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Some strong sufficient conditions for cyclic homogeneous polynomial inequalities of degree four in nonnegative variables

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`Dedicated to the memory of Professor Viorel Radu`

`Communicated by Adrian Petrusel`

Abstract

We establish some strong sufficient conditions that the inequality $f_4(x, y, z) \geq 0$ holds for all nonnegative real numbers x, y, z, where $f_4(x, y, z)$ is a cyclic homogeneous polynomial of degree four. In addition, in the case $f_4(1, 1, 1) = 0$ and also in the case when the inequality $f_4(x, y, z) \geq 0$ does not hold for all real numbers x, y, z, we conjecture that the proposed sufficient conditions are also necessary that $f_4(x, y, z) \geq 0$ for all nonnegative real numbers x, y, z. Several applications are given to show the effectiveness of the proposed methods.

Keywords: Cyclic homogeneous polynomial; strong sufficient conditions; necessary and sufficient conditions; nonnegative real variables.
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1. Introduction

Consider first the third degree cyclic homogeneous polynomial

$$f_3(x, y, z) = \sum x^3 + Bxyz + C\sum x^2y + D\sum xy^2,$$

where B, C, D are real constants, and $\sum$ denotes a cyclic sum over x, y and z. In [6], Pham Kim Hung gives the necessary and sufficient conditions that $f_3(x, y, z) \geq 0$ for any nonnegative real numbers x, y, z.

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Theorem 1.1. *The cyclic inequality $f_3(x, y, z) \geq 0$ holds for all nonnegative real numbers x, y, z if and only if*

$$f_3(1, 1, 1) \geq 0$$

and

$$f_3(x, 1, 0) \geq 0$$

for all nonnegative real x.

Consider now the fourth degree cyclic homogeneous polynomial

$$f_4(x, y, z) = \sum x^4 + A \sum x^2y^2 + Bxyz \sum x + C \sum x^3y + D \sum xy^3,$$

where A, B, C, D are real constants.

The following two theorems in [4] express the necessary and sufficient conditions that $f_4(x, y, z) \geq 0$ for any real numbers x, y, z.

Theorem 1.2. *The cyclic inequality $f_4(x, y, z) \geq 0$ holds for all real numbers x, y, z if and only if*

$$f_4(t + k, k + 1, kt + 1) \geq 0$$

for all real t, where $k \in [0, 1]$ is a root of the equation

$$(C - D)k^3 + (2A - B - C + 2D - 4)k^2 - (2A - B + 2C - D - 4)k + C - D = 0.$$

Theorem 1.3. *The cyclic inequality*

$$f_4(x, y, z) \geq 0$$

holds for all real numbers x, y, z if and only if $g_4(t) \geq 0$ for all $t \geq 0$, where

$$g_4(t) = 3(2 + A - C - D)t^4 - Ft^3 + 3(4 - B + C + D)t^2 + 1 + A + B + C + D,$$

$$F = \sqrt{27(C - D)^2 + E^2}, \qquad E = 8 - 4A + 2B - C - D.$$

In the particular case $f_4(1, 1, 1) = 0$, from Theorem 1.3 we get the following corollary (see [1] and [3]):

Corollary 1.4. *If*

$$1 + A + B + C + D = 0,$$

then the cyclic inequality $f_4(x, y, z) \geq 0$ holds for all real numbers x, y, z if and only if

$$3(1 + A) \geq C^2 + CD + D^2.$$

The following propositions in [4] give the equality cases of the inequality $f_4(x, y, z) \geq 0$ in Theorem 1.2 and Theorem 1.3, respectively.

Proposition 1.5. *The cyclic inequality $f_4(x, y, z) \geq 0$ in Theorem 1.2 becomes an equality if*

$$\frac{x}{t + k} = \frac{y}{k + 1} = \frac{z}{kt + 1}$$

(or any cyclic permutation), where $k \in (0, 1]$ is a root of the equation

$$(C - D)k^3 + (2A - B - C + 2D - 4)k^2 - (2A - B + 2C - D - 4)k + C - D = 0$$

and $t \in \mathbb{R}$ is a root of the equation

$$f_4(t + k, k + 1, kt + 1) = 0.$$

Proposition 1.6. *For $F > 0$, the cyclic inequality $f_4(x, y, z) \geq 0$ in Theorem 1.3 becomes an equality if and only if x, y, z satisfy*

$$(C - D)(x + y + z)(x - y)(y - z)(z - x) \geq 0$$

and are proportional to the real roots w_1, w_2 and w_3 of the equation

$$w^3 - 3w^2 + 3(1 - t^2)w + \frac{2E}{F}t^3 + 3t^2 - 1 = 0,$$

where t is any double nonnegative real root of the polynomial $g_4(t)$.

The following theorem in [5] expresses some strong sufficient conditions that the inequality $f_4(x, y, z) \geq 0$ holds for any real numbers x, y, z.

Theorem 1.7. *Let*

$$G = \sqrt{1 + A + B + C + D},$$
$$H = 2 + 2A - B - C - D - C^2 - CD - D^2.$$

The cyclic inequality $f_4(x, y, z) \geq 0$ holds for all real numbers x, y, z if the following two conditions are satisfied:

(a) $1 + A + B + C + D \geq 0$;

(b) *there exists a real number $t \in (-\sqrt{3}, \sqrt{3})$ such that $f(t) \geq 0$, where*

$$f(t) = 2Gt^3 - (6 + 2A + B + 3C + 3D)t^2 + 2(1 + C + D)Gt + H.$$

In this paper, we will establish some very strong sufficient conditions that the inequality

$$f_4(x, y, z) \geq 0$$

holds for all nonnegative real numbers x, y, z.

2. Main Results

The main result of this paper is given by the following two theorems.

Theorem 2.1. *The inequality $f_4(x, y, z) \geq 0$ holds for all nonnegative real numbers x, y, z if*

$$1 + A + B + C + D \geq 0$$

and one of the following two conditions is fulfilled:

(a) $3(1 + A) \geq C^2 + CD + D^2$;

(b) $3(1 + A) < C^2 + CD + D^2$, $\quad 5 + A + 2C + 2D \geq 0$, $\quad f_4(x, 1, 0) \geq 0$, $\quad h_3(x) \geq 0$ *for all* $x \geq 0$, *where*

$$h_3(x) = (4 + C + D)(x^3 + 1) + (A + 2C - D - 1)x^2 + (A - C + 2D - 1)x.$$

Theorem 2.2. *The inequality $f_4(x, y, z) \geq 0$ holds for all nonnegative real numbers x, y, z if*

$$1 + A + B + C + D \geq 0$$

and one of the following two conditions is fulfilled:

(a) $3(1 + A) \geq C^2 + CD + D^2$;

(b) $3(1 + A) < C^2 + CD + D^2$, *and there is $t \geq 0$ such that*

$$(C + 2D)t^2 + 6t + 2C + D \geq 2\sqrt{(t^4 + t^2 + 1)(C^2 + CD + D^2 - 3 - 3A)}.$$

Remark 2.3. If the sufficient conditions in Theorem 2.1 or Theorem 2.2 are fulfilled, then the following sharper inequality holds for all $x, y, z \geq 0$:

$$f_4(x, y, z) \geq (1 + A + B + C + D)xyz \sum x.$$

This claim is true because $B \geq -1 - A - C - D$ and, on the other hand, Theorems 2.1 and 2.2 remain valid by replacing B with $-1 - A - C - D$. Therefore, if

$$1 + A + B + C + D > 0$$

and the other sufficient conditions in Theorem 2.1 or Theorem 2.2 are fulfilled, then the inequality $f_4(x, y, z) \geq 0$ becomes an equality only when one of x, y, z is zero; that is, for $x = \beta y$ and $z = 0$ (or any cyclic permutation), where β is a double positive root of the polynomial $f_4(x, 1, 0)$ (see the proof of Theorem 2.1) or

$$h_4(t) = [(C + 2D)t^2 + 6t + 2C + D]^2 - 4(t^4 + t^2 + 1)(C^2 + CD + D^2 - 3 - 3A)$$

(see the proof of Theorem 2.2).

Remark 2.4. Consider the main case when

$$1 + A + B + C + D = 0.$$

In the case (a) of Theorem 2.1 and Theorem 2.2, the inequality $f_4(x, y, z) \geq 0$ holds for all real numbers x, y, z, and the equality conditions (including the case $x = y = z$) are given by Proposition 1.5 and Proposition 1.6.

In the case (b) of Theorem 2.1 and Theorem 2.2, the inequality $f_4(x, y, z) \geq 0$ holds for all nonnegative real numbers x, y, z, but does not hold for all real numbers x, y, z. Equality holds for $x = y = z$, and for $x = \beta y$ and $z = 0$ (or any cyclic permutation), where β is a double positive root of the polynomial $f_4(x, 1, 0)$ (see the proof of Theorem 2.1) or $h_4(t)$ (see the proof of Theorem 2.2).

Conjecture 2.5. *If $1 + A + B + C + D = 0$, then the conditions in Theorem 2.1 and Theorem 2.2 are necessary and sufficient to have $f_4(x, y, z) \geq 0$ for all $x, y, z \geq 0$.*

Conjecture 2.6. *If the inequality $f_4(x, y, z) \geq 0$ does not hold for all real numbers x, y, z, then the conditions in Theorem 2.1 and Theorem 2.2 are necessary and sufficient to have $f_4(x, y, z) \geq 0$ for all $x, y, z \geq 0$.*

3. Proof of Theorem 2.1

Let us define

$$\bar{f}_4(x, y, z) = \sum x^4 + A \sum x^2y^2 - (1 + A + B + C + D)xyz \sum x + C \sum x^3y + D \sum xy^3.$$

Since

$$f_4(x, y, z) \geq \bar{f}_4(x, y, z)$$

for all $x, y, z \geq 0$, it suffices to prove that $\bar{f}_4(x, y, z) \geq 0$. Assume that $x = \min\{x, y, z\}$, and use the substitution $y = x + p$, $z = x + q$, where $p, q \geq 0$. From

$$\sum x^4 = 3x^4 + 4(p + q)x^3 + 6(p^2 + q^2)x^2 + 4(p^3 + q^3)x + p^4 + q^4,$$

$$\sum x^2y^2 = 3x^4 + 4(p + q)x^3 + 2(p + q)^2x^2 + 2pq(p + q)x + p^2q^2,$$

$$xyz \sum x = 3x^4 + 4(p + q)x^3 + (p^2 + 5pq + q^2)x^2 + pq(p + q)x,$$

$$\sum x^3y = 3x^4 + 4(p + q)x^3 + 3(p^2 + pq + q^2)x^2 + (p^3 + 3p^2q + q^3)x + p^3q,$$

$$\sum xy^3 = 3x^4 + 4(p+q)x^3 + 3(p^2+pq+q^2)x^2 + (p^3+3pq^2+q^3)x + pq^3,$$

we get

$$\bar{f}_4(x,y,z) = A_1(p,q)x^2 + B_1(p,q)x + C_1(p,q) := h(x),$$

where

$$A_1(p,q) = (5+A+2C+2D)(p^2-pq+q^2),$$
$$B_1(p,q) = (4+C+D)(p^3+q^3) + (A+2C-D-1)p^2q + (A-C+2D-1)pq^2,$$
$$C_1(p,q) = p^4 + Cp^3q + Ap^2q^2 + Dpq^3 + q^4.$$

As we have shown in [3], the inequality $h(x) \geq 0$ holds for all real x and all $p,q \geq 0$ if $3(1+A) \geq C^2+CD+D^2$. Assume now that $3(1+A) < C^2+CD+D^2$. Clearly, the inequality $h(x) \geq 0$ holds for all nonnegative real x if $A_1(p,q) \geq 0$, $B_1(p,q) \geq 0$ and $C_1(p,q) \geq 0$ for all $p,q \geq 0$. Clearly, these inequality are respectively equivalent to $5+A+2C+2D \geq 0$, $h_3(x) \geq 0$ for all $x \geq 0$ and $f_4(x,1,0) \geq 0$ for all $x \geq 0$.

4. Proof of Theorem 2.2

(a) By Corollary 1.4, if

$$3(1+A) \geq C^2+CD+D^2$$

and

$$B = -1-A-C-D,$$

then $f_4(x,y,z) \geq 0$ for all real numbers x,y,z, so the more for all nonnegative real numbers x,y,z. Since the polynomial f_4 is increasing in B, the inequality $f_4(x,y,z) \geq 0$ holds also for all $B \geq -1-A-C-D$.

(b) The main idea is to find a sharper cyclic homogeneous inequality of degree four

$$\sum x^4 + A_1\sum x^2y^2 + B_1xyz\sum x + C_1\sum x^3y + D_1\sum xy^3 \geq 0,$$

such that

$$1+A_1+B_1+C_1+D_1 = 0.$$

Let us define

$$\bar{f}_4(x,y,z) = f_4(x,y,z) - g(x,y,z),$$

where

$$g(x,y,z) = yz(px+qy-qtz)^2 + zx(py+qz-qtx)^2 + xy(pz+qx-qty)^2,$$

with

$$t \geq 0,$$
$$q = \sqrt[4]{\frac{C^2+CD+D^2-3-3A}{t^4+t^2+1}} > 0,$$
$$p = q(t-1) + \sqrt{1+A+B+C+D}.$$

Since $g(x,y,z) \geq 0$, it suffices to prove that $\bar{f}_4(x,y,z) \geq 0$. We can write $\bar{f}_4(x,y,z)$ in the form

$$\bar{f}_4(x,y,z) = \sum x^4 + A_1\sum x^2y^2 + B_1xyz\sum x + C_1\sum x^3y + D_1\sum xy^3,$$

where

$$A_1 = A + 2q^2t, \quad B_1 = B - p(p+2q-2qt),$$
$$C_1 = C - q^2, \quad D_1 = D - q^2t^2.$$

Since

$$1+A_1+B_1+C_1+D_1 = 1+A+B+C+D-(p+q-qt)^2 = 0,$$

according to Corollary 1.4, it suffices to show that $3(1+A_1) \geq C_1^2 + C_1D_1 + D_1^2$. Write this inequality as

$$(C+2D)t^2 + 6t + 2C + D \geq q^2(t^4+t^2+1) + \frac{1}{q^2}(C^2+CD+D^2-3-3A),$$

$$(C+2D)t^2 + 6t + 2C + D \geq 2\sqrt{(t^4+t^2+1)(C^2+CD+D^2-3-3A)}.$$

By the hypothesis in (b), there is $t \geq 0$ such that the last inequality is true. Thus, the proof is completed.

5. Applications

Application 5.1. *Let x, y, z be nonnegative real numbers. If $k \geq 0$, then ([2] and [7])*

$$\sum x^4 + (k^2-2)\sum x^2y^2 + (1-k^2)xyz\sum x \geq 2k(\sum x^3y - \sum xy^3).$$

Proof. Write the inequality as $f_4(x, y, z) \geq 0$, where

$$A = k^2-2, \quad B = 1-k^2, \quad C = -2k, \quad D = 2k, \quad 1+A+B+C+D = 0.$$

First Solution. We will show that the condition (b) in Theorem 2.1 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1+A) = k^2 + 3 > 0$$

and

$$5 + A + 2C + 2D = k^2 + 3 > 0,$$

we only need to show that $f_4(x, 1, 0) \geq 0$ and $h_3(x) \geq 0$ for all $x \geq 0$. We have

$$f_4(x, 1, 0) = x^4 - 2kx^3 + (k^2-2)x^2 + 2kx + 1 = (x^2 - kx - 1)^2 \geq 0,$$

$$h_3(x) = 4(x^3+1) + (k^2-6k-3)x^2 + (k^2+6k-3)x.$$

For $0 \leq x < 1$, we get

$$h_3(x) = 4(x^3+1) + (k^2-3)x(1+x) + 6kx(1-x) \geq 4(x^3+1) + (k^2-3)x(1+x)$$

$$\geq 4(x^3+1) - 4x(1+x) = 4(x+1)(x-1)^2 > 0.$$

Also, for $x \geq 1$, we get

$$h_3(x) = 4(x-1)^3 + (k-3)^2x^2 + (k^2+6k-15)x + 8$$

$$= 4(x-1)^3 + (k-3)^2(x-1)^2 + 3(k-1)^2x - k^2 + 6k - 1$$

$$= 4(x-1)^3 + (k-3)^2(x-1)^2 + 3(k-1)^2(x-1) + 2(k^2+1) > 0.$$

The polynomial $f_4(x, 1, 0)$ has the double positive real root $\beta = \dfrac{k+\sqrt{k^2+4}}{2}$. Therefore, according to Remark 2.4, equality holds for $x = y = z$, and also for $x = 0$ and $\dfrac{y}{z} = \dfrac{k+\sqrt{k^2+4}}{2}$ (or any cyclic permutation).

Second Solution. We will show that the condition (b) in Theorem 2.2 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1+A) = k^2 + 3 > 0,$$

we only need to show that there exists $t \geq 0$ such that

$$kt^2 + 3t - k \geq \sqrt{(k^2+3)(t^4+t^2+1)}.$$

This is true if

$$kt^2 + 3t - k \geq 0$$

and $h_4(t) \geq 0$, where

$$h_4(t) = (kt^2 + 3t - k)^2 - (k^2 + 3)(t^4 + t^2 + 1) = -(t^2 - kt - 1)^2.$$

Clearly, for

$$t = \frac{k + \sqrt{k^2 + 4}}{2},$$

we have $h_4(t) = 0$ and

$$kt^2 + 3t - k = k(kt + 1) + 3t - k = (k^2 + 3)t > 0.$$

Since the polynomial $h_4(t)$ has the double positive real root $\beta = \dfrac{k + \sqrt{k^2 + 4}}{2}$, according to Remark 2.4, equality holds for $x = y = z$, and also for $x = 0$ and $\dfrac{y}{z} = \dfrac{k + \sqrt{k^2 + 4}}{2}$ (or any cyclic permutation).

Remark. For $k = 1$, we get the inequality

$$x^4 + y^4 + z^4 - x^2y^2 - y^2z^2 - z^2x^2 \geq 2(x^3y + y^3z + z^3x - xy^3 - yz^3 - zx^3),$$

with equality for $x = y = z$, and for $x = 0$ and $\dfrac{y}{z} = \dfrac{1 + \sqrt{5}}{2}$ (or any cyclic permutation).

Also, for $k = \sqrt{2}$, we get the inequality

$$x^4 + y^4 + z^4 - xyz(x + y + z) \geq 2\sqrt{2}(x^3y + y^3z + z^3x - xy^3 - yz^3 - zx^3),$$

with equality for $x = y = z$, and for $x = 0$ and $\dfrac{y}{z} = \dfrac{\sqrt{2} + \sqrt{6}}{2}$ (or any cyclic permutation).

□

Application 5.2. *If x, y, z are nonnegative real numbers, then ([2])*

$$x^4 + y^4 + z^4 + 5(x^3y + y^3z + z^3x) \geq 6(x^2y^2 + y^2z^2 + z^2x^2).$$

Proof. Write the inequality as $f_4(x, y, z) \geq 0$, where

$$A = -6, \quad B = 0, \quad C = 5, \quad D = 0, \quad 1 + A + B + C + D = 0.$$

First Solution. We will show that the condition (b) in Theorem 2.1 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1 + A) = 40$$

and $5 + A + 2C + 2D = 9$, we only need to show that $f_4(x, 1, 0) \geq 0$ and $h_3(x) \geq 0$ for all $x \geq 0$. We have

$$f_4(x, 1, 0) = x^4 + 5x^3 - 6x^2 + 1 = (x - 1)^4 + x(3x - 2)^2 > 0$$

and

$$h_3(x) = 3(3x^3 + x^2 - 4x + 3).$$

For $0 \leq x < 1$, we get

$$3x^3 + x^2 - 4x + 3 \geq (x - 1)(x - 3) > 0,$$

and for $x \geq 1$, we get

$$3x^3 + x^2 - 4x + 3 \geq 4x(x - 1) + 3 > 0.$$

Since the polynomial $f_4(x, 1, 0)$ has no double positive real root, equality holds only for $x = y = z$ (see Remark 2.4).

Second Solution. We will show that the condition (b) in Theorem 2.2 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1 + A) = 40,$$

we only need to show that there is $t \geq 0$ such that

$$10t^2 + 6t + 5 \geq 2\sqrt{40(t^4 + t^2 + 1)}.$$

Indeed, for $t = 3/2$, we get

$$10t^2 + 6t + 5 - \sqrt{40(t^4 + t^2 + 1)} = \frac{73}{2} - \sqrt{1330} = \frac{9}{2(73 + 2\sqrt{1330}} > 0.$$

According to Remark 2.4, equality holds for $x = y = z$.

□

Application 5.3. *If x, y, z are nonnegative real numbers, then*

$$3(x^4 + y^4 + z^4) + 4(xy^3 + yz^3 + zx^3) \geq 7(x^3y + y^3z + z^3x).$$

Proof. Write the inequality as $f_4(x, y, z) \geq 0$, where

$$A = 0, \quad B = 0, \quad C = -\frac{7}{3}, \quad D = \frac{4}{3}, \quad 1 + A + B + C + D = 0.$$

First Solution. We will prove that the condition (b) in Theorem 2.1 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1 + A) = \frac{10}{9}$$

and $5 + A + 2C + 2D = 2$, we only need to show that $f_4(x, 1, 0) \geq 0$ and $h_3(x) \geq 0$ for all $x \geq 0$. We have

$$f_4(x, 1, 0) = x(x + 1)(3x - 5)^2 + 5\left(x - \frac{13}{10}\right)^2 + \frac{11}{20} > 0,$$

and

$$h_3(x) = 3x^3 - 7x^2 + 4x + 3.$$

For $0 \leq x \leq 1$ and $x \geq \frac{4}{3}$, we get

$$3x^3 - 7x^2 + 4x + 3 > 3x^3 - 7x^2 + 4x = x(x - 1)(3x - 4) \geq 0,$$

and for $1 \leq x \leq \frac{3}{2}$, we get

$$3x^3 - 7x^2 + 4x + 3 \geq -4x^2 + 4x + 3 = (2x + 1)(3 - 2x) \geq 0.$$

Since the polynomial $f_4(x, 1, 0)$ has no double positive real root, equality holds only for $x = y = z$ (see Remark 2.4).

Second Solution. We will prove that the condition (b) in Theorem 2.2 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1 + A) = \frac{10}{9}$$

we only need to show that there exists $t \geq 0$ such that

$$t^2 + 18t - 10 \geq 2\sqrt{10(t^4 + t^2 + 1)}.$$

Indeed, for $t = 2$, we get

$$t^2 + 18t - 10 - \sqrt{10(t^4 + t^2 + 1)} = 30 - 2\sqrt{210} = \frac{609}{30 + 2\sqrt{210}} > 0.$$

According to Remark 2.4, equality holds for $x = y = z$.

□

Application 5.4. *If x, y, z are nonnegative real numbers, then ([1])*

$$x^4 + y^4 + z^4 + \left(\frac{4}{\sqrt[4]{27}} - 1\right) xyz(x + y + z) \geq \frac{4}{\sqrt[4]{27}}(x^3y + y^3z + z^3x).$$

Proof. Write the inequality as $f_4(x, y, z) \geq 0$, where

$$A = 0, \quad B = \frac{4}{\sqrt[4]{27}} - 1, \quad C = -\frac{4}{\sqrt[4]{27}}, \quad D = 0, \quad 1 + A + B + C + D = 0.$$

First Solution. We will show that the condition (b) in Theorem 2.1 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1 + A) = \frac{16}{3\sqrt{3}} - 3 > 0,$$

and

$$5 + A + 2C + 2D = 5 - \frac{8}{\sqrt[4]{27}} > 0,$$

we only need to show that $f_4(x, 1, 0) \geq 0$ and $h_3(x) \geq 0$ for all $x \geq 0$. We have

$$f_4(x, 1, 0) = x^4 - \frac{4}{\sqrt[4]{27}}x^3 + 1 = (x - \sqrt[4]{3})^2 \left(x^2 + \frac{2}{\sqrt[4]{27}}\, x + \frac{1}{\sqrt{3}}\right) \geq 0$$

and

$$h_3(x) = 4x^3 - x^2 - x + 4 - \frac{4}{\sqrt[4]{27}}(x^3 + 2x^2 - x + 1).$$

Since

$$x^3 + 2x^2 - x + 1 \geq x^2 - x + 1 > 0$$

and

$$\frac{4}{\sqrt[4]{27}} < \frac{9}{5},$$

we get

$$5h_3(x) > 5(4x^3 - x^2 - x + 4) - 9(x^3 + 2x^2 - x + 1) = 11x^3 - 23x^2 + 4x + 11$$
$$= 11x\left(x - \frac{3}{2}\right)^2 + 10x^2 - \frac{83}{4}x + 11 \geq 10x^2 - \frac{83}{4}x + 11$$
$$= 10\left(x - \frac{83}{80}\right)^2 + \frac{251}{640} > 0.$$

The polynomial $f_4(x, 1, 0)$ has the double positive real root $\beta = \sqrt[4]{3}$. Therefore, according to Remark 2.4, equality holds for $x = y = z$, and also for $x = 0$ and $\frac{y}{z} = \sqrt[4]{3}$ (or any cyclic permutation).

Second Solution. We will show that the condition (b) in Theorem 2.2 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1 + A) = \frac{16}{3\sqrt{3}} - 3 > 0,$$

we only need to show that there exists $t \geq 0$ such that

$$-2t^2 + 3\sqrt[4]{27}t - 4 \geq \sqrt{(16 - 9\sqrt{3})(t^4 + t^2 + 1)}.$$

This is true if

$$-2t^2 + 3\sqrt[4]{27}t - 4 \geq 0$$

and $h_4(t) \geq 0$, where

$$h_4(t) = (-2t^2 + 3\sqrt[4]{27}t - 4)^2 - (16 - 9\sqrt{3})(t^4 + t^2 + 1).$$

Since

$$h_4(t) = 3(t - \sqrt[4]{3})^2[(3\sqrt{3} - 4)t^2 - 2\sqrt[4]{3}(4 - \sqrt{3})t + 3],$$

we have $h_4(t) = 0$ for $t = \sqrt[4]{3}$, when

$$-2t^2 + 3\sqrt[4]{27}t - 4 = 5 - 2\sqrt{3} > 0.$$

The polynomial $h_4(t)$ has the double positive real root $\beta = \sqrt[4]{3}$. Therefore, according to Remark 2.4, equality holds for $x = y = z$, and also for $x = 0$ and $\frac{y}{z} = \sqrt[4]{3}$ (or any cyclic permutation).

□

Application 5.5. *If x, y, z are nonnegative real numbers, then ([7])*

$$x^4 + y^4 + z^4 + 15(x^3y + y^3z + z^3x) \geq \frac{47}{4}(x^2y^2 + y^2z^2 + z^2x^2).$$

Proof. Write the inequality as $f_4(x, y, z) \geq 0$, where

$$A = \frac{-47}{4}, \quad B = 0, \quad C = 15, \quad D = 0, \quad 1 + A + B + C + D = \frac{17}{4}.$$

First Solution. We will show that the condition (b) in Theorem 2.1 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1 + A) = \frac{1029}{4},$$

and

$$5 + A + 2C + 2D = \frac{93}{4},$$

we only need to show that $f_4(x, 1, 0) \geq 0$ and $h_3(x) \geq 0$ for all $x \geq 0$. We have

$$f_4(x, 1, 0) = x^4 + 15x^3 - \frac{47}{4}x^2 + 1 = \frac{1}{4}(2x - 1)^2(x^2 + 16x + 4) \geq 0.$$

and

$$h_3(x) = 19(x^3 + 1) + \frac{69}{4}x^2 - \frac{111}{4}x > 14 + 14x^2 - 28x = 14(x - 1)^2 \geq 0.$$

According to Remark 2.3, since the polynomial $f_4(x, 1, 0)$ has the double nonnegative real root $\beta = \frac{1}{2}$, equality holds for $x = 0$ and $2y = z$ (or any cyclic permutation).

Second Solution. We will show that the condition (b) in Theorem 2.2 is fulfilled. Since

$$C^2 + CD + D^2 - 3(1+A) = \frac{1029}{4},$$

we only need to show that there is $t \geq 0$ such that

$$15t^2 + 6t + 30 \geq \sqrt{1029(t^4 + t^2 + 1)}.$$

This is true if $h_4(t) \geq 0$, where

$$h_4(t) = (15t^2 + 6t + 30)^2 - 1029(t^4 + t^2 + 1).$$

Since

$$h_4(t) = -3(2t-1)^2(67t^2 + 52t + 43),$$

we have $h_4(t) = 0$ for $t = \frac{1}{2}$.

According to Remark 2.3, since the polynomial $h_4(t)$ has the double nonnegative real root $\beta = \frac{1}{2}$, equality holds for $x = 0$ and $2y = z$ (or any cyclic permutation).

□

Application 5.6. *If x, y, z are nonnegative real numbers such that*

$$x^2 + y^2 + z^2 = \frac{5}{2}(xy + yz + zx),$$

then

$$x^4 + y^4 + z^4 \geq \frac{17}{8}(x^3y + y^3z + z^3x).$$

Proof. We see that equality holds for $x = 0$, $y = 2$, $z = 1$ (or any cyclic permutation). Since

$$\begin{aligned} x^4 + y^4 + z^4 &\geq (x^2 + y^2 + z^2)^2 - 2(xy + yz + zx)^2 \\ &= \frac{17}{4}(xy + yz + zx)^2, \end{aligned}$$

it suffices to show that

$$2(xy + yz + zx)^2 \geq x^3y + y^3z + z^3x.$$

In addition, since

$$36(xy + yz + zx)^2 = [6(xy + yz + zx)^2 = [2(x^2 + y^2 + z^2) + xy + yz + zx]^2,$$

it suffices to show that

$$[2(x^2 + y^2 + z^2) + xy + yz + zx]^2 \geq 18(x^3y + y^3z + z^3x),$$

which is equivalent to

$$4\sum x^4 + 9\sum x^2y^2 + 6xyz\sum x + 4\sum xy^3 \geq 14\sum x^3y.$$

It suffices to show that $f_4(x, y, z) \geq 0$, where

$$f_4(x, y, z) = 4\sum x^4 + 9\sum x^2y^2 - 3xyz\sum x - 14\sum x^3y + 4\sum xy^3.$$

with

$$A = \frac{9}{4}, \quad B = \frac{-3}{4}, \quad C = \frac{-7}{2}, \quad D = 1, \quad 1 + A + B + C + D = \frac{9}{4}.$$

Since

$$3(1+A) - C^2 - CD - D^2 = 0,$$

the condition (a) in Theorem 2.1 and Theorem 2.2 is fulfilled.

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