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Involute Curves Of Timelike Biharmonic Reeb Curves $(LCS)_3$ - Manifolds

Talat Körpinar*, Essin Turhan† and Vedat Asil ‡

Fırat University, Department of Mathematics 23119, Elazığ, Turkey

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Abstract: In this paper, we study involute timelike biharmonic Reeb curves in $(LCS)_3$-manifold. We characterize curvatures of timelike biharmonic Reeb curves in $(LCS)_3$-manifold. We obtain parametric equation involute curves of the timelike biharmonic Reeb curves in $(LCS)_3$-manifold.

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1. Introduction

A smooth map $\phi : N \longrightarrow M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathcal{T}(\phi)|^2 \, dv_h,$$

where $\mathcal{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler–Lagrange equation of the bienergy is given by $\mathcal{T}_2(\phi) = 0$. Here the section $\mathcal{T}_2(\phi)$ is defined by

$$\mathcal{T}_2(\phi) = -\Delta_\phi \mathcal{T}(\phi) + \mathrm{tr} R(\mathcal{T}(\phi), d\phi) \, d\phi, \tag{1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study timelike biharmonic Reeb curves in $(LCS)_3$-manifold. We characterize curvatures of timelike biharmonic Reeb curves in $(LCS)_3$-manifold. Several

* Email:talatkorpinar@gmail.com
† Email:essin.turhan@gmail.com
‡ Email:vasil@firat.edu.tr

interesting results on a $(LCS)_3$-manifold are obtained in terms of timelike biharmonic Reeb curves. Finally, we obtain parametric equation of the timelike biharmonic Reeb curves in $(LCS)_3$-manifold.

2. (LCS)$_3$-Manifolds

Definition 2.1. In a Lorentzian manifold (M, g) a vector field P defined by

$$g(X, P) = A(X)$$

for any $X \in \chi(M)$ is said to be a concircular vector field if

$$(\nabla_X A)(Y) = \alpha\{g(X, Y) + \omega(X)A(Y)\},$$

where α is a non-zero scalar and ω is a closed 1-form [15].

Let M^3 be a Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

$$g(\xi, \xi) = -1. \tag{2}$$

Since ξ is a unit concircular vector field, it follows that there exists a non-zero 1-form η such that for

$$g(X, \xi) = \eta(X), \tag{3}$$

the equation of the following form holds

$$(\nabla_X \eta)(Y) = \alpha\{g(X, Y) + \eta(X)\eta(Y)\} \ \ (\alpha \neq 0)$$

for all vector fields X, Y, where ∇ denotes the operator of covariant differentiation with respect to the Lorentzian metric g and α is a non-zero scalar function satisfies

$$\nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X),$$

ρ being a certain scalar function given by $\rho = -\,(\xi\alpha)$. If we put

$$\phi X = \frac{1}{\alpha}\nabla_X \xi,$$

then from (2.3) and (2.5) we have

$$\phi X = X + \eta(X)\xi,$$

from which it follows that ϕ is a symmetric (1) tensor and called the structure tensor of the manifold. Thus the Lorentzian manifold M^3 together with the unit timelike concircular vector field ξ, its associated 1-form η and (1) tensor field ϕ is said to be a Lorentzian concircular structure manifold (briefly $(LCS)_3$- manifold) [16]. In a $(LCS)_3$-manifold, the following relations hold [15]:

$$\eta(\xi) = -1,$$
$$\phi\xi = 0,$$
$$\eta(\phi X) = 0, \tag{4}$$
$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{5}$$
$$\eta(R(X,Y)Z) = \left(\alpha^2 - \rho\right)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)], \tag{6}$$
$$S(X,\xi) = (n-1)\left(\alpha^2 - \rho\right)\eta(X), \tag{7}$$
$$R(X,Y)\xi = \left(\alpha^2 - \rho\right)[\eta(Y)X - \eta(X)Y], \tag{8}$$
$$(\nabla_X \phi)(Y) = \alpha[g(X,Y)\xi + 2\eta(X)\eta(Y)\xi + \eta(Y)X], \tag{9}$$

for all vector fields X, Y, Z, where R, S denote respectively the curvature tensor and the Ricci tensor of the manifold.

3. Biharmonic Reeb Curves in the(LCS)$_3$-Manifold

Let γ be a timelike curve on the $(LCS)_3$- manifold parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the 3-dimensional Kenmotsu manifold along γ defined as follows:

T is the unit vector field γ' tangent to γ, **N** is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and **B** is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \tag{10}$$

where κ is the curvature of γ and τ its torsion and

$$g(\mathbf{T},\mathbf{T}) = -1, \; g(\mathbf{N},\mathbf{N}) = 1, \; g(\mathbf{B},\mathbf{B}) = 1, \tag{11}$$
$$g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0.$$

Theorem 3.1. *[13] Let (M, ϕ, ξ, η, g) be an 3-dimensional $(LCS)_3$- manifold and unit vector field X orthogonal to the Reeb vector field ξ. Then,*

$$R(\xi, X)\xi = \left(\alpha^2 - \rho\right)X, \tag{12}$$
$$R(X,\xi)X = -\left(\alpha^2 - \rho\right)\xi. \tag{13}$$

Theorem 3.2. *[13] γ is a timelike biharmonic Reeb curve which are either tangent or normal to the Reeb vector field in $(LCS)_3$- manifold then*

$$\begin{aligned} \kappa &= constant \neq 0, \\ \kappa^2 - \tau^2 &= \alpha^2 - \rho, \\ \tau &= constant. \end{aligned} \tag{14}$$

Proof. Using (12) and Frenet formulas (10), we have (16).

Corollary 3.3. *[13] If γ is a timelike biharmonic Reeb curve which orthogonal to the Reeb vector field ξ in $(LCS)_3$- manifold, then γ is a helix.*

We consider the three-dimensional $(LCS)_3$-manifold

$$\mathbb{M}=\left\{(x,y,z)\in\mathbb{R}^3 : x\neq\pm\sqrt{2}z^2,\ x\neq 0,\ z\neq 0\right\},$$

where (x,y,z) are the standard coordinates in $\mathbb{R}^3$. Let $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ be linearly independent global frame on $\mathbb{M}$ given by [17]

$$\mathbf{e}_1=z\frac{\partial}{\partial x},\quad \mathbf{e}_2=zx\frac{\partial}{\partial y},\quad \mathbf{e}_3=\frac{\partial}{\partial z}. \tag{15}$$

Let g be the Riemannian metric defined by

$$\begin{aligned} g(\mathbf{e}_1,\mathbf{e}_1) &= g(\mathbf{e}_2,\mathbf{e}_2)=1,\ g(\mathbf{e}_3,\mathbf{e}_3)=-1,\\ g(\mathbf{e}_1,\mathbf{e}_2) &= g(\mathbf{e}_2,\mathbf{e}_3)=g(\mathbf{e}_1,\mathbf{e}_3)=0. \end{aligned}$$

Let η be the 1-form defined by

$$\eta(U)=g(U,\mathbf{e}_3)\text{ for any }U\in\chi(\mathbb{M}).$$

Let ϕ be the (1) tensor field defined by

$$\phi(\mathbf{e}_1)=\mathbf{e}_1,\ \phi(\mathbf{e}_2)=\mathbf{e}_2,\ \phi(\mathbf{e}_3)=0.$$

Then using the linearity of ϕ and g we have

$$\eta(\mathbf{e}_3)=-1,$$

$$\phi^2(U)=U-\ \eta(U)\mathbf{e}_3,$$

$$g(\phi U,\phi W)=g(U,W)-\eta(U)\eta(W),$$

for any $U,W\in\chi(\mathbb{M})$.

Let ∇ be the Levi-Civita connection with respect to g. Then, we have

$$[\mathbf{e}_1,\mathbf{e}_2]=\frac{z}{x}\mathbf{e}_2,\quad [\mathbf{e}_1,\mathbf{e}_3]=-\frac{1}{z}\mathbf{e}_1,\quad [\mathbf{e}_2,\mathbf{e}_3]=-\frac{1}{z}\mathbf{e}_2.$$

Taking $\mathbf{e}_3=\xi$ and using the Koszul's formula, we obtain

$$\begin{array}{lll} \nabla_{\mathbf{e}_1}\mathbf{e}_1=-\frac{1}{z}\mathbf{e}_3, & \nabla_{\mathbf{e}_1}\mathbf{e}_2=0, & \nabla_{\mathbf{e}_1}\mathbf{e}_3=-\frac{1}{z}\mathbf{e}_1,\\ \nabla_{\mathbf{e}_2}\mathbf{e}_1=-\frac{z}{x}\mathbf{e}_2, & \nabla_{\mathbf{e}_2}\mathbf{e}_2=\frac{z}{x}\mathbf{e}_1-\frac{1}{z}\mathbf{e}_3, & \nabla_{\mathbf{e}_2}\mathbf{e}_3=-\frac{1}{z}\mathbf{e}_2,\\ \nabla_{\mathbf{e}_3}\mathbf{e}_1=0, & \nabla_{\mathbf{e}_3}\mathbf{e}_2=0, & \nabla_{\mathbf{e}_3}\mathbf{e}_3=0. \end{array} \tag{16}$$

Moreover we put

$$R_{ijk}=R(\mathbf{e}_i,\mathbf{e}_j)\mathbf{e}_k,$$

where the indices i, j, k take the values $1, 2$ and 3.

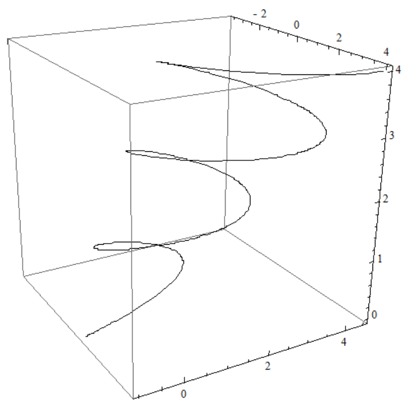
$$R_{122} = -[2(\frac{z}{x})^2 - \frac{1}{z^2}]\mathbf{e}_{1,}, \quad R_{133} = -\frac{2}{z^2}\mathbf{e}_{1,}, \; R_{233} = -\frac{2}{z^2}\mathbf{e}_2.$$

Theorem 3.4. *[13] Let $\gamma : I \longrightarrow \mathbb{M}$ be a unit speed timelike biharmonic Reeb curve which orthogonal to the Reeb vector field ξ in $(LCS)_3$- manifold $\mathbb{M}$. Then, the parametric equations of γ are*

$$\begin{aligned} x(s) &= \frac{\sinh\varphi}{\wp^2}\sqrt{\sinh^2\varphi - 1}\cos(\wp s + \sigma) \\ &+ \frac{1}{\wp}(\sinh\varphi s + c_1)\sqrt{\sinh^2\varphi - 1}\sin(\wp s + \sigma) + c_2, \\ y(s) &= \frac{1}{12\wp^4}\left(\sinh^2\varphi - 1\right)(2\wp^3 s(3c_1^2 + 3\sinh\varphi c_1 s + \sinh^2\varphi s^2) \\ &- 6\wp\sinh\varphi(\sinh\varphi s + c_1)\cos[2(\wp s + \sigma)] \\ &- 3(\wp^2 c_1^2 + 2\wp^2\sinh\varphi c_1 s + \sinh^2\varphi(-1 + \wp^2 s^2))\sin[2(\wp s + \sigma)]) \\ &+ \frac{c_2\sinh\varphi}{\wp^2}\sqrt{\sinh^2\varphi - 1}\sin(\wp s + \sigma) \\ &- \frac{c_2}{\wp}(\sinh\varphi s + c_1)\sqrt{\sinh^2\varphi - 1}\cos(\wp s + \sigma) + c_3, \\ z(s) &= \sinh\varphi s + c_1, \end{aligned} \tag{17}$$

where σ, c_1, c_2, c_3 are constants of integration.

Using Mathematica in above theorem, we have



4. Involute Curves of Biharmonic Reeb Curves in the(LCS)$_3$-Manifold

Definition 4.1. *Let unit speed timelike curve $\gamma : I \longrightarrow \mathbb{M}$ and the curve $\Theta : I \longrightarrow \mathbb{M}$ be given. For $\forall s \in I$, then the curve Θ is called the involute of the curve γ, if the tangent* *at the point $\gamma(s)$ to the curve γ passes through the tangent at the point $\Theta(s)$ to the curve Θ and*

$$g\left(\mathbf{T}^{*}\left(s\right),\mathbf{T}\left(s\right)\right)=0. \tag{18}$$

Let the Frenet-Serret frames of the curves γ and ζ be $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ and $\{\mathbf{T}^{*},\mathbf{N}^{*},\mathbf{B}^{*}\}$, respectively.

Theorem 4.2. *Let $\gamma : I \longrightarrow \mathbb{M}$ be a unit speed timelike biharmonic Reeb curve which are either tangent or normal to the Reeb vector field in $(LCS)_3$- manifold $\mathbb{M}$, Θ its involute curve.Then, the parametric equations of Θ are*

$$\begin{aligned}
x\left(s\right) &= \frac{\sinh\varphi}{\wp^{2}}\sqrt{\sinh^{2}\varphi-1}\cos\left(\wp s+\sigma\right)\\
&+\frac{1}{\wp}\left(\sinh\varphi s+c_{1}\right)\sqrt{\sinh^{2}\varphi-1}\sin\left(\wp s+\sigma\right)\\
&+\left(C-s\right)\left(\sinh\varphi s+c_{1}\right)\sqrt{\sinh^{2}\varphi-1}\cos\left(\wp s+\sigma\right)+c_{2},\\
y\left(s\right) &= \frac{1}{12\wp^{4}}\left(\sinh^{2}\varphi-1\right)(2\wp^{3}s(3c_{1}^{2}+3\sinh\varphi c_{1}s+\sinh^{2}\varphi s^{2})\\
&-6\wp\sinh\varphi\left(\sinh\varphi s+c_{1}\right)\cos[2\left(\wp s+\sigma\right)]\\
&-3(\wp^{2}c_{1}^{2}+2\wp^{2}\sinh\varphi c_{1}s+\sinh^{2}\varphi(-1+\wp^{2}s^{2}))\sin[2\left(\wp s+\sigma\right)])\\
&+\frac{c_{2}\sinh\varphi}{\wp^{2}}\sqrt{\sinh^{2}\varphi-1}\sin\left(\wp s+\sigma\right)\\
&-\frac{c_{2}}{\wp}\left(\sinh\varphi s+c_{1}\right)\sqrt{\sinh^{2}\varphi-1}\cos\left(\wp s+\sigma\right)\\
&+\left(C-s\right)\left(\sinh\varphi s+c_{1}\right)(\frac{\sinh\varphi}{\wp^{2}}\sqrt{\sinh^{2}\varphi-1}\cos\left(\wp s+\sigma\right)\\
&+\frac{1}{\wp}\left(\sinh\varphi s+c_{1}\right)\sqrt{\sinh^{2}\varphi-1}\sin\left(\wp s+\sigma\right)\\
&+c_{2})\sqrt{\sinh^{2}\varphi-1}\sin\left(\wp s+\sigma\right)+c_{3},\\
z\left(s\right) &= C\sinh\varphi+c_{1},
\end{aligned} \tag{19}$$

where σ, c_1, c_2, c_3 are constants of integration.

Proof. We assume that $\gamma : I \longrightarrow Heis^3$ be a unit speed timelike biharmonic curve and Θ its involute curve on Heis^3. We find that the parametric equations of Θ.

The involute curve of timelike biharmonic curve may be given as

$$\Theta(s)=\gamma(s)+u(s)\mathbf{T}\left(s\right). \tag{20}$$

From (4.3), then we have

$$\Theta'(s)=\left(1+u'(s)\right)\mathbf{T}\left(s\right)+u(s)\kappa\left(s\right)\mathbf{N}\left(s\right). \tag{21}$$

Since the curve Θ is involute of the curve γ, $g\left(\mathbf{T}^{*}\left(s\right), \mathbf{T}\left(s\right)\right)=0$. Then, we get

$$1+u'(s)=0 \text{ or } u(s)=C-s, \tag{22}$$

where C is constant of integration.

Substituting (22) into (20), we get

$$\Theta(s)=\gamma(s)+\left(C-s\right)\mathbf{T}\left(s\right). \tag{23}$$

On the other hand, (19) and (23), imply

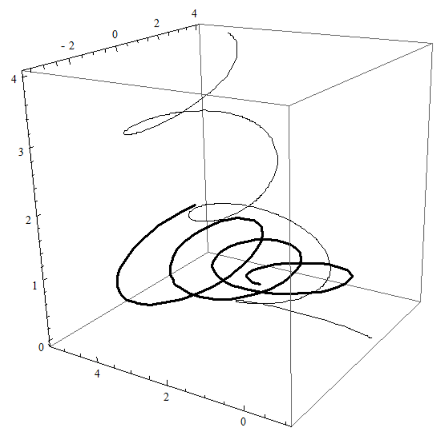
$$\mathbf{T}=\sqrt{\sinh^2\varphi-1}\cos\left(\wp s+\sigma\right)\mathbf{e}_1+\sqrt{\sinh^2\varphi-1}\sin\left(\wp s+\sigma\right)\mathbf{e}_2+\sinh\varphi\mathbf{e}_3. \tag{24}$$

Therefore, from (23) and (24) we have

$$\begin{aligned}\mathbf{T}=&\left(\left(\sinh\varphi s+c_1\right)\sqrt{\sinh^2\varphi-1}\cos\left(\wp s+\sigma\right),\right.\\&\left(\sinh\varphi s+c_1\right)\left(\frac{\sinh\varphi}{\wp^2}\sqrt{\sinh^2\varphi-1}\cos\left(\wp s+\sigma\right)\right.\\&+\frac{1}{\wp}\left(\sinh\varphi s+c_1\right)\sqrt{\sinh^2\varphi-1}\sin\left(\wp s+\sigma\right)\\&\left.\left.+c_2\right)\sqrt{\sinh^2\varphi-1}\sin\left(\wp s+\sigma\right),\sinh\varphi\right).\end{aligned} \tag{25}$$

If we substitute (17) and (25) into (23), we have (19). This concludes the proof of Theorem.

We show that γ and Θ in terms of Mathematica as follows:



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References

[1] K. Arslan, R. Ezentas, C. Murathan, T. Sasahara, *Biharmonic submanifolds 3-dimensional* (κ,μ)*-manifolds,* Internat. J. Math. Math. Sci. 22 (2005), 3575-3586.

[2] D. E. Blair: *Contact Manifolds in Riemannian Geometry*, Lecture Notes in Mathematics, Springer-Verlag 509, Berlin-New York, 1976.

[3] R. Caddeo, S. Montaldo and C. Oniciuc: *Biharmonic submanifolds of* $\mathbb{S}^n$, Israel J. Math., to appear.

[4] R. Caddeo, S. Montaldo and P. Piu: *Biharmonic curves on a surface*, Rend. Mat., to appear.

[5] B. Y. Chen: *Some open problems and conjectures on submanifolds of finite type,* Soochow J. Math. 17 (1991), 169–188.

[6] S. Degla, L. Todjihounde: *Biharmonic Reeb curves in Sasakian manifolds,* arXiv:1008.1903.

[7] J. Eells and L. Lemaire: *A report on harmonic maps,* Bull. London Math. Soc. 10 (1978), 1–68.

[8] J. Eells and J. H. Sampson: *Harmonic mappings of Riemannian manifolds,* Amer. J. Math. 86 (1964), 109–160.

[9] T. Hasanis and T. Vlachos: *Hypersurfaces in* $\mathbb{E}^4$ *with harmonic mean curvature vector field,* Math. Nachr. 172 (1995), 145–169.

[10] G. Y.Jiang: *2-harmonic isometric immersions between Riemannian manifolds,* Chinese Ann. Math. Ser. A 7(2) (1986), 130–144.

[11] G. Y. Jiang: *2-harmonic maps and their first and second variational formulas,* Chinese Ann. Math. Ser. A 7(4) (1986), 389–402.

[12] K. Kenmotsu: *A class of almost contact Riemannian manifolds,* Tohoku Math. J., 24(1972), 93-103.

[13] T. Körpınar, E. Turhan: *Construction of timelike biharmonic reeb curves* $(LCS)_3$*-manifolds,* (submitted).

[14] B. O'Neill: *Semi-Riemannian Geometry*, Academic Press, New York (1983).

[15] A. A. Shaikh: *Some results on* $(LCS)_n$*- manifolds,* J. Korean Math. Soc. 46 (3) (2009), 449-461

[16] A. A. Shaikh: *On Lorentzian almost paracontact manifolds with a structure of the concircular type,* Kyungpook Math. J. 43 (2) (2003), 305-314.

[17] A. A. Shaikh, T. Basu, S. Eyasmin: *On existence of* $\phi-$*recurrent* $(LCS)_n$*- manifolds,* Extracta Mathematicae 23 (1) (2008), 71-83.