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ON THE DYNAMICS OF THE RECURSIVE SEQUENCE

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma \sum_{k=1}^{t} x_{n-2k}^{p} \prod_{k=1}^{t} x_{n-2k}^{q}}\,^{*}$$

Abstract. In this paper, we investigate the global behavior of the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma \sum_{k=1}^{t} x_{n-2k}^{p} \prod_{k=1}^{t} x_{n-2k}^{q}}, \quad n = 0, 1, \ldots,$$

where β is a positive parameter and α, γ are non-negative parameters and non-negative initial conditions.

Key words: difference equations, recursive sequences, oscillation, global asymptotic behavior, period two solutions, semicycles.

AMS Mathematics Subject Classification: 39A10.

1. Introduction

Consider the higher-order difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma \sum_{k=1}^{t} x_{n-2k}^{p} \prod_{k=1}^{t} x_{n-2k}^{q}}, \quad n = 0, 1, \ldots \tag{1}$$

where the parameters, β is positive and α, γ are non-negative real numbers and the initial conditions $x_{-2t}, \ldots, x_{-2}, x_{-1}$ and x_0 are non-negative real numbers such that

$$0 < \beta+\gamma \sum_{k=1}^{t} x_{n-2k}^{p} \prod_{k=1}^{t} x_{n-2k}^{q}, \quad n = 0, 1, \ldots .$$

and if $\alpha = 0$ the equation $x_{n+1} = 0$ is trivial, if $\gamma = 0$ the equation $x_{n+1} = \frac{\alpha}{\beta} x_{n-1}$ is linear. We assume that all parameters in equations are positive.

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We investigate the global asymptotic behavior and the periodic character of the solutions of the difference (1), by generalizing the results due to El-Owaidy et al. [1] corresponding to the difference equation

$$x_{n+1} = \frac{\alpha x_{n-1}}{\beta + \gamma x_{n-2}^{p}}, \quad n = 0, 1, \dots .$$

where the parameters α, β and γ are positive real numbers and the initial conditions x_{-2}, x_{-1} and x_0 are arbitrary non-negative real numbers. Similar recursive sequences were studied previously; for example, see Refs. [1-22].

We need the following definitions and theorem [23]:

Definition 1. *Let I be an interval of the real numbers and let $f : I^{2t+1} \longrightarrow I$ be a continuously differantiable function. Consider the difference equation*

$$x_{n+1} = f\left(x_n, x_{n-1}, x_{n-2}, ..., x_{n-2t}\right), \quad n = 0, 1, \dots . \tag{2}$$

with x_{-i} for $i = 0, 1, ..., 2t \in I$. Let $\overline{x}$ be the equilibrium point of (2). The linearized equation of (2) about the equilibrium point $\overline{x}$ is

$$y_{n+1} = c_1 y_n + c_2 y_{n-1} + ... + c_{2t+1} y_{n-2t}, \quad n = 0, 1, \dots \tag{3}$$

where

$$\begin{aligned} c_1 &= \frac{\partial f}{\partial x_n}\left(\overline{x}, \overline{x}, ..., \overline{x}\right), \\ c_2 &= \frac{\partial f}{\partial x_{n-1}}\left(\overline{x}, \overline{x}, ..., \overline{x}\right), \\ &\vdots \\ c_{2t+1} &= \frac{\partial f}{\partial x_{n-2t}}\left(\overline{x}, \overline{x}, ..., \overline{x}\right). \end{aligned}$$

The characteristic equation of (3) is

$$\lambda^{2t+1} - c_1\lambda^{2t} - \dots - c_{2t-1}\lambda^2 - c_{2t}\lambda - c_{2t+1} = 0 \tag{4}$$

Definition 2. *Let $\overline{x}$ be an equilibrium point of (2).*

(*a*) *The equilibrium $\overline{x}$ is called* locally stable *if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x_0, \dots, x_{-2t} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-2t} - \overline{x}| < \delta$, then $|x_n - \overline{x}| < \varepsilon$, for all $n \geq -2t$.*

(*b*) *The equilibrium $\overline{x}$ is called* locally asymptotically stable *if it is locally stable and if there exists $\gamma > 0$ such that if $x_0, \dots, x_{-2t} \in I$ and $|x_0 - \overline{x}| + \cdots + |x_{-2t} - \overline{x}| < \gamma$, then $\lim_{n\to\infty} x_n = \overline{x}$.*

(c) *The equilibrium* $\overline{x}$ *is called* global attractor *if for every* $x_0, \dots, x_{-2t} \in I$ *we have* $\lim\limits_{n\to\infty} x_n = \overline{x}$.

(d) *The equilibrium* $\overline{x}$ *is called* globally asymptotically stable *if it is locally stable and is a global attractor.*

Definition 3. *A positive semicycle of* $\{x_n\}_{n=-2t}^{\infty}$ *of (2) consists of a 'string' of terms* $\{x_l, x_{l+1}, \dots, x_m\}$, *all greater than or equal to* $\overline{x}$, *with* $l \geq -2t$ *and* $m < \infty$ *and such that either* $l = -2t$ *or* $l > -2t$ *and* $x_{l-1} < \overline{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} < \overline{x}$.

A negative semicycle of $\{x_n\}_{n=-2t}^{\infty}$ *of (2) consists of a 'string' of terms* $\{x_l, x_{l+1}, \dots, x_m\}$ *all less than* $\overline{x}$, *with* $l \geq -2t$ *and* $m < \infty$ *and such that either* $l = -2t$ *or* $l > -2t$ *and* $x_{l-1} \geq \overline{x}$ *and either* $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} \geq \overline{x}$.

Definition 4. *A solution* $\{x_n\}_{n=-2t}^{\infty}$ *of (2) is called nonoscillatory if there exists* $N \geq -2t$ *such that either*

$$x_n > \overline{x} \text{ or } x_n < \overline{x} \quad for \ \ \forall n \geq N$$

and it is called oscillatory if it is not nonoscillatory.

Theorem 1. (*i*) *If all roots of (4) have absolute values less than one, then the equilibrium point* $\overline{x}$ *of (2) is locally asymptotically stable.*

(*ii*) *If at least one of the roots of (4) has absolute value greater than one, then the equilibrium point* $\overline{x}$ *of (2) is unstable.*

(*iii*) *The equilibrium point* $\overline{x}$ *of (2) is called saddle point if (4) has roots both inside and outside the unit disk.*

2. Dynamics of equation (1)

In this section, we investigate the dynamics of (1) under the assumptions that all parameters in the equation are positive and the initial conditions are non-negative.

The change of variables $x_n = (\beta/\gamma)^{1/qt+p}\, y_n$ reduces (1) to the difference equation

$$(5) \qquad y_{n+1} = \frac{r y_{n-1}}{1 + y_{n-2}^{q+p} y_{n-4}^{q} \cdots y_{n-2t}^{q} + y_{n-2}^{q} y_{n-4}^{q+p} \cdots y_{n-2t}^{q} + \dots + y_{n-2}^{q} y_{n-4}^{q} \cdots y_{n-2t}^{q+p}}$$

where $r = \alpha/\beta > 0$ and $n = 0, 1, \dots$.

Note that $\overline{y}_1 = 0$ is always an equilibrium point of (5). When $r > 1$, (5) also possesses the unique positive equilibrium $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$.

Theorem 2. *The following statements are true.*

(*i*) *If* $r < 1$, *then the equilibrium point* $\overline{y}_1 = 0$ *of (5) is locally asymptotically stable.*

(ii) If $r > 1$, then the equilibrium point $\overline{y}_1 = 0$ of (5) is a saddle point.

(iii) When $r > 1$, then the positive equilibrium point $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ of (5) is unstable.

Proof. The linearized equation of (5) about the equilibrium point $\overline{y}_1 = 0$ is

$$z_{n+1} = rz_{n-1}, \quad n = 0, 1, \ldots$$

so, the characteristic equation of (5) about the equilibrium point $\overline{y}_1 = 0$ is

$$\lambda^{2t+1} - r\lambda^{2t-1} = 0$$

hence the proof of (i) and (ii) follows Theorem 1.

For (iii) we assume that $r > 1$; then the linearized equation of (5) about the equilibrium point $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ has the form

$$\begin{aligned} z_{n+1} &= z_{n-1} - \frac{(qt+p)}{t}\frac{(r-1)}{r}z_{n-2} - \frac{(qt+p)}{t}\frac{(r-1)}{r}z_{n-4} \\ &\quad - \ldots - \frac{(qt+p)}{t}\frac{(r-1)}{r}z_{n-2t} = 0 \end{aligned}$$

where $n = 0, 1, \ldots$. So the characteristic equation of (5) about the equilibrium point $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ is

$$\lambda^{2t+1} - \lambda^{2t-1} + \frac{(qt+p)}{t}\frac{(r-1)}{r}\lambda^{2t-2} + \ldots + \frac{(qt+p)}{t}\frac{(r-1)}{r} = 0. \tag{6}$$

It is clear that (6) has a root in the interval $(-\infty, -1)$ and so $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ is an unstable equilibrium point. This completes the proof. ■

Theorem 3. *Assume that $r > 1$. Let $\{y_n\}_{n=-2t}^{\infty}$ be a solution of (5) such that*

$$y_{-2t}, \ldots y_0 \geq \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}, \quad y_{-2t+1}, \ldots, y_{-1} < \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p} \tag{7}$$

or

$$y_{-2t}, \ldots y_0 < \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}, \quad y_{-2t+1}, \ldots, y_{-1} \geq \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p} \tag{8}$$

Then $\{y_n\}_{n=-2t}^{\infty}$ oscillates about $\overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p}$ with semicycle of length 1.

Proof. Assume that (7) holds. (The case where (8) holds is similar and will be omitted.) Then,

$$\begin{aligned} y_1 &= \frac{ry_{-1}}{1 + y_{-2}^{q+p}y_{-4}^{q}\cdots y_{-2t}^{q} + y_{-2}^{q}y_{-4}^{q+p}\cdots y_{-2t}^{q} + y_{-2}^{q}y_{-4}^{q}\cdots y_{-2t}^{q+p}} \\ &< \frac{r\overline{y}_2}{1 + t\overline{y}_2^{qt+p}} = \frac{r\overline{y}_2}{1 + r - 1} = \overline{y}_2, \\ &< \overline{y}_2 \end{aligned}$$

and

$$\begin{aligned} y_2 &= \frac{ry_0}{1 + y_{-1}^{q+p} y_{-3}^q \ldots y_{-2t+1}^q + y_{-1}^q y_{-3}^{q+p} \ldots y_{-2t+1}^q + y_{-1}^q y_{-3}^q \ldots y_{-2t+1}^{q+p}} \\ &\geq \frac{r\overline{y}_2}{1 + t\overline{y}_2^{qt+p}} = \frac{r\overline{y}_2}{1 + r - 1} = \overline{y}_2, \\ &\geq \overline{y}_2 \end{aligned}$$

then the proof follows by induction. ■

Theorem 4. *Assume that $r < 1$; then the equilibrium point $\overline{y}_1 = 0$ of (5) is globally asymptotically stable.*

Proof. We know by Theorem 2 that the equilibrium point $\overline{y}_1 = 0$ of (5) is locally asymptotically stable. So let $\{y_n\}_{n=-2t}^{\infty}$ be a solution of (5). It suffices to show that

$$\lim_{n \to \infty} y_n = 0$$

Since

$$\begin{aligned} y_{n+1} &= \frac{ry_{n-1}}{1 + y_{n-2}^{q+p} y_{n-4}^q \cdots y_{n-2t}^q + y_{n-2}^q y_{n-4}^{q+p} \cdots y_{n-2t}^q + y_{n-2}^q y_{n-4}^q \cdots y_{n-2t}^{q+p}} \\ y_{2n-1} &< r^n y_{-1} \quad \text{and} \quad y_{2n} < r^n y_0 \end{aligned}$$

we have

$$\lim_{n \to \infty} y_n = 0.$$

This completes the proof. ■

Theorem 5. *Assume that $r = 1$; then (5) possesses the prime period 2 solutions*

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots \tag{9}$$

with $\Phi > 0$. Furthermore, every solution of (5) converges to a period 2 solution (9) with $\Phi \geq 0$.

Proof. Let

$$\ldots, \Phi, \Psi, \Phi, \Psi, \ldots$$

be a period two solution of (5). Then

$$\Phi = \frac{r\Phi}{1 + \Psi^{q+p}\Psi^q \ldots \Psi^q + \Psi^q\Psi^{q+p} \ldots \Psi^q + \Psi^q\Psi^q \ldots \Psi^{q+p}}$$

and

$$\Psi = \frac{r\Psi}{1 + \Phi^{q+p}\Phi^q \ldots \Phi^q + \Phi^q\Phi^{q+p} \ldots \Phi^q + \Phi^q\Phi^q \ldots \Phi^{q+p}}.$$

So

$$t\Phi\Psi = \frac{(\Phi - \Psi)(r-1)}{\Psi^{qt+p-1} - \Phi^{qt+p-1}} \geq 0,$$

which implies that $r - 1 \leq 0$.

If $r < 1$, then this implies that $\Phi < 0$ or $\Psi < 0$, which is impossible, so $r = 1$. If $r > 1$, then this implies that $\Phi = \Psi = \left(\frac{r-1}{t}\right)^{1/qt+p} \neq 0$, which contradicts that

$\Phi \neq \Psi$, so $r = 1$. To complete the proof, assume that $r = 1$ and let $\{y_n\}_{n=-2t}^{\infty}$ be a solution of (5); then

$$\begin{aligned}
&y_{n+1} - y_{n-1} \\
&= \frac{-y_{n-1}y_{n-2}^{q+p}y_{n-4}^{q}\cdots y_{n-2t}^{q} - y_{n-1}y_{n-2}^{q}y_{n-4}^{q+p}\cdots, y_{n-2t}^{q} - y_{n-1}y_{n-2}^{q}y_{n-4}^{q}\cdots y_{n-2t}^{q+p}}{1 + y_{n-2}^{q+p}y_{n-4}^{q}\cdots y_{n-2t}^{q} + y_{n-2}^{q}y_{n-4}^{q+p}\cdots, y_{n-2t}^{q} + y_{n-2}^{q}y_{n-4}^{q}\cdots y_{n-2t}^{q+p}} \\
&y_{n+1} - y_{n-1} \leq 0.
\end{aligned}$$

So, the even terms of this solution decrease to a limit (say $\Phi \geq 0$) and the odd terms decrease to a limit (say $\Psi \geq 0$). Thus

$$\Phi = \frac{\Phi}{1 + \Psi^{q+p}\Psi^{q}...\Psi^{q} + \Psi^{q}\Psi^{q+p}\ldots\Psi^{q} + \Psi^{q}\Psi^{q}\ldots\Psi^{q+p}}$$

and

$$\Psi = \frac{\Psi}{1 + \Phi^{q+p}\Phi^{q}\ldots\Phi^{q} + \Phi^{q}\Phi^{q+p}\ldots\Phi^{q} + \Phi^{q}\Phi^{q}\ldots\Phi^{q+p}},$$

which implies that

$$t\Phi\Psi^{qt+p} = 0 \ \textit{and} \ t\Psi\Phi^{qt+p} = 0.$$

This completes the proof. ■

Theorem 6. *Assume that $r > 1$; then (5) possesses an unbounded solution.*

Proof. From Theorem 3, we can assume without loss of generality that the solution $\{y_n\}_{n=-2t}^{\infty}$ of (5) is such that

$$y_{2n-1} < \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p} \quad \text{and} \quad y_{2n} > \overline{y}_2 = \left(\frac{r-1}{t}\right)^{1/qt+p},$$

for $n \geq 0$. Then

$$\begin{aligned}
&y_{2n+2} \\
&= \frac{ry_{2n}}{1 + y_{2n-1}^{q+p}y_{2n-3}^{q}...y_{2n-2t+1}^{q} + y_{2n-1}^{q}y_{2n-3}^{q+p}\cdots y_{2n-2t+1}^{q} + y_{2n-1}^{q}y_{2n-3}^{q}\cdots y_{2n-2t+1}^{q+p}} \\
&y_{2n+2} > \frac{ry_{2n}}{1+(r-1)} = y_{2n}
\end{aligned}$$

and

$$\begin{aligned}
y_{2n+3} &= \frac{ry_{2n+1}}{1 + y_{2n}^{q+p}y_{2n-2}^{q}\cdots y_{2n-2t+2}^{q} + y_{2n}^{q}y_{2n-2}^{q+p}\cdots y_{2n-2t+2}^{q} + y_{2n}^{q}y_{2n-2}^{q}\cdots y_{2n-2t+2}^{q+p}} \\
y_{2n+3} &< \frac{ry_{2n+1}}{1+(r-1)} = y_{2n+1}
\end{aligned}$$

from which it follows that

$$\lim n \to \infty y_{2n} = \infty \quad \text{and} \quad \lim_{n\to\infty} y_{2n+1} = 0.$$

Then, the proof is complete. ■

References

[1] El-Owaidy H.M., Ahmed A.M., Youssef A.M., The dynamics of recursive $x_{n+1} = \frac{\alpha x_{n-1}}{\beta+\gamma x_{n-2}^p}$, *Applied Mathematics Letters,* (2005), 1013-1018.

[2] Amleh A.M., Kirk V., Ladas G., On the dynamics of $x_{n+1} = \frac{a+bx_{n-1}}{A+Bx_{n-2}}$, *Math. Sci. Res. Hot-Line,* 5(2001), 1-15.

[3] Gibbons C., Kulenovic M., Ladas G., On the reqursive sequence $y_{n+1} = \frac{\alpha+\beta y_{n-1}}{\gamma+y_n}$, *Math. Sci. Res. Hot-Line,* 4(2)(2000), 1-11.

[4] Kocic V.L., Ladas G., *Global Behavior of Nonlinear Difference Equations of Higher Order with Applications,* Kluwer Academic, Dordrecht, 1993.

[5] Kocic V.L., Ladas G., Rodrigues I., On the rational recursive sequences, *J. Math. Anal. Appl.,* 173(1993), 127-157.

[6] El-Owaidy H.M., Ahmed A.M., Mousa M.S., On the recursive sequences $x_{n+1} = \frac{-\alpha x_{n-1}}{\beta\pm x_n}$, *J. Appl. Math. Comput.,* 145(2003), 747-753.

[7] El-Owaidy H.M., Ahmed A.M., Elsady Z., Global attractivity of the recursive sequence $x_{n+1} = \frac{\alpha-\beta x_{n-1}}{\gamma+x_n}$, *J. Appl. Math. Comput.,* 151(2004), 827-833.

[8] Stevic S., On the recursive sequence $x_{n+1} = \frac{ax_{n-2m+1}^p}{b+cx_{n-2k}^{p-1}}$, *J. Appl. Math. Comput.,* 21(2006), 223-232.

[9] El-Owaidy H.M., Ahmed A.M., Youssef A.M., On the dynamics of $x_{n+1} = \frac{bx_{n-1}^2}{A+Bx_{n-2}}$, *Rostock Math Kollog.,* 59(2005), 3-10.

[10] Hamza A.E., El-Sayed M.A., Stability problem of some nonlinear difference equations, *Int. T.Math. and Math. Sci.,* 2(1998), 331-340.

[11] Zhang D.C., Shi B., Gai M.T., A rational recursive sequence, *Computers and Math. with Applications,* 41(2001), 301-306.

[12] Chen D., Li X., Dynamics for nonlinear difference equation $x_{n+1} = \frac{\alpha x_{n-k}}{\beta+\gamma x_{n-l}^p}$, *Advances in Difference Equations,* 2009, Article ID 235691.

[13] Hamza A.E., Khalaf-Allah R., Global behavior of higher order difference equation, *J. Math. Stat.,* 3(1)(2007), 17-20.

[14] Hamza A.E., Khalaf-Allah R., On the recursive sequence $x_{n+1} = A\times \prod_{i=1}^{k} x_{n-2i-1}/(B+C\prod_{i=1}^{k-1} x_{n-2i})$, *Comput. Math. Appl.,* 56(2008), 1726-1731.

[15] El-Metwally H., Qualitative properties of some higher order difference equations, *Comput. Math. Appl.,* 58(2009), 686-692.

[16] Yalcinkaya I., Cinar C., On the dynamics of the difference equation $x_{n+1} = (ax_{n-k})\,/\,(b+cx_n^p)$, *Fasc. Math.,* 42(2009), 141-148.

[17] Dehghan M., Rastegar N., On the global behavior of a high-order rational difference equation, *Comput. Phys. Comm.,* 180(6)(2009), 873-878.

[18] Battaloglu N., Cinar C., Yalcinkaya I., The dynamics of the difference equation $x_{n+1} = (\alpha x_{n-k})\ /\ (\beta+\gamma x_{n-(k+1)}^p)$, *ARS Combinatoria,* 97(2010), 281-288.

[19] Lİ X., Zhu D., Global asymptotic stability for two recursive difference equations, *Appl. Math. Comput.,* 150(2)(2004), 481-492.

[20] Yang X., Chen B., Megson G.M., Evans D.J., Global attactivity in a recursive sequence, *Appl. Math. Comput.,* 158(2004), 667-682.

[21] Erdogan M.E., Cinar C., Yalcinkaya I., On the dynamics of the recursive

Sentence $x_{n+1} = \alpha x_{n-1}/\beta + \gamma x_{n-2}^2 x_{n-4} + \gamma x_{n-2} x_{n-4}^2$, *Comput. Math. Appl.*, 61(2011), 533-537.

[22] Kulenovic M.R.S., Merino O., *Discrete Dynamical Systems and Difference Equations with Mathematica,* Chapman & Hall / CRC Press, Boca Raton / London, 2002.

[23] Erdogan M.E., Cinar C., Yalcinkaya I., On the dynamics of the recursive Sequence $x_{n+1} = \alpha x_{n-1}/\beta + \gamma \sum_{k=1}^{t} x_{n-2k} \prod_{k=1}^{t} x_{n-2k}$, *Math. Comput. Model.,* 54(2011) 1481-1485.

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