-Hawking proposal of "no-boundary" condition removes the initial singularity and allows to calculate the Universe wave function (Hartle-Hawking, 1989). In fact, it is possible – as in the usual QFT- to calculate the path integrals by using a Wick rotation as "Euclidization" procedure. In such way also the essential characteristics of the inflationary hypotheses are englobed (A. Borde, A. Guth and A. Vilenkin, 2003). The derived formalism is similar to that used in the ordinary QM for the tunnel effect, an analogy which should explain the physics at its bottom (Vilenkin, 1982; S.W. Hawking and I.G. Moss, 1982).

The group extension method provides this procedure with a solid foundation, because the De Sitter space, maximally symmetric and simply connected, is univocally individuated by the group structure, and consequently is directly linked to the space-time homogeneity and isotropy principle with respect to physical laws. The original Hartle-Hawking formulation operates a mix of topologies hardly justified both on the formal level and the conceptual one. The "no-boundary" condition is only valid if we works with imaginary time, and the theory does not contain a strict logical procedure to explain the passage to real time. This corresponds to a quite vague attempt to conciliate an hyper-spherical description at imaginary time with an evolutive one at real time according to the traditional Big-Bang scenario.In fact, it has been observed that the Hartle-Hawking condition is the same as to substitute a singularity with a "nebulosity".

The spontaneous proposal, at this point, is considering the Hartle-Hawking conditions on primordial space-time as a consequence of a global charaterization of the hyper-sphere and directly developing quantum physics on S^4.Which thing does not contradict the quantum mechanics formulation and its fundamental spirit, which is to say the Feynman path integrals. In other words, quantum mechanics has not to be applied to cosmology for the Universe smallness at its beginning, but because each physical system – without exception- gets quantum histories with amplitude interferences. We point out that such view is in perfect consonance with the so-called quantum mechanics Many Worlds Interpretation (Halliwell, 1994). The "by nothing creation" means that we cannot "look inside" an istanton (hyper-spherical space), but we have to recourse to an "evolutionary" description which separates space from time. The projective methods tell us how to do it.

An analogous problem– to some extent – is that of the Weyl Tensor Hypothesis. Recently, Roger Penrose has suggested a condition on the initial singularity that, within the GR, ties entropy and gravity and makes a time arrow emerge (Penrose,1989). It is known that the W_{ABCD} Weyl conformal tensor describes the freedom degrees of the gravitational field. The Penrose Hypothesis is that $W_{ABCD} \to 0$ in the Big-Bang, while $W_{ABCD} \to \infty$ in the Big-Crunch. The physical reason is that in the Universe's initial state we have an highly uniform matter distribution at low entropy (entalpic order), while in Big-Crunch, just like a black hole, we have an high entropy situation. This differentiates the two singularities and provides a time arrow. In an hyper-spherical Universe there is no "beginning" and "ending", but only quantum transitions.Consequently, the Penrose

Hypothesis can only be implemented in terms of projective representation within the ambit of PGR.

Finally, we can take into consideration the possibility to build a Quantum Field Theory on S^4. A QFT, for T tending towards zero, is a limit case of a theory describing some physical fields interacting with an external environment at T temperature. Without this external environment we could not speak of dechoerence , could not introduce concepts such as like dissipation, chaos, noise and, obviously, the possibility to describe phase transitions would vanish too. Therefore, it is of paramount importance to write a QFT on De Sitter background metric and then studying it in projective representation. If we admit decoherence processes on S^4, it is possible to interpret the Weyl Principle as a form of Anthropic Principle: the "classical" and observable Universes are the ones where it can be operated a description at real time.

In conclusion, it is possible to delineate an alternative, but not incompatible with traditional cosmology scenario.The Universe is the quantum configuration of the quantum fields on S^4.Thus developing a Quantum Cosmology coincides with developing a Quantum Field Theory on a space free of singularities.The Big-Bang is a by vacuum nucleation in an hyper-spherical background at imaginary time, and so the concepts of "beginning", "expansion" and "ending" belong to the space-time foreground and gain their meaning only by means of a suitable representation which defines a family of cosmological observers.

Acknowledgements:

I owe my knowledge of the group extension method to the regretted Prof. G. Arcidiacono (1927 - 1998), during our intense discussions while strolling throughout Rome.

Special thanks to my friends E. Pessa and L. Chiatti for the rich exchange of viewpoints and e-mails.

References

[1] Arcidiacono, G.(1956), *Rend. Accad. Lincei,*20, 4

[2] Arcidiacono, G. (1976), *Gen. Rel. And Grav.*, 7, 885

[3] Arcidiacono, G.(1977), *Gen. Rel. And Grav.*,7, 865

[4] Arcidiacono, G.(1984), in De Sabbata,V. & T.M.Karade (eds), *Relativistic Astrophysics and Cosmology,* World Scientific, Singapore

[5] Arcidiacono, G. (1986), *Projective Relativity,Cosmology and Gravitation*, Hadr.Press,Cambridge,USA

[6] Borde, A. , Guth,A., Vilenkin, A. (2003),*Phys. Rev. Lett.* 90

[7] Ellis, G.F.R. & Wiliams,R. (1988), *Flat and Curved Space-Times*, Clarendon Press

[8] Fantappié, L. (1954), *Rend. Accad. Lincei*,17,5

[9] Fantappié, L. (1959), *Collectanea Mathematica*,XI, 77

[10] Frolov, V.P., Markov,M.A., Mukhanov,V.F.(1989), *Phys.Lett.,*B216,272

[11] Halliwell,J.J. (1994), in Greenberger,D. & Zeilinger,A.(eds), *Fundamental Problems in Quantum Theory*, New York Academy of Sciences,NY

[12] Hartle, J.B. & Hawking,S.W. (1983), *Phys.Rev.D*,28,12

[13] Hawking,S.W. & Ellis, G.F.R.,(1973) *The Large Scale Structure of Space-Time,*Cambridge Univ.Press

[14] Hawking,S.W. (1975), *Commun.Math.Phys.*,43

[15] Hawking, S.W. & Moss, I.G. (1982), *Phys.Lett.*,B110,35

[16] Pessa,E. (1973), *Collectanea Mathematica*,XXIV,2

[17] Rajaraman,R.(1982), *Solitons and Istantons*,North-Holland Publ.,NY

[18] Snyder, H.S. (1947), *Phys.Rev.*, 51,38

[19] Strominger, A. (1992), *Phys.Rev.D*,46,10

[20] Susskind, L. (1995), *J. Math. Phys.*,36

[21] Vilenkin, A. (1982), *Phys.Lett.*,117B,1.

EJTP **3, No. 10** (2006) 225–238 *Electronic Journal of Theoretical Physics*

Majorana and the Investigation of Infrared Spectra of Ammonia

E. Di Grezia[1,2] *

[1] *Dipartimento di Scienze Fisiche, Università di Napoli "Federico II" Complesso Universitario di Monte S. Angelo, Via Cinthia, I-80126 Napoli, Italy*
[2] *Istituto Nazionale di Fisica Nucleare, Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cinthia, I-80126 Napoli, Italy*

Received 22 February 2006, Published 28 May 2006

Abstract: An account is given on the first studies on the physics of ammonia, focusing on the infrared spectra of that molecule. Relevant contributions from several authors, in the years until 1932, are pointed out, discussing also an unknown study by E.Majorana on this topic.

Keywords: Ettore Majorana, Infrared Spectra of Ammonia
PACS (2006): 33.20.Ea, 32.30.30.Bv, 36.20.Ng

1. Introduction

Because of the intensity and richness of its spectrum, ammonia has played a great role in the development of microwave spectroscopy. It has provided a large number of observable lines on which to try both experimental techniques and the theory. NH_3 provides the simplest and most thoroughly worked out example of a class of spectra which occupied and puzzle microwave spectroscopists for many years. In the paper of 1932 Fermi [1] discusses the influence of the ammonia molecule's rotation on the doubling of its levels. This doubling originates -according Dennis and Hardy [2]- in the oscillation by which the nitrogen atom crosses the plane determined by the three hydrogens, i.e., due to inversion respect the plane of the three atoms of H influenced by the rotation of molecule and he compared the theoretical results with the experimental results and he found accords (inversion problem). This paper on NH_3, together with other three articles on the accidental degeneracy of the carbon dioxide molecule's frequencies of oscillation on the Raman effect in crystals, constitutes a series of investigations from the period

* Elisabetta.Digrezia@na.infn.it

1931-33 in which Fermi attempted to explain various molecular phenomena. His interest in these studies is reflected in the book "Molecules and crystals" [3]. Fermi's interest in this type of problem was to give a quantitative explanation to experimental observation at the center of Institutes by Rasetti, which in the period 1929-1930 studied the Raman effect in diatomic gas O_2, N_2. In particular the paper on the NH_3 molecule is connected to experiments conducted in Rome during the same period by E. Amaldi on the Raman Effects with theoretical contribute by G. Palczek [4], and theoretical research by G. Placzek and E. Teller [5] on molecular spectra for CO_2 and NH_3.
In reality W.W. Coblentz [6] in 1905 investigated the positions and fine structure of the infrared bands of polyatomic molecules CO_2, NH_3. Coblentz observed two very intense bands at $\lambda = 10.7\mu, 6.14\mu$, with a considerable weaker band at $\lambda = 2.97\mu$.
Lately K. Schierklok [7] has re-examined the ammonia infra-red (IR) spectrum and in addition to the bands found by Coblentz, he has found a band at $\lambda = 2.22\mu$. Beyond this he found two bands at $\lambda = 1.94\mu, 1.49\mu$ whose intensity was about half that of the two previous bands; so Schierklok observed six bands.

Historically the study of systems with several atoms, combining or not to form a molecule, has interested chemists for many years through the rules of valence. But only at the beginning of "900 the physicists interpret these rules in the light of quantum mechanics and the behavior of the constituents of atom with the spectral analysis of radiation emitted by the atom. Heitler and London [8] connected the valence in the formation of homopolar diatomic molecules with symmetry character of wave functions of the outer electrons in each atom. The problem of the vibration groups of atoms possessing geometric symmetry has been considered for the first time by C. J. Brester [9]. A number of models representing particular molecules have been treated making use of various assumptions to obtain the potential energy function. Historically the first example is that of CO_2 [10]. Hund [12] and Kornfeld [13] examined the spectra of H_2O, H_2S, CO_3 ion, NH_3 [14]. Dennison [15] found the normal vibrations for models of NH_3 and CH_4 assuming the forces to be central and Nielsen made a like treatment of the CO_3 ion. In all these investigations, the molecule was assumed to have a certain geometry symmetry in its equilibrium configuration. So another more simple way to analyze these models is to use the theory of vibrations [16]. In particular our interest will be on the NH_3- molecule which Coblentz in 1905 will start to study. In the present article we will investigate how the NH_3- molecule, in particular, has been studied qualitatively by character of the vibration of symmetrical polyatomic molecules through the theory of vibrations [12], [15], approximatively using the Wentzel-Kramers-Brillouin method of approximation. And a quantitative analysis with exact solution for a two-minima problem of the ammonia molecule solving secular equation.
We will analyze the historical development of important works in studying molecular spectra by means of the quantum theory, and in obtaining information about the structure of the molecule through an examination of the positions and fine structure of the IR bands. Then we will present briefly the experiments in observations of vibrational and rotational transitions in the cases of gas NH_3 and the comparison of theoretical and experimental

amounts of vibrational frequencies.

Different activities of the Fermi group at the Physics Institute in Rome from 1930 to 1932 was devoted to this subject. In this contest is the contribute of Majorana.

2. Theoretical Analysis of Ammonia Spectra.

2.1 A Brief Introduction to the Theory of Vibrations.

The classical theory of vibrations about an equilibrium configuration has developed from Galileo's study of small oscillations of a pendulum. In the first half of the eighteenth century Brook Taylor, D'Alambert, Euler, Daniel Bernoulli investigated the vibrations of a stretched cord. In 1753 Bernoulli enunciated the principle of the resolution of all compound types of vibration into independent modes. In $1762-1765$ Lagrange gave the general theory of the vibrations of a dynamical system with a finite number of degrees of freedom. One considers a vibrating system defined by its kinetic energy T and its potential energy V and its position is specified by a set of coordinates $(q_1, q_2, .., q_n)$, giving the displacements from equilibrium. The problem of vibrations around an equilibrium configuration is to solve Lagrangian equations of motion in which the kinetic T (a positive definite form with $|a_{ij}| \neq 0$) and a potential energies V (Taylor expansion in powers of $q_1, q_2, ..., q_n$) are homogenous quadratic forms in velocities and coordinates respectively, with constant coefficients:

$$T = \frac{1}{2}\left(a_{11}\dot{q}_1^2 + ... a_{22}\dot{q}_2^2 + \cdots + a_{nn}\dot{q}_n^2 + 2a_{12}\dot{q}_1\dot{q}_2 + 2a_{13}\dot{q}_1\dot{q}_3 + +2a_{n-1}\dot{q}_{n-1}\dot{q}_n\right) \quad (1)$$

$$V = \frac{1}{2}\left(b_{11}q_1^2 + b_{22}q_2^2 + ... + b_{nn}q_n^2 + 2b_{12}q_1q_2 + 2b_{13}q_1q_3 + 2b_{n-1}q_{n-1}q_n\right) \quad (2)$$

The equation of motion are:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_r}\right) = -\frac{\partial V}{\partial q_r} \quad (r = 1, 2, \cdots n) \quad (3)$$

If T and V has the form (1), (2) (following the method of Jordan [18]), it is always possible to find a

linear transformation of coordinates $q_i = \sum_{k=1}^{n} c_{ik}x_k$ such that the kinetic and potential energies, expressed in terms of the new coordinates, called *normal (principal) coordinates*, have the form:

$$T = \frac{1}{2}\left(\dot{x}_1^2 + ... \dot{x}_2^2 + \cdots + \dot{x}_n^2\right) \quad (4)$$

$$V = \frac{1}{2}\left(\lambda_1 x_1^2 + \lambda_2 x_2^2 + ... + \lambda_n x_n^2\right) \quad (5)$$

where the constants $\lambda_1, \cdots, \lambda_n$, which occur as coefficients of the squares of x_k in V, are the n distinct or multiple roots of the determinant $det(a_{ik}\lambda - b_{ik}) = 0$, and a_{ik}, b_{ik} are

the coefficients in the original expressions of T and V energies. The Lagrangian equation of motion is therefore:

$$\ddot{x}_r + \lambda_r x_r = 0 \quad r = 1, 2, \cdots, n. \tag{6}$$

Thus the classical theory of small oscillations shows that system will vibrate as an aggregation of n independent mode of vibration of the system, provided the corresponding constant λ_r is positive (stable equilibrium configuration), with normal or characteristic frequencies $\nu_i = \lambda_i^{1/2}/2\pi$. Moreover every conceivable vibration of the system may be regarded as the superposition of n independent normal vibrations according Daniel Bernoulli's principle [19]

2.2 Ammonia Molecule Analysis with Theory of Vibrations.

The vibration spectra of polyatomic molecules, in particular of NH_3, has been investigated in great details both theoretically and experimentally. For this system one has a number $s = 4$ of atomic nuclei which one assumes to have a possible equilibrium position. Dealing with the internal or vibrational degrees of freedom, the whole system has $n = 3s - 6 = 6$ degrees of freedom. The ammonia molecule is like a one-dimensional system of a particle moving in a potential field consisting of two equal minima and was first treated qualitatively by Dennison and Hund [12], [15]. Dennison and Hund assumed that the behavior of the nuclei in the neighborhood of their equilibrium positions may be described by means of central forces acting between them in the case of a polyatomic molecule with certain limitations in regard to the character of the equilibrium of the system.

The assumption for a molecule of the type XY_3 is that in the normal state of the molecule the $X-$atom is equidistant from each of the $Y-$atoms which themselves lie at the corners of an equilateral triangle. It is further assumed that the $X-$atom does only four frequencies, as indeed will any model which posses an axis of symmetry (Hund) and so X is at the apex of a regular pyramid with an equilateral triangle as a base. Experimentally four fundamental ν are found (without axial symmetry are found six fundamental frequencies).

Then they assumed that the four independent active frequencies are four fundamental absorption bands because of their fine structure.

Their study of NH_3 was motivated from IR spectroscopy measures and Raman spectra for polyatomic molecules $CO_2, N_2O, NH_3, CH_4, C_2H_4$ during the period $1905 - 1935$. Dennison and Hund, separately, showed, for molecules H_2O, NH_3, CH_4, that the vibrational levels which lie below the potential maximum occur in pairs. To find the normal vibrations they used the wave mechanical treatment of vibration spectrum of NH_3-molecule and to obtain their properties they investigated the geometric symmetry of NH_3 in its equilibrium configuration. Let there be chosen a set of coordinates $q_1, .., q_6$ giving the displacements from equilibrium. In considering the system either in classical mechanics or in wave mechanics, the first step is to find the Hamiltonian. To the approximation in which the motions of the atoms are small compared with the inter-atomic distances,

the system may absorb or emit radiation with a series of frequencies. These frequencies are the so-called normal frequencies and may be computed with the classical theory of small oscillations (theory of vibrations [16] we have summarized in the above section), for which in first approximation the kinetic and potential energies assume a simple form:

$$T = \frac{1}{2}\left(a_{11}\dot{q}_1^2 + ... + a_{66}\dot{q}_6^2 + 2a_{12}\dot{q}_1\dot{q}_2 + ...\right) \tag{7}$$

$$V = \frac{1}{2}\left(b_{11}q_1^2 + ... + b_{66}q_6^2 + 2b_{12}q_1q_2 + ...\right) \tag{8}$$

where the $a's$ and $b's$ are constants.
Then a linear transformation to normal-coordinates:

$$q_1 = \sum_{k=1}^{n} c_{ik}x_k \tag{9}$$

whereby T and V are diagonal. The λ's are the n roots, distinct or multiple, of the determinant:

$$det(a_{ik} - b_{ik}) = 0 \tag{10}$$

The Hamiltonian may be then written:

$$H = H_1 + ... + H_6 \tag{11}$$

where

$$H_i = \frac{1}{2}p_i^2 + \frac{1}{2}\lambda_j x_i^2 \tag{12}$$

So one has an aggregation of 6 independent simple harmonic oscillators, i.e., in the language of wave mechanics, the wave function of the whole system is the product of the wave function for the individual oscillators and characteristic value is the sum of the individual eigenvalues. This method is allowed because the system is separable in 6 normal coordinates.

The properties of 6 normal fundamental vibrations frequencies related to λ can be obtained following the *Hund* 's analysis.
In the investigation, the molecule NH_3 is assumed to have a certain geometric symmetry in its equilibrium configuration. In fact in considering the vibration spectrum of a tetratomic molecule of the general type XY_3 (i.e. NH_3), the assumption is that in the normal state of the molecule the $X(N)$-atom at

equilibrium position is equidistant, i.e., at the center of gravity, from each of $Y(H)$-atoms which themselves lie at the corners of an equilateral triangle, not in the same plane in which X-atom is. So a regular pyramid is the normal configuration of NH_3. The approximation is that the force fields between the X-atoms is strong and those connecting the X and the Y atoms is weak. In this case the potential function energy is assumed to have the same symmetry as the geometric configuration of the molecule. Then will be two frequencies ν_1 and ν_2 corresponding to the mutual vibrations of the $Y_3(H_3)$ group alone which have just the properties of triatomic molecule [17]. In ν_1 the Y atoms remain at the corners of an equilateral triangle throughout the motion. This oscillation

is along the symmetry axis so it is called a || vibration. While ν_2 is a double frequency due to an isotropic vibration of N-atom in a plane perpendicular to the symmetry axis, it is a $\perp$ vibration. The remaining normal vibrations of the system may be determined by considering the motion of the Y_3 group, taken as a rigid triangle, relative to the X atom. The vibration will consist of two sorts, a vibration ν_3 in which the triangle and the point X oscillate with respect to each other, the triangle plane remaining always parallel to itself. Then ν_3 is a single and a || vibration. The last frequencies ν_4 is represented by a typing motion of the triangle relative to the X-point. It is a double $\perp$ vibration frequency. So there are four independent active frequencies, two || and two $\perp$. Since the latter are double, there are six degrees of internal freedom corresponding to the formula of internal degrees of freedom for four atoms we have seen $n = 3s - 6$ where s is the number of atomic nuclei which one assumes to have a possible equilibrium position.
So this qualitative discussion done by Dennison, Hund allowed them to predict the essential features of IR spectrum of the XY_3 molecule. There will be four fundamental absorption bands. The intensity will be different depending upon the force fields, i.e., the configuration of the molecule. The fine structure of the band ν_1 is similar to the fine structure of the band ν_3 since they both correspond to a vibration along the symmetry-axis. The pair of bands ν_2 and ν_4 will have a similar fine structure because $\perp$ to the symmetry axis and will be unlike to the pair ν_1 and ν_3. Questions with regard to fine structure arise when one discusses experimental spectra by spectrometer analysis.

2.3 Ammonia Molecule Analysis with Theory of Groups

The ammonia infrared spectrum is an example of the application of group theory [11] to physics. Molecules absorb and emit electromagnetic radiation in wide areas of the spectrum. If electrons change state, the radiation may be in the visible region. Molecular ultraviolet spectra are rather rare, since molecules fall apart at these high energies. Changes in vibrational states are associated with infrared wavelengths, and changes in rotational states with the far infrared. There are even finer energy differences that cause spectra even in the radio-frequency region. All of these generally consist of a great number of lines, sometimes not resolved individually, forming bands and such.

Infrared spectra are a valuable tool for determining the structure of molecules. An infrared band is simpler than the band spectra in the visible, but still rather complex, consisting of several series of lines corresponding to transitions between different rotational states. Two methods are generally used, absorption spectra that study the transitions from the ground state to excited states, and Raman spectra that studies the changes in wavelength in scattered radiation. Raman spectroscopy can be done in the visible region with its more convenient experimental conditions, and with the powerful beams of lasers.

Quantum mechanics is necessary for the understanding of molecular spectra, which it perfectly explains. Then there is a relation of group theory to quantum mechanics. Symmetry is a powerful tool in the quantum mechanics of molecules, and the ammonia molecule furnishes a good example. One can consider what infrared and Raman spectra

are to be expected if the molecule is a symmetrical pyramid, which is indeed the case using the character analysis. The symmetrical pyramid has the symmetry group C_{3v}, whit its character table[1]:

C_{3v}	E	$2C$	3σ	$basis functions$
A_1	1	1	1	T_z, x^2+y^2, z^2
A_2	1	-1	1	R_z
E	2	-1	0	$(T_x, T_y)(R_x, R_y)(x^2-y^2, xy)(xz, yz)$

T and R are the representations to which components of the translation and rotation displacements belong; these are vectors and axial vectors, respectively. T also shows the representations of the dipole moment operator which produces the infrared spectrum. Then there are the quadratic functions which transform like the molecular polarizability, the operator which produces the Raman spectrum.

We assign three displacement coordinates to each atom, 12 in all for the four atoms. The first thing to do is to find the characters of this representation. The character for E is 12, since the identity transforms each coordinate into itself. The rotations about the axis leave only the displacements on the nitrogen in the same place, and the character is the same as that of the three T components, or 1 - 1 = 0. Reflections in a vertical plane leave the nitrogen and one hydrogen unmoved, and the character is easily seen to be 2 - 1 = 1 for each atom. Therefore, the characters of the reducible representation of the displacements is 12, 0, 2. This must include the representations of the translation and rotation of the molecule as a whole, $A_1 + A_2 + 2E$. Therefore, we subtract the characters $6, 0, 0$ to find the character of the vibrations, $6, 0, 2$. By character analysis, we find that this gives $2A_1 + 2E$. Ammonia, therefore, should exhibit four fundamentals, all active in both infrared and Raman spectra. This is exactly what is observed. The Raman spectra of the E fundamentals ought to be faint, and they were not observed (or were not until lasers came in). If the ammonia molecule were planar, two more fundamentals would be expected, and they are not observed. Herzfeld gives the four modes as follows. There is a very strong band at $1627.5 cm^{-1}$ (infrared spectroscopists use the reciprocal of the wavelength, since it is proportional to the frequency and the quantum energy), about 6.1μ, and is a so-called perpendicular band, which would be expected from the x and y

[1] This table defines the abstract group Ci, which has many representations, or concrete realizations. Let the symbol σ stands for the transformation x = -x. C is either of the rotations. In three dimensions, this would be a reflection in the yz-plane. We can use σ as an operator: $\sigma f(x) = f(-x)$, E is the identity operator, such that $Ef(x) = f(x)$ for any f(x).One calls the elements E and A_i, that all obey the same multiplication table.

components of the dipole moment. This is one of the doubly-degenerate E fundamentals, a symmetric bending of two of the hydrogens to or away from each other. The asymmetric bending is of higher frequency, $3414cm^{-1}$, and difficult to observe. These are the two E modes. There is a strong parallel band at $931.58cm^{-1}$ and $968.08cm^{-1}$, about 10.6μ corresponding to an A_1 representation. This band is double, and the reason is curious. The ammonia molecule can turn itself inside-out; that is, the nitrogen can pass through the plane of the hydrogens. This isn't easy, but the nitrogen can tunnel through, and the doubling is the result. The states divide into those symmetrical with respect to this inversion, and those that are antisymmetrical (change sign). The selection rules on the rotational transitions make the band separations the sum of the inversion splitting in the two cases. In the Raman spectrum, the separation is the difference of the splitting. The Raman bands are observed at $934.0cm^{-1}$ and $964.3cm^{-1}$. Finally, there is a strong band at $3335.9cm^{-1}$and $3337.5cm^{-1}$, and a Raman shift at $3334.2cm^{-1}$ (about 3.0μ) corresponding to the other A_1 fundamental. In this mode, the bond lengths lengthen and shorten symmetrically. The two A modes can be called bending and stretching, respectively.

ND_3, with the heavier deuterium substituted for the protons, gives somewhat different (lower) frequencies, and the shifts can be used to nail down the identification of the vibrational frequencies, confirming the conclusion that ammonia is a symmetrical pyramid. The inversion doubling is a very interesting phenomenon. It turns out to be possible to separate molecules in even and odd inversion states, and this led to the ammonia maser, the first of its kind. Although one can form a good picture of ammonia as if it were a macroscopic object, try to picture it with the nitrogen partly on both sides of the hydrogens!

Using the group theory F. Hund (1925) [12] studies the equilibrium of the molecule of ammonia, and he shows that, if the electronic configuration around the nitrogen, originally central, is capable of a polarization induced by the hydrogen nuclei, the molecule in the normal state have just the axial symmetrical form. So he assumed that the molecule NH_3 has a regular pyramid equilibrium configuration. The nitrogen atom at equilibrium position is equidistant, i.e. at the center of gravity, from each of hydrogen atoms, which lie at the corners of an equilateral triangle, not in the same plane of $N-$atom. In considering the vibrations of such a molecule he erroneously states that there exist only three active characteristic frequencies, whereas, unless the particles all lie in the same plane, there must in general exist four, as shown by Dennison [15]. Hund gives a table of harmonic and combinations bands of NH_3 with three fundamental frequencies $\nu_1 = 970cm^{-1}, \nu_2 = 1700cm^{-1}, \nu_3 = 4500cm^{-1}$, that may be changed by allowing the band at $\lambda = 97\mu$ i.e. $\nu \sim 3300cm^{-1}$ to become the fourth fundamental band.

2.4 Approximate Analysis of NH_3 with WKB Method.

Dennison and Uhlenbeck [20] compute the level separation of NH_3, using the Wentzel-Kramers-Brillouin (WKB) method of approximation for a one-dimensional system of a

particle moving in a potential field, consisting of two equal minima. Then they make an application of the results to the ammonia molecule to determine its form. The WKB method yields an approximate solution of the wave equation whose form depends upon whether the region considered lies within or without the region of classical motion, that is, the region where the kinetic energy is positive. In the first case the solution is oscillatory, in the second or non-classical region the solution consists of a linear combination of an increasing and a decreasing exponential. At each boundary or critical point are valid the so-called Kramers connection formulae [21]. These formulae furnish a method by which one may approximate to any solution of the wave equation.
They show that the infrared spectrum of the ammonia molecule exhibits features which may be directly related to the one dimensional problem of two equal minima. The parallel type vibration bands for example are observed to be composed of two nearly superimposed bands, depending upon the fact that there are two equivalent positions of equilibrium for the nitrogen nucleus. Symmetrical molecules of the NH_3 type which are not coplanar exhibit that all vibrational levels ar double, depending upon the fact that there are two exactly equivalent positions of equilibrium for N atom, one above the plane of the H atoms, and the other at an equal distance below. A quantum mechanical treatment reveals that it causes the vibrational level become double. The doublet separation is small compared with the spacing of vibrational levels (inversion problem related to rotational spectrum).

The physical origin and theoretical description of this doubling is presented, followed by a description of the experimental measurement. The inversion doubling of about 35 cm-1 represents an excellent coupling of a simple infrared measurement with a quantum mechanical description involving many aspects of the wave nature of vibrations. The normal modes of Ammonia are $\nu_2 = 950cm^{-1}$,(symmetric bend), $\nu_{4a} = 1627cm^{-1}$ (asymmetric bend), $\nu_{4b} = 1627cm^{-1}$ (asymmetric bend), $\nu_1 = 3336cm^{-1}$ (symmetric stretch), $\nu_{3a} = 3414cm^{-1}$ (asymmetric stretch), $\nu_{3b} = 3414cm^{-1}$ (asymmetric stretch); ν_{3a}, ν_{3b} are degenerate modes, as are ν_{4a}, ν_{4b}. All six normal modes are IR active.

2.5 Exact Analysis of NH_3.

Rosen and Morse [22] give an analysis of the vibration of the nitrogen in the ammonia molecule using an exact solution of the wave equation for a form of one-dimensional potential energy. The potential energy for this molecule has two minima at distance $2x_m$ apart, separated by a "hill" of height H.
They describe another solution, for a form of potential field different from that of Dennison [17] and they give an example of its application to the vibrational states of NH_3. Due to the symmetry of the molecule there are two equivalent positions of equilibrium for the nitrogen, at equal distances above and below the plane of the three hydrogens. This equivalence of the two minima makes every vibrational level a doublet, a result which is found experimentally. To analyze the vibrational behavior one separates off the coordinates of the center of the gravity of the molecule and the Euler angles fixing its

orientation in space, and deal only with the coordinates fixing the relative positions of the atoms.
One of these coordinates is x, the distance of the nitrogen (N) atom from the plane of the hydrogen. The other five coordinates z_1, z_2, z_3, z_4, z_5, can be chosen that the positions of the two equilibrium configurations are at $z_1 = z_2 = z_3 = z_4 = z_5 = 0, x = \pm x_m$. The potential function $V(x, z_1, z_2, z_3, z_4, z_5)$ therefore has its two minima at these two points. They justify the use of x as a "normal" coordinate (i.e. splitting from the general six-dimensional problem to a one-dimensional problem in x alone) by the following method. From considerations of symmetry all the wave functions are symmetric or antisymmetric about the nodal hypersurface $x = 0$. They give a two minima potential field $V(x)$ which is amenable of exact solution. For each level of the one minimum problem there is a pair of levels for the double minimum case. The separation between the levels in a pair is small compared to the energy difference between different pairs as long as the levels are below the top of the intermediate hill.

Salant and Rosenthal in 1932 [23] derive expressions for the effects of isotopy on the normal frequencies, following Dennison's [17] general, noncentral force treatment of the normal modes of vibration of symmetrical triatomic and tetratomic molecules.
Sanderson and Silverman in 1933 [24], following the procedure of Dennison [17], calculate the positions of the fundamental vibrations of molecule ND_3.
Rosenthal [25] summarizes briefly the general procedure for obtaining the normal vibration frequencies of a molecule of any type of symmetry, without the use of group theory. He writes the expression for the kinetic energy T in terms of the displacements of the various atoms from their equilibrium positions. The potential energy, V, is written in terms of the mutual displacements of the atoms as the most general quadratic form consistent with geometrical symmetry. As the next step, linear combinations of the original displacements are introduced and both T and V are transformed to them. The normal vibration frequencies, ω, or rather $\lambda = 4\pi^2\omega^2$ are then obtained as the roots of $|\lambda T - V| = 0$. For n degrees of internal freedom, the expansion of this nth order determinant will give rise to an equation in λ of the nth degree. He gives a discussion of the vibration frequencies and isotopic shifts of tetratomic molecules, with a discussion of various intramolecular forces and the physical meaning of the results, for pyramidal and coplanar molecules.
Manning [26] chooses an expression for the potential energy of $NH_3(ND_3)$ which has the correct general characteristics of geometry symmetry of NH_3 and which permits an exact solution of the Schrodinger equation. Making substitutions they obtain the indicial equation from Schrodinger equation and make quantitative calculations of the behavior of the energy levels, those below the top of the center of the hill of V are double according data (Wright, Randall [27]).

3. On the Oscillations Bands of Ammonia by Majorana

Majorana studied the NH_3 spectra [28] and obtained results in agree with the experimental results, i.e., two simple vibrations and two double vibrations. He considered the

symmetry of the NH_3. The three atoms H occupy the vertices of equilateral triangle; the atom N is on the axes out of the plane. The independent displacements which contribute to elastic forces are six and they obtain from the twelve displacements of the four atoms with the condition that the resultant of applied vectors δP_i at the rest points P_i' is zero.

He defines the displacements $q_1 = 1$, $q_2 = q_3 = \ldots = q_6 = 0$ as those in which the atom H^1 moves in direction NH^1 of $M_N/(M_N + M_H)$ and the atom N in the opposite direction of length $M_H/(M_N + M_H)$. Similarly one defines the displacements $q_i = \delta_{i2}$ e $q_i = \delta_{i3}$. Then we define as displacement $q_i = \delta_{i4}$ that in which the atom H^3 shifts of $1/2$ in the direction H^2H^3 and the atom H^2 of $1/2$ in the opposite direction; for circular permutation he puts the displacements $q_i = \delta_{i5}$ e $q_i = \delta_{i6}$.

Indicating α the angle (in the equilibrium position) $\widehat{NH^1H^2}$ and with β the angle $\widehat{H^1NH^2}$ the kinetic energy is:

$$\begin{aligned} T = \frac{1}{2}\Bigg[& \frac{M_H^2 M_N}{(M_N+M_H)^2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2 + 2\dot{q}_1\dot{q}_2\cos\beta + 2\dot{q}_2\dot{q}_3\cos\beta\right. \\ & \left. + 2\dot{q}_3\dot{q}_1\cos\beta\right) + \frac{M_N^2 M_H}{(M_N+M_H)^2}\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right) \\ & + \frac{M_N M_H}{M_N+M_H}\cos\alpha\left(\dot{q}_1\dot{q}_5 + \dot{q}_1\dot{q}_6 + \dot{q}_2\dot{q}_6 + \dot{q}_2\dot{q}_4 + \dot{q}_3\dot{q}_4 + \dot{q}_3\dot{q}_5\right) \\ & + \frac{1}{2}M_H\left(\dot{q}_4 + \dot{q}_5 + \dot{q}_6 + \frac{1}{2}\dot{q}_4\dot{q}_5 + \frac{1}{2}\dot{q}_5\dot{q}_6 + \frac{1}{2}\dot{q}_6\dot{q}_4\right)\Bigg]. \end{aligned} \tag{13}$$

then he defines the potential energy

$$V = \frac{1}{2}\sum_{ik} a_{ik}\, q_1 q_k \tag{14}$$

and he performs a canonical transformation [28]. He obtains a new expression of the kinetic energy in the new coordinates Q_i, similarly for the potential. He obtains then two simple vibrations relative to coordinates Q_1 and Q_2 and two double vibrations relative to coordinates Q_3 and Q_4 with the square of angular velocity:

$$\lambda = 4\pi^2\,\nu^2 \tag{15}$$

4. Brief Experimental Investigation until 1932

Now we will give a brief chronology of the experiments on NH_3.

Fox studies (1928) the IR region of the spectrum of NH_3 using the Prism spectrometer [29]. Sir Robert Robertson and J.J. Fox in 1928 used a small infra red prism spectrometer, filled of ammonia gas. They took a source of energy constant, for calibrating the mechanism for reading wave lengths.

At $\Delta V = (100 \pm 200)V$ and $T \sim 18C$ they used Nernst filaments as source of Radiation, since those gave the most uniform supply having regard to the intensity at different regions of the spectrum. There are source radiation - tubes observation - spectrometer. As the full radiation contains light of short wave length, it may affect chemically the gas

NH_3 under observation. The results is a measure of position of bands of NH_3 and their intensity. He made a preparation of Ammonia generated in the little flask A by warming a mixture of damp solid ammonia and 50% KOH solution was allowed to escape at the two-way top x, until samples were completely absorbed by water.
He confirmed that the view that the NH_3 is a tetrahedron was acceptable.
Rasetti [30] et al. have photographed the Raman spectra in 1929 of gaseous CO_2, NO_2, NH_3, CH_4, C_2H_4 using the line $\lambda = 2536$ of mercury as the exciting radiation. They have observed vibrational transitions in all the gases, and rotational transitions in the NH_3 and CH_4.
Berker in 1929 [31] analyzes the NH_3 absorption band extending at 3.0μ and 1.9μ and from 8μ to 24μ interpreting the double character of the 10μ band to be a consequence of the close proximity of the two equilibrium positions for the N atom, one of either side of the plane formed by the H atoms.
Dennison and Hardy in 1932 [2] make an experimental search for the doubling of the 3.0μ band using an IR spectrometer of high resolving power. The experimental results furnishes a strong argument for the theory of the doubling of the ammonia bands. They discuss the form of ammonia molecule with the theory . And then they prove that those states of ammonia existing in nature have vibration-rotation-nuclear spin wave functions which are antisymmetrical for an interchange of two of the hydrogen atoms.

5. Conclusions

In this paper we have depicted the genesis and the first developments of the study of spectra of NH_3 analyzed for the first time by Coblentz. Far from being complete, our account has focused on the results achieved from 1905 to 1932, as given evidence by many articles published in widespread journals. We have also pointed out the practically unknown contribution to spectral analysis Majorana, who was introduced to the subject by studies and experiments in Rome. The result reached by Majorana as early as in the beginning of 1930 is to find the right number of fundamental frequencies of spectrum of NH_3. Wide room has been made to different approaches to study the spectrum qualitatively and quantitatively and experimentally too. A theoretical analysis of ammonia spectra has been reported in Sect. 2, with a brief account of Theory of Vibrations and Theory of Groups. In the same sections we have showed an approximate analysis of NH_3 with WKB method and an exact analysis with a particular form of potential. Particular attention has been given to the approach of Majorana for the analysis of IR spectra of NH_3 in Sect. 3. Early experiments of ammonia, essentially dealt with atomic spectroscopy, have been discussed above in Sect.4. From what discussed here, it is then evident the interest to study the ammonia spectra by Majorana and its contribute to find the exact solution.

Acknowledgments

The author is indebted with S.Esposito for fruitful discussions.

References

[1] E. Fermi *Nuovo Cimento*, 9:277 1932.

[2] D.M. Dennison and J. D. Hardy, *Physical Review*, 15:938, 1932.

[3] E. Fermi Molecole e cristalli, Ed. Zanichelli, Bologna, 1934.

[4] E. Amaldi and G. Placzek, *Naturwissenschaften*, 20, 521, 1932; *Zeitschrift fur Physik*, 81:259 1932.

[5] G. Placzek and E. Teller, *Zeitschrift fur Physik*, 81:209, 1932.

[6] W.W. Coblentz,*Investigations of Infra-red Spectra* Part. I. Carnegie Institute of Washington, D.C., 1905.

[7] K. Schierkolk, *Zeitschrift fur Physik*, 29:277, 1924.

[8] W. Heitler and F. London, *Zeitschrift fur Physik* 44:455, 1927.

[9] C. J. Brester, *Kristallsymmetric und Reststrahlen*, Ultrecht, 1923.

[10] N. Bjerrum, Verh. d. D., *Phys. Ges.* 16:737, 1914.

[11] E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra* New York: Academic Press, 1959.

[12] F. Hund, *Zeitschrift fur Physik*, 31:81, 1925.

[13] H. Kornfeld, *Zeitschrift fur Physik*, 26:205, 1924.

[14] F. Hund *Zeitschrift fur Physik*, 43:805, 1927.

[15] D.M. Dennison, *Philosophical Magazine*, 1:195, 1926.

[16] E.T. Whittaker *A treatise on the Analytical Dynamics of Particles and Rigid Bodies*, III Ed. Cambridge, University Press, 1927.

[17] D.M. Dennison, *Review Modern Physics*, 3:280, 1931.

[18] C. W. Jordan *Comptes Rendus*, 74:1395, 1872.

[19] C. W.Jordan *Histoire de l'Academie de Berlin*, pg. 147, 1753.

[20] D.M. Dennison and G. E. Uhlenbeck, *Physical Review*, 41:313, 1932.

[21] L.A. Young and G. Uhlenbeck *Physical Review*, 36:1154, 1930.

[22] N. Rosen and P. M. Morse *Physical Review*, 42:210, 1932.

[23] E. O. Salant and J. E.Rosenthal *Physical Review*, 42:812, 1932.

[24] S.Silverman and J. A. Sanderson *Physical Review*, 44:1032 1933.

[25] J. E.Rosenthal *Physics Review*, 47:235, 1935.

[26] M. F. Manning *Journal of Chemical Physics*, 3:136, 1935.

[27] Wright and Randall *Physical Review*, 44:391, 1933.

[28] S. Esposito, E. Majorana jr, A. van der Merwe and E. Recami (eds.), *Ettore Majorana: Notes on Theoretical Physics* (Kluwer, New York, 2003).

[29] J. J. Fox et al., Proc. Roy. Soc. 1120:128, 1928.

[30] R. G. Dickinson, R. T. Dillon and F. Rasetti *Physical Review*, 34:582, 1929.

[31] G. A. Stinchcomb and E. F. Barker *Physical Review*, 33:305, 1929.

EJTP **3, No. 10** (2006) 239–247 *Electronic Journal of Theoretical Physics*

Exact Solution of Majorana Equation via Heaviside Operational Ansatz

Valentino A. Simpao *

Mathematical Consultant Services,
108 Hopkinsville St.,
Greenville, KY 42345 USA

Received 20 April 2006, Published 28 May 2006

Abstract: In context of a transformation between Majorana and Dirac wavefunctions, it suffices to solve the related interactive Dirac problem and then apply the transformation of variables on the Dirac wavefunction in order to obtain the Majorana wavefunction of the given Majorana equation. Clearly, this connection between solutions continues to hold if the free Majorana and Dirac equations are each coupled to an external gauge field[e.g., Electromagnetism] via the minimum coupling prescription. Applying the formal solution scheme Heaviside Operational Ansatz[heretofore, HOA] put forward in [EJTP 1 (2004), 10-16], provides an exact quadrature solution for the massive minimum-coupled Dirac equation, which may then be transformed into the solution of the corresponding massive minimum-coupled Majorana equation.

Keywords: Majorana equation, Heaviside operators, Quantum Dynamics, Phase Space, Differential Equations, Analytical Solution
PACS (2006): 04.50.+h, 03.65.Pm, 03.85.Db, 02.30.Qy, 04.20.Jb, 02.30.Vv, 02.70.-c
AMS Mathematical Subject Classification: 44A45, 44A35,44A10, 81Q05

1. Motivation and Layout

In this brief note, after a short recap of HOA methods, we shall apply the scheme to obtain quadrature solutions of the Majorana equation[ref.1] minimal-coupled to an external gauge field(e.g., electromagnetism). Crucial to this will be using the HOA methods to solve the related minimal-coupled Dirac Equation. Once the Dirac solutions are known, the Majorana solutions follow directly. The details of transforming of the Dirac wavefunctions into solutions of the Majorana Equation are elaborated in great detail elsewhere[see

* mcs007@muhlon.com

for example the excellent review article by Valle ref.2] and will not be reproduced here.

2. Recap of HOA

To recall the full details of HOA results, see the original work[EJTP 1 (2004), 10-16]. As pointed out therein,

'Notwithstanding its quantum mechanical origins, the HOA scheme takes on a life of its own and transcends the limits of quantum applications to address a wide variety of purely formal mathematical problems as well. Among other things, the result provides a formula for obtaining an exact solution to a wide variety of variable-coefficient integro-differential equations. Since the functional dependence of the Hamiltonian operator as considered is in general arbitrary upon its arguments(i.e., independent variables, derivative operator symbols[including negative powers thereof, thus the possible integral character]), then its multivariable extension can be interpreted as the most general variable coefficient partial differential operator. Moreover, it is not confined to being a scalar or even vector operator, but may be generally construed an arbitrary rank matrix operator. In all cases of course, its rank dictates the matrix rank of the wavefunction solution.'

In the present case of the Majorana equation and related Dirac equation, we shall be dealing with such a 4x4 matrix Hamiltonian structure and the solution wavefunction will be of a 4-dimensional column vector character.

Recalling the fundamental HOA results, we let x, p, t respectively denote the configuration space, momentum and time variables. The $\hat{}$ denotes the operators, with H and Ψ denoting the Hamiltonian and wavefunction of the phase space representation, respectively. Also the α, γ are otherwise free parameters as specified therein the original work.

Hence for a given Hamiltonian and Initial-State, the configuration space solution obtains from the quantum phase space solution as

$$\hat{H}_{\text{configuration space}}\left(x_1, ..., x_n, -i\hbar\partial_{x_1}, ..., -i\hbar\partial_{x_n}, t\right)\Psi_{\text{configuration space}}\left(x_1, ..., x_n, t\right)$$
$$= i\hbar\partial_t\Psi_{\text{configuration space}}\left(x_1, ..., x_n, t\right)$$

$$\Psi_{\text{configuration space}}\left(x_1, ..., x_n, t\right)$$
$$= \int_{-\infty}^{\infty} \frac{e^{\frac{ix_1p_1}{2\hbar}}}{\sqrt{4\pi\hbar}} .... \int_{-\infty}^{\infty} \frac{e^{\frac{ix_np_n}{2\hbar}}}{\sqrt{4\pi\hbar}} L^{-1}\left(\begin{array}{l}(\bar{x}_1, ..., \bar{x}_n)\\ \\ \rightarrow (x_1, ..., x_n)\end{array}\right)\left[L^{-1}\left(\begin{array}{l}(\bar{p}_1, ..., \bar{p}_n)\\ \\ \rightarrow (p_1, ..., p_n)\end{array}\right)\left[\begin{array}{l} e^{\frac{-i}{\hbar}\int_0^t \hat{H}\left(\begin{array}{l} i\hbar\bar{p}_1 + \alpha_1 x_1, ..., i\hbar\bar{p}_n + \alpha_n x_n;\\ \\ -i\hbar\bar{x}_1 + \gamma_1 p_1, ..., -i\hbar\bar{x}_n + \gamma_n p_n; u\end{array}\right)du}\\ \\ \times \\ \\ \breve{\Psi}_0\left(\bar{x}_1, ..., \bar{x}_n; \bar{p}_1, ..., \bar{p}_n; t = 0\right)\end{array}\right]\right] dp_1..dp_n \quad (1)$$

where

$$\hat{H}_{\text{configuration space}}\left(\begin{array}{l} x_1,....,x_n, \\ -i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n},t \end{array}\right)_{\substack{(x_1,....,x_n)\mapsto(i\hbar\partial_{p_1}+\alpha_1 x_1,....,i\hbar\partial_{p_n}+\alpha_n x_n) \\ (-i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n})\mapsto(-i\hbar\partial_{x_1}+\gamma_1 p_1,....,-i\hbar\partial_{x_n}+\gamma_n p_n)}}$$

$$\equiv \hat{H}\left(\begin{array}{l} i\hbar\partial_{p_1}+\alpha_1 x_1,....,i\hbar\partial_{p_n}+\alpha_n x_n; \\ -i\hbar\partial_{x_1}+\gamma_1 p_1,....,-i\hbar\partial_{x_n}+\gamma_n p_n;t \end{array}\right) \tag{2}$$

To wit, via HOA the configuration space solution becomes

$$\Psi_{\text{configuration space}}(x_1,....,x_n,t)$$

$$=\int_{-\infty}^{\infty}\frac{e^{\frac{ix_1p_1}{2\hbar}}}{\sqrt{4\pi\hbar}}....\int_{-\infty}^{\infty}\frac{e^{\frac{ix_np_n}{2\hbar}}}{\sqrt{4\pi\hbar}}L^{-1}\left(\begin{array}{l}(\bar{x}_1,....,\bar{x}_n;\bar{p}_1,....,\bar{p}_n) \\ \rightarrow(x_1,....,x_n;p_1,....,p_n)\end{array}\right)\left[\begin{array}{l} e^{\frac{-i}{\hbar}\int_0^t \hat{H}_{\text{configuration space}}\quad du} \\ \times \\ \breve{\Psi}_{0\,\text{configuration space}}(\bar{x}_1,....,\bar{x}_n;;t=0)\end{array}\right]dp_1..dp_n \tag{3}$$

where

$$\hat{H}_{\text{configuration space}}=\hat{H}_{\text{configuration space}}\left(\begin{array}{l} x_1,....,x_n, \\ -i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n},u \end{array}\right)_{\substack{(x_1,....,x_n)\mapsto(i\hbar\bar{p}_1+\alpha_1 x_1,....,i\hbar\bar{p}_n+\alpha_n x_n) \\ (-i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n})\mapsto(-i\hbar\bar{x}_1+\gamma_1 p_1,....,-i\hbar\bar{x}_n+\gamma_n p_n)}}$$

With that said, a relatively simplistic prescription results for actually using the Ansatz to solve the problem,

Given the function $\hat{H}(\hat{x}_1,....,\hat{x}_n;\hat{p}_1,....,\hat{p}_n,t)$

[respectively $\hat{H}(x_1,....,x_n;-i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n},t)$] replace

$(\hat{x}_1,....,\hat{x}_n;\hat{p}_1,....,\hat{p}_n,t)$[respectively $(x_1,....,x_n;-i\hbar\partial_{x_1},....,-i\hbar\partial_{x_n},t)$] with

$(i\hbar\bar{p}_1+\alpha_1 x_1,....,i\hbar\bar{p}_n+\alpha_n x_n;-i\hbar\bar{x}_1+\gamma_1 p_1,....,-i\hbar\bar{x}_n+\gamma_1 p_n,t)$ in equation (3)

The result of course is the quantum phase space[respectively configuration space] wavefunction for the quantum dynamics wave equation. Before addressing the Majorana equation directly, just a comment on the α and γ parameters in the above formulae. From the HOA, they are otherwise arbitrary except for the condition $\alpha+\gamma=1$. This is explained therein as a consequence of the arbitrary phase shift associated with the quantum phase space wavefunction. Further, any choice of the parameters thus constrained yields a Hamiltonian, which is dynamically equivalent[describes the same physics] as any other choice. However, it is shown in therein that the Hamiltonian operator $\hat{H}(i\hbar\partial_p+\alpha x,-i\hbar\partial_x+\gamma p,t)$, $\ni \alpha+\gamma=1$ takes on the symmetric canonical form

when $\alpha = \gamma = \frac{1}{2}$ thusly $\hat{H}\left(i\hbar\partial_p + \frac{x}{2}, -i\hbar\partial_x + \frac{p}{2}, t\right)$, $\ni \alpha + \gamma = 1$. Notwithstanding this and with an eye towards computational simplifications for particular classes of applications, it has been found that other choices than $\alpha = \gamma = \frac{1}{2}$ greatly facilitates evaluation of the integral transforms. Unless otherwise directed, the convention for α and γ shall be specified for particular cases, presently and elsewhere.

3. HOA Solution of Majorana Equation with Minimum-Coupled Electromagnetic Gauge Field

First consider the related Dirac equation with minimum-coupled electromagnetic gauge field
$\mathbf{A}(x_1, x_2, x_3, t) = A_1(x_1, x_2, x_3, t)\,\mathbf{e_{x_1}} + A_2(x_1, x_2, x_3, t)\,\mathbf{e_{x_2}} + A_3(x_1, x_2, x_3, t)\,\mathbf{e_{x_3}}$, $A_0(x_1, x_2, x_3, t)$ interaction

$$\mathbf{H}_{\mathrm{Dirac}_{4\times 4}}\Psi_D = \left(mc^2 a_0 + \sum_{j=1}^{3}\left(a_j\left(p_j - eA_j\right)c + eA_0\right)\right)\Psi_D = i\hbar\partial_t\Psi_D$$

$$\mathbf{A}(x, y, z, t) = A_1(x, y, z, t)\,\mathbf{e_x} + A_2(x, y, z, t)\,\mathbf{e_y} + A_3(x, y, z, t)\,\mathbf{e_z},\ A_0(x, y, z, t)$$

$$\Psi_D \equiv \begin{pmatrix} \Psi_{D1} \\ \Psi_{D2} \\ \Psi_{D3} \\ \Psi_{D4} \end{pmatrix} : \text{ 4-component Dirac wavefunction}$$

$$\Psi_{D_0} \equiv \begin{pmatrix} \Psi_{D1_0} \\ \Psi_{D2_0} \\ \Psi_{D3_0} \\ \Psi_{D4_0} \end{pmatrix} : \text{ 4-component Dirac Initial State} \tag{4}$$

$$a_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},\ a_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$a_2 = \begin{bmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{bmatrix},\ a_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

Similarly for the Majorana equation minimal-coupled to the EM gauge field.

$$
\begin{aligned}
&-imc\sigma^2\rho_M^* + (i\hbar\sigma^\mu\partial_\mu - eA_\mu)\,\rho_M = 0\\
&A_\mu \equiv (\mathbf{A}\,(x_1,x_2,x_3,t) = A_1\,(x_1,x_2,x_3,t)\,\mathbf{e_{x_1}} + A_2\,(x_1,x_2,x_3,t)\,\mathbf{e_{x_2}} + A_3\,(x_1,x_2,x_3,t)\,\mathbf{e_{x_3}}\,,\ A_0\,(x_1,x_2,x_3,t))\\
&\rho_M = \begin{pmatrix}\rho_{M1}\\ \rho_{M2}\end{pmatrix} : 2-\text{component Majorana wavefunction}\\
&\rho_{M_0} = \begin{pmatrix}\rho_{M1_0}\\ \rho_{M2_0}\end{pmatrix} : 2-\text{component Majorana Initial state}\\
&\sigma^\mu : \text{usual } 2\times 2 \text{ Pauli spin matrices } \sigma^{1,2,3}, \sigma^0 = -i\mathbf{I_{2\times 2}}
\end{aligned}
\tag{5}
$$

Now the connection between the Majorana (5) ρ_M and Dirac [4] $\Psi_D$wavefunctions subject to the Majorana self-conjugacy condition $\Psi_D = \Psi_D^c$ is thoroughly discussed in the excellent review article by Valle [ref 2]; only some key relationships between them are reproduced here for convenience

$$
\begin{aligned}
&\Psi_D \equiv \begin{pmatrix}\Psi_{D1}\\ \Psi_{D2}\\ \Psi_{D3}\\ \Psi_{D4}\end{pmatrix} = \begin{pmatrix}\chi_D\\ \sigma_2\phi_D^*\end{pmatrix}\,, \quad \Psi_D^c = \begin{pmatrix}\sigma_2 & 0\\ 0 & \sigma_2\end{pmatrix}\Psi_D^{*Transpose} = \begin{pmatrix}\phi_D\\ \sigma_2\chi_D^*\end{pmatrix}\,,\\
&\chi_D = \begin{pmatrix}\Psi_{D1}\\ \Psi_{D2}\end{pmatrix}\,, \quad \sigma_2\phi_D^* = \begin{pmatrix}\Psi_{D3}\\ \Psi_{D4}\end{pmatrix}\,,\ \phi_D = \begin{pmatrix}i\Psi_{D4}^*\\ -i\Psi_{D3}^*\end{pmatrix}\,,\\
&\text{Majorana Self-Conjugacy condition } \Psi_D = \Psi_D^c\\
&\chi_D = \tfrac{1}{\sqrt{2}}\,(\rho_{M_2} + i\rho_{M_1})\ ,\ \rho_{M_2} = \tfrac{1}{\sqrt{2}}\,(\chi_D + \phi_D) = \tfrac{1}{\sqrt{2}}\begin{pmatrix}\Psi_{D1} + i\Psi_{D4}^*\\ \Psi_{D2} - i\Psi_{D3}^*\end{pmatrix}\\
&\phi_D = \tfrac{1}{\sqrt{2}}\,(\rho_{M_2} - i\rho_{M_1})\ ,\ \rho_{M_1} = \tfrac{i}{\sqrt{2}}\,(\chi_D - \phi_D) = \tfrac{i}{\sqrt{2}}\begin{pmatrix}\Psi_{D1} - i\Psi_{D4}^*\\ \Psi_{D2} + i\Psi_{D3}^*\end{pmatrix}\\
&\rho_M = \begin{pmatrix}\rho_{M_1}\\ \rho_{M_2}\end{pmatrix}
\end{aligned}
\tag{6}
$$

So by way of (6), given the related Dirac wavefunction and subject to the Majorana self-conjugacy condition $\Psi_D = \Psi_D^c$, the Majorana wavefunction ascends naturally. Moreover, by way of the HOA method, substituting the Dirac Hamiltonian of (4) gives the quantum phase space dynamics of the Dirac system for initial conditions and EM gauge

fields of general form.

$$\hat{\mathbf{H}}_{\mathrm{Dirac}_{\mathbf{4\times4}}}\left(\begin{array}{l} i\hbar\partial_{p_1}+\alpha_1x_1, i\hbar\partial_{p_2}+\alpha_2x_2, i\hbar\partial_{p_3}+\alpha_3x_3; \\ -i\hbar\partial_{x_1}+\gamma_1p_1, -i\hbar\partial_{x_2}+\gamma_2p_2, -i\hbar\partial_{x_3}+\gamma_3p_3; t \end{array}\right)\Psi_D\left(x_1,x_2,x_3;p_1,p_2,p_3;t\right)$$

$$=i\hbar\partial_t\Psi_D\left(x_1,x_2,x_3;p_1,p_2,p_3;t\right)$$

$$\left(m_c^2a_0+\sum_{j=1}^{3}\left(\begin{array}{l} a_j\left(-i\hbar\partial_{x_j}+\gamma_jp_j-eA_j\left(i\hbar\partial_{p_1}+\alpha_1x_1, i\hbar\partial_{p_2}+\alpha_2x_2, i\hbar\partial_{p_3}+\alpha_3x_3,t\right)\right)c \\ +eA_0\left(i\hbar\partial_{p_1}+\alpha_1x_1, i\hbar\partial_{p_2}+\alpha_2x_2, i\hbar\partial_{p_3}+\alpha_3x_3;,t\right) \end{array}\right)\right)\Psi_D=i\hbar\partial_t\Psi_D$$

$$\mathbf{A}\left(i\hbar\partial_{p_1}+\alpha_1x_1, i\hbar\partial_{p_2}+\alpha_2x_2, i\hbar\partial_{p_3}+\alpha_3x_3,t\right)=$$

$$A_1\left(i\hbar\partial_{p_1}+\alpha_1x_1, i\hbar\partial_{p_2}+\alpha_2x_2, i\hbar\partial_{p_3}+\alpha_3x_3,t\right)\mathbf{e}_{\mathbf{x}_1}$$

$$+A_2\left(i\hbar\partial_{p_1}+\alpha_1x_1, i\hbar\partial_{p_2}+\alpha_2x_2, i\hbar\partial_{p_3}+\alpha_3x_3,t\right)\mathbf{e}_{\mathbf{x_2}}$$

$$+A_3\left(i\hbar\partial_{p_1}+\alpha_1x_1, i\hbar\partial_{p_2}+\alpha_2x_2, i\hbar\partial_{p_3}+\alpha_3x_3,t\right)\mathbf{e}_{\mathbf{x_3}}$$

$$,A_0\left(i\hbar\partial_{p_1}+\alpha_1x_1, i\hbar\partial_{p_2}+\alpha_2x_2, i\hbar\partial_{p_3}+\alpha_3x_3,t\right)$$

$$\Psi_D\equiv\left(\begin{array}{l}\Psi_{D1}\\ \Psi_{D2}\\ \Psi_{D3}\\ \Psi_{D4}\end{array}\right):\ \text{4-component Dirac wavefunction}$$

(7)

Hence the configuration space dynamics for the related minimal-coupled Dirac system

$$\Psi_{\text{D configuration space}}\left(x_1,x_2,x_3,t\right)=$$

$$\left(\begin{array}{l}\Psi_{D1}\left(x_1,x_2,x_3,t\right)\\ \Psi_{D2}\left(x_1,x_2,x_3,t\right)\\ \Psi_{D3}\left(x_1,x_2,x_3,t\right)\\ \Psi_{D4}\left(x_1,x_2,x_3,t\right)\end{array}\right)_{\text{configuration space}}=$$

$$=\int_{-\infty}^{\infty}\frac{e^{\frac{ix_1p_1}{2\hbar}}}{\sqrt{4\pi\hbar}}\int_{-\infty}^{\infty}\frac{e^{\frac{ix_2p_2}{2\hbar}}}{\sqrt{4\pi\hbar}}\int_{-\infty}^{\infty}\frac{e^{\frac{ix_3p_3}{2\hbar}}}{\sqrt{4\pi\hbar}}L^{-1}\left(\begin{array}{l}\left(\begin{array}{l}\bar{x}_1,\ldots,\bar{x}_n;\\ \bar{p}_1,\ldots,\bar{p}_n\end{array}\right)\\ \rightarrow\left(\begin{array}{l}x_1,\ldots,x_n;\\ p_1,\ldots,p_n\end{array}\right)\end{array}\right)e^{\left(\frac{-i}{\hbar}\int_0^t\left(\begin{array}{l} mc^2a_0 \\ +\sum_{j=1}^{3}\left(\begin{array}{l} a_j\left(\begin{array}{l}-i\hbar\bar{x}_j+\gamma_jp_j\\ -eA_j\left(\begin{array}{l}i\hbar\bar{p}_1+\alpha_1x_1,\\ i\hbar\bar{p}_2+\alpha_2x_2,\\ i\hbar\bar{p}_3+\alpha_3x_3,u\end{array}\right)\end{array}\right)c \\ +eA_0\left(\begin{array}{l}i\hbar\bar{p}_1+\alpha_1x_1,\\ i\hbar\bar{p}_2+\alpha_2x_2,\\ i\hbar\bar{p}_3+\alpha_3x_3,u\end{array}\right)\end{array}\right)\end{array}\right)du\right)}\left(\begin{array}{l}\breve{\Psi}_{D0_1}\\ \breve{\Psi}_{D0_2}\\ \breve{\Psi}_{D0_3}\\ \breve{\Psi}_{D0_4}\end{array}\right)dp_1dp_2dp_3$$

$$\breve{\Psi}_{D0}\left(\begin{array}{l}\bar{x}_1,\bar{x}_2,\bar{x}_3;\\ \bar{p}_1,\bar{p}_2,\bar{p}_3;t=0\end{array}\right)=\left(\begin{array}{l}\breve{\Psi}_{D0_1}\\ \breve{\Psi}_{D0_2}\\ \breve{\Psi}_{D0_3}\\ \breve{\Psi}_{D0_4}\end{array}\right):\ \text{Transformed Initial-condition vector}$$

(8)

where the explicit form of the $\hat{\mathbf{H}}_{\mathrm{Dirac}_{\mathbf{4\times4}}}$ in the exponent 4x4 matrix integral is supplied

as

$$
\begin{pmatrix}
\begin{pmatrix} mc^2 \\ +ecA_0 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \end{pmatrix} & 0 & \begin{pmatrix} c\,(-i\hbar\bar{x}_3+\gamma_3 p_3) \\ -eA_3 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \end{pmatrix} & \begin{pmatrix} c\,(-i\hbar\bar{x}_1+\gamma_1 p_1) \\ -eA_1 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \\ -i(c\,(-i\hbar\bar{x}_2+\gamma_2 p_2) \\ -eA_2 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix}) \end{pmatrix} \\
0 & \begin{pmatrix} mc^2 \\ +ecA_0 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \end{pmatrix} & \begin{pmatrix} c\,(-i\hbar\bar{x}_1+\gamma_1 p_1) \\ -eA_1 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \\ +i(c\,(-i\hbar\bar{x}_2+\gamma_2 p_2) \\ -eA_2 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix}) \end{pmatrix} & - \begin{pmatrix} c\,(-i\hbar\bar{x}_3+\gamma_3 p_3) \\ -eA_3 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \end{pmatrix} \\
\begin{pmatrix} c\,(-i\hbar\bar{x}_3+\gamma_3 p_3) \\ -eA_3 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \end{pmatrix} & \begin{pmatrix} c\,(-i\hbar\bar{x}_1+\gamma_1 p_1) \\ -eA_1 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \\ -i(c\,(-i\hbar\bar{x}_2+\gamma_2 p_2) \\ -eA_2 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix}) \end{pmatrix} & \begin{pmatrix} -mc^2 \\ +ecA_0 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \end{pmatrix} & 0 \\
\begin{pmatrix} c\,(-i\hbar\bar{x}_1+\gamma_1 p_1) \\ -eA_1 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \\ +i(c\,(-i\hbar\bar{x}_2+\gamma_2 p_2) \\ -eA_2 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix}) \end{pmatrix} & - \begin{pmatrix} c\,(-i\hbar\bar{x}_3+\gamma_3 p_3) \\ -eA_3 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \end{pmatrix} & 0 & \begin{pmatrix} -mc^2 \\ +ecA_0 \begin{pmatrix} i\hbar\bar{p}_1+\alpha_1 x_1, \\ i\hbar\bar{p}_2+\alpha_2 x_2, \\ i\hbar\bar{p}_3+\alpha_3 x_3, t \end{pmatrix} \end{pmatrix}
\end{pmatrix} \tag{9}
$$

yields by way of (6), the associated Majorana wavefunction for the dynamics of the minimal-coupled system with arbitrary profile EM interaction and initial-conditions, in terms of the quadrature solutions for the related Dirac system just calculated above in (8)

$$\begin{aligned}
\rho_{M_2} &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{D1}(x_1,x_2,x_3,t) + i\Psi^*_{D4}(x_1,x_2,x_3,t) \\ \Psi_{D2}(x_1,x_2,x_3,t) - i\Psi^*_{D3}(x_1,x_2,x_3,t) \end{pmatrix} \\
\rho_{M_1} &= \tfrac{i}{\sqrt{2}} \begin{pmatrix} \Psi_{D1}(x_1,x_2,x_3,t) - i\Psi^*_{D4}(x_1,x_2,x_3,t) \\ \Psi_{D2}(x_1,x_2,x_3,t) + i\Psi^*_{D3}(x_1,x_2,x_3,t) \end{pmatrix} \\
\rho_M \quad \text{configuration space} &= \begin{pmatrix} \rho_{M_1} \\ \rho_{M_2} \end{pmatrix}_{\text{configuration space}} \\
\rho_{M_0} \quad \text{configuration space} &= \begin{pmatrix} \rho_{M_{10}} \\ \rho_{M_{20}} \end{pmatrix}_{\text{configuration space}} : \text{Majorana Initial-State} \\
\rho_{M_{20}} &= \tfrac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{D1_0}(x_1,x_2,x_3,t) + i\Psi^*_{D4_0}(x_1,x_2,x_3,t) \\ \Psi_{D2_0}(x_1,x_2,x_3,t) - i\Psi^*_{D3_0}(x_1,x_2,x_3,t) \end{pmatrix} \\
\rho_{M_{10}} &= \tfrac{i}{\sqrt{2}} \begin{pmatrix} \Psi_{D1_0}(x_1,x_2,x_3,t) - i\Psi^*_{D4_0}(x_1,x_2,x_3,t) \\ \Psi_{D2_0}(x_1,x_2,x_3,t) + i\Psi^*_{D3_0}(x_1,x_2,x_3,t) \end{pmatrix}
\end{aligned} \tag{10}$$

References

[1] E. Majorana, Nuovo Cimento **9**, P335 (1932)

[2] J.W.F.Valle, Prog.Part.Nucl.Phys.26,pp91-171 (1991)

EJTP **3, No. 10** (2006) 249–263 *Electronic Journal of Theoretical Physics*

A Logical Analysis of Majorana's Papers of Theoretical Physics

A. Drago [1] * and S. Esposito [2] †

[1] *Università di Pisa, Pisa, Italy*
[2] *Università di Napoli "Federico II" ,Naples, Italy*

Received 28 April 2006, Published 28 May 2006

Abstract: We study two celebrated Majorana's papers through a method of investigation which relies upon the recently recognised distinction between classical logic and *grosso modo* the several kinds of non-classical logics, i.e. the failure of the double negation law. This law fails when a double negated sentence is not equivalent to the corresponding positive sentence, owing to the lack of scientific evidence of the latter one. All recognised double negated sentences inside the text of each paper are listed; the mere sequence of such sentences gives the logical thread of Majorana's arguing. This one is recognised to be of the Lagrangian kind, which mixes logical arguing and mathematical calculation; i.e. the author puts a fundamental problem which is solved by anticipating the mathematical hypothesis which is capable of solving it, and then by drawing from this hypothesis the mathematical consequences in order to obtain the desired result. Furthermore Majorana's rhetoric of presentation results to be a juridical one, owing to his style of presenting the laws to which an ideal theoretical physicist has to conform himself in order to solve the problem at issue.

Keywords: Ettore Majorana, Logical Analysis
PACS (2006): 01.65.+g,01.30.Tt,02.10.-v

1. Introduction

Recently the interest for the human and scientific figure of Ettore Majorana acquired vigour. Apart several biographies and books (about his life and in particular about the unsolved problem of his disappearance [1]), the following books have been edited: the last lectures at Naples University in Italian language [2]; and, in English language, his "Volumetti" [3], where Majorana was accustomed to solve for his own interest scientific

* drago@unina.it
† sesposito@na.infn.it

problems of various kinds; it seems that his accuracy and precision was the higher one with respect to any other scientist's notebook. This element supports what Fermi spoke about him:

"...in the world there are various categories of scientists: people of a secondary or tertiary standing, who do their best but do not go very far. There are also those of high standing, who come to discoveries of great importance, fundamental for the development of science' (and here I had the impression that he placed himself in that category). 'But then there are geniuses like Galileo and Newton. Well, Ettore was one of them. Majorana had what no one else in the world had ..." [4].

Maybe such an evaluation appears somewhat apologetic; but Fermi saw some other elements. In the following we want to analyse a characteristic feature of Majorana papers - i.e. the kind of logic in his arguments - which was missed in previous analysis.

2. Majorana as a Theoretical Physicist: Papers and Publications

The scientific production by E. Majorana includes nine printed publications [5]. In last years some studies wanted to put such papers inside a historical background, then to interpret them and possibly to develop them for achieving new results.

Some decades ago E. Amaldi wrote a review on Majorana's scientific production [6]; he first offered a detailed valuations of them; it constitutes a first characterisation of Majorana's work in theoretical physics.

Furthermore in last years some improvements have been considered of the ideas that Majorana started to develop and then have been forgotten - though independently recuperated by others -; in particular, his solutions of those differential equations which are useful in many physical problems [7].A first valuation for Majorana's work on symmetries has been given; along with Giovanni Gentile jr, for the first time he introduced this mathematical tool in the Italian theoretical physics scenario [8]. It is also interesting to note that he was almost the unique (though remote) Weyl's follower in Weyl's attitude of founding quantum mechanics and the field theory through group theory only [9].

The problem is open whether it is possible to reconstruct Majorana's scientific production according to a line of development which represents a conception of theoretical physics as a whole, for instance a conception similar to H. Weyl's.

The present paper is aimed to improve the comprehension of Majorana's papers by means of a technique of mathematical logic. Among his papers we have chosen two of the most important papers in theoretical physics: *Teoria relativistica di particelle con momento intrinseco arbitrario* (Relativistic theory of particles with arbitrary intrinsic momentum) of 1932 and *Teoria simmetrica dell'elettrone e del positrone* (Symmetric theory of electron and positron) of 1937 [10]. The historical background of these papers is offered by the above mentioned Amaldi's review.

3. Analysis of a Scientific Writing by Means of Non-Classical Logic

In the above-mentioned text Majorana illustrates a theory which is based upon a very general problem, i.e. the relativistic generalisation of Dirac theory of the electron for particles of arbitrary spin. For solving this problem, a deductive method of solution does not *a priori* exist. The final result is not deductively drawn from some a priori principle; rather, the arguments proceed in a heuristic way, without absolute certainty.

Elsewhere, one of us (A. D.) suggested the model of theoretical organisation aimed to search a new scientific method capable to solve a given universal problem [11]. Inside this kind of organisation, called a problem-based one, several double negated sentences (DNS) play a crucial role, each one being not equivalent to the corresponding positive sentence, owing to the lack of (either experimental or theoretical) scientific evidence of the latter one. For instance, in the historical development of thermodynamics very often the following sentence occurs: "It is impossible the perpetual (= without an end) motion" (we underlined the negations for evidentiating them to the reader; the same will be done in the following DNSs); the corresponding positive statement is "Each motion has an end"; in order to state it we have to know either the instant or the place where the given motion will end; this knowledge is impossible, owing to our ignorance of both the friction function which affects the motion and the length of the path along which friction is exerted upon the body in motion. Hence, the positive sentence is not supported by scientific facts.

In 20^{th} Century the studies upon the foundations of mathematical logic resulted in establishing more precisely the borderline between classical logic and *grosso modo* non-classical logics; this bordeline is the failure of rather than the law of excluded middle (either A or *not-A*), the double negation law (*not-not-A* $=A$) [12]. This change in the attention is very relevant for, since inside a scientific writing a scholar easily recognises a sentence *not-not-A* and then easily he tests whether in this sentence the double negation logical law holds true or not, by testing whether the corresponding sentence A is supported by scientific evidence or not.

According to what occurs in the problem-based theories, in Majorana's paper too occur DNSs; hence, Majorana was surely out of a full deductive theory in classical logic. Hence, it is useful to analyse his writings under this logical aspect. To this aim it is necessary first to list all DNSs included by Majorana's paper.

4. Inspection of All DNSs in *Teoria Relativistica* (1932)

Let us analyse the paper by recognising the DNSs inside it. In order to make easier the reader to recognise the DNS nature of each sentence listed in the following, the corresponding positive sentence is written inside square brackets, after a sign $\neq$.

There exists several kinds of DNSs. Beyond the plain DNSs (either a sentence or a word including two negations say, in-variant, etc.) there are the "weak" DNSs (in the

following: wDNS), that is those including a modality, as "may", "allow", "want", "must" (in the following each of these words is written in Italic and the order number of its DNS is marked by an asterisk*). The appearance of these words is quite improper in a scientific text, meant in a traditional sense, i.e. as a deductive exposition; because this exposition has to be based upon either experimental or deductive certainties; if instead such modalities occur, they show that the arguing is heuristic in nature; in such a case they have to be expressed by means of DNSs; for ex. "*may* = it is not true that it is not".

Furthermore, by taking into account that Majorana ignored at all non-classical logic, we will consider as a DNS also sentences which are DNS in a more general sense, for example the words "equivalent", "analogous", similar, each one to be meant as "not unequal".

1*. (p. 335) [...] A relativistic generalisation of the previous theory *must* [= it is not true that is not] satisfy [≠ satisfies] to the following conditions when its degree of accuracy is growing up:

2*. (*a*) The theory *allows* the study of [≠ studies] particles having velocity almost determined in magnitude and direction,...

3. ... giving results which are equivalent [= that are not different from] [≠ equal] to the non relativistic theory,...

4. (p. 336) without however the necessity of choosing [= differentiate] a particular reference system [≠ for all systems].

5*. (*b*) The theory moreover *allows* to study [≠ studies] processes where the velocities of the particles vary slowly, but in an extended range, owing to weak external fields.

6. (*c*) The theory is valid in general, whatever be undetermined [≠ for an arbitrary determination of] the velocity of the particles.

7. [...] On the contrary, it is likely [= it is not true that it is not] [≠ is true] that...

8. ... a theory satisfying (*b*) and only in part (*c*) does not hurt in subtantial difficulties [≠ is valid].. .

9*. ... its physical content *can* be [= it is not true that it is not] [≠ is] essentially the same justifying the Schroedinger equation.

10. The most remarkable example of such generalisations is just offered by DIRAC's theory, but since this applies only to particles with intrinsic momentum s=1/2, I looked for *analogous* [= not different] [≠ equal] in form equations to DIRAC's, though somewhat more cumbersome,...

11*. ... which *allow* to consider [≠ consider] particles with arbitrary momentum, also for a vanishing value of it

12*. [...] Equations of this kind [that is Dirac-like] present a difficulty of principle. The operator -1 indeed *must* transform [≠transforms] as the temporal component of a 4-vector...

13*. ... and so β *cannot be* [≠ is] merely a multiple of the unity matrix,...

14*. ... but *must* have [≠ has] at least two different eigenvalues, say β_1 and β_2 ;...

15*. ... but from this fact follows that the rest-energy of the particles, which is

obtained from the (1) by putting $p = 0$, *must have* [≠ has] at least two different values, $\beta_1 mc^2$ and $\beta_2 mc^2$.

16*. According to the DIRAC equation the possible values of the rest mass are, as it is well-known, $+m$ and $-m$, from which, due to the relativistic invariance follows that the energy *can have* [≠ has] two different values owing to the sign of each value of p : $W = \pm\sqrt{m^2c^4 + c^2p^2}$.

17. The indeterminacy in the sign of energy can be overcame [= cancelled] [≠is determined], by using equations of the fundamental kind (1),...

18. (p. 337) ... only if the wave function has infinite components which *not* [Italic in the text] allow to be broken [= not-reducible] [≠ elementary] in finite tensors or finite spinors.

19*. 1. The equation (1) *can be* [≠ is] deduced from the variational principle [a formula follows].

20. One of the conditions of relativistic in-variance [≠ constancy]...

21. ... has of course to be in-variant [≠ constant] the form [a formula follows].

22*. If now we want that the rest-energy result [≠ when the rest-energy is] always positive,...

23*. ... the eigenvalues of β *must be* [≠ are] all positive ones and the form [a formula follows] will be positive definite.

24*. Then it is *possible* to reduce [≠ reduces] through a non-unitary transformation $\psi \to \varphi$ such a form to the unity form: [a formula follows].

25. By substituting in (2) for ψ his expression by means of φ one will have: [a formula follows] from which the equations equivalent [≠ egual] to (1) follow: [a formula follow].

26*. We now *must* determine the law of transformation of φ according to a LORENTZ rotation and moreover expression of matrices...

27. ... in such a way that the relativistic in-variance [≠ constancy] of the variational principle be satisfied (4)...

28. ... and hence the function to be integrated be invariant [≠ constant].

29*. Let us start by stating the law of transformation of φ by observing first of all that the invariance of [a formula follow] means that we *must consider* [≠ we consider] unitary transformations only.

30*. (p. 338) The operators *a* e *b must be* [≠ are] Hermitian in a unitary representation and *viceversa*;...

31*. ... furthermore, in order to the infinitesimal transformations to be integrable *must satisfy* [≠ satisfy] some commutation relations which are deduced from both (6) and (7): (formulas follow) and the orther ones which are obtained by permutation of x, y and z.

32. (p. 339) The denomination " zero-index " means that is zero the in-variant [= the constant operator]: [a formula follows].

33*. [...] 2. We now *must* determine [≠ we determine] the operators [a formula follows]...

34. ... in such a way that (4) be invariant [≠ constant].

35*. Because we consider unitary transformations only, the above-mentioned operators transform as the Hermitian forms related to them, and hence owing to the invariance [≠ constancy] of the integrand function in (4)...

36. ... it is *necessary* that they constitute [≠ constitute] a covariant vector [a formula follows].

37*. [...] The operators γ *must satisfy* [≠ satisfy] the exchange relations [formulas follow] and some others obtained by cyclical permutation of x, y, z.

38. (p. 342) [...] Without difficulties [≠ easily] it is verified that...

39*. [...] These states *can be regarded* [≠ are] as belonging to the imaginary value ik of the mass.

40*. We now *want to consider* [≠ consider] quickly the introduction of the electromagnetic field in equation (16).

41. (p. 343) For instance we may add invariant [≠ constant] terms,...

42. ... analogous [≠ equal] to those introduced by Pauli in the theory of the magnetic neutron, which considers the field strength instead of the electromagnetic potentials,...

43. ... so that to do not unsettle [≠ maintain as a constant]...

44. ... the invariance [≠ the constancy] of the equations according to the indeterminacy of the potentials.

In total, in 10 pages of this paper we recognised 21 DNSs (among them 6 are due to the word "invariant") and 23 wDNSs.

5. Analysis of the Arguing in *Teoria Relativistica* (1932)

The same kind of analysis when applied to scientific texts written by some other authors, gives each time a list of DNSs, whose sequence is sufficient to manifest the logical thread of the presentation at issue. This is the case for instance of the discursive part of S. Carnot's booklet, i.e. the part originating most of modern thermodynamics. His theory is a problem-based theory since it puts a universal problem: whether a maximum efficiency in converting heat into work there exists.

Its sequence of DNSs can be broken into some units of arguing; each unit starts from a DNS setting a problem, then some DNSs establish methodological principles leading to construct a result which, in a rigorous way, is obtained by means of an *ad absurdum* argument, such as it is the celebrated S. Carnot's theorem in thermodynamics [13]. But this is not the case of Majorana's paper, because the DNSs are intermixed with mathematical derivations; where each mathematical formula includes an exact equality; hence, the negation of the negation of the formula gives again the same formula, in agreement with classical logic and contrarily to the case of a DNS.

However, a more sophisticated case than S. Carnot's thermodynamics is given by Lagrange's mechanics. This theory too wanted to solve a great problem, which Lagrange himself stated, i.e., how to develop a theory of mechanical phenomena including constraints. Also inside Lagrange's book (*Mécanique analytique*) we recognise DNSs; it is even possible to point out some units of arguing; each one of them starting from a new

problem; which then is solved. In this case, however, the solving argument belongs to classical logic, being developed by means of mathematical derivations. Hence, the cycle of arguing links the non-classical logic of DNSs with the classical logic of mathematical formulas. In which way are all they linked together inside a consistent logical development?

Lagrange suggests a formalism constituted by a set of differential equations from which one achieves the wanted solution; which is derived in such an easy way ("a regular and uniform march", claims Lagrange in p. *vi* of his book) to give us the feeling of making use of a "magic wand" [14]. Let us analyse the deep logical structure of this formalism.

Lagrange declares that his theory develops a general principle, capable to govern the whole theoretical physics. Then Lagrange presents his celebrated differential equations. Although he tried to justify them by means of pseudo-derivations, one recognises that these equations are no less than a generalisation of the principle of virtual works [15]. Hence, these differential equations are put *a priori* by him. Which is their theoretical import?

Actually, the Principle of virtual works results from a problem-based theory, aimed to solve the constraints problem. Its core actually is a DNS, i.e. "No work from nothing"; or, in more specific terms "It is impossible that the not real forces of the constraints furnish positive work". Here we recognise the last step of an *ad absurdum* argument. Its mathematical formula, translating an essentially DNS, includes an inequality (likely as the mathematical formula for the variations of entropy in thermodynamics). It is a methodological principle, addressing to obtain specific solutions. But, the long practice of its effectiveness led theorists to rise up it to an *a priori* principle; from a philosophical viewpoint, owing to its rejection of an absurdity, it constitutes a general reality principle. This idealisation of a methodological principle in a so general *a priori* principle opens the door to a more idealistic operation.

Let us consider the same argument leading to the principle of virtual works from a classical logic viewpoint. A logical scheme of natural deduction may help the reader [16].

Cycle of arguing in discursive terms	***Translation of the cycle in mathematical terms***
Common Knowledge: (X), (Y), (Z),...	General axiom-principle ($= \neg\bot$)
$\neg$ T	Specific principles: X, Y, Z,...
Methodological Principles	Conditions (= PM mathematised)
AA	Mathematical derivations
----------	----------
$\bot \rightarrow \neg\neg$ T	T

Legenda: (X, Y, Z) = physical conditions; X, Y, Z = the corresponding mathematised hypotheses. MP = Methodological Principle; AA = *Ad absurdum* Argument; $\bot$ =

Absurd. To draw consequences is represented by the vertical line.

Notice that whereas the first column shows the (non-classical) implication $\neg$ T $\rightarrow \perp$, the second column shows the classical converse of this implication (let us recall that if M $\rightarrow$ N, the classical converse (which yet is rejected by non-classical logic) is $\neg$ N $\rightarrow \neg$ M), i.e. $\neg\perp \rightarrow \neg\neg$ T $\rightarrow$ T ; just what classical logic allows, i.e. to reverse through a conversion laws an *ad absurdum* argument in an affirmative argument, however the conclusion may be idealistic in nature.

Let us reverse the path of the previous argument leading to the methodological principle of the virtual works, so to start from its final step; then the negated *absurdum* is equivalent a positive sentence. Which by a mathematical step, can be referred no more to the constraints reactions, but to the resulting active forces; then it, by concerning these forces which can be *a posteriori* experimented, seems a full physical principle. Hence, its philosophical nature of a reality principle together with the (semi-)physical nature of its mathematical formula qualify it as an instance of theoretical omniscience ("all the necessary equations for the solution of each problem", p. 12).

When it is applied no more to forces, but, through the Lagrange's celebrated step $\Sigma(f_i\text{-}ma_i)=0$, to velocities it becomes an omniscient principle on dynamical equilibria. In total, Lagrange generalises the negated *absurdum* to a very general mathematical formula, playing the role of an omniscient principle.

However, the nature of the theory to be a problem-based one, is not lost. The last step, the solution of the problem at issue, is obtained not in forces causing effects (trajectories), but in in-variants (about energy, momentum, etc.), i.e. the same results originated by each theory based upon a general problem.

Majorana's paper, though including DNSs, does not conform to Carnot's problematic model but to Lagrange's; indeed, *ad absurdum* arguments are absent. In order to solve the problem at issue he introduces a sophisticated formalism (variational calculus applied upon the action principle, theory of group representations), which in fact is the Lagrangian formalism. Then, he develops the arguments in a full mathematical way; once he recognises the suitable hypotheses (which actually suggest the restrictions under which the problem is solvable) he puts them as mathematical premises; then the subsequent development merely verify that they effectively solve the problem at issue. We conclude that his arguing is similar to Lagrange's.

6. The Scheme of Arguing in *Teoria Relativistica* (1932)

From both the direct reading of the paper (without taking in account DNSs) and by the knowledge of the scientific subject dealt with, one recognises the following thread of the speech.

[*General problem*] (p. 335) To generalise Schroedinger's mechanics to the relativistic framework

The generalisation has to satisfy three conditions: (a), (b) e (c) [see DNSs no.s 2, 5 and 6]

The third condition is not easy to be satisfied

Dirac equation has already solved the problem under the conditions (b) and in part (c)

But Dirac equation holds only for spin $s = 1/2$.

[S*pecific problem*] (p. 336) A generalisation is required for arbitrary spin

In Dirac equation the mass operator assumes two eigenvalues, a positive and a negative one

[*Methodological principle*: it seems an unnatural fact the negative mass or, equivalently (in the theory of relativity), negative energy]

[*Methodological principle*] (p. 336) Looking for an equation which is similar to Dirac's, but dealing with only positive mass.

[*Anticipation of the solution in a mathematical form*] (pp. 336-337) The equation can be obtained by introducing a condition upon the wave function: this has an infinite number of components forming an irreducible representation of the Lorentz group; Majorana thus makes use of unitary infinite-dimensional representations of the Lorentz group.

[*Having apperceived by intuition the possible final solution, the development comes back to the problem and a mathematical step is performed*] (p. 337) In order to go from the wave function pertaining to Dirac equation to that pertaining to the novel equation to be recognised, a non-unitary transformation is required.

[*Position of the principle of mathematical omnisolution from which to draw the desired solution*] (p. 337) Variational principle on the action function.

From this principle Dirac equation can be obtained.

In Dirac equation the operator $\beta\ m\ c\hat{}2$ [$=\beta\ m\ c^2$] is the only one including the mass; it is an invariant.

[*Anticipation of the mathematical solution*] (p. 337) In order to obtain only positive mass values, it is necessary that the matrix β has only positive eigenvalues.

[*Methodological principle*] (p. 337) This can be achieved by changing the β operator by means of a non-unitary transformation.

[*Result*] (p. 337) The general form of the Majorana equations are obtained, as a modification of the Dirac's one.

[*Subordinate Problem*] The subsequent task is to establish the invariance of the wave function under the Lorentz group, being the matrices γ occurring in the Majorana equation undefined.

[*Methodological principle*] (p. 337) An infinitesimal Lorentz transformation does it. A Hermitian infinite-dimensional representation of the above group is considered.

[*Result*] (p. 339) Infinite spinors and tensors.

[*Subordinate Problem*] (p. 339) Majorana looks for the invariance of γ matrices [*Methodological principle*] under the Lorentz group; he obtains an explicit expression for them.

[*General result*] (p. 342) Hence, the obtained infinite-dimensional representations of the γ's gives the complete determination of his equations.

[*Further problem*] (p. 342) Generalisation in order to include electromagnetic inter-

action.

[*Methodological principle, chosen among various other methodological principles*] (p. 343) A transformation on both W and p introducing electric charge, scalar and potential vectors; the new equations are obtained and discussed.

Let us now come back to the list of DNS.

The interpretative analysis of the paper in units of non-classical logic arguing, expressed by DNSs, is as follows:

1 ° unit: The DNSs 1-11 describe the theoretical background; the general problem is put by (DNS 1,10). This unit will be closed by means of the 4 ° unit.

2 ° unit: The specific problem of eliminating negative values of mass is expressed (DNSs 12-16); some methodological principles for solving it follow, i.e. the use of Dirac-like equation, variational principle (DNSs 17-19) and Lorentz invariance of the equation (DNSs 20-21); the problem is then reiterated (DNS 22), the condition for the invariance of the coefficients is introduced in a mathematised form (DNS 24) and the result (Majorana equations) is obtained (DNS 25).

3 ° unit: The problem of the transformation law of the wave function φ is put (DNSs 26-28) and after the methodological principle of unitarian representations (DNS 29), the result is obtained (DNS 30-32).

4 ° unit: After both the problem of determining the operators for the Lorentz invariance of the equations (DNSs 33-34), and the methodological principle (DNS 35-36), the result is obtained and discussed (DNS 37-38)

5 ° unit: After the problem (DNS 40), he chooses among many others a methodological principle for introducing electromagnetic fields in analogy to Pauli theory in the magnetic neutron (DNSs 41-44).

In total, we have five units of arguing; but they are not cycles of arguing, because, by relying the arguing upon mathematical deduction, it never ends by means of an *ad absurdum* theorem; i.e. that kind of arguing which has to close a correct arguing in non-classical logic. It is manifest to the reader that this lack is due to the introduction of the mathematical hypotheses which anticipate the final point of the arguing and, owing to the mathematical formalism, have to be verified as (even partial) solutions only of the problem set at the beginnings.

7. Analysis of *Teoria Simmetrica* (1937)

An accurate reading of this paper suggests that the deductive thread of thinking in Majorana can be organised according to the following steps:

- Variational principle using real variables
- Method of quantization
- Dirac equation in terms of real variables
- Relativistic invariance according to the Jordan-Wigner method

Instead, his exposition is a complex one:

- *Problem*: Replacement of the Dirac equation

- *Problem*: generalization of variational principle
- *Methodological principle*: the method of quantization;

from which he obtains an undetermined formula for the equation of motion.

- These equations can be represented by an Hamiltonian if a certain relationships on his

magnitudes hold true

- Then he shows that Dirac equations (written in terms of real variables) can be obtained from the variational principle (using real variables)
- By means of the mentioned method, these equations may be quantized
- They are relativistically invariant because they reiterate a well-known scheme
- Applications

The reader can see that again the exposition is not organised in a deductive one; rather Majorana puts a problem and then a more general problem; after the solution of the latter one, he can solve his starting problem. This method of resolution starts from a variational principle in the Lagrangian formalism and then he puts the Hamilton equations as the closing step of the presentation.

Here we could reiterate the same interpretative analysis as above. First, to list the DNSs in the text and then to group them according to units of arguing. But for not boring the reader, we leave out the list of DNSs, whose number however is almost the double of the previous one: 82 DNSs in 14 pages of the paper; 43 of which are normal DNSs and 41 wDNSs.

The set of cycles of arguing shows one time more that his theory is a problem-based theory. The analysis through cycles of arguing is already apparent from the previous scheme: a first cycle sets the problem and then the more general problem is stated. A second cycle applies the method of quantization in order to obtain the formula of the equations of motion, which are then compared with the Hamiltonian and thus declared valid (actually this argument can be considered an implicit *ad absurdum* arguing). After this step, he closes the first cycle of arguing by deriving the solution of his problem from the solution obtained for the general problem (variational principle in real variables). The remaining part is devoted to verifications and logical consequences.

8. Global Considerations on the Two Papers

Let us consider the distributions of the DNSs and the wDNSs. Notice that there in both paper the recourse to DNSs is frequent (a mean which varies from 2 per page to 3 per page). Moreover, there is not a substantial difference between the two distributions of DNSs and wDNSs; hence, by summing up the two kinds of DNS, weak or not, we obtain a double number of DNS (around 5) per page.

Summing up, the two papers illustrate a consistent way of arguing in the particular organisation of a physical theory, the problem-based one. The examination of the DNSs, including the wDNSs, shows that by making use of non-classical logic Majorana well recognises and formalises all the problems he faces to. On the other hand he states

TAB. THE DISTRIBUTIONS OF DNSs INSIDE THE TWO MAJORANA PAPERS

	DNS	**wDNS**				DNS per page
		necessity	permission	possibility	want	
Teoria relativistica	21	10	3	8	1	2, 1 DNSs and 2,3 wDNS per page (in 10 pages)
Teoria simmetrica	43	14	5	20	-	3 DNS and 2,9 wDNS per page (in 14 pages)
Total	64	24	8	28	1	2,6 DNS and 2,5 wDNS per page (in 24 pages)

the action principle, from which he derives all mathematical consequences useful for his scopes. In this way he fashions his theory as a kind of Lagrangian theory, just the principle to which he makes appeal.

9. The Weak DNS and the Rhetoric of the Scientific Presentation by E. Majorana

The following kinds of rhetoric in presenting a scientific novelty seem to come out:

1) Historical report of the way the scientific discovery happened in the mind of the researcher or in the laboratory; hence, the set of the data are described by listing them in the way they occurred along the time of the discovery.

2) Exposition of formal laws which hold true for both the obtained data and the calculations (hence, the set of data and calculations: must, obey, result, etc.).

3) The list of obligations and potentialities that the " ideal solver" wants in order to reiterate the experience to solve the problem (hence the solver: must, it is necessary, see, it is possible to him, he is allowed, etc.).

4) A programmatic arguing which then leaves the developments as the executive part.

5) The deductive or axiomatic exposition of the novelty, obtained as a consequence of a known theory (hence the verbs are: is, holds true, follows, etc.).

In the paper at issue (but also inside the "Volumetti") Ettore Majorana's rhetoric is apparently the third one; he presents the laws holding true for the ideal scientist; hence, it is the "juridical" way of speaking to a scientist. About this point, one can show that for instance in the second paper the DNSs can be grouped in different sets according to which they refer to Nature laws N, to calculation laws C, to the attitude asked to the theoretical

physicist F: the maximum number is for those of kind F: 10, plus 34 wDNSs. We mean that Majorana imposes these laws to each scientist; in this sense his presentation is more than a "juridical" one; it can be considered a "royal" one.

References

[1] E. Recami: *Il caso Majorana*, Di Renzo, Roma, 2004.

[2] The reproduction of the original manuscripts is in E. Majorana: *Lezioni all'Università di Napoli*, Bibliopolis, Napoli, 1987; for the critical edition see E. Majorana: *Lezioni di Fisica Teorica*, edited by S. Esposito, Bibliopolis, Napoli, 2006.

[3] S. Esposito, E. Majorana jr, E. Recami, A. van der Merwe (eds.): *Ettore Majorana: Notes on Theoretical Physics*, Dordrecht, Kluwer, 2003. For the original Italian edition see S. Esposito and E. Recami (eds.): *Ettore Majorana: Appunti inediti di Fisica Teorica*, Zanichelli, Bologna. 2006.

[4] Reported in the letter from G. Cocconi to E. Amaldi (18 July, 1965). The letter is quoted in E. Recami: *op. cit.* p. 23.

[5] They are reported, for example, in the books in reference 3.

[6] E. Amaldi: *La Vita e l'Opera di E. Majorana*, Accad. Naz. dei Lincei, Roma, 1966.

[7] S. Esposito: Majorana solution of the Thomas-Fermi equation, *Am. J. Phys.* **70** (2002) 852; Majorana transformation for differential equations, *Int. J. Theor. Phys.* **41** (2002) 2417; Covariant Majorana formulation of electrodynamics, *Found. Phys.* **28** (1998) 231; The "Volumetti" by Ettore Majorana, in L. Gariboldi and P. Tucci (eds.): *History of Physics and Astronomy in Italy in the 19th and 20th Centuries. Sources, themes and international context*, Milano, 2004, 37-46.

[8] L. Bonolis: Giovanni Gentile ed Ettore Majorana: La comparsa della teoria dei gruppi nella fisica teorica italiana, in M. Leone et al. (eds.): *Atti XXII Congr. Naz. Storia Fis. e Astr.*, Microart, Recco GE, 2004, 117-135.

[9] A. Drago e S. Esposito: Following Weyl on Quantum Mechanics: The Contribution of Ettore Majorana, *Found. Phys.*, **34**, 2004, 871-887.

[10] E. Majorana: Teoria relativistica di particelle con momento intrinseco arbitrario, *Nuovo Cimento*, **9** (1932) 335-344; Teoria simmetrica dell'elettrone e del positrone, *Nuovo Cimento*, **14**, 1937, 171.184.

[11] A. Drago: *Le due opzioni*, La Meridiana, Molfetta BA, 1991; in particular, pp. 164-167. This kind of organisation has been recognised in several past physical, mathematical and chemical theories; its ideal model has been illustrated in A. Drago e A. Perno: La teoria geometrica delle parallele impostata coerentemente su un problema (I), *Periodico di Mat.*, ser. VIII, vol. 4, ott.-dic. 2004, 41-52 and in A. Drago: There exist two models of organization of a scientific theory, *Mind and Society* (in press).

[12] A. Drago: Incommensurable scientific theories: The rejection of the double negation logical law, in D. Costantini e M. G. Galavotti (eds.): *Nuovi problemi della logica e della filosofia della scienza,* CLUEB, Bologna, 1991, vol. I, 195-202.

[13] A. Drago e R. Pisano: Interpretazione e ricostruzione delle *Réflexions* di Sadi Carnot mediante la logica non classica, *Giornale di Fisica*, **41** (2000) 195-215. Interpretation and reconstruction of S. Carnot's *Réflexions* according to non-classical logic, *Atti Fond. Ronchi*, **58** (2003) 195-215.

[14] L. D. Scott: "Can a projection method of obtaining equations of motion compete with Lagrange's equations?", *Am. J. Phys.*, 56 (1988) 451-456, p. 451.

[15] This conclusion was achieved by Poinsot, after forty years of unsuccesfull attempts by the major scientists of that times, of proving the principle of virtual works:

P. Bailhache, *L. Poinsot: La théorie générale de l'équilibre et du mouvement des systèmes*, Vrin, Paris, 1975. See also A. Drago: La Meccanica nella formulazione di Lagrange: un esame storico-critico, in P. Tucci, A. Garuccio e M. Nigro (edd.): *Atti XXIII Congresso Società Italiana Storia Fisica e Astronomia,* Progredit, Bari, 2003, 165-179.

[16] D. Prawitz: *Natural Deduction: A Proof-Theoretical Study*, Almqvist & Wiksell, Stockholm, 1965.

EJTP **3, No. 10** (2006) 265–283 *Electronic Journal of Theoretical Physics*

Four Variations on Theoretical Physics by Ettore Majorana

S. Esposito *

Dipartimento di Scienze Fisiche, Università di Napoli "Federico II" &
I.N.F.N. Sezione di Napoli,
Complesso Universitario di M. S. Angelo, Via Cinthia, 80126 Naples, Italy

Received 22 February 2006, Published 28 May 2006

Abstract: An account is given of some topical unpublished work by Ettore Majorana, revealing his very deep intuitions and skillfulness in Theoretical Physics. The relevance of the quite unknown results obtained by him is pointed out as well

Keywords: Ettore Majorana,Thomas-Fermi equation,Electrodynamics,Lorentz Group
PACS (2006): 25.30.Pt, 13.35.Hb, 31.15.Bs, 0.3.50.De, 12.20. m

1. Introduction

Probably, the highest appraisal received by the work of Ettore Majorana was expressed by the Nobel Prize Enrico Fermi in several occasions [1], but such opinions could appear as overstatements or unjustified (especially because they are expressed by a great physicist as Fermi), when compared with the spare (known) Majorana's scientific production, just 9 published papers. However, today the name of Majorana is largely known to the nuclear and subnuclear physicist's community: Majorana neutrino, Majorana-Heisenberg exchange forces, and so on are, in fact, widely used concepts.

In this paper, we focus on the less-known (or completely unknown) work by this scientist, aimed to shed some light on the peculiar abilities of Majorana that were well recognized by Fermi and his coworkers. The wide unpublished scientific production by Majorana is testified by a large amount of papers [2], almost all deposited at the Domus Galilaeana in Pisa; those known, in Italian, as "Volumetti" has been recently collected and translated in a book [3], and we refer the interested reader to this book for further

* Salvatore.Esposito@na.infn.it

study.

Here we have chosen to discuss only four topics dealt with by Majorana in different areas of Physics, just to give a sample of his very deep intuitions and skilfulness, together with the relevance of the results obtained.

We start with a discussion of a peculiar approach to Quantum Mechanics, as deduced by a manuscript [4] which probably corresponds to the text for a seminar delivered at the University of Naples in 1938, where Majorana lectured on Theoretical Physics [5]. Some passages of that manuscript reveal a physical interpretation of the Quantum Mechanics, which anticipates of several years the Feynman approach in terms of path integrals, independently of the underlying mathematical formulation. The main topic of that dissertation was the application of Quantum Mechanics to the theory of molecular bonding, but the present scientific interest in it is more centered on the interpretation given by Majorana about some topics of the novel, for that time, Quantum Theory (namely, the concept of quantum state) and the direct application of this theory to a particular case (that is, precisely, the molecular bonding). It not only discloses a peculiar cleverness of the author in treating a pivotal argument of the novel Mechanics, but, keeping in mind that it was written in 1938, also reveals a net advance of at least ten years in the use made of that topic.

In the second topic, we report on a more applicative subject, discussing an original method that leads to a semi-analytical series solution of the Thomas -Fermi equation, with appropriate boundary conditions, in terms of only one quadrature [6]. This was developed by Majorana in 1928, just when starting to collaborate (still as a University student) with the Fermi group in Rome, and reveals an outstanding ability to solve very involved mathematical problems in a very interesting and clear way. The whole work performed on the Thomas-Fermi model is contained in some spare sheets, and diligently reported by the author himself in his notebooks [3]. From these it is evident the considerable contribution given by Majorana even in the achievement of the statistical model [7], anticipating, in many respects, some important results reached later by leading specialists. But the major finding by Majorana was his solution (or, rather, methods of solutions) of the Thomas-Fermi equation, which remained completely unknown, until recent times, to the Physics community, who ignored that the non-linear differential equation relevant for atoms and other systems could even be solved semi-analytically. The method proposed by Majorana can also be extended to an entire class of particular differential equations [8].

Afterwards we discuss a subject that was repeatedly studied by Majorana in his research notebooks; namely that of a formulation of Electrodynamics in terms of the electric and magnetic fields, rather than the potentials, which is suitable for a quantum generalization, in a complete analogy with the Dirac theory [9] [10]. This argument was already faced in 1931 by Oppenheimer [11], who only supposed the analogy of the photon case with that described by Dirac, but Majorana explicitly deduced a Dirac-like equation for the photon, thus building up the presumed analogy.

Finally, we report on another topic particularly loved by Majorana, after the appearance (at the end of 1928) of the seminal book by Hermann Weyl [12], that is the Group

Theory and its application to physical problem. As testified by the large number of unpublished manuscript pages of the Italian physicist, the Weyl approach greatly influenced the scientific thought and work of Majorana [13]. In fact, when Majorana became aware of the great relevance of the Weyls application of the Group Theory to Quantum Mechanics, he immediately grabbed the Weyl method and developed it in many applications. In one of his notebooks [3] we find, for example, a preliminary study of what will be one of the most important (published) papers by Majorana on a generalization of the Dirac equation to particles with arbitrary spin [14]. In particular, in 1932 Majorana obtained the infinite-dimensional unitary representations of the Lorentz group that will be re-discovered by Wigner in his 1939 and 1948 works [15], and the entire theory was re-invented by Soviet mathematicians (in particular Gel'fand and collaborators) in a series of articles from 1948 to 1958 [16] and finally applied by physicists years later.

What presented here is, necessarily, a very short account of what Majorana really did in his few years of work (about ten years), but, we hope, it serves in the centennial year at least to understand the very relevant role played by him in the advancement of Physics.

2. Path-Integral Approach to Quantum Mechanics

The usual quantum-mechanical description of a given system is strongly centered on the role played by the hamiltonian H of the system and, as a consequence, the time variable plays itself a key role in this description. Such a dissymmetry between space and time variables is, obviously, not satisfactory in the light of the postulates of the Theory of Relativity. This was firstly realized in 1932 by Dirac [17], who put forward the idea of reformulating the whole Quantum Mechanics in terms of lagrangians rather than hamiltonians. The starting point in the Dirac thought is that of exploiting an analogy, holding at the quantum level, with the Hamilton principal function S in Classical Mechanics, thus writing the transition amplitude from one space-time point to another as an (imaginary) exponential of S. However, the original Dirac formulation was not free from some unjustified assumptions, leading also to wrong results, and the correct mathematical formulation and the physical interpretation of it came only in the forties with the work by Feynman [18]. In practice, in the Feynman approach to Quantum Mechanics, the transition amplitude between an initial and a final state can be expressed as a sum of the factor $e^{iS[q]/\hbar}$ over all the paths q with fixed end-points, not just those corresponding to classical dynamical trajectories, for which the action is stationary.

In 1938 Majorana was appointed as full professor of Theoretical Physics at the University of Naples, where probably delivered a general conference mentioning also his particular viewpoint on some basic concepts on Quantum Mechanics (see Ref. [4]). Fortunately enough, we have some papers written by him on this subject, and few crucial points, anticipating the Feynman approach to Quantum Mechanics, will be discussed in the following. However, we firstly note that such papers contain *nothing* of the mathematical aspect of that peculiar approach to Quantum Mechanics, but it is quite evident

as well the presence of the *physical* foundations of it. This is particularly impressive if we take into account that, in the known historical path, the interpretation of the formalism has only followed the mathematical development of the formalism itself.

The starting point in Majorana is to search for a meaningful and clear formulation of the concept of quantum state. And, obviously, in 1938 the dispute is opened with the conceptions of the Old Quantum Theory.

> "According to the Heisenberg theory, a quantum state corresponds not to a strangely privileged solution of the classical equations but rather to a set of solutions which differ for the initial conditions and even for the energy, i.e. what it is meant as precisely defined energy for the quantum state corresponds to a sort of average over the infinite classical orbits belonging to that state. Thus the quantum states come to be the minimal statistical sets of classical motions, slightly different from each other, accessible to the observations. These minimal statistical sets cannot be further partitioned due to the uncertainty principle, introduced by Heisenberg himself, which forbids the precise simultaneous measurement of the position and the velocity of a particle, that is the determination of its orbit."

Let us note that the "solutions which differ for the initial conditions" correspond, in the Feynman language of 1948, precisely to the different integration paths. In fact, the different initial conditions are, in any case, always referred to the same initial time (t_a), while the determined quantum state corresponds to a fixed end time (t_b). The introduced issue of "slightly different classical motions" (the emphasis is given by Majorana himself), according to what specified by the Heisenberg's uncertainty principle and mentioned just afterwards, is thus evidently related to that of the sufficiently wide integration region required in the Feynman path-integral formula for quantum (rather than classical) systems. In this respect, such a mathematical point is intimately related to a fundamental physical principle.

The crucial point in the Feynman formulation of Quantum Mechanics is, as well-known, to consider not only the paths corresponding to classical trajectories, but *all* the possible paths joining the initial point with the end one. In the Majorana manuscript, after a discussion on an interesting example on the harmonic oscillator, the author points out:

> "Obviously the correspondence between quantum states and sets of classical solutions is only approximate, since the equations describing the quantum dynamics are in general independent of the corresponding classical equations, but denote a real modification of the mechanical laws, as well as a constraint on the feasibility of a given observation; however it is better founded than the representation of the quantum states in terms of quantized orbits, and can be usefully employed in qualitative studies."

And, in a later passage, it is more explicitly stated that the wave function "corresponds in Quantum Mechanics to any possible state of the electron". Such a reference, that only superficially could be interpreted, in the common acceptation, that all the information on the physical systems is contained in the wave function, should instead be considered in the meaning given by Feynman, according to the comprehensive discussion made by

Majorana on the concept of state.

Finally we point out that, in the Majorana analysis, a key role is played by the symmetry properties of the physical system.

> "Under given assumptions, that are verified in the very simple problems which we will consider, we can say that every quantum state possesses all the symmetry properties of the constraints of the system."

The relationship with the path-integral formulation is made as follows. In discussing a given atomic system, Majorana points out how from one quantum state S of the system we can obtain another one S' by means of a symmetry operation.

> "However, differently from what happens in Classical Mechanics for the single solutions of the dynamical equations, in general it is no longer true that S' will be distinct from S. We can realize this easily by representing S' with a set of classical solutions, as seen above; it then suffices that S includes, for any given solution, even the other one obtained from that solution by applying a symmetry property of the motions of the systems, in order that S' results to be identical to S."

This passage is particularly intriguing if we observe that the issue of the redundant counting in the integration measure in gauge theories, leading to infinite expressions for the transition amplitudes, was raised (and solved) only after much time from the Feynman paper.

3. Solution of the Thomas-Fermi Equation

The main idea of the Thomas-Fermi atomic model is that of considering the electrons around the nucleus as a gas of particles, obeying the Pauli exclusion principle, at the absolute zero of temperature. The limiting case of the Fermi statistics for strong degeneracy applies to such a gas. Then, in this approximation, the potential V inside a given atom of charge number Z at a distance r from the nucleus may be written as

$$V(r) = \frac{Ze}{r}\varphi(r). \tag{1}$$

With a suitable change of variable, $r = \mu x$ and

$$\mu = \frac{1}{2}\left(\frac{3\pi}{4}\right)^{2/3} \frac{\hbar^2}{m_e e^2} Z^{-1/3}, \tag{2}$$

the Thomas-Fermi function φ satisfies the following non-linear differential equation (for $\varphi > 0$):

$$\varphi'' = \frac{\varphi^{3/2}}{\sqrt{x}} \tag{3}$$

(a prime denotes differentiation with respect to x) with the boundary conditions:

$$\begin{gathered} \varphi(0) = 1, \\ \\ \varphi(\infty) = 0. \end{gathered} \tag{4}$$

The Fermi equation (3) is a universal equation which does not depend neither on Z nor on physical constants $(\hbar, m, e)$. Its solution gives, from Eq. (1), as noted by Fermi himself, a screened Coulomb potential which at any point is equal to that produced by an effective charge

$$Ze\,\varphi\left(\frac{r}{\mu}\right). \tag{5}$$

As was immediately realized, in force of the independence of Eq. (3) on Z, the method gives an effective potential which can be easily adapted to describe any atom with a suitable scaling factor, according to Eq. (5).

The problem of the theoretical calculation of observable atomic properties is thus solved, in the Thomas-Fermi approximation, in terms of the function $\varphi(x)$ introduced in Eq. (1) and satisfying the Fermi differential equation (3). By using standard but involved mathematical tools, in his paper [19] Thomas got an exact, "singular" solution of his differential equation satisfying only the second condition (4). This was later (in 1930) considered by Sommerfeld [20] as an approximation of the function $\varphi(x)$ for large x (and is indeed known as the "Sommerfeld solution" of the Fermi equation),

$$\varphi(x) = \frac{144}{x^3}, \tag{6}$$

and Sommerfeld himself obtained corrections to the above quantity in order to approximate in a better way the function $\varphi(x)$ for not extremely large values of x.

Until recent times it has been believed that the solution of such equation satisfying both the appropriate boundary conditions in (4) cannot be expressed in closed form, and some effort has been made, starting from Thomas [19], Fermi [21], [22] and others, in order to achieve the numerical integration of the differential equation. However, we now know [6], [7] that Majorana in 1927-8 found a semi-analytical solution of the Thomas-Fermi equation by applying a novel exact method [8]. Before proceeding, we will indulge here on an anecdote reported by Rasetti [23], Segrè [24] and Amaldi [25]. According to the last author, "Fermi gave a broad outline of the model and showed some reprints of his recent works on the subject to Majorana, in particular the table showing the numerical values of the so-called Fermi universal potential. Majorana listened with interest and, after having asked for some explanations, left without giving any indication of his thoughts or intentions. The next day, towards the end of the morning, he again came into Fermi's office and asked him without more ado to draw him the table which he had seen for few moments the day before. Holding this table in his hand, he took from his pocket a piece of paper on which he had worked out a similar table at home in the last twenty-four hours, transforming, as far as Segrè remembers, the second-order Thomas-Fermi non-linear differential equation into a Riccati equation, which he had then integrated numerically."

The whole work performed by Majorana on the solution of the Fermi equation, is contained in some spare sheets conserved at the Domus Galilaeana in Pisa, and diligently reported by the author himself in his notebooks [3]. The reduction of the Fermi equation to an Abel equation (rather than a Riccati one, as confused by Segrè) proceeds as follows.

Let's adopt a change of variables, from (x, φ) to (t, u), where the formula relating the two sets of variables has to be determined in order to satisfy, if possible, both the boundary conditions (4). The function φ in Eq. (6) has the correct behavior for large x, but the wrong one near $x = 0$, so that we could modify the functional form of φ to take into account the first condition in (4). An obvious modification is $\varphi = (144/x^3)f(x)$, with $f(x)$ a suitable function which vanishes for $x \to 0$ in order to account for $\varphi(x = 0) = 1$. The simplest choice for $f(x)$ is a polynomial in the novel variable t, as it was also considered later, in a similar way, by Sommerfeld [20]. The Majorana choice is as follows:

$$\varphi(x) = \frac{144}{x^3}(1-t)^2, \tag{7}$$

with $t \to 1$ as $x \to 0$. From Eq. (7) we can then obtain the first relation linking t to x, φ. The second one, involving the dependent variable u, is that typical of homogeneous differential equations (like the Fermi equation) for reducing the order of the equation, i.e. exponentiation with an integral of $u(t)$. The transformation relations are thus:

$$\begin{aligned} t &= 1 - \frac{1}{12}\sqrt{x^3\varphi}, \\ \varphi &= e^{\int_1^t u(t)dt}. \end{aligned} \tag{8}$$

Substitution into Eq. (3) leads to an Abel equation for $u(t)$,

$$\frac{du}{dt} = \alpha(t) + \beta(t)\, u + \gamma(t)\, u^2 + \delta(t)\, u^3, \tag{9}$$

with

$$\begin{aligned} \alpha(t) &= \frac{16}{3(1-t)}, \\ \beta(t) &= 8 + \frac{1}{3(1-t)}, \\ \gamma(t) &= \frac{7}{3} - 4t, \\ \delta(t) &= -\frac{2}{3}t(1-t). \end{aligned} \tag{10}$$

Note that both the boundary conditions in (4) are automatically verified by the relations (8). We have reported the derivation of the Abel equation (9) mainly for historical reasons (nevertheless, it is quite important since, in this way, all the theorems on the Abel equation may thus be applied to the non-linear Thomas-Fermi equation too); the precise numerical values for the Fermi function $\varphi(x)$ were obtained by Majorana by

solving a different first-order differential equation. Instead of Eq. (7), Majorana chooses $\varphi(x)$ of the form

$$\varphi(x) = \frac{144}{x^3}\, t^6. \tag{11}$$

Now the point with $x = 0$ corresponds to $t = 0$. In order to obtain again a first order differential equation for $u(t)$, the transformation equation for the variable u involves φ and its first derivative. Majorana then introduced the following formulas:

$$\begin{aligned} t &= 144^{-1/6}\, x^{1/2}\, \varphi^{1/6}, \\ u &= -\left(\frac{16}{3}\right)^{1/3} \varphi^{-4/3}\varphi' \ . \end{aligned} \tag{12}$$

By taking the t-derivative of the last equation in (12) and inserting Eq. (3) in it, one gets:

$$\frac{du}{dt} = -\left(\frac{16}{3}\right)^{1/3} \dot{x}\varphi^{-4/3}\left[-\frac{4}{3}\frac{\varphi'^2}{\varphi} + \frac{\varphi^{3/2}}{x^{1/2}}\right]. \tag{13}$$

By using Eqs. (12) to eliminate $x^{1/2}$ and φ'^2, the following equation results:

$$\frac{du}{dt} = \left(\frac{4}{9}\right)^{1/3} \frac{tu^2 - 1}{t}\dot{x}\varphi^{1/3}. \tag{14}$$

Now the quantity $\dot{x}\varphi^{1/3}$ can be expressed in terms of t and u by making use again of the first equation in (12) (and its t-derivative). After some algebra, the final result for the differential equation for $u(t)$ is:

$$\frac{du}{dt} = 8\,\frac{tu^2 - 1}{1 - t^2 u}. \tag{15}$$

The obtained equation is again non-linear but, differently from the original Fermi equation (3), it is first-order in the novel variable t and the degree of non-linearity is lower than that of Eq. (3). The boundary conditions for $u(t)$ are easily taken into account from the second equation in (12) and by requiring that for $x \to \infty$ the Sommerfeld solution (Eq. (11) with $t = 1$) be recovered:

$$\begin{aligned} u(0) &= -\left(\frac{16}{3}\right)^{1/3} \varphi_0', \\ u(1) &= 1. \end{aligned} \tag{16}$$

Here we have denoted with $\varphi_0' = \varphi'(x = 0)$ the initial slope of the Thomas-Fermi function $\varphi(x)$ which, for a neutral atom, is approximately equal to -1.588.

The solution of Eq. (15) was achieved by Majorana in terms of a series expansion in powers of the variable $\tau = 1 - t$:

$$u = a_0 + a_1\tau + a_2\tau^2 + a_3\tau^3 + \ldots \ . \tag{17}$$

Substitution of Eq. (17) (with the conditions in Eq. (16)) into Eq. (15) results into an iterative formula for the coefficients a_n (for details see Ref. [6]). It is remarkable that the series expansion in Eq. (17) is uniformly convergent in the interval $[0, 1]$ for τ, since the series $\sum_{n=0}^{\infty} a_n$ of the coefficients converges to a finite value determined by the initial slope φ_0'. In fact, by setting $\tau = 1$ ($t = 0$) in Eq. (17) we have from Eq. (16):

$$\sum_{n=0}^{\infty} a_n = -\left(\frac{16}{3}\right)\varphi_0' \tag{18}$$

Majorana was aware [3] of the fact that the series in Eq. (17) exhibits geometrical convergence with $a_n/a_{n-1} \sim 4/5$ for $n \to \infty$. Given the function $u(t)$, we now have to look for the Thomas-Fermi function $\varphi(x)$. This was obtained in a parametric form $x = x(t)$, $\varphi = \varphi(t)$ in terms of the parameter t already introduced in Eq. (12), and by writing $\varphi(t)$ as

$$\varphi(t) = e^{\int_0^t w(t)dt} \tag{19}$$

(with this choice, $\varphi(t = 0) = 1$ and the first condition in (4) is automatically satisfied). Here $w(t)$ is an auxiliary function which is determined in terms of $u(t)$ by substituting Eq. (19) into Eq. (12). As a result, the parametric solution of Eq. (3), with boundary conditions (4), takes the form:

$$\begin{aligned} x(t) &= 144^{1/3}\, t^2\, e^{2\mathcal{I}(t)} \\ \varphi(t) &= e^{-6\,\mathcal{I}(t)} \end{aligned} \tag{20}$$

with

$$\mathcal{I}(t) = \int_0^t \frac{ut}{1 - t^2 u}\, dt \tag{21}$$

Remarkably, the Majorana solution of the Thomas-Fermi equation is obtained with only one quadrature and gives easily obtainable numerical values for the electrostatic potential inside atoms. By taking into account only 10 terms in the series expansion for $u(t)$, such numerical values approximate the values of the exact solution of the Thomas-Fermi equation with a relative error of the order of 0.1%.

The intriguing property in the Majorana derivation of the solution of the Thomas-Fermi equation is that his method can be easily generalized and may be applied to a large class of particular differential equations, as discussed in [8].

Several generalizations of the Thomas-Fermi method for atoms were proposed as early as in 1928 by Majorana, but they were considered by the physics community, ignoring the Majorana unpublished works, only many years later.

Indeed, in Sect. 16 of Volumetto II [3], Majorana studied the problem of an atom in a weak external electric field E, i.e. atomic polarizability, and obtained an expression for the electric dipole moment for a (neutral or arbitrarily ionized) atom.

Furthermore, he also started to consider the application of the statistical method to molecules, rather than single atoms, studying the case of a diatomic molecule with

identical nuclei (see Sect. 12 of Volumetto II [3]). The effective potential in the molecule was cast in the form:

$$V = V_1 + V_2 - \alpha \frac{2V_1V_2}{V_1 + V_2}, \tag{22}$$

V_1 and V_2 being the potentials generated by each of the two atoms. The function α must obey the differential equation for V,

$$\nabla^2 V = -kV^{3/2} \tag{23}$$

(k is a suitable constant), with appropriate boundary conditions, discussed in [3]. Majorana also gave a general method to determine V when the equipotential surfaces are approximately known (see Sect. 12 of Volumetto III [3]). In fact, writing the approximate expression for the equipotential surfaces, as functions of a parameter p, as

$$f(x, y, z) = p, \tag{24}$$

he deduced a thorough equation from which it is possible to determine $V(\rho)$, when the boundary conditions are assigned. The particular case of a diatomic molecule with identical nuclei was, again, considered by Majorana using elliptic coordinates in order to illustrate his original method [3].

Finally, our author also considered the second approximation for the potential inside the atom, beyond the Thomas-Fermi one, with a generalization of the statistical model of neutral atoms to those ionized n times, including the case $n = 0$ (see Sect. 15 of Volumetto II [3]). As recently pointed out, the approach used by Majorana to this end is rather similar to that now adopted in the renormalization of physical quantities in modern gauge theories [26].

4. Majorana Formulation of Electrodynamics

In 1931, in his "note on light quanta and the electromagnetic field" [11], Oppenheimer developed an alternative model to the theory of Quantum Electrodynamics, starting from an analogy with the Dirac theory of the electron. Such a formulation was particularly held dear by Majorana, who studied it in some of his unpublished notebooks [9].

Majorana's original idea was that if the Maxwell theory of electromagnetism has to be viewed as the wave mechanics of the photon, then it must be possible to write the Maxwell equations as a Dirac-like equation for a probability quantum wave ψ, this wave function being expressible by means of the physical $\vec{\mathcal{E}}$, $\vec{\mathcal{B}}$ fields. This can be, indeed, realized introducing the quantity

$$\vec{\psi} \;=\; \vec{\mathcal{E}} \;-\; i\,\vec{\mathcal{B}} \tag{25}$$

since $\vec{\psi}^* \cdot \vec{\psi} \;=\; \vec{\mathcal{E}}^2 \;+\; \vec{\mathcal{B}}^2$ is directly proportional to the probability density function for a photon [1]. In terms of $\vec{\psi}$, the Maxwell equations in vacuum then write

$$\vec{\nabla} \cdot \vec{\psi} \;=\; 0 \tag{26}$$

[1] If we have a beam of n equal photons each of them with energy ϵ (given by the Planck relation),

$$\frac{\partial \vec{\psi}}{\partial t} \;=\; i\,\vec{\nabla}\times\vec{\psi} \tag{27}$$

By making use of the correspondence principle

$$E \rightarrow i\,\frac{\partial}{\partial t} \tag{28}$$

$$\vec{p} \rightarrow -i\,\vec{\nabla} \tag{29}$$

these equations effectively can be cast in a Dirac-like form

$$(E \;-\; \alpha{\cdot}\vec{p})\;\vec{\psi} \;=\; 0 \tag{30}$$

with the transversality condition

$$\vec{p}{\cdot}\vec{\psi} \;=\; 0 \tag{31}$$

where the 3x3 hermitian matrices $(\alpha_i)_{lm} \;=\; i\,\epsilon_{ilm}$

$$\alpha^1 \;=\; \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix} \qquad \alpha^2 \;=\; \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad \alpha^3 \;=\; \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{32}$$

satisfying

$$[\alpha_i \;,\; \alpha_j] \;=\; -i\,\epsilon_{ijk}\,\alpha_k \tag{33}$$

have been introduced.

The probabilistic interpretation is indeed possible given the "continuity equation" (Poynting theorem)

$$\frac{\partial \rho}{\partial t} \;+\; \vec{\nabla}{\cdot}\vec{j} \;=\; 0 \tag{34}$$

where

$$\rho \;=\; \frac{1}{2}\,\vec{\psi}^{*}{\cdot}\vec{\psi} \qquad\qquad \vec{j} \;=\; -\,\frac{1}{2}\,\psi^{*}\,\alpha\,\psi \tag{35}$$

are respectively the energy and momentum density of the electromagnetic field.

It is interesting to observe that, differently from Oppenheimer, who started from a mere, presumed analogy with the electron case, Majorana built on analytically the analogy with the Dirac theory, at a dynamical level, by *deducing* the Dirac-like equation for the photon from the Maxwell equations with the introduction of a complex wave field. As noted by Giannetto in Ref. [10], the Majorana formulation is algebraically equivalent to the standard one of Quantum Electrodynamics and, in addition, also some relevant problems concerning the negative energy states, that induced Oppenheimer to abandon his model, may be elegantly solved by using the method envisaged in a later work [27], thus giving further physical insight into Majorana theory.

since $\frac{1}{2}\,(\vec{\mathcal{E}}^2 \,+\, \vec{\mathcal{B}}^2)$ is the energy density of the electromagnetic field, then $\frac{1}{n\epsilon}\,\frac{1}{2}\,(\vec{\mathcal{E}}^2 \,+\, \vec{\mathcal{B}}^2)\,dS\,dt$ gives the probability that each photon has to be detected in the area dS in the time dt. The generalization to photons of different energies (i.e. of different frequencies) is obtained with the aid of the superposition principle.

5. Lorentz Group and its Applications

The important role of symmetries in Quantum Mechanics was established in the third decade of the XX century, when it was discovered the special relationships concerning systems of identical particles, reflection and rotational symmetry or translation invariance. Very soon it was discovered that the systematic theory of symmetry resulted to be just a part of the mathematical theory of groups, as pointed out, for example, in the reference book by H. Weyl [12]. A particularly intriguing example is that of the Lorentz group which, as well known, underlies the Theory of Relativity, and its representations are especially relevant for the Dirac equation in Relativistic Quantum Mechanics. In the mentioned book, however, although the correspondence between the Dirac equation and the Lorentz transformations is pointed out, the group properties of this connection are not highlighted. Moreover, only a particular kind of such representations are considered (those related to the two-dimensional representations of the group of rotations, according to Pauli), but an exhaustive study of this subject was still lacking at that time.

The situation changes [13] quite sensibly with several (unpublished) papers by Majorana [3], where he gives a detailed deduction of the relationship between the representations of the Lorentz group and the matrices of the (special) unitary group in two dimensions, and a strict connection with the Dirac equation is always taken into account. Moreover the *explicit* form of the transformations of every bilinear in the spinor field Ψ is reported. For example, Majorana obtains that some of such bilinears behave as the 4-position vector (ct, x, y, z) or as the components of the rank-2 electromagnetic tensor $(\vec{E}, \vec{H})$ under Lorentz transformations, according to the following rules:

$$\begin{gathered}
\Psi^\dagger\Psi \sim -i\Psi^\dagger\alpha_x\alpha_y\alpha_z\Psi \sim ct, \\
-\Psi^\dagger\alpha_x\Psi \sim i\Psi^\dagger\alpha_y\alpha_z\Psi \sim x, \\
-\Psi^\dagger\alpha_y\Psi \sim i\Psi^\dagger\alpha_z\alpha_x\Psi \sim y, \\
-\Psi^\dagger\alpha_z\Psi \sim i\Psi^\dagger\alpha_x\alpha_y\Psi \sim z, \\
i\Psi^\dagger\beta\alpha_x\Psi \sim E_x, \quad i\Psi^\dagger\beta\alpha_y\Psi \sim E_y, \quad i\Psi^\dagger\beta\alpha_z\Psi \sim E_z, \\
i\Psi^\dagger\beta\alpha_y\alpha_z\Psi \sim H_x, \quad i\Psi^\dagger\beta\alpha_z\alpha_x\Psi \sim H_y, \quad i\Psi^\dagger\beta\alpha_x\alpha_y\Psi \sim H_z, \\
\Psi^\dagger\beta\Psi \sim \Psi^\dagger\beta\alpha_x\alpha_y\alpha_z\Psi \sim 1,
\end{gathered}$$

where $\alpha_x, \alpha_y, \alpha_z, \beta$ are Dirac matrices. But, probably, the most important result achieved by Majorana on this subject is his discussion of **infinite-dimensional** unitary representations of the Lorentz group, giving also an *explicit* form for them. Note that such representations were independently discovered by Wigner in 1939 and 1948 [15] and were thoroughly studied only in the years 1948-1958 [16]. Lucky enough, we are able to reconstruct the reasoning which led Majorana to discuss the infinite-dimensional representations. In Sect. 8 of Volumetto V we read [3]:

> "The representations of the Lorentz group are, except for the identity representation, essentially not unitary, i.e., they cannot be converted into unitary representations by some transformation. The reason for this is that the Lorentz group is an open group. However, in contrast to what happens for closed groups, open groups may have irre-

ducible representations (even unitary) in infinite dimensions. In what follows, we shall give two classes of such representations for the Lorentz group, each of them composed of a continuous infinity of unitary representations."

The two classes of representations correspond to integer and half-integer values for the representation index j (angular momentum). Majorana begins by noting that the group of the real Lorentz transformations acting on the variables ct, x, y, z can be constructed from the infinitesimal transformations associated to the matrices:

$$
\begin{aligned}
S_x &= \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&-1 \\ 0&0&1&0 \end{pmatrix}, \quad S_y = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&1 \\ 0&0&0&0 \\ 0&-1&0&0 \end{pmatrix}, \\
S_z &= \begin{pmatrix} 0&0&0&0 \\ 0&0&-1&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix}, \\
T_x &= \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad T_y = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \\
T_z &= \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 1&0&0&0 \end{pmatrix},
\end{aligned}
\tag{36}
$$

from which he deduces the general commutation relations satisfied by the S and T oper-

ators acting on generic (even infinite) tensors or spinors:

$$\begin{aligned} S_x\,S_y\,-\,S_y\,S_x &= S_z, \\ T_x\,T_y\,-\,T_y\,T_x &= -\,S_z, \\ S_x\,T_x\,-\,T_x\,S_x &= 0, \\ S_x\,T_y\,-\,T_y\,S_x &= T_z, \\ S_x\,T_z\,-\,T_z\,S_x &= -\,T_y, \\ &\text{etc.} \end{aligned} \tag{37}$$

Next he introduces the matrices

$$a_x \;=\; i\,S_x, \quad b_x \;=\; -i\,T_x, \quad \text{etc.} \tag{38}$$

which are Hermitian for unitary representations (and viceversa), and obey the following commutation relations:

$$\begin{aligned} [a_x, a_y] &= i\,a_z, \\ [b_x, b_y] &= -i\,a_z, \\ [a_x, b_x] &= 0, \\ [a_x, b_y] &= i\,b_z, \\ [b_x, a_y] &= i\,b_z, \\ &\text{etc.} \end{aligned} \tag{39}$$

By using only these relations he then obtains (algebraically [2]) the explicit expressions of the matrix elements for given j and m [3] [14]. The non-zero elements of the infinite matrices a and b, whose diagonal elements are labelled by j and m, are as follows:

$$\begin{aligned} < j,m\,|\,a_x - ia_y\,|\,j,m+1 > &= \sqrt{(j+m+1)(j-m)}, \\ < j,m\,|\,a_x + ia_y\,|\,j,m-1 > &= \sqrt{(j+m)(j-m+1)}, \\ < j,m\,|\,a_z\,|\,j,m > &= m, \\ < j,m\,|\,b_x - ib_y\,|\,j+1,m+1 > &= -\,\frac{1}{2}\,\sqrt{(j+m+1)(j+m+2)}, \\ < j,m\,|\,b_x - ib_y\,|\,j-1,m+1 > &= \frac{1}{2}\,\sqrt{(j-m)(j-m-1)}, \\ < j,m\,|\,b_x + ib_y\,|\,j+1,m-1 > &= \frac{1}{2}\,\sqrt{(j-m+1)(j-m+2)}, \\ < j,m\,|\,b_x + ib_y\,|\,j-1,m-1 > &= -\,\frac{1}{2}\,\sqrt{(j+m)(j+m-1)}, \\ < j,m\,|\,b_z\,|\,j+1,m > &= \frac{1}{2}\,\sqrt{(j+m+1)(j-m+1)}, \\ < j,m\,|\,b_z\,|\,j-1,m > &= \frac{1}{2}\,\sqrt{(j+m)(j-m)}. \end{aligned} \tag{40}$$

[2] The algebraic method to obtain the matrix elements in Eq. (40) follows closely the analogous one for evaluating eigenvalues and normalization factors for angular momentum operators, discovered by Born, Heisenberg and Jordan in 1926 and reported in every textbook on Quantum Mechanics (see, for example, [28]).

The quantities on which a and b act are infinite tensors or spinors (for integer or half-integer j, respectively) in the given representation, so that Majorana effectively constructs, for the first time, infinite-dimensional representations of the Lorentz group. In [14] the author also picks out a physical realization for the matrices a and b for Dirac particles with energy operator H, momentum operator $\vec{p}$ and spin operator $\vec{\sigma}$:

$$a_x \,=\, \frac{1}{\hbar}\,(yp_z - zp_y) + \frac{1}{2}\sigma_x, \quad b_x \,=\, \frac{1}{\hbar}x\frac{H}{c} + \frac{i}{2}\alpha_x, \quad \text{etc.}, \tag{41}$$

where $\alpha_x, \alpha_y, \alpha_z$ are the Dirac α-matrices.

Further development of this material then brought Majorana to obtain a relativistic equation for a wave-function ψ with infinite components, able to describe particles with arbitrary spin (the result was published in 1932 [14]). By starting from the following variational principle:

$$\delta \int \overline{\psi}\left(\frac{H}{c} + \vec{\alpha}\cdot\vec{p} - \beta mc\right)\psi\, d^4x \,=\, 0, \tag{42}$$

By requiring the relativistic invariance of the variational principle in Eq. (42), Majorana deduces both the transformation law for ψ under an (infinitesimal) Lorentz transformation and the explicit expressions for the matrices $\vec{\alpha}$, β. In particular, the transformation law for ψ is obtained directly from the corresponding ones for the variables ct, x, y, z by means of the matrices a and b in the representation (41). By using the same procedure leading to the matrix elements in (40), Majorana gets the following expressions for the elements of the (infinite) Dirac $\vec{\alpha}$ and β matrices:

$$\begin{aligned}
&< j,m\,|\,\alpha_x - i\alpha_y\,|\,j+1, m+1 > \,= -1/2\sqrt{\frac{(j+m+1)(j+m+2)}{(j+1/2)\,(j+3/2)}},\\
&< j,m\,|\,\alpha_x - i\alpha_y\,|\,j-1, m+1 > \,= -1/2\sqrt{\frac{(j-m)(j-m-1)}{(j-1/2)\,(j+1/2)}},\\
&< j,m\,|\,\alpha_x + i\alpha_y\,|\,j+1, m-1 > \,= 1/2\sqrt{\frac{(j-m+1)(j-m+2)}{(j+1/2)\,(j+3/2)}},\\
&< j,m\,|\,\alpha_x + i\alpha_y\,|\,j-1, m-1 > \,= 1/2\sqrt{\frac{(j+m)(j+m-1)}{(j-1/2)\,(j+1/2)}},\\
&< j,m\,|\,\alpha_z\,|\,j+1, m > \,= -1/2\sqrt{\frac{(j+m+1)(j-m+1)}{(j+1/2)\,(j+3/2)}},\\
&< j,m\,|\,\alpha_z\,|\,j-1, m > \,= -1/2\sqrt{\frac{(j+m)(j-m)}{(j-1/2)\,(j+1/2)}},\\
&\beta = \frac{1}{j+1/2}.
\end{aligned} \tag{43}$$

The Majorana equation for particles with arbitrary spin has, then, the same form of the Dirac equation:

$$\left(\frac{H}{c} + \vec{\alpha}\cdot\vec{p} - \beta mc\right)\psi \,=\, 0, \tag{44}$$

but with different (and infinite) matrices α and β, whose elements are given in Eqs. (43). The rest energy of the particles thus described has the form:

$$E_0 = \frac{mc^2}{s+1/2}, \tag{45}$$

and depends on the spin s of the particle. We here stress that the scientific community of that time was convinced that only equations of motion for spin 0 (Klein-Gordon equation) and spin 1/2 (Dirac equation) particles could be written down. The importance of the Majorana work was first realized by van der Waerden [29] but, unfortunately, the paper remained unnoticed until recent times.

Acknowledgments

The author warmly thanks I. Licata for his kind spur to write the present paper and E. Recami for fruitful discussions.

References

[1] See the contribution by E. Recami in this volume.

[2] The list of the unpublished scientific manuscripts by Majorana can be found in E. Recami, Quaderni di Storia della Fisica, **5** (1999) 19.

[3] S. Esposito, E. Majorana jr, A. van der Merwe and E. Recami (eds.) *Ettore Majorana: Notes on Theoretical Physics* (Kluwer, New York, 2003).

[4] S. Esposito, Quaderni di Storia della Fisica, preprint; arXiv:physics/0512174.

[5] E. Majorana, *Lezioni all'Università di Napoli*, (Bibliopolis, Napoli, 1987). E. Majorana, *Lezioni di Fisica Teorica*, edited by S. Esposito (Bibliopolis, Napoli, 2006). S. Esposito, Il Nuovo Saggiatore, **21** (2005) 21; A. Drago and S. Esposito, e-print arXiv:physics/0503084.

[6] S. Esposito, Am. J. Phys. **70** (2002) 852; arXiv:physics/0111167.

[7] E. Di Grezia and S. Esposito, Found. Phys. **34** (2004) 1431; e-print arXiv:physics/0406030.

[8] S. Esposito, Int. J. Theor. Phys. **41** (2002) 2417; e-print arXiv:math-ph/0204040.

[9] R. Mignani, E. Recami and M. Baldo, Lett. Nuovo Cim. **11** (1974) 568; see also E. Giannetto, Lett. Nuovo Cim. **44** (1985) 140 and 145. S. Esposito, Found. Phys. **28** (1998) 231; e-print arXiv:hep-th/9704144.

[10] E. Giannetto, in F. Bevilacqua (ed.) *Atti IX Congresso Naz.le di Storia della Fisica* (Milan, 1988) p.173.

[11] J.R. Oppenheimer, Phys. Rev. **38** (1931) 725.

[12] H. Weyl, *Gruppentheorie und Quantenmechanik*, first edition (Hirzel, Leipzig, 1928); English translation from the second German edition in *The Theory of Groups and Quantum Mechanics* (Dover, New York 1931).

[13] A. Drago and S. Esposito, Found. Phys. **34** (2004) 871; arXiv:physics/0503084.

[14] E. Majorana, Nuovo Cim. **9** (1932) 335.

[15] E. Wigner, Ann. Math. **40** (1939) 149.

[16] I.M. Gel'fand, A.M. Yaglom, Zh. Ehksp. Teor. Fiz. **18** (1948) 703; I.M. Gel'fand, R.A. Minlos, Z.Ya. Shapiro, *Representations of the Rotation and Lorentz Groups and their Applications* (Pergamon Press, Oxford, 1963).

[17] P. A. M. Dirac, Phys. Zeits. Sowjetunion **3** (1933) 64.

[18] R. P. Feynman, *The Principle of Least Action in Quantum Mechanics* (Princeton University, Ann Arbor, 1942; University Microfilms Publication No. 2948). R. P. Feynman, Rev. Mod. Phys. **20** (1948) 367.

[19] L.H. Thomas, Proc. Camb. Phil. Soc. **23** (1926) 542.

[20] A. Sommerfeld, Rend. Lincei **15** (1932) 788.

[21] E. Fermi, Rend. Lincei **6** (1927) 602.

[22] E. Fermi, *Falkenhagen, Quantentheorie und Chemie*, Leipziger Vortraege **95** (1928).

[23] F. Rasetti in E. Fermi *Collected Papers (Note e Memorie)* (University of Chicago Press, Chicago, 1962) p. 277.

[24] E. Segré, *A Mind Always in Motion* (University of California Press, Berkeley, 1993).

[25] E. Amaldi, *Ettore Majorana: Man and Scientist*, in A. Zichichi (ed.) *Strong and Weak Interactions. Present problems* (Academic Press, New York, 1966).

[26] S. Esposito, preprint arXiv:physics/0512259.

[27] E. Majorana, Nuovo Cim. **14** (1937) 171.

[28] J.J. Sakurai, *Modern Quantum Mechanics* (Benjamin-Cummings, Menlo Park, 1985).

[29] Letter from Leipzig by Ettore Majorana to his father dated February 18, 1933, quoted by Recami in Ettore Majorana, *Lezioni all'Università di Napoli* in [5].

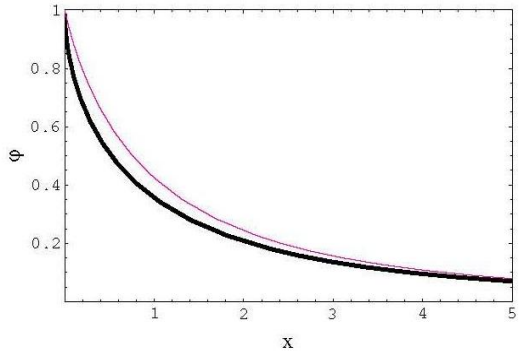


Fig. 1 The Thomas-Fermi function $\varphi(x)$ and the Majorana apprimation of it. Then thin (upper) line refers to the exact (numerical) solution of Eq. (3) while the thick (lower) one corresponds to the parametric solution obtained from Eqa. (20)-(21).

EJTP **3, No. 10** (2006) 285–292 *Electronic Journal of Theoretical Physics*

The Majorana Oscillator

Eliano Pessa *

Centro Interdipartimentale di Scienze Cognitive,
Università di Pavia.Piazza Botta 6,27100 Pavia, Italy

Received 26 April 2006, Published 28 May 2006

Abstract: At present the expression 'Majorana oscillator' does not appear to be in use in theoretical physics. However, the author of this paper heard it in the Seventies, during private conversations with the late Prof. B.Touschek. This little contribution tries to explore the possible meanings of this expression and introduces a new field equation, generalizing the one already introduced by Majorana himself, which could describe a hypothetical 'Majorana oscillator'.

Keywords: Majorana Oscillator, Majorana Equation
PACS (2006): 03.65.Pm, 11.10.-z, 03.70.+k

1. Introduction

The Majorana oscillator came into my life around 1976. At that time I was student at the postgraduate school in Physics of the University of Rome, Italy. In this regard I need to remind younger physicists of the lack, within the university system holding in Italy at that time, of the PhD degree. The highest attainable degree was the one given by postgraduate schools, typically after two years of hard work and the discussion of a final thesis. In Italy, however, there were very few schools of this kind, and limited to a small number of disciplinary domains. Fortunately the latter included Physics and such a circumstance had given me the opportunity of attending the courses of postgraduate school existing in the University of Rome (named 'Scuola di Perfezionamento in Fisica'). The scientific level of the latter was very high, more or less comparable to the one of present PhD schools. From 1974 to 1976 I worked at my final thesis, dealing with the process of particle creation by a quantum field embedded in a Riemannian space, under the supervision of the late Prof. Bruno Touschek. His name being widely known among

* eliano.pessa@unipv.it

physicists, I don't need an illustration of his outstanding contributions to the development of Physics (the introduction of storage rings and the theory of neutrino would be enough), which are not the subject of this paper. I will limit myself to mention the fact that throughout these two years (and even later) I was engaged in frequent and long discussions with Prof. Touschek about the subject of my thesis and, chiefly, about the very foundations of Quantum Field Theory. At the beginning of 1976 I had completed my work and finished all computations. However, I was not satisfied with my results, which I considered only as approximate, owing to the fact that I had realized a deep conceptual incompleteness of Quantum Field Theory itself. The latter was the subject of a number of discussions with Prof. Touschek, which took place up to the Avril 1976 (when I discussed my thesis), in which, typically, I started by criticizing specific aspects of Quantum Field Theory, so as to stimulate him to answering me, or by showing the inconsistence of my arguments or by making new proposals for overcoming the limits of theoretical schemata used at that time. In particular, I remember that one day (I forgot the exact date) I started by claiming that Quantum Field Theory could not be considered as consistent with Special Relativity, as it was heavily based on a concept of 'oscillator' entirely drawn from traditional classical mechanics. To be more coherent with Special Relativity – I maintained – one should start by relying on a theory of special-relativistic oscillators and then shaping on it the whole formalism. It was during the ensuing discussion that Prof. Touschek used the term 'Majorana oscillator'. At that time I was young and – as it is often the case – very ignorant. Of course, I knew very well the name of Majorana, but not his theory of neutrino. So I thought that the expression 'Majorana oscillator' were denoting a concept widely known among theoretical physicists, but unknown to me (a circumstance, the latter, which it was better not to underline), and in this way I missed the occasion for asking Prof. Touschek about this subject. Long after, when talking with other physicists, I was surprised in realizing that nobody knew the meaning of the expression 'Majorana oscillator'. On the contrary, my colleagues asked me for more information about this concept. When I undertook a search, on books, journals and even on the Web, about Majorana oscillator, nothing emerged. Unfortunately, the untimely death of Prof. Touschek in 1978 prevented me from asking him more details about this subject.

This long introduction explains why now the only possibility I have for clarifying what is a Majorana oscillator is to speculate about the possible meaning of the expression used by Prof. Touschek, taking into account his knowledge, his theoretical purposes and schemata (as emerging from our discussions), as well as the ideas put forward by Majorana itself. This modest contribution contains the outcomes of this speculation, heavily relying on the findings of a number of other authors which, after the publication of the last paper of Ettore Majorana in 1937, tried to draw the conclusions ensuing from his hypotheses. For this reason the second section of the present paper will be devoted to some considerations about Majorana equation and its usefulness in describing some kind of 'oscillator'. Within the third section, on the contrary, we will introduce a new field equation, generalizing the usual Majorana one, designed in an explicit way so as to

describe a new kind of oscillator, which could tentatively be called 'Majorana oscillator'. The temporary conclusions of this paper, contained in the fourth section, point that probably the expression used by Prof. Touschek was introduced to denote the solutions of Majorana equation. On the contrary, the generalization of the latter introduced in the third section, while being worthy of exploration, raises a number of problems still difficult to solve.

2. The Majorana Equation

As it is well known, the so-called Majorana equation was introduced by Ettore Majorana in his last paper, entitled "Teoria simmetrica dell'elettrone e del positrone", written in italian language and published in 1937 on the journal "Il Nuovo Cimento" (Majorana, 1937). In it Majorana showed that fermions obeying his equation were undistinguishable from antifermions, and suggested that it was useful to describe neutrinos, by assuming the latter were identical to antineutrinos. Such a 'Majorana neutrino theory' led to a large number of experimental and theoretical investigations, whose temporal trend appears to be characterized by two pronounced peaks (at least in terms of the number of published papers), one more or less in correspondence to the middle Fifties (the epoch of discovery of parity nonconservation), and another corresponding to the last then years (in concomitance with the development of new measuring devices). However, while stressing the importance of the consequences of Majorana hypotheses for elementary particle physics, astrophysics and cosmology, we will fully neglect this subject, by limiting ourselves to quote very few references (Mohapatra & Pal, 1991; Fukugita & Yanagida, 2003; Kang & Kim, 2004; Mohapatra *et al.*, 2005). We will instead focus our attention on some formal properties of Majorana equation.

The departure point is constituted by the well known fact that Majorana equation derives from a suitable representation of matrices used in writing Dirac equation, a representation often denoted as *Majorana representation*. In this regard, we remark that different authors use different conventions and different measure units. Within this paper the generic form of Dirac equation will be written as:

$$i\hbar\partial\psi/\partial t = c\underline{\alpha}\underline{p}\psi + mc^2\beta\psi \tag{1}$$

where ψ is a 4-component spinor, $\underline{\alpha}$ denotes a 3-component vector, each element of which consists in a 4x4 matrix α_k (k = 1, 2, 3), $\underline{p}$ is another 3-component vector, each element of which consists in an operator p_k (k = 1, 2, 3) given by:

$$p_k = (\hbar/i)\partial/\partial x_k \tag{2}$$

and finally β is a 4x4 matrix. Moreover, we will use a space-time metric with signature (1, -1, -1, -1). The α_k and β are anticommuting matrices whose square is the unit matrix and satisfy a set of algebraic relationships which will not be listed here, being widely known among physicists.

The well known consequences of Dirac equation, extensively described in every textbook on relativistic quantum mechanics or on Quantum Field Theory, stem from a particular choice of the form of matrices α_k and β, denoted as *Dirac representation*. However, there exist many other different representations of these matrices, keeping invariant the form of their algebraic relationships. Among them, the most interesting for our purposes are the *Majorana representations*. Here we speak of 'representations' instead of 'representation' because there many different ways, including the one introduced by Majorana in his original paper, for choosing matrices such that all coefficients of Dirac equation (1) become *real numbers*. In the following, in agreement with the previous choice of metric signature, we will adopt the particular choice (see, for instance, Itzykson & Zuber, 1985, p. 694):

$$\beta = \begin{vmatrix} 0 & \sigma 2 \\ \sigma 2 & 0 \end{vmatrix} \quad \alpha 1 = \begin{vmatrix} 0 & -\sigma 1 \\ -\sigma 2 & 0 \end{vmatrix} \quad \alpha 2 = \begin{vmatrix} I & 0 \\ 0 & -I \end{vmatrix} \quad \alpha 3 = \begin{vmatrix} 0 & -\sigma 3 \\ -\sigma 3 & 0 \end{vmatrix} \tag{3}$$

where **I** and **0** are, respectively, the unit and zero 2x2 matrices, while σ_k ($k = 1, 2, 3$) are the usual 2x2 Pauli matrices.

By substituting (3) and (2) into (1) we obtain an explicit form of Majorana equation. If we denote by ψ_s ($s = 1, ..., 4$) the components of the 4-spinor ψ, straightforward computations show that these latter obey the following system of 4 partial differential equations, whose coefficients are all real numbers:

$$\partial\psi_1/\partial t = c[(\partial\psi_4/\partial x_1) - (\partial\psi_1/\partial x_2) + (\partial\psi_3/\partial x_3)] - (mc^2/\hbar)\psi_4 \tag{4a}$$

$$\partial\psi_2/\partial t = c[(\partial\psi_3/\partial x_1) - (\partial\psi_2/\partial x_2) - (\partial\psi_4/\partial x_3)] + (mc^2/\hbar)\psi_3 \tag{4b}$$

$$\partial\psi_3/\partial t = c[(\partial\psi_2/\partial x_1) + (\partial\psi_3/\partial x_2) + (\partial \quad \psi_1/\partial x_3)] - (mc^2/\hbar)\psi_2 \tag{4c}$$

$$\partial\psi_4/\partial t = c[(\partial\psi_1/\partial x_1) + (\partial\psi_4/\partial x_2) - (\partial\psi_2/\partial x_3)] + (mc^2/\hbar)\psi_1 \tag{4d}$$

An inspection of these equations lets us understand that they don't allow an easy separation of dependent variables into small and large components, as occurring when choosing Dirac representation (a proof of this fact, though based on a choice of a Majorana representation differing from the one depicted in (3), has been given by Weaver, 1976). On the other hand, the whole system appears as naturally composed of two interacting subsystems: the first including the equations (4.a) and (4.d), and the second including the equations (4.b) and (4.c). Such a subdivision has been already noticed in the past by other authors (see, for instance, McLennan, 1957). It becomes very useful if we go to the non-relativistic limit of (4.a)-(4.d). Namely in this case all terms containing spatial derivatives can be neglected with respect to the ones associated to the coefficients $mc^2/\hbar$, so that we obtain the simple non-interacting subsystems:

$$\partial\psi_1/\partial t = -(mc^2/\hbar)\psi_4, \quad \partial\psi_4/\partial t = (mc^2/\hbar)\psi_1 \tag{5a}$$

$$\partial\psi_2/\partial t = (mc^2/\hbar)\psi_3, \quad \partial\psi_3/\partial t = -(mc^2/\hbar)\psi_2 \tag{5b}$$

each one of which can be immediately recognized as describing an harmonic oscillator with frequency $\omega = mc^2/\hbar$.

Could such a simple circumstance be the origin for the expression 'Majorana oscillator', introduced by Prof. Touschek? Of course, nobody can answer this question, even if two facts induce to think that the answer to it could be positive. First of all, he was deeply involved in investigations about the theory of neutrino (as evidenced by his important contributions to this subject; see, for instance, Radicati & Touschek, 1957; Cini & Touschek, 1958). This implies that he was well-informed about all papers appeared in the Fifties and Sixties dealing with Majorana equation and the underlying mathematics. In second place I remember that in most cases our discussions ended in examples drawn from theory of neutrino, and that in these circumstances he showed off a considerable mathematical ability, producing then and there results and formulae not present in standard textbooks and (perhaps) never proved before.

3. The Majorana Oscillator

Despite the previous arguments, another possible source of inspiration for the concept of Majorana oscillator should be taken into account. It is connected to the publication of a paper by Itô, Mori and Carrieri, appeared in 1967 on *Il Nuovo Cimento*, in which the authors introduced a generalization of usual Dirac equation, describing a system denoted as *Dirac oscillator* (see Itô, Mori & Carrieri, 1967). Such a paper was followed by a number of subsequent papers (for a complete list see Benítez *et al.*, 1990b), all dealing with the same subject and appeared from 1969 to 1986. It can reasonably be supposed that Prof. Touschek knew some of them, though I have no proof of this fact. This topic did not attract much attention, and, as a matter of fact, Moshinsky and Szczepaniak rediscovered it in 1989 (see Moshinsky & Szczepaniak, 1989; further papers are, among others, Benítez *et al.*, 1990a, Szmytkowski & Gruchowski, 2001).

The key idea underlying the introduction of Dirac oscillator consists in adding to the usual Dirac equation a new term whose form be such that, squaring the new Dirac Hamiltonian (remember that, in a sense, Dirac equation was obtained by taking the 'square root' of usual Hamiltonian), or iterating the Dirac operator, one can obtain an expression containing terms like the ones appearing in harmonic oscillator Hamiltonian, that is having the form $p^2 + m^2\ \Omega^2 \quad r^2$, where Ω is a suitable frequency. In order to reach this goal, all authors previously quoted modify the Dirac equation by introducing the transformation:

$$p \rightarrow p - im\Omega r\beta \tag{6}$$

where the components of 3-vector $\boldsymbol{r}$ are given by x_k ($k = 1, 2, 3$), and β is the 4x4 matrix already defined in (3). However, when taking into consideration the Majorana equation, the transformation (7) could give rise to troubles, as it introduces, within a system of equations whose coefficients are all real numbers, new terms containing coefficients which are pure imaginary. Therefore, it seems that a better proposal, in the case of Majorana equation, would consist in introducing the transformation:

$$p \to p - m\Omega r\beta \tag{7}$$

which gives rise to a generalized Majorana equation in which all coefficients are still real numbers.

If we substitute (7) into (1), with the aid of (2) and (3), elementary computations show that the explicit form of the generalized equations describing what we could denote as a *Majorana oscillator*, is:

$$\begin{aligned}\partial\psi_1/\partial t = c[(\partial\psi_4/\partial x_1) - (\partial\psi_1/\partial x_2) + (\partial\psi_3/\partial x_3)] \\ -(mc/\hbar)\Omega(-x_1\psi_1 - x_2\psi_4 - x_3\psi_2) - (mc^2/\hbar)\psi_4\end{aligned} \tag{8a}$$

$$\begin{aligned}\partial\psi_2/\partial t = c[(\partial\psi_3/\partial x_1) - (\partial\psi_2/\partial x_2) - (\partial\psi_4/\partial x_3)] \\ -(mc/\hbar)\Omega(x_1\psi_2 + x_2\psi_3 + x_3\psi_1) + (mc^2/\hbar)\psi_3\end{aligned} \tag{8b}$$

$$\begin{aligned}\partial\psi_3/\partial t = c[(\partial\psi_2/\partial x_1) + (\partial\psi_3/\partial x_2) + (\partial\psi_1/\partial x_3)] \\ -(mc/\hbar)\Omega(-x_1\psi_3 + x_2\psi_2 + x_3\psi_4) - (mc^2/\hbar)\psi_2\end{aligned} \tag{8c}$$

$$\begin{aligned}\partial\psi_4/\partial t = c[(\partial\psi_1/\partial x_1) + (\partial\psi_4/\partial x_2) - (\partial\psi_2/\partial x_3)] \\ -(mc/\hbar)\Omega(x_1\psi_4 - x_2\psi_1 + x_3\psi_3) + (mc^2/\hbar)\psi_1\end{aligned} \tag{8d}$$

The search for solutions of (8.a)-(8.d) is a very difficult affair. Namely it can be shown that these equations do not allow plane wave solutions. To this end it is enough to make an ansatz of the form:

$$\psi_s = a_s F(pr - \omega t) \tag{9}$$

where F is a periodic function and a_s are suitable constant coefficients. By substituting (9) into (8.a)-(8.d), one obtains or a single homogeneous linear system of algebraic equations in the unknown coefficients a_s or two systems of this kind, according to the choice made for F (the last alternative occurs, for instance, when Fcoincides with simple trigonometric functions like *sin* or *cos*). In order to have an infinite number of nonzero solutions, the determinant of this system (or the determinants of these systems) must vanish. In turn, this condition gives rise to a number of relationships to be fulfilled by $\boldsymbol{p}$, $\boldsymbol{r}$ and ω, as well as by oscillator frequency Ω. In all cases these relationships lead to mathematical or physical inconsistencies. To make an example, we will show here the explicit form of the two relationships obtained, when F coincides with the function *sin*. If we put:

$$e = \omega^2 = E^2/\hbar^2, \quad x = c^2(p_1)^2/\hbar^2, \quad y = c^2(p_2)^2/\hbar^2, \quad z = c^2(p_3)^2/\hbar^2, \tag{10}$$

then the two obtained relationships are:

$$(e - x - y - z)^2 - 2z(e - x - y + z) = 0, \quad x = \Omega^2[(x_1)^2 + (x_2)^2 + (x_3)^2] - c^2 = 0 \tag{11}$$

The second of (11) is clearly inconsistent with the assumption that Ω be a constant quantity. As regards the first of (11), it is satisfied when $p_3 = 0$ (a very strange requirement!), in which case the relationship itself takes the well known form $E^2 = c^2p^2$, characterizing ultrarelativistic particles of zero mass (like the neutrino). However, this implies, in turn, that the contribution to (8.a)-(8.d) of the new term proportional to $\boldsymbol{r}$ be vanishing (so that the Majorana oscillator is destroyed).

Without pursuing further this topic, which, in any case, it is worth exploring, we remark that the above considerations tend to exclude the possibility that Prof. Touschek, when speaking of Majorana oscillator, could have in mind a system like the one introduced in this section.

4. Conclusion

The simple arguments introduced in this paper propose two possible ways for defining the Majorana oscillator: either an oscillatory behavior of solutions of Majorana equation in the nonrelativistic case, or a new equation, obtained through the introduction of a suitable (squared root) oscillatory potential, and generalizing the usual Majorana equation. While the second alternative could become *per se* a subject for future mathematical and physical investigations, it seems that the expression 'Majorana oscillator', when introduced by Prof. Touschek, were rather connected to the first alternative. In any case the considerations made in this contribution evidence how much the world of Majorana representations (and of their physical content) be still widely unexplored. It is to be hoped that future investigations will help to fill this gap.

Acknowledgements

The author thanks Prof. Erasmo Recami, of University of Bergamo, Italy, and Prof. Carlo Bernardini, of University of Rome "La Sapienza", Italy, for their stimulating inputs about the subject of this paper.

References

[1] Benítez, J., Martínez y Romero, R.P., Núñez-Yépez, H.N., & Salas-Brito, A.L. (1990a). *Physical Review Letters*, **64**, 1643.

[2] Benítez, J., Martínez y Romero, R.P., Núñez-Yépez, H.N., & Salas-Brito, A.L. (1990b). *Physical Review Letters*, **65**, 2085.

[3] Cini, M., & Touschek, B. (1958). *Il Nuovo Cimento*, **7**, 422.

[4] Fukugita, M., & Yanagida, T. (2003). *Physics of Neutrinos*. Springer, Berlin.

[5] Itô, D., Mori, K., & Carrieri, E. (1967). *Il Nuovo Cimento*, **51A**, 1119.

[6] Itzykson, C., & Zuber, J.-B. (1985). *Quantum Field Theory*. McGraw-Hill, Singapore.

[7] Kang, S.N., & Kim, C.S. (2004). ArXiv: hep-ph/0403059 v1.

[8] Majorana, E. (1937). *Il Nuovo Cimento*, **14**, 171.

[9] McLennan, J.A. (1957). *Physical Review*, **106**, 821.

[10] Mohapatra, R.N., & Pal, P.B. (1991). *Massive Neutrinos in Physics and Astrophysics*. World Scientific, Singapore.

[11] Mohapatra, R.N., *et al.* (2005). ArXiv: hep-ph/0510213.

[12] Moshinsky, M., & Szczepaniak, A. (1989). *Journal of Physics A*, **22**, L817.

[13] Radicati, L.A., & Touschek, B. (1957). *Il Nuovo Cimento*, **5**, 1623.

[14] Szmytkowski, R., & Gruchowski, M. (2001). *Journal of Physics A*, **34**, 4991.

[15] Weaver, D.L. (1976). *Journal of Mathematical Physics*, **17**, 485.

EJTP **3, No. 10** (2006) 293–303 *Electronic Journal of Theoretical Physics*

Scattering of an α Particle by a Radioactive Nucleus

E. Majorana

Written 1928 *published 2006*

Abstract: *In the following we reproduce, translated into English, a section of Volumetto II, a notebook written by Majorana in 1928 when he was still a Physics student at the University of Rome (see S. Esposito, E. Majorana jr, A. van der Merwe and E. Recami (eds.)* Ettore Majorana: Notes on Theoretical Physics*, Kluwer, New York, 2003). This study was performed by the author when he was preparing his Thesis work on "The Quantum Theory of Radioactive Nuclei" (unpublished), whose supervisor was E. Fermi.*

S. Esposito

Keywords: Ettore Majorana, Unpublished work

Let us consider the emission of an α particle by a radioactive nucleus and assume that such a particle is described by a quasi-stationary wave. As Gamov has shown, after some time this wave scatters at infinity. In other words, the particle spends some time near the nucleus but eventually ends up far from it. We now begin to study the features of such a quasi-stationary wave, and then address the *inverse* of the problem studied by Gamov.[1] Namely, we want to determine the probability that an α particle, colliding with a nucleus that has just undergone an α radioactive transmutation, will be captured by the nucleus so as to reconstruct a nucleus of the element preceding the original one in the radioactive genealogy. This issue has somewhat been addressed by Gudar, although not deeply enough. It is directly related to our hypothesis according to which, under conditions rather different from the ones we are usually concerned with, a process can take place that reconstitutes the radioactive element.

Following Gamov, let us suppose that spherical symmetry is realized, so that the azimuthal quantum of the particle near the nucleus is zero. For simplicity, we neglect for the moment the overall motion of the other nuclear components. The exact formulae will have to take account of that motion, and thus the formulae that we shall now derive will have to be modified; but this does not involve any major difficulty. For the spherically

[1] The author is referring here to G.Gamov, *Z. Phys.* **41** (1928) 204. [N.d.T]

symmetric stationary states, setting, as usual, $\psi = \chi/x$, we shall have

$$\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}\,(E - U)\,\chi \;=\; 0. \tag{1}$$

Beyond a given distance R, which we can assume to be of the order of the atomic dimensions, the potential U practically vanishes. The functions χ will then be symmetric for $E > 0$. For definiteness, we require U to be exactly zero for $x > R$, but it will be clear that no substantial error is really introduced in this way in our calculations. For the time being, let us consider the functions χ to depend only on position, and —as it is allowed— to be real. Furthermore, we use the normalization condition

$$\int_0^R \chi^2\,\mathrm{d}x \;=\; 1. \tag{2}$$

Let us now imagine that it exists a quasi-stationary state such that it is possible to construct a function u_0 which vanishes for $x > R$, satisfies the constraint

$$\int_0^R |u_0|^2\,\mathrm{d}x \;=\; 1, \tag{3}$$

and approximately obeys[2] the differential equation (1) at the points where its value is large. This function u_0 will be suited to represent the α particle at the initial time. It is possible to expand it in terms of the functions χ that are obtained by varying E within a limited range. Let us then set

$$E \;=\; E_0 + W. \tag{4}$$

The existence of such a quasi-stationary state is revealed by the fact that for $x < R$ the functions χ, normalized according to Eq. (2), and their derivatives are small for small W.

In first approximation, we can set, for $x < R$,

$$\begin{aligned} \chi_W &= \chi_0 + W\,y(x), \\ \chi'_W &= \chi'_0 + W\,y'(x), \end{aligned} \tag{5}$$

and these are valid (as long as U has a reasonable behavior) with great accuracy and for all values of W in the range of interest. In particular, for $x = R$:

$$\begin{aligned} \chi_W(R) &= \chi_0(R) + W\,y(R), \\ \chi'_W(R) &= \chi'_0(R) + W\,y'(R). \end{aligned} \tag{6}$$

[2] For an approximately determined value of t, while being almost real.

Bearing in mind that Eq. (1) simply reduces for $x > R$ to

$$\frac{\mathrm{d}^2\chi_W}{\mathrm{d}x^2} + \frac{2m}{\hbar^2}\,(E_0 + W)\,\chi_W = 0, \tag{7}$$

for $x > R$ we get

$$\begin{aligned}\chi_W = (a + bW)\,\cos\frac{1}{\hbar}\sqrt{2m(E_0+W)}(x-R)\\ + (a_1 + b_1 W)\,\sin\frac{1}{\hbar}\sqrt{2m(E_0+W)}(x-R),\end{aligned} \tag{8}$$

having set

$$\begin{aligned}a = \chi_0(R), \quad b = y(R),\\ a_1 = \frac{\hbar\,\chi_0'(R)}{\sqrt{2m(E_0+W)}}, \quad b_1 = \frac{\hbar\,y'(R)}{\sqrt{2m(E_0+W)}}.\end{aligned} \tag{9}$$

Note that a_1 and b_1 are not strictly constant but, to the order of approximation for which our problem is determined, we can consider them as constant and replace them with

$$a_1 = \frac{\hbar\,\chi_0'(R)}{\sqrt{2mE_0}}, \quad b_1 = \frac{\hbar\,y'(R)}{\sqrt{2mE_0}}. \tag{10}$$

Moreover, since E_0 is not completely determined, we shall fix it in order to simplify Eq. (8); with this aim, we can shift R by a fraction of wavelength $h/\sqrt{2mE_0}$. It will then be found that Eq. (8) can always be replaced with the simpler one

$$\begin{aligned}\chi_W = \alpha\,\cos\sqrt{2m(E_0+W)}\,(x-R)/\,\hbar\\ + \beta W\,\sin\sqrt{2m(E_0+W)}\,(x-R)/\,\hbar.\end{aligned} \tag{11}$$

We set

$$\sqrt{2m(E_0+W)}\,/\,\hbar = \sqrt{2mE_0}\,/\,\hbar + 2\pi\,\gamma = C + 2\pi\,\gamma, \tag{12}$$

and, in first approximation, the following will hold:

$$2\pi\,\gamma \simeq \frac{W}{\hbar\sqrt{2E_0/m}} = \frac{W}{\hbar v}, \tag{13}$$

v being the (average) speed of the emitted α particles. On substituting into Eq. (11), we approximately find

$$\begin{aligned}\chi_W = \alpha\,\cos(C + 2\pi\gamma)(x-R)\\ + \beta'\,\gamma\,\sin(C+2\pi\gamma)(x-R),\end{aligned} \tag{14}$$

with

$$\beta' \;=\; \beta\, 2\pi\hbar\, \sqrt{2E_0/m}. \tag{15}$$

For the moment, the χ_W functions are normalized as follows:

$$\int_0^R \chi_W^2 \,\mathrm{d}x \;=\; 1.$$

We denote by η_W the same eigenfunctions normalized with respect to $\mathrm{d}\gamma$. For $x > R$, we then get

$$\begin{aligned} \eta_W \;=\; & \frac{2}{\sqrt{\alpha^2+\beta'^2\gamma^2}}\,\left[\alpha\,\cos(C+2\pi\gamma)(x-R)\right. \\ & \left. +\;\beta'\,\gamma\,\sin(C+2\pi\gamma)(x-R)\right] \;=\; \frac{2}{\sqrt{\alpha^2+\beta'^2\gamma^2}}\,\chi_W. \end{aligned} \tag{16}$$

We expand u_0, which represents the α particle at the initial time, as a series in η_W, and get

$$u_0 \;=\; \int_{-\infty}^{\infty} K_\gamma\,\eta_W\,\mathrm{d}\gamma. \tag{17}$$

Now, since $u_0 = \chi_W$ for $x \le R$ and therefore

$$K_\gamma \;=\; \int_0^{\infty} \eta_W\,u_0\,\mathrm{d}x \;=\; \frac{2}{\sqrt{\alpha^2+\beta'^2\gamma^2}}\int_0^R \chi_W^2\,\mathrm{d}x \;=\; \frac{2}{\sqrt{\alpha^2+\beta'^2\gamma^2}}, \tag{18}$$

on substituting into Eq. (17), we obtain

$$u_0 \;=\; \int_{-\infty}^{\infty} \frac{4\,\chi_W}{\alpha^2+\beta'^2\gamma^2}\,\mathrm{d}\gamma. \tag{19}$$

For small values of x, the different functions χ_W actually coincide and are also equal to u_0; it must then be true that

$$1 \;=\; \int_{-\infty}^{\infty} \frac{4}{\alpha^2+\beta'^2\gamma^2}\,\mathrm{d}\gamma \;=\; -\,\frac{4\pi}{\alpha\beta'}, \tag{20}$$

and, consequently,

$$\beta' \;=\; -\,\frac{4\pi}{\alpha} \tag{21}$$

must necessarily hold. Because of Eq. (13), if we introduce the time dependence, we approximately get

$$u \;=\; \mathrm{e}^{iEt/\hbar}\int_{-\infty}^{\infty} \frac{4\chi_W\,\exp\left\{2\pi i\sqrt{2E_0/m}\;\gamma t\right\}}{\alpha^2+16\pi^2\gamma^2/\alpha^2}\,\mathrm{d}\gamma. \tag{22}$$

For small values of x the χ_W's can be replaced with u_0, and we have

$$u \;=\; u_0\,\mathrm{e}^{iE_0t/\hbar}\,\exp\left\{-\alpha^2\sqrt{2E_0/m}\,t/2\right\}. \tag{23}$$

This can be written as

$$u \;=\; u_0\, \mathrm{e}^{iE_0 t/\hbar}\, \mathrm{e}^{-t/2T}, \tag{24}$$

quantity T denoting the time-constant ("mean-life")

$$T \;=\; \frac{1}{\alpha^2\sqrt{2E_0/m}} \;=\; \frac{1}{\alpha^2 v}. \tag{25}$$

In this way, and using also Eq. (21), both α and β' can be expressed in terms of T:

$$\alpha = \frac{\pm 1}{\sqrt{vT}} \;=\; \frac{\pm 1}{\sqrt[4]{2(E/m)T^2}}, \tag{26}$$

$$\beta' = \mp 4\pi\sqrt{vT} \;=\; \mp 4\pi\sqrt[4]{2(E/m)T^2}. \tag{27}$$

It will be clear that only one stationary state corresponds to a hyperbolic-like orbit in the classical theory. The revolution period or, more precisely, the time interval between two intersections of the orbit with the spherical surface of radius r, is given by

$$P_W \;=\; \frac{4}{(\alpha^2+\beta'^2\gamma^2)v}, \tag{28}$$

and the maximum value is reached for $W = 0$:

$$P_W \;=\; \frac{4}{\alpha^2 v} \;=\; 4T. \tag{29}$$

As a purely classical picture suggests, the probabilities for the realization of single stationary states are proportional to the revolution periods (see Eq. (18)), and T itself can be derived from classical arguments. Indeed, if a particle is on an orbit W and inside the sphere of radius R, on average it will stay in this orbit for a time $T_W = (1/2)P_W = (2/v)/(\alpha^2+\beta'^2\gamma^2)$, and the mean value of T_W will be

$$\overline{T_W} \;=\; \int_0^\infty T_W^2\,\mathrm{d}\gamma \Big/ \int_0^\infty T_W\,\mathrm{d}\gamma \;=\; \frac{1}{\alpha^2 v} \;=\; T. \tag{30}$$

However, we must caution that, by pushing the analogy even further to determine the expression for the survival probability, we would eventually get a wrong result.

The eigenfunction u takes the form in Eq. (23) only for small values of x. Neglecting what happens for values of x that are not too small, but still lower than R, and considering, moreover, even the case $x > R$, from Eqs. (15) and (19) we have

$$\begin{aligned} u = \mathrm{e}^{iE_0 t/\hbar} \Bigg[& \int_0^\infty \frac{4\alpha\cos(C+2\pi\gamma)(x-R)}{\alpha^2+\beta'^2\gamma^2}\,\mathrm{e}^{2\pi i v\gamma t}\,\mathrm{d}\gamma \\ & - \int_0^\infty \frac{4\beta'\gamma\sin(C+2\pi\gamma)(x-R)}{\alpha^2+\beta'^2\gamma^2}\,\mathrm{e}^{2\pi i v\gamma t}\,\mathrm{d}\gamma \Bigg], \end{aligned} \tag{31}$$

where α and β' depend on T according to Eqs. (26), (27). Equation (31) can be written as

$$\begin{aligned} u = \mathrm{e}^{iE_0 t/\hbar} \Bigg[& \mathrm{e}^{iC(x-R)} \int_0^\infty \frac{(2\alpha - 2i\beta'\gamma)}{\alpha^2+\beta'^2\gamma^2}\,\mathrm{e}^{2\pi i(vt+x-R)\gamma}\,\mathrm{d}\gamma \\ & + \mathrm{e}^{-iC(x-R)} \int_0^\infty \frac{(2\alpha + 2i\beta'\gamma)}{\alpha^2+\beta'^2\gamma^2}\,\mathrm{e}^{2\pi i[vt-(x-R)]\gamma}\,\mathrm{d}\gamma \Bigg]. \end{aligned} \tag{32}$$

Since α and β' have opposite signs and, for $t > 0$ and $x > R$, one has $vt + x - R > 0$, the first integral is zero, while the second one equals

$$\int_0^\infty \frac{(2\alpha + 2i\beta'\gamma)}{\alpha^2 + \beta'^2\gamma^2}\, \mathrm{e}^{2\pi i[vt-(x-R)]\gamma}\, \mathrm{d}\gamma \;=\; 2\int_0^\infty \frac{\mathrm{e}^{2\pi i[vt-(x-R)]\gamma}}{\alpha - i\beta'\gamma}\, \mathrm{d}\gamma$$

$$= \begin{cases} -\dfrac{4\pi}{\beta'}\, \mathrm{e}^{2\pi(\alpha/\beta')[vt-(x-R)]} \;=\; -\dfrac{4\pi}{\beta'}\, \mathrm{e}^{-(\alpha^2/2)[vt-(x-R)]}, \\ \\ 0, \end{cases} \tag{33}$$

for $vt - (x - R) > 0$ and $vt - (x - R) < 0$, respectively. On substituting into Eq. (34) and recalling that, from Eq. (12), $C = mv/\hbar$, we finally find

$$u \;=\; \begin{cases} \alpha\, \mathrm{e}^{iE_0t/\hbar}\, \mathrm{e}^{-imv(x-R)/\hbar}\, \mathrm{e}^{-t/2T}\, \mathrm{e}^{(x-R)/(2vT)}, \\ \\ 0, \end{cases} \tag{34}$$

for $vt-(x-R) > 0$ and $vt-(x-R) < 0$, respectively. Let us now assume that the nucleus has lost the α particle; this means that, initially, it is $u_0 = 0$ near the nucleus. We now evaluate the probability that such a nucleus will re-absorb an α particle when bombarded with a parallel beam of particles. To characterize the beam we'll have to give the intensity per unit area, the energy per particle, and the duration of the bombardment. The only particles with a high absorption probability are those having energy close to E_0, with an uncertainty of the order h/T. On the other hand, in order to make clear the interpretation of the results, the duration τ of the bombardment must be small compared to T. Then it follows that, from the uncertainty relations, the energy of the incident particles will be determined with an error greater than h/T. Thus, instead of fixing the intensity per unit area, it is more appropriate to give the intensity per unit area and unit energy close to E_0; so, let N be the total number of particles incident on the nucleus during the entire duration of the bombardment, per unit area and unit energy.

Suppose that, initially, the incident plane wave is confined between two parallel planes at distance d_1 and $d_2 = d_1 + \ell$ from the nucleus, respectively. Since we have assumed that the initial wave is a plane wave, it will be

$$u_0 \;=\; u_0(\xi), \tag{35}$$

ξ being the abscissa (distance from the nucleus) of a generic plane that is parallel to the other two. Then, for $\xi < d_1$ or $\xi > d_2$, it is $u_0 = 0$. Furthermore, we'll suppose $d_1 > R$ and, without introducing any further constraint,

$$\ell \;=\; \frac{h\rho}{m\sqrt{2E_0/m}} \;=\; \frac{h\rho}{mv} \;=\; \rho\,\lambda, \tag{36}$$

with ρ an integer number and λ the wavelength of the emitted α particle. We can now expand ψ_0 between d_1 and d_2 in a Fourier series and thus as a sum of terms of the kind

$$k_\sigma \, \mathrm{e}^{\sigma 2\pi i(\xi - d_1)/\ell}, \tag{37}$$

with integer σ. The terms with negative σ roughly represent outgoing particles, and thus we can assume them to be zero. Let us concentrate on the term

$$k_\rho \, \mathrm{e}^{\rho 2\pi i(\xi - d_1)/\ell} \;=\; k_\rho \, \mathrm{e}^{imv(\xi - d_1)/\hbar} \tag{38}$$

and let us set[3]

$$u_0 \;=\; \psi_0 \,+\, k_\rho \, \mathrm{e}^{imv(\xi - d_1)/\hbar}. \tag{39}$$

The eigenfunctions of a free particle moving perpendicularly to the incoming wave, normalized with respect to $\mathrm{d}E$, are

$$\psi_\sigma \;=\; \frac{1}{\sqrt{2hE/m}} \, \mathrm{e}^{i\sqrt{2mE}(\xi - d_1)/\hbar}. \tag{40}$$

Note that the square root at the exponent must be considered once with the positive sign and once with the negative sign, and E runs twice between 0 and ∞. However, only the eigenfunctions with the positive square root sign are of interest to us, since they represent the particles moving in the direction of decreasing ξ. We can set

$$\psi_0 \;=\; \int_0^\infty c_E \, \psi_\rho \, \mathrm{d}E, \tag{41}$$

wherein

$$c_E \;=\; \int_{d_1}^{d_2} \psi_0 \, \psi_\rho^* \, \mathrm{d}\xi. \tag{42}$$

In particular, we put

$$c_{E_0} \;=\; \int_{d_1}^{d_2} \psi_0 \, \frac{1}{\sqrt{hv}} \, \mathrm{e}^{-imv(\xi - d_1)/\hbar} \, \mathrm{d}\xi \;=\; \frac{k_\rho \ell}{\sqrt{hv}}. \tag{43}$$

Since, evidently,

$$N \;=\; c_{E_0}^2, \tag{44}$$

one finds

$$N \;=\; \frac{k_\rho^2 \ell^2}{hv}. \tag{45}$$

Let us now expand u_0 in terms of the eigenfunctions associated with the central field produced by the remaining nuclear constituents. Since only the spherically symmetric eigenfunctions having eigenvalues very close to E_0 are significantly different from zero near the nucleus, we shall concentrate only on these. For $x > R$, the expression of these

[3] Note that the author split the wavefunction of the incident particles into a term related to the principal energy E_0 (the second term in Eq. (39)) plus another term which will be expanded according to Eq. (41). [N.d.T]

eigenfunctions is given in Eqs. (16), (26), (27). Actually, the η_W given by Eq. (16) are the eigenfunctions relative to the problem reduced to one dimension. In order to have the spatial eigenfunctions, normalized with respect to γ, we must consider

$$g_W \;=\; \frac{\eta_W}{\sqrt{4\pi x}}. \tag{46}$$

In this way we will set

$$\psi_0 \;=\; \int_0^\infty p_\gamma \, g_W \, \mathrm{d}\gamma \;+\; \ldots, \tag{47}$$

wherein

$$p_\gamma \;=\; \iiint \mathrm{d}S \, g_W \, \psi_0 \;=\; \int_{d_1}^{d_2} 2\pi \, x \, g_W \, \mathrm{d}x \int_{d_1}^{x} \psi_0 \, \mathrm{d}\xi. \tag{48}$$

We can set

$$g_W \;=\; \frac{1}{\sqrt{4\pi x}} \left[A_\gamma \, \mathrm{e}^{i(C+2\pi\gamma)(x-d_1)} \;+\; B_\gamma \, \mathrm{e}^{-i(C+2\pi\gamma)(x-d_1)} \right], \tag{49}$$

and, from Eq. (16),

$$\begin{aligned} A_\gamma &= \frac{\alpha - i\beta'\gamma}{\sqrt{\alpha^2 + \beta'^2\gamma^2}} \, \mathrm{e}^{i(C+2\pi\gamma)(d_1-R)}, \\ B_\gamma &= \frac{\alpha + i\beta'\gamma}{\sqrt{\alpha^2 + \beta'^2\gamma^2}} \, \mathrm{e}^{-i(C+2\pi\gamma)(d_1-R)}. \end{aligned} \tag{50}$$

We can now assume that d_1, and thus d_2, is arbitrarily large; but $\ell = d_2 - d_1$ has to be small because the duration of the bombardment, which is of the order ℓ/v, must be negligible with respect to T. Since $2\pi\gamma$ is of the same order as α^2, that is to say, of the same order as $1/vT$ (see Eq. (25)), $2\pi\gamma\ell$ is absolutely negligible. For $d_1 < x < d_2$ it is then possible to rewrite Eq. (49) as

$$g^W \;=\; \frac{1}{\sqrt{4\pi x}} \left[A_\gamma \, \mathrm{e}^{imv(x-d_1)/\hbar} \;+\; B_\gamma \, \mathrm{e}^{-imv(x-d_1)/\hbar} \right], \tag{51}$$

given Eqs. (50).

Let us now substitute this into Eq. (48), taking into account Eqs. (39) and (45). We'll simply have

$$\begin{aligned} p_\gamma &= \frac{2\pi B_\gamma}{\sqrt{4\pi}} \int_{d_1}^{d_2} \mathrm{e}^{-imv(x-d_1)/\hbar} \mathrm{d}x \int_{d_1}^{x} \mathrm{e}^{imv(\xi-d_1)/\hbar} \mathrm{d}\xi \\ &= \frac{hB_\gamma k_\rho \ell}{i\sqrt{4\pi}\, m\, v} \;=\; \frac{B_\gamma h^{3/2}\sqrt{N}}{i\sqrt{4\pi}\, m\, \sqrt{v}} \;=\; q\, B_\gamma, \end{aligned} \tag{52}$$

with

$$q \;=\; \frac{h^{3/2} N^{1/2}}{i\, m\, v^{1/2} \sqrt{4\pi}}. \tag{53}$$

Substituting into Eq. (47), one gets

$$\psi_0 \;=\; q \int_0^\infty B_\gamma g_W \, \mathrm{d}\gamma \;+\; \ldots \tag{54}$$

and, at an arbitrary time,

$$\psi \;=\; \mathrm{e}^{iE_0t/\hbar}\, q \int_0^\infty B_\gamma\, g_W\, \mathrm{e}^{2\pi i v\gamma t}\, \mathrm{d}\gamma \;+\; \ldots, \tag{55}$$

or, taking into account Eqs. (46) and (16),

$$\psi \;=\; \mathrm{e}^{iE_0t/\hbar}\, \frac{q}{\sqrt{4\pi x}} \int_0^\infty \frac{2B_\gamma}{\sqrt{\alpha^2+\beta'^2\gamma^2}}\, \chi_W\, \mathrm{e}^{2\pi i v\gamma t}\, \mathrm{d}\gamma \;+\; \ldots. \tag{56}$$

We now want to investigate the behavior of ψ near the nucleus. There, assuming that other quasi-stationary state different from the one we are considering do not exist, the terms we have not written down in the expansion of ψ can contribute significantly only during a short time interval after the scattering of the wave. If this is the case, ψ will have spherical symmetry near the nucleus. We set

$$\psi \;=\; \frac{u}{\sqrt{4\pi x}}, \tag{57}$$

so that the number of particles that will eventually be captured is

$$\int |u^2|\, \mathrm{d}x \tag{58}$$

(the integration range should extend up to a reasonable distance, for example up to R). Substituting into Eq. (56), and noting that for small values of x we approximately have $\chi_W = \chi_0$, one obtains

$$u \;=\; q\,\chi_0\, \mathrm{e}^{iE_0t/\hbar} \int_0^\infty \frac{2}{\alpha - i\beta'\gamma}\, \mathrm{e}^{2\pi i[vt-(d_1-R)]\gamma}\, \mathrm{d}\gamma. \tag{59}$$

Since, as we already noted, $\alpha\beta' < 0$, and setting $d = d_1 - R$, from Eqs. (26), we find

$$u \;=\; \begin{cases} q\,\alpha\,\chi_0\, \mathrm{e}^{iE_0t/\hbar}\, \mathrm{e}^{-\dfrac{t-d/v}{2T}} = q\,\alpha\, \mathrm{e}^{-iCd}\, \mathrm{e}^{-\dfrac{t-d/v}{2T}}, & \text{for } t > \dfrac{d}{v}, \\[2ex] 0, \quad \text{for } t < \dfrac{d}{v}. \end{cases} \tag{60}$$

The meaning of these formulae is very clear: The α-particle beam, which by assumption does not last for a long time, reaches the nucleus at the time $t = d/v$, and there is a probability $|q\alpha|^2$ that a particle is captured (obviously, $q^2\alpha^2 \ll 1$). The effect of the beam then ceases and, if a particle has been absorbed, it is re-emitted on the time scale predicted by the laws of radioactive phenomena. If we set $n = |q\alpha|^2$, then from Eqs. (25) and (53) we get

$$n \;=\; \frac{2\pi^2\hbar^3}{m^2v^2T}\, N, \tag{61}$$

which tells us that the absorption probabilities are completely independent of any hypothesis on the form of the potential near the nucleus, and that they only depend on the

time-constant T.[4]

Equation (61) has been derived using only mechanical arguments but, as a matter of fact, we can get the same result using thermodynamics. Let us consider one of our radioactive nuclei in a bath of α particles in thermal motion. To the degree of approximation we have treated the problem so far, we can consider the nucleus to be at rest. Due to the assumed spherical symmetry of the system, a particle in contact with the nucleus is in a quantum state with a simple statistical weight. Such a state, of energy E_0, is not strictly stationary, but has a finite half-life; this should be considered, as in all similar cases, as a second-order effect. Assuming that the density and the temperature of the gas of α particles is such that there exist D particles per unit volume and unit energy near E_0, then, in an energy interval $\mathrm{d}E$, we will find

$$D\,\mathrm{d}E \tag{67}$$

particles per unit volume. Let us denote by p the momentum of the particles, so that we have

$$p = \sqrt{2mE_0}, \tag{68}$$

$$\mathrm{d}p = \sqrt{m/2E_0}\,\mathrm{d}E. \tag{69}$$

[4] The original manuscript then continues with two large paragraphs which have however been crossed out by the author. The first one reads as follows:
"Since only the particles with energy near E_0 are absorbed, we can think, with some imagination, that every energy level $E_0 + W$ is associated with a different absorption coefficient ℓ_W, and that such ℓ_W is proportional to the probability that a particle in the quasi-stationary state has energy $E_0 + W$. From (13), (21), (25), and (18), we then have

$$\ell_W \;=\; \frac{D}{1 + 4T^2W^2/\hbar^2}. \tag{62}$$

Since the number of incident particles per unit area and unit energy with energy between $(E_0 + W)$ and $(E_0 + W) + \mathrm{d}W$ is $N\mathrm{d}W$, we must have

$$n \;=\; N\int_{-\infty}^{\infty} \ell_W\,\mathrm{d}W \;=\; N\,D\,\frac{\pi\hbar}{2T}, \tag{63}$$

from which, comparing with (61),

$$D \;=\; \frac{1}{\pi}\,\frac{h^2}{m^2v^2} \;=\; \frac{\lambda^2}{\pi}. \tag{64}$$

This is a very simple expression for the absorption cross section of particles with energy E_0, i.e., the particles with the greatest absorption coefficient. If we set

$$N' \;=\; N\,\frac{\pi\hbar}{2T}, \tag{65}$$

then Eq. (61) becomes

$$n \;=\; \frac{\lambda^2}{\pi}\,N', \tag{66}$$

which means that the absorption of N' particles of energy E_0 is equivalent to the absorption of N particles per unit energy." The second paragraph is not reproduced here since it appears to be incomplete. [N.d.T]

E_0 appears instead of E in the previous equations because we are considering particles with energy close to E_0. The $D\mathrm{d}E$ particles fill a unitary volume in ordinary space, and in momentum space they fill the volume between two spheres of radii p and $p + \mathrm{d}p$, respectively. Thus, in phase space they occupy a volume

$$4\pi\, p^2\, \mathrm{d}p \;=\; 4\pi\, m^2\, \sqrt{2E_0/m}\, \mathrm{d}E \;=\; 4\pi\, m^2\, v\, \mathrm{d}E. \tag{70}$$

This volume contains

$$\frac{m^2 v}{2\pi^2\hbar^3}\, \mathrm{d}E \tag{71}$$

quantum states. Therefore, on average, we have

$$D\, \frac{2\pi^2\hbar^3}{m^2 v} \tag{72}$$

particles in every quantum state with energy close to E_0. This is also the mean number of particles inside the nucleus, provided that the expression (72) is much smaller than 1, so that we can neglect the interactions between the particles. Since the time-constant ("mean-life") of the particles in the nucleus is T, then

$$n \;=\; \frac{2\pi^2\hbar^3 D}{m^2 v T} \tag{73}$$

particles will be emitted per unit time and, in order to maintain the equilibrium, the same number of particles will be absorbed. Concerning the collision probability with a nucleus, and then the absorption probability, D particles per unit volume and energy are equivalent to a parallel beam of $N = Dv$ particles per unit area, unit energy and unit time. On substituting, we then find

$$n \;=\; \frac{2\pi^2\hbar^3}{m^2 v^2 T}\, N, \tag{74}$$

which coincides with Eq. (61).

EJTP **3, No. 10** (2006) 305–314 *Electronic Journal of Theoretical Physics*

Comments on a Paper by Majorana Concerning Elementary Particles*

D. M. Fradkin

Department of Physics, Wayne State University, Detroit, Michigan

Received 13 October 1965, Published April 1966

Abstract: An early paper (1932) by Majorana, that has received but scant attention, is reexamined in light of later developments. This pioneering paper constructs a relativistically invariant theory of arbitrary spin particles, develops and utilizes infinite dimensional representations of the homogeneous Lorentz group, and provides a mass spectrum for elementary particles. The relevance of Majorana's approach and results to later and current research is explained.

We reproduce here the historical D. M. Fradkin 1966 paper whose role among the physicists of high energy was decisive; since then espressions like "Majorana mass", "Majorana spinors" and "Majorana neutrino" have become usual. The paper is based upon the work Teoria di Particelle con Momento Intrinseco Arbitrario, translated by Italiam from Edoardo Amaldi.

Ignazio Licata†‡

Keywords: Ettore Majorana, Elementary Particles
PACS (2006): 01.30.Tt, 01.65.+g, 11.30.-j

The paper "Teoria relativistica di particelle con momento intrinseco arbitrario" by Ettore Majoran (1932) [1,2] is remarkable for the following reasons : (i) it is apparently the first construction of a relativistically invariant theory of arbitrary half integer or integer spin particle; (ii) it is apparently the first recognition, development, and application of the infinite dimensional representations of the Lorentz group; (iii) it is a theory

† Ignazio.licata@ejtp.info

‡ ignazio.licata@ixtucyber.org

that provides a mass spectrum for elementary particles. This pioneering paper , dealing with topics that in later years (and now!) have excited major interest, attracted little or no attention at its publication, and since then it has received only scant mention in several places [3-5]. The purpose of this present note is to rectify historical neglect by describing to a larger audience the content of Majorana's paper and placing his work in the the context of later research.

The starting point in Majorana's article is the single linear differential equation of Dirac form

$$(\frac{W}{c} + \boldsymbol{\alpha} \cdot \mathbf{p} - \beta Mc)\psi = 0, \tag{1}$$

Where $\boldsymbol{\alpha}$ and β are numerical matrices, W is the energy, $\mathbf{p}$ is the momentum operator and ψ is the multicomponent wavefunction. Unlike Dirac (1928) [6], he does not require the dispersion relation

$$(\frac{W}{c})^2\psi = [p^2 + (Mc)^2]\psi, \tag{2}$$

To be satisfied for each component. As is well known, the imposition of both requirements, Eqs. (1) and (2), led Dirac [6] to certain commutation relation for $\boldsymbol{\alpha}$ and β -relations that Majorana does not assume. Many of the (finite dimensional) later higher spin theories [Petiau (1936), Duffin (1938), Kemmer (1939), and Fierz and Pauli (1939)] [3, 7-12] also have required the dispersion relation. In particular, the Dirac-Fierz -Pauli arbitrary spin theories [10-12] essentially factorize Eq. (2) into two coupled first-oder equations. For spin higher than one (spin value related to the number of components of ψ), these coupled first-order equations not only yield the dispersion relation, which describes the time development, but also yield additional restrictions on the wavefunction. These restrictions are known as subsidiary conditions. By insisting only on the linear form [Eq. (1)] but not the dispersion relation, one may avoid subjecting the wavefunction to subsidiary conditions. For finite representational theory, this point was explained years later by Bhabha [13]. An over- all brief summary of the invariant equations for finite dimensions is given by Umezawa (1956)[14].

At the time that Majorana wrote his article, it was an embarrassing fact that the original Dirac theory of the electron [6] introduced negative masses, i.e., that β has eigenvalues ± 1. In order to avoid this, Majorana set for himself the following problem : Is it possible to have a relativistic invariant linear theory for which the eigenvalues of β are all positive? In obtaining an affirmative answer, he was led to the infinite dimensional representations of the Lorentz group.

Majorana's arguments goes something like this: One of the conditions for relativistic invariance of the Dirac-form equation is that the Lagrangian density (which, in conjunction with the variational principle, can be used to derive this equation) must be invariant. A term of the density is $\psi^*\beta\psi$, so this is a relativistic invariant. But if β has all positive eigenvalues, the quantity $\psi^*\beta\psi$ must be positive definite. Hence, it is possible to make a transformation $\psi \rightarrow \phi$ so that $\psi^*\beta\psi \rightarrow \phi^*\phi$ (this transformation is nonunitary and, in the representation where β is diagonal is obviously accomplished by weighting the components of ψ with the reciprocal square roots of the appropriate eigenvalues of β).

In terms of ϕ then, relativistic invariance of the Lagrangian density implies that $\phi^*\phi$ is an invariant to all Lorentz transformations. This is certainly true if one can represent the Lorentz transformations in terms of unitary operators, i.e., if the generators of the infinitesimal Lorentz transformations have a specific Hermitian character.

With this in mind, Majorana displays the generators of the infinitesimal Lorentz transformations in the 4×4 space appropriate to operation on the coordinates, exhibits the generator commutation relations which must be satisfied in any representation, and then immediately gives the infinite dimensional Hermitian representations of these generators. In detail, an infinitesimal Lorentz transformation on the coordinates [15] is given by

$$x_{\mu}^{'} = (\delta_{\mu\nu} + \xi_{\mu\nu})x_{\nu} = (1 - \frac{1}{2}i\xi_{\alpha\beta}I_{\alpha\beta})_{\mu\nu}x_{\nu}, \tag{3}$$

Where here $\xi_{\mu\nu}$ are antisymmetric infinitesimals and $I_{\alpha\beta}$ represent the six independent group generators [16]. In this coordinate-base representation, the matrix components of the generators are

$$(I_{\alpha\beta})_{\mu\nu} = i(\delta_{\alpha\mu}\delta_{\beta\nu} - \delta_{\alpha\mu}\delta_{\beta\nu}) \tag{4}$$

Defining the space-space generators **a** and the space-time generators **b** by the relations

$$a_i = -\frac{1}{2}\varepsilon_{ijk}I_{jk}, b_i = iI_{i4}, \tag{5}$$

One obtains by direct manipulation the commutation relations

$$\begin{aligned} [a_i, a_j] &= i\varepsilon_{ijk}a_k, \\ [b_i, b_j] &= -i\varepsilon_{ijk}a_k, \\ [a_i, b_j] &= i\varepsilon_{ijk}b_k. \end{aligned} \tag{6}$$

For the infinite dimensional representations Majorana gives the matrix elements

$$(j, m|a_1 - i\varepsilon a_2|j, m + \varepsilon) = [(j + \varepsilon m + 1)(j - \varepsilon m)]^{\frac{1}{2}} \tag{7}$$

$$(j, m|a_3|j, m) = m$$

and

$$(j, m|b_1 - i\varepsilon b_2|j + \varepsilon\lambda, m + \varepsilon) = -\tfrac{1}{2}\lambda\{[(j + \lambda(m + \varepsilon)][j + 1 + \lambda(m + \varepsilon)\}^{\frac{1}{2}} \tag{8}$$

$$(j, m|b_3|j + \lambda, m) = \tfrac{1}{2}\{[j + m + \tfrac{1}{2}(\lambda + 1)][j - m + \tfrac{1}{2}(\lambda + 1)]\}^{\frac{1}{2}}$$

Here, ε and λ are independently ± 1, and non-indicated matrix element are zero. The index m ranges from j to –j, and the range of j is either from 0 to ∞ by integer steps from $\frac{1}{2}$ to ∞ by integer steps.

Matrix elements identical with Majorana's infinite dimensional Lorentz representations [Eqs. (7) and (8)] were given earlier by Weyl (1928) [17] in connection with selection and intensity rules for the Schrödinger quantum mechanical problem of electric dipole transitions for an atom. How ever, it seems probable that Majorana derived these results himself. The fact that the three space-space generators **a** have the algebra of angular momenta suggests the angular momentum basis for their representation which Majorana employed [Eq. (7)]. The remaining three space-time generators **b** transform under spatial rotation like a three vector, so in effect they carry one unit of angular momentum. Thus, these generators have possible nonvanishing matrix elements between a basis with angular momentum j and bases j, j$\pm$1 [18]. Consequently, the representation for all six generators involves all j values an integer apart (hence the fact that the representation is of infinite dimension). The ones exhibited by Majorana are not the only infinite dimensional representations [18] nor does he make this claim, but he merely states (accurately) that they are the most simple.

The problem of unitary infinite-dimensional representations of the inhomogeneous Lorentz group was discussed years later by Wigner (1939,1948) [4] from a different standpoint. Instead of proceeding from a definite wave equation and a postulated set of eigenfunctions on which the infinitesimal generators of the group act, Wigner assumed only the existence of a linear relativistically invariant manifold, i.e., a set of states which map into a superposition of themselves under the influence of a Lorentz transformation. This invariant –theoretic approach, though more general and certainly more rigorous than Majorana's development, gives somewhat less physical information (especially kinetic relation) and is of greater algebraic complexity. Following Wigner's first paper on the subject, the question of the infinite representation was investigated so little that in a later paper, reviving the study, Dirac (1945) [19] wrote: "....The group has also some infinite representations which are unitary, These do not seem to have been studied much, in spite of their possible importance for physical application....." Dirac and subsequent authors, with the exception of Wigner [4] and Corson [6], make no reference to Majorana's pioneering work even though many of them employ essentially the same approach. Further detailed investigations of the infinite representations have been made by Gelfand and Naimark (1946) [20], Harish –Chandra (1947) [21], Bargmann (1947) [22], and Gelfand and Yaglom (1948) [23].

It is of interest to note that not only did Majorana give the infinite dimensional representations involving the series of integer j (special case of j bounded from below by 0) and the series of half –integer basis of angular momentum eigenfunctions of the Dirac electron theory. Moreover, Majorana recognized that this representations were special cases associated with a particular value (zero) of an invariant, namely $\mathbf{a} \cdot \mathbf{b}$, of the homogenous Lorentz group. In fact, there are two invariants characterizing a representation

of the homogenous Lorentz group. These are the quantities

$$\frac{1}{2}I_{\alpha\beta}I_{\alpha\beta} = (\mathbf{a}\cdot\mathbf{a} - \mathbf{b}\cdot\mathbf{b}), \tag{9}$$

$$-\frac{1}{8}\varepsilon_{\mu\nu\alpha\beta}I_{\mu\nu}I_{\alpha\beta} = (\mathbf{a}\cdot\mathbf{b}),$$

which can easily be shown to commute with all the generators **a** and **b**. Consequently, in a given representation, these invariants have a fixed value and may be used to characterize the representation [21]. Thus, although Majorana recognized the role of one invariant with reference to a generalization of his results, he did not pursue the question of the group invariants to the extent of exhausting the possibility for generalization.

After Majorana developed the Hermition infinite dimensional representations of the Lorentz transformation generators, he went back to the Lagrangian density expressed in terms of the wavefunction ϕ, and again insisted on invariance of the density under Lorentz transformations. This invariance requirement reduces to the insistence that the wave equation [24]

$$(\gamma_\mu p_\mu^{'} - Mc)\phi^{'}(x^{'}) = 0, \tag{10}$$

be form invariant under the simultaneous transformations

$$\phi^{'}(x^{'}) = \{\exp[-\frac{1}{2}i\xi_{\alpha\beta}I_{\alpha\beta}]\}\phi(x), \tag{11}$$

$$p_\mu^{'} = (\delta_{\mu\nu} + \xi_{\mu\nu})p_\nu.$$

From this requirement, it follows that

$$[I_{\alpha\beta}, \gamma_\pi] = i(\delta_{\beta\pi}\gamma_\alpha - \delta_{\alpha\pi}\gamma_\beta). \tag{12}$$

These commutation relations together with the previously given infinite dimensional representations of $I_{\alpha\beta}$ (in terms of **a** and **b**) essentially determine the matrices γ_π .

Majorana gives the results

$$(j, m|\gamma_4|j, m) = -i(j + \frac{1}{2}), \tag{13}$$

$$(j, m|\gamma_1 - i\varepsilon\gamma_2|j + \varepsilon\lambda, m + \varepsilon) = -\frac{1}{2}i\varepsilon\{[j + \lambda(m + \varepsilon)][j + 1 + \lambda(m + \varepsilon)]\}^{\frac{1}{2}},$$

$$(j, m|\gamma_3|j + \lambda, m) = \frac{1}{2}i\lambda\{[j + m + \frac{1}{2}(\lambda + 1)][j - m + \frac{1}{2}(\lambda + 1)]\}^{\frac{1}{2}},$$

where again ε and λ are independently ± 1, and nonindicated matrix elements are zero.

Finally, Majorana transformed the wave equation written in terms of ϕ [Eq. (10) without the primes] into the Hamiltonian form written in terms of ψ [Eq. (1)]. For this purpose, he utilized a nonunitary transformation

$$\phi = T\psi, \tag{14}$$

where

$$(j, m|T|j, m) = (j + \frac{1}{2})^{-\frac{1}{2}}, \tag{15}$$

nonindicated matrix elements being zero. In this way, he recovered Eq. (1), for which **a** and β are given in terms of the γ's and T [Eqs. (13) and (15)] by the relations

$$\alpha_i = T\gamma_i T, \tag{16}$$

$$\beta = T^2, i.e., (j, m|\beta|j, m) = (j + \frac{1}{2})^{-1}.$$

Thus, Majorana had solved the problem of obtaining an explicit form of a wave equation which was relativistically invariant and involved only positive energy eigenvalues.

The eigenvalues of β in Majorana's infinite dimensional theory are $(j + \frac{1}{2})^{-1}$, where j ranges over –all values in appropriate integer or half integer series. Consequently, the rest mass eigenvalues for the infinite number of eigenfunctions in this theory are $M(j + \frac{1}{2})^{-1}$, where M is an assigned constant. Apparently, Majorana was not interested in the idea of a mass spectrum, and this disinterest was perhaps understandable in view of the limited number of elementary particles known at that time [25]. Instead, Majorana emphasized a particular eigenfunction corresponding to the rest –mass eigenvalue characterized by j=s (s fixed) and he disregarded the other eigenfunctions as as being unphysical. He admitted that sufficiently strong interactions would cause transitions to these other unphysical states, so consequently he restricted the domain of applicability of the theory to interactions sufficiently weak so as to cause no transitions. This line of argument is exactly the same as the one used in Dirac electron theory at the time to prohibit transitions to the (unphysical) negative energy states. Also, for the physical eigenfunction of rest mass $M(j + \frac{1}{2})^{-1}$, Majorana demonstrated that in the nonrelativistic limit , all the components of the eigenfunction labeled by j different from s vanished by order v/c or grater. Again, this demonstration is parallel to the Dirac electron theory for which the "small" components of the positive energy eigenfunction vanish in the nonrelativistic limit [26]. Thus in the nonrelativistic limit, the infinite dimensional eigenfunction has only 2s+1 nonvanishing components in agreement with the ad hoc Pauli modification of Schrödinger quantum mechanics. To have only 2s+1 components in the nonrelativistic was one of the stated aims of Majorana's theory. Moreover, this fact means that in the rest system, The wavefunction transformed like an irreducible finite representation of the rotation group corresponding to spin s, and since spin is essentially defined by rest-system properties, the theory of Majorana was indeed one for a single pure spin associated with his selected mass eigenvalue.

Majorana completed his article by mentioning the existence of imaginary mass eigenvalues in the theory, developing the expression for the eigenfunctions in a plane wave state, and discussing the incorporation of an electromagnetic interaction by means of the usual replacement $p_\mu \to p_\mu - (e/c)A_\mu$. In this connection, he discussed how one may add a Pauli -type term (in the non-Hamiltonian form of the wave equation, essentially a term of the form $\gamma_\mu\gamma_\nu f_{\mu\nu}$ where $f_{\mu\nu}$ Is the electromagnetic field tensor) in order to provide for an anomalous magnetic moment [27]. His discussion of Pauli's procedure involves the only reference to published work that Majorana makes in his article, and even this is indirect as he refers to Pauli as cited by Oppenheimer! [28].

One may speculate why this article by Majorana occasioned little or no comment, and practically disappeared into the archives. First of all, just shortly before its publication, the positron was discovered [25] and this invested the negative mass eigenvalues (reinterpreted)of the Dirac electron theory with physical significance. Consequently, a theory which had mass eigenvalues of only one sign and did not incorporate the feature of antiparticles held little appeal. Furthermore, second quantization showed how to deal with transitions between different mass eigenvalues, so the weak field limit restriction was lifted. This meant that all of the mass eigenstates of Majorana's theory were accessible. But his mass spectrum was of the form $M(s+\frac{1}{2})^{-1}$, which says that the higher the spin, the lower the mass. This relation was undesirable since it indicated that higher spin particles should be the more stable ones. Here, again, any interest in Majorana's theory would be inhibited.

Also, the infinite dimensional representations of the Dirac matrices are unwieldy and this is a disadvantage. Moreover , most physicists were unfamiliar with group theory, and its study was not much in fashion, so the exhibition of infinite dimensional representation of the homogeneous Lorents group did not find a very avid audience. Furthermore, the later development of the Dirac-Fierz-Pauli [10-12] theory which was based on a finite dimensional representation, had antiparticles, and proceeded from a factorized familiar dispersion relation may have saturated the interest of those pursuing the nonexsistent higher spin particles. Finally, the fact that Majorana's ,written in terse, condensed , somewhat cryptic style, was published in Italian in a journal not very widely read at the time [29] probably also contributed to its obscurity.

In recent years, Majorana's theory has been reconstructed, discussed, and generalized (always without reference to him). In this connection, one might mention the works of Gelfand and Yaglom (1948) [23], Ginzburg (1956) [30], Gelfand, Minlos, and Shapiro (1958) [31] and Naimark (1958) [32]. Some attempts have been made to combine a number of irreducible dimensional representations in order to produce a mass spectrum for which the values increase with increasing spin [33]

Acknowledgment

The author wishes to thank Professor E.Amaldi, at whose suggestion this review was undertaken, for introducing him to Majorana's paper and for many informative discussions. This work was initiated at the Istituto di Fisica, Universita di Rome, while the author was a 1964-65 NATO Fellow there.

References

[1] E. Mojorana, Nuovo Cimento **9** , 335-344 (1932)

[2] The brilliant Italian physicist Ettore Majorana was born 5 August 1906 in Catania, Sicily and disappeared under unknown circumstances off coast of Naples on 26 March 1938. A fascinating account of Majorana's life and professional contributions has been given by E. Amaldi (to be published).

[3] N. Kemmer, " The Particle Aspect of Meson Theory", Proc. Roy. Soc. **173A**, 91-116 (1939). Kemmer conjectured that existing higher spin theories, including Majorana's might be simplified by extensions of his (Kemmer) work.

[4] E. P. Wigner, " On Unitary Representations of the Inhomogeneous Lorentz Group", Ann. Math. (Princeton) **40**, 149-204 (1939); " Relativistiche Wellengleichungen", Z. Physik **124**, 665-684 (1948).

[5] E. M. Corson, Intoduction to Tensors, Spinors and Relativistic Wave –Equations (Blackie and Son, Ltd., Glasgow, 1953), Chap. 5, pp 140 and 199.

[6] P. A. M. Dirac, " The Quantum Theory of the Electron", Proc. Roy. Soc. (London) **117A**, 610-624 (1928).

[7] G. Petiau, " Contribution á la Théorie des Equations d'Ondes corpusculaires" Thesis, Univ. Paris (1936) . Published in Mem. Acad. Sci. Roy. Belgique **16**, No. 2, 1-116 (1936).

[8] R. J. Duffin, " On the Characteristic Matrices of Covariant" , Phys. Rev. **54**, 1114 (1938)

[9] A. Proca , " Sur les equations fondamebtales des particules élémentaires", Compt Rend. **202**, 1490-1492 (1936); " Sur la théorie ondulatoire des électrons positives et négatifs" J. Phys. Radium **7**, 347-353 (1936).

[10] P. A. M. Dirac , "Relativistic Wave Equations", Proc. Roy. Soc. (London) **155A**, 447-459 (1936).

[11] M. Fierz, " ber die relativistische theorie Kraftefreier Teilchen mit beliebigem Spin", Helv. Phys. Acta **12**, 3-37 (1939).

[12] M. Fierz and W. Pauli, "On Relativistic Wave Equations of Particles of Arbitrary Spin in an Electromagnetic Field" , Proc. Roy. (London) **173A**, 211-232 (1939).

[13] H. J. Bhabha, " Relativistic Wave Equations for the Elementary Particles", Rev. Mod. Phys. **17**, 200-216 (1945). See also: E. Wild, "On First Order Wave Equations for Elementary Particles without Subsidiary Conditions", Proc. Roy. Soc. (London) **191A,** 253-268 (1947); K. J. LeCouter, "The Structure of Linear Relativistic Wave Equations I, II", Proc. Roy. Soc. (London) **202A**, 284-300, 394-407 (1950).

[14] H. Umezaawa, Quantum Field Theory (North-Holland Publishing Co., Amsterdam, 1956).

[15] Here, we *deviate* from Majorana's covariant –controvariant notation and take $x_4 = ict$, thus avoiding the use of an explicit metric. In the following, Greek indices range from 1 to 4, Latin from 1 to 3, and summation convention on repeated indices is employed.

[16] As a consequence of the reality properties of the infinitesimals $\xi_{\mu\nu}$, a unitary representation here is one for which I_{jk} is Hermitian and $I_{i4}$is anti –Hermitian.

[17] H. Weyl, *Gruppentheorie und Quantenmechanik* (S. Hirzel Verlag, Leipzing, 1928). 1^{st} ed., P. 160. Since it is known that Majorana was very familiar with Weyl's Book (see Ref. 2), it is possible that Majorana took over Weyl's equations directly, having recognized that they also applied in his problem.

[18] The fact that the matrix elements of the b's[Eq. (8)] do not connect bases with the same j, but only connect with j±1, merly reflects the special cases discussed by Majorana which have j lower bounds of 0 or $\frac{1}{2}$.

[19] P. A. M. Dirac, " Unitary Representations of the Lorentz Group", Proc. Roy. Soc. (London) 183A, 284-295 (1945).

[20] I. M. Gelfand and M. A. Naimark, "Unitary Representations of the Lorentz Group", J. Phys. (USSR) **10**, 93-94 (1946).

[21] Harish –Chandra, "Infinite Irreducible Representations of the Lorentz Group", Proc. Roy. Soc. (London) **189A**, 372-401 (1947).

[22] V. Bargmann, "Irreducible Unitary Representations of the Lorentz Group", Ann. Math. **48**, 568-640 (1947).

[23] I. M. Gelfand and A. M. Yaglom, "General Relativistic –Invariant Equations and Infinite –Dimensional Representations of the Lorentz Group" (in Russian), Zh. Eksperim. I Teor. Fiz. **18**, 707-733 (1948) [English transl.: TT-345, National Research Council of Canada (1953)].

[24] In Majorana's notation, $p_4 = iW/c$ and $\gamma_4 = -i\gamma_0$.

[25] The neutron had been discovered early in 1932 [J. Chadwick, "Possible Existence of a Neutron", Nature **129**, 312 (1932); "The Existence of a Neutron", Proc. Roy. Soc. (London) **136A**, 692-708 (1932)] and the positron had just been discovered [C. D. Anderson, "The Apparent Existence of Easily Deflectable Positives" , Science **76**, 238-239 (1932)] several months before Majorana's article was published. However news of the latter event had not yet arrived in Rome when Majorana was preparing his paper.

[26] See, for example, A. Messiah, *Quantum Mechanics* (North- Holland Publishing Co., Amsterdam, 1962), Vol. 2, p933.

[27] This technique of Pauli is discussed and generalized by L. L. Foldy, "Electromagnetic Properties of Dirac Particles", Phys. Rev. **87**, 688-693 (1952).

[28] J. F. Carlson and J. R. Oppenheimer, " The Impact of Fact Electrons and Magnetic Neutrons" , Phys. Rev. 41, 763-792 (1932). They refer to Pauli's 1931 Ann Arbor lectures.

[29] In 1932, Nuovo Cimento was not renowned Italian international journal that it is today. In fact, Science Abstracts (Section A, Physics), the English language abstracting service, did not abstract from Nuovo Cimento until 1946. Majorana's article was given a several line abstract by contemporary German abstract service [Physikalische Berichte, 1933 –I, P. 548] but the abstractor, whose major field was florescence of salts and crystal studies, failed to assess its significance or even mention the occurrence of the infinite dimensional representations.

[30] V. L. Ginzburg, " On Relativistic Wave Equations with a Mass Spectrum", Acta Phys. Polon. **15**, 163-175 (1956).

[31] I. M. Gelfand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the Rotation and Lorentz Group and their Applications* (Pergamon Press, Inc., Oxford, 1963), translated by G. Cummins and T. Boddington from Russian (Fizmatgiz, Moscow, 1958).

[32] M. Naimark, *Linear Representations of the Lorentz Group* (Pergamon Press, Inc., Oxford, 1964), translated by A. Swinfen and O. J. Marstrand from Russian (Fizmatgiz, Moscow, 1958).

[33] See English edition of Ref. 31, p. 340.