Partition Statistics and q-Bell Numbers ($q = -1$)

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Abstract

We study three types of q-Bell numbers that arise as generating functions for some well known statistics on the family of partitions of a finite set, evaluating these numbers when $q = -1$. Among the numbers that arise in this way are (1) Fibonacci numbers and (2) numbers occurring in the study of fermionic oscillators.

1 Introduction

The notational conventions of this paper are as follows: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{P} = \{1, 2, \dots\}$, $[0] = 0$, and $[n] = \{1, \dots, n\}$ for $n \in \mathbb{P}$. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. The letter q denotes an indeterminate, with $0_q := 0$, $n_q := 1 + q + \cdots + q^{n-1}$ for $n \in \mathbb{P}$, $0\underset{q}{!} := 1$, and $n\underset{q}{!} := 1_q 2_q \cdots n_q$ for $n \in \mathbb{P}$. The binomial coefficient $\binom{n}{k}$ is equal to zero if k is a negative integer or if $0 \leqslant n < k$.

Let Δ be a finite set of discrete structures, with $I : \Delta \to \mathbb{N}$. The generating function

$$G(I, \Delta; q) := \sum_{\delta \in \Delta} q^{I(\delta)} = \sum_{k} |\{\delta \in \Delta : \ I(\delta) = k\}| \, q^k \tag{1.1}$$

is an effective tool for studying the statistic I. Elementary examples include the binomial theorem,

$$(1+q)^n = \sum_{S \subset [n]} q^{|S|} = \sum_{k=0}^{n} \binom{n}{k} q^k, \tag{1.2}$$

and

$$n \underset{q}{!} = \sum_{\sigma \in \mathcal{S}_n} q^{i(\sigma)}, \tag{1.3}$$

where $\mathcal{S}_n$ is the set of permutations of $[n]$ and $i(\sigma)$ is the number of inversions in the permutation $\sigma = i_1 i_2 \dots i_n$, i.e., the number of pairs (r, s) with $1 \leqslant r < s \leqslant n$ and $i_r > i_s$ [7, Corollary 1.3.10].

Of course, $G(I, \Delta; 1) = |\Delta|$. On the other hand,

$$G(I, \Delta; -1) = |\{\delta \in \Delta : \ I(\delta) \text{ is even}\}| - |\{\delta \in \Delta : \ I(\delta) \text{ is odd}\}|. \tag{1.4}$$

Hence if $G(I, \Delta; -1) = 0$, the set Δ is "balanced" with respect to the parity of I. In particular, setting $q = -1$ in (1.2) yields the familiar result that a finite nonempty set has as many subsets of odd cardinality as it has subsets of even cardinality. Setting $q = -1$ in (1.3) reveals that if $n \geqslant 2$, then among the permutations of $[n]$ there are as many with an odd number of inversions as there are with an even number of inversions.

A word of explanation of our use of the term "statistic" may be in order. One can always regard Δ as being equipped with the uniform probability distribution. Then I is a statistic, i.e., random variable, in the traditional sense and $G(I, \Delta; q)/G(I, \Delta; 1)$ is the so-called *probability generating function* of I, which is useful in calculating various moments of I. In particular,

$$E(I) = G'(I, \Delta; 1)/G(I, \Delta; 1). \tag{1.5}$$

In this note we consider three q-generalizations of Stirling numbers of the second kind, denoted $S_q^*(n,k)$, $S_q(n,k)$, and $\tilde{S}_q(n,k)$. These polynomials are generating functions for three closely related statistics on the set of partitions of $[n]$ with k blocks. Most of the properties of these q-Stirling numbers, to be established below in § 3, have appeared in the literature in various contexts, Carlitz [1] having apparently been the first to construe these numbers as generating functions for partition statistics. See also [4], [9], [3], and [8]. Our aim here is to offer a compact, unified treatment of these numbers. Interestingly, each of the three types turns out to be suited to elucidating a particular subset of their more-or-less common properties. Our analysis is greatly facilitated by a powerful formal algebraic result of Comtet [2].

We conclude in § 4 with new results on the evaluation of $S_q^*(n,k)$, $S_q(n,k)$, and $\tilde{S}_q(n,k)$ and their associated q-Bell numbers (gotten by summing q-Stirling numbers over k for fixed n) when $q=-1$. Apart from the interpretation of these results in terms of (1.4), the evaluation of $S_{-1}(n,k)$ and its associated Bell numbers may be of additional interest, since these numbers arise in the study of fermionic oscillators [6].

2 Preliminaries

This section reviews some material to be used later in the paper.

2.1. *Comtet Numbers.* The following theorem, due to Comtet [2] greatly facilitates the analysis of many combinatorial arrays:

Theorem 2.1. *Let D be an integral domain. If $(u_n)_{n\geqslant 0}$ is a sequence in D and x is an indeterminate over D, then the following are equivalent characterizations of an array $(U(n,k))_{n,k\geqslant 0}$:*

$$U(n,k) = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i\in\mathbb{N}}} u_0^{d_0}u_1^{d_1}\cdots u_k^{d_k}, \qquad \forall\ n,k\in\mathbb{N}, \tag{2.1}$$

$$\sum_{n\geqslant 0} U(n,k)x^n = \frac{x^k}{(1-u_0x)(1-u_1x)\cdots(1-u_kx)}, \qquad \forall\ k\in\mathbb{N}, \tag{2.2}$$

$$U(n,k) = U(n-1,k-1) + u_kU(n-1,k), \qquad \forall\ n,k\in\mathbb{P}, \tag{2.3}$$

with $U(n,0)=u_0^n$ and $U(0,k)=\delta_{0,k}$ $\forall\ n,k\in\mathbb{N}$, and

$$x^n = \sum_{k=0}^{n} U(n,k)p_k(x), \qquad \forall\ n\in\mathbb{N}, \tag{2.4}$$

where $p_0(x):=1$ and $p_k(x):=(x-u_0)\cdots(x-u_{k-1})$ for $k\in\mathbb{P}$.

Proof. Straightforward algebraic exercise. □

In what follows, we call the numbers $U(n,k)$ the *Comtet numbers associated with the sequence $(u_n)_{n\geqslant 0}$*.

2.2 *Partitions of a Set.* A *partition* of a set S is a set of nonempty, pairwise disjoint subsets (called *blocks*) of S, with union S. For all $n, k \in \mathbb{N}$, let $S(n, k) :=$ the number of partitions of $[n]$ with k blocks. Then $S(0, 0) = 1$, $S(n, 0) = S(0, k) = 0$, $\forall\ n, k \in \mathbb{P}$, and

$$S(n, k) = S(n-1, k-1) + kS(n-1, k), \qquad \forall\ n, k \in \mathbb{P}, \tag{2.5}$$

$S(n-1, k-1)$ enumerating those partitions in which n is the sole element of one of the blocks, and $kS(n-1, k)$ those in which the block containing n contains at least one other element of $[n]$. From (2.5) it follows that the numbers $S(n, k)$, called *Stirling numbers of the second kind*, are the Comtet numbers associated with the sequence $(0, 1, 2, \dots)$. Hence by Theorem 2.1

$$S(n, k) = \sum_{\substack{d_1 + \cdots + d_k = n-k \\ d_i \in \mathbb{N}}} 1^{d_1} 2^{d_2} \cdots k^{d_k}, \qquad \forall\ n, k \in \mathbb{N}, \tag{2.6}$$

$$\sum_{n \geqslant 0} S(n, k) x^n = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)}, \qquad \forall\ k \in \mathbb{N}, \tag{2.7}$$

and

$$x^n = \sum_{k=0}^{n} S(n, k) x^{\underline{k}}, \qquad \forall\ n \in \mathbb{N}, \tag{2.8}$$

where $x^{\underline{0}} := 1$ and $x^{\underline{k}} := x(x-1)\cdots(x-k+1)$ for $k \in \mathbb{P}$.

The total number of partitions of $[n]$ is given by the *Bell number* B_n, where

$$B_n = \sum_{k=0}^{n} S(n, k). \tag{2.9}$$

Clearly, $B_0 = 1$, and

$$B_{n+1} = \sum_{k=0}^{n} \binom{n}{k} B_k, \tag{2.10}$$

$\binom{n}{k} B_k$ enumerating those partitions of $[n+1]$ for which the size of the block containing the element $n+1$ is $n-k+1$.

In the next section we consider three statistics on the set of partitions of $[n]$ with k blocks, and analyze their associated generating functions, each of which furnishes a q-generalization of the Stirling number $S(n, k)$.

2.3 *Restricted Sums of Binomial Coefficients.* As we have already noted in § 1, setting $q = 1$ and $q = -1$ in (1.2) yields the well known result

$$\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}, \qquad \forall\, n \in \mathbb{P}. \tag{2.11}$$

Here we recall, for readers not familiar with it, a method for evaluating sums such as

$$\sum_{k \equiv 0 \pmod 3} \binom{n}{k}. \tag{2.12}$$

Let ω be either of the two complex cube roots of 1, e.g., $\omega = (-1 + i\sqrt{3})/2$. Then

$$\begin{aligned}(1+x)^n + (1+\omega x)^n + (1+\omega^2 x)^n &= \sum_{k=0}^{n} \binom{n}{k} x^k \left(1 + \omega^k + \omega^{2k}\right) \\ &= 3 \sum_{k \equiv 0 \pmod 3} \binom{n}{k} x^k, \qquad (2.13)\end{aligned}$$

since $k \equiv 0 \pmod 3 \Rightarrow 1 + \omega^k + \omega^{2k} = 3$ and $k \equiv 1 \text{ or } 2 \pmod 3 \Rightarrow 1 + \omega^k + \omega^{2k} = 1 + \omega + \omega^2 = 0$. Setting $x = 1$ in (2.13) yields

$$\sum_{k \equiv 0 \pmod 3} \binom{n}{k} = \frac{1}{3}\left(2^n + (1+\omega)^n + (1+\omega^2)^n\right). \tag{2.14}$$

3 Partition Statistics and q-Stirling Numbers

Let $\Pi(n,k)$ denote the set of all partitions of $[n]$ with k blocks. Given a partition $\pi \in \Pi(n,k)$, let $(E_1, \ldots, E_k)$ be the unique *ordered* partition of $[n]$ comprising the same blocks as π, arranged in increasing order of their smallest elements, and define statistics w^*, w, and $\tilde{w}$ by

$$w^*(\pi) := \sum_{i=1}^{k} i|E_i|, \tag{3.1}$$

$$w(\pi) := \sum_{i=1}^{k} (i-1)|E_i| = w^*(\pi) - n, \tag{3.2}$$

and

$$\tilde{w}(\pi) := \sum_{i=1}^{k}(i-1)(|E_i|-1) = w^*(\pi) - n - \binom{k}{2}. \tag{3.3}$$

If elements of $[n]$ are regarded as labels on n unit masses, then $w^*(\pi)$ is the moment about $x = 0$ of the mass configuration in which the masses with labels in E_i are placed at $x = i$. The statistics $w(\pi)$ and $\tilde{w}(\pi)$ admit of similar interpretations.

We wish to study the generating functions

$$S_q^*(n,k) := \sum_{\pi \in \Pi(n,k)} q^{w^*(\pi)}, \tag{3.4}$$

$$S_q(n,k) := \sum_{\pi \in \Pi(n,k)} q^{w(\pi)} = q^{-n} S_q^*(n,k), \tag{3.5}$$

and

$$\tilde{S}_q(n,k) := \sum_{\pi \in \Pi(n,k)} q^{\tilde{w}(\pi)} = q^{-n-\binom{k}{2}} S_q^*(n,k). \tag{3.6}$$

Each of these polynomials furnishes a q-generalization of $S(n,k)$, reducing to the latter when $q = 1$. As closely related as these q-Stirling numbers appear to be, it might be thought that one could carry out an analysis of any one of them, chosen arbitrarily, with properties of the others derived as easy corollaries. Interestingly, it turns out that each is best suited to elucidating a particular subset of their more-or-less common properties. We consider first the matter of recursive formulas.

Theorem 3.1. *The q-Stirling numbers $S_q^*(n,k)$ are generated by the recurrence relation*

$$S_q^*(n,k) = q^k S_q^*(n-1,k-1) + qk_q S_q^*(n-1,k), \qquad \forall\ n,k \in \mathbb{P}, \tag{3.7}$$

with $S_q^*(0,0) = 1$ and $S_q^*(n,0) = S_q^*(0,k) = 0$, $\forall\ n,k \in \mathbb{P}$.

Proof. The boundary conditions are obvious. To establish the recurrence (3.7), let

$$c(n,k,t) := |\{\pi \in \Pi(n,k) :\ w^*(\pi) = t\}|\,.$$

Then,

$$c(n,k,t) = c(n-1,k-1,t-k) + \sum_{i=1}^{k} c(n-1,k,t-i), \qquad \forall\, n,k \in \mathbb{P}. \quad (3.8)$$

For if $w^*(\pi) = t$, with $(E_1, \ldots, E_k)$ being the ordered partition associated with π, then the number $n \in [n]$ is either (i) in E_k alone (there are clearly $c(n-1,k-1,t-k)$ such π's) or (ii) in some E_i, where $1 \leqslant i \leqslant k$, with at least one element of $[n-1]$ (there are clearly $c(n-1,k,t-i)$ such π's). From (3.8) it follows that

$$\begin{aligned} S_q^*(n,k) &= \sum_t c(n,k,t)q^t \\ &= \sum_r c(n-1,k-1,r)q^{r+k} + \sum_{i=1}^{k} q^i \sum_r c(n-1,k,r)q^r \\ &= q^k S_q^*(n-1,k-1) + qk_q S_q^*(n-1,k). \end{aligned}$$

□

Recurrence relations for $S_q(n,k)$ and $\tilde{S}_q(n,k)$ follow immediately from (3.7), along with (3.5) and (3.6), respectively. We have

$$S_q(n,k) = q^{k-1} S_q(n-1,k-1) + k_q S_q(n-1,k), \qquad \forall\, n,k \in \mathbb{P}, \quad (3.9)$$

and

$$\tilde{S}_q(n,k) = \tilde{S}_q(n-1,k-1) + k_q \tilde{S}_q(n-1,k), \qquad \forall\, n,k \in \mathbb{P}. \quad (3.10)$$

By (3.10), the numbers $\tilde{S}_q(n,k)$ are the Comtet numbers associated with the sequence $(n_q)_{n \geqslant 0}$. By Theorem 2.1 it follows immediately that

$$\tilde{S}_q(n,k) = \sum_{\substack{d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} (1_q)^{d_1}(2_q)^{d_2}\cdots(k_q)^{d_k}, \qquad \forall\, n,k \in \mathbb{N}, \quad (3.11)$$

$$\sum_{n \geqslant 0} \tilde{S}_q(n,k)x^n = \frac{x^k}{(1-1_q x)(1-2_q x)\cdots(1-k_q x)}, \qquad \forall\, k \in \mathbb{N}, \quad (3.12)$$

and

$$x^n = \sum_{k=0}^{n} \tilde{S}_q(n,k)\phi_k(x), \qquad \forall\ n \in \mathbb{N}, \tag{3.13}$$

where $\phi_0(x) := 1$ and $\phi_k(x) := x(x-1_q)\cdots(x-(k-1)_q)$, $\forall\ k \in \mathbb{P}$.

Variants of (3.11)–(3.13) that hold for $S_q(n,k)$ and $S_q^*(n,k)$ follow immediately from the relations $S_q(n,k) = q^{\binom{k}{2}}\tilde{S}_q(n,k)$ and $S_q^*(n,k) = q^n S_q(n,k)$. To cite a few examples, we have

$$\sum_{n\geqslant 0} S^*(n,k)x^n = \frac{q^{\binom{k+1}{2}}x^k}{(1-qx)(1-qx-q^2x)\cdots(1-qx-\cdots-q^kx)}, \qquad \forall\ k \in \mathbb{N}, \tag{3.14}$$

and

$$x^n = \sum_{k=0}^{n} S_q(n,k)\psi_k(x) = \sum_{k=0}^{n} S_q^*(n,k)\psi_k\left(\frac{x}{q}\right), \tag{3.15}$$

where $\psi_k(x) := q^{-\binom{k}{2}}\phi_k(x)$.

Using the method of linear functionals [5, pp. 89–90] one can derive from (3.15) the recurrence [8, Theorem 5.4]

$$S_q(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} q^j S_q(j,k-1), \qquad \forall\ n \in \mathbb{N},\ k \in \mathbb{P}, \tag{3.16}$$

from which the variant recurrences

$$S_q^*(n+1,k) = q^{n+1}\sum_{j=0}^{n} \binom{n}{j} S_q^*(j,k-1), \qquad \forall\ n \in \mathbb{N},\ k \in \mathbb{P}, \tag{3.17}$$

and

$$\tilde{S}_q(n+1,k) = \sum_{j=0}^{n} \binom{n}{j} q^{j-k+1}\tilde{S}_q(j,k-1), \qquad \forall\ n \in \mathbb{N},\ k \in \mathbb{P} \tag{3.18}$$

follow immediately.

Summing the q-Stirling numbers $S_q^*(n,k)$, $S_q(n,k)$ and $\tilde{S}_q(n,k)$ over k yields the respective q-Bell numbers $B_q^*(n)$, $B_q(n)$, and $\tilde{B}_q(n)$. From (3.16) it follows that

$$B_q(n+1) = \sum_{j=0}^{n} \binom{n}{j} q^j B_q(j), \qquad \forall\ n \in \mathbb{N}. \tag{3.19}$$

Since $B_q^*(n) = q^n B_q(n)$, the recurrence (3.19) yields

$$B_q^*(n+1) = q^{n+1} \sum_{j=0}^{n} \binom{n}{j} B_q^*(j), \qquad \forall\ n \in \mathbb{N}. \tag{3.20}$$

Due to the factor q^{-k} in (3.18), we do not get any recurrence for $\tilde{B}_q(n)$ analogous to (3.19) and (3.20), this being the single exception to the general parallelism between properties of the three q-Stirling numbers under consideration. The uniqueness of $\tilde{B}_q(n)$ is further manifested when $q=-1$, as we shall see in the next section.

4 The Case $q = -1$

In this section we derive simple expressions for the foregoing q-Stirling and q-Bell numbers when $q=-1$.

Theorem 4.1. *The number $\tilde{S}_{-1}(n,k)$ is given by the formula*

$$\tilde{S}_{-1}(n,k) = \binom{n - \lfloor \frac{k}{2} \rfloor - 1}{\lfloor \frac{k-1}{2} \rfloor}, \qquad 1 \leqslant k \leqslant n. \tag{4.1}$$

Proof. Note that

$$i_q\big|_{q=-1} = \omega_i := \begin{cases} 1, & \text{if } i \text{ is odd;} \\ 0, & \text{if } i \text{ is even.} \end{cases} \tag{4.2}$$

Hence by (3.11), if $1 \leqslant m \leqslant \lfloor n/2 \rfloor$,

$$\tilde{S}_{-1}(n,2m) = \sum_{\substack{d_1+d_3+\cdots+d_{2m-1}=n-2m \\ d_i \in \mathbb{N}}} 1 = \binom{n-m-1}{m-1}, \tag{4.3}$$

since the number of sequences $(t_1, \dots, t_m)$ of nonnegative integers summing to s is $\binom{s+m-1}{m-1}$ [7, p. 15]. Similarly, if $0 \leqslant m \leqslant \lfloor (n-1)/2 \rfloor$,

$$\tilde{S}_{-1}(n, 2m+1) = \binom{n-m-1}{m}. \tag{4.4}$$

Formula (4.1) incorporates (4.3) and (4.4). □

In tabulating the numbers $\tilde{S}_{-1}(n,k)$ it is of course more efficient to use the recurrence

$$\tilde{S}_{-1}(n,k) = \tilde{S}_{-1}(n-1, k-1) + \omega_k \tilde{S}_{-1}(n-1,k), \tag{4.5}$$

representing the case $q = -1$ of (3.10).

Let $F_0 = F_1 = 1$, with $F_n = F_{n-1} + F_{n-2}$ if $n \geqslant 2$. As is well known,

$$F_n = \sum_{m=0}^{\lfloor n/2 \rfloor} \binom{n-m}{m}, \qquad \forall n \in \mathbb{N}. \tag{4.6}$$

Theorem 4.2. *For all $n \in \mathbb{N}$,*

$$\tilde{B}_{-1}(n) := \sum_{k=0}^{n} \tilde{S}_{-1}(n,k) = F_n. \tag{4.7}$$

Proof. It is easy to check that (4.7) holds for $n = 0, 1$. If $n \geqslant 2$, then by (4.3) and (4.4),

$$\begin{aligned}
\tilde{B}_{-1}(n) &= \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \binom{n-m-1}{m} + \sum_{m=1}^{\lfloor n/2 \rfloor} \binom{n-m-1}{m-1} \\
&= \sum_{m=0}^{\lfloor (n-1)/2 \rfloor} \binom{(n-1)-m}{m} + \sum_{m=0}^{\lfloor (n-2)/2 \rfloor} \binom{(n-2)-m}{m} \\
&= F_{n-1} + F_{n-2} = F_n.
\end{aligned}$$

□

From (4.1) and the fact that $S_q^*(n,k) = q^{\binom{k}{2}+n} \tilde{S}_q(n,k)$, we have

$$S_{-1}^*(n,k) = (-1)^{\binom{k}{2}+n} \binom{n - \lfloor \frac{k}{2} \rfloor - 1}{\lfloor \frac{k-1}{2} \rfloor}, \qquad 1 \leqslant k \leqslant n. \tag{4.8}$$

On the other hand, the Bell numbers $B_{-1}^*(n)$ are quite different from the numbers $\tilde{B}_{-1}(n)$.

Theorem 4.3. *For all $n \in \mathbb{N}$,*

$$B^*_{-1}(n) := \sum_{k=0}^{n} S^*_{-1}(n,k) = \begin{cases} 1, & \textit{if } n \equiv 0 \pmod 3; \\ -1, & \textit{if } n \equiv 1 \pmod 3; \\ 0, & \textit{if } n \equiv 2 \pmod 3. \end{cases} \tag{4.9}$$

Proof. Noting that $B^*_{-1}(0) = 1$, we prove (4.9) by course-of-values induction on n. In what follows

$$b_r(n) := \sum_{k \equiv r \pmod 3} \binom{n}{k}. \tag{4.10}$$

From (3.20) with $q = -1$, we have

$$\begin{aligned} B^*_{-1}(n+1) &= (-1)^{n+1} \sum_{j=0}^{n} \binom{n}{j} B^*_{-1}(j) = (-1)^{n+1} b_0(n) + (-1)^n b_1(n) \\ &= (-1)^{n+1} b_0(n) + (-1)^n b_0(n-1) + (-1)^n b_1(n-1). \end{aligned} \tag{4.11}$$

Similarly, $B^*_{-1}(n) = (-1)^n b_0(n-1) + (-1)^{n-1} b_1(n-1)$, and so

$$\begin{aligned} B^*_{-1}(n+1) &= (-1)^{n+1} b_0(n) + 2(-1)^n b_0(n-1) - B^*_{-1}(n) \\ &\frac{1}{3}\left[\omega^{2n-1} - \omega^{2n-2} + \omega^{n+1} - \omega^{n-1}\right] - B^*_{-1}(n), \end{aligned} \tag{4.12}$$

by (2.14), where ω is either of the two complex cube roots of 1. Taking $n+1 = 3m$, $3m+1$, and $3m+2$, respectively, in (4.12) yields

$$B^*_{-1}(3m) = 1 - B^*_{-1}(3m-1) = 1, \tag{4.13}$$

$$B^*_{-1}(3m+1) = 0 - B^*_{-1}(3m) = -1, \quad \text{and} \tag{4.14}$$

$$B^*_{-1}(3m+2) = -1 - B^*_{-1}(3m+1) = 0. \tag{4.15}$$

□

It is easy to check that one can write (4.9) more compactly as

$$B^*_{-1}(n) = \frac{1}{1-\omega}\omega^n - \frac{\omega}{1-\omega}\omega^{2n}, \tag{4.16}$$

from which we get the nice exponential generating function

$$\sum_{n=0}^{\infty} B^*_{-1}(n)\frac{x^n}{n!} = \frac{1}{1-\omega}e^{\omega x} - \frac{\omega}{1-\omega}e^{\omega^2 x}. \tag{4.17}$$

From (4.1) and the fact that $S_q(n,k) = q^{\binom{k}{2}}\tilde{S}_q(n,k)$, we have

$$S_{-1}(n,k) = (-1)^{\binom{k}{2}}\binom{n-\lfloor\frac{k}{2}\rfloor-1}{\lfloor\frac{k-1}{2}\rfloor}, \qquad 1 \leqslant k \leqslant n. \tag{4.18}$$

By (3.5),

$$B_{-1}(n) := \sum_{k=0}^{n} S_{-1}(n,k) = (-1)^n B^*_{-1}(n),$$

and so by (4.9)

$$B_{-1}(n) = \begin{cases} (-1)^n, & \text{if } n \equiv 0 \pmod 3; \\ (-1)^{n+1}, & \text{if } n \equiv 1 \pmod 3; \\ 0, & \text{if } n \equiv 2 \pmod 3, \end{cases} \tag{4.19}$$

and by (4.17)

$$\sum_{n=0}^{\infty} B_{-1}(n)\frac{x^n}{n!} = \frac{1}{1-\omega}e^{-\omega x} - \frac{\omega}{1-\omega}e^{-\omega^2 x}. \tag{4.20}$$

Formulas (4.19) and (4.20) furnish answers to questions posed by Schork [6] in a paper which showed how the numbers $S_{-1}(n,k)$ and $B_{-1}(n)$ arise in the study of fermionic oscillators.

To conclude this section we remark that the list of partition statistics (3.1)–(3.3) might have been rounded out to include the statistic

$$\hat{w}(\pi) := \sum_{i=1}^{k} i(|E_i|-1) = \tilde{w}(\pi) + n - k, \tag{4.21}$$

with generating function

$$\hat{S}_q(n,k) := \sum_{\pi\in\Pi(n,k)} q^{\hat{w}(\pi)} = q^{(n-k)}\tilde{S}_q(n,k). \tag{4.22}$$

Formula (4.22) and Theorem 4.1 yield an easy evaluation of $\hat{S}_{-1}(n,k)$. As for

$$\hat{B}_{-1}(n) := \sum_{k=0}^{n} \tilde{S}_{-1}(n,k), \tag{4.23}$$

we have $\hat{B}_{-1}(0) = \hat{B}_{-1}(1) = 1$, $\hat{B}_{-1}(2) = 0$, and

$$\hat{B}_{-1}(n) = (-1)^{n-1} F_{n-3}, \qquad \forall n \geqslant 3, \tag{4.24}$$

the proof of which we leave to interested readers.

5 Bijective Proofs

We conclude by returning to the opening theme of this paper. If $G(I,\Delta;-1) = 0$, then, as already noted, $|\Delta_0| = |\Delta_1|$, where $\Delta_i = \{\delta \in \Delta : I(\delta) \equiv i \pmod 2\}$. In such a case it is enlightening, if sometimes difficult, to exhibit a bijection from Δ_0 to Δ_1. A familiar example, related to (1.2), is the map

$$S \mapsto \begin{cases} S \cup \{1\}, & \text{if } 1 \notin S; \\ S - \{1\}, & \text{if } 1 \in S, \end{cases}$$

which is a bijection from the family of subsets of $[n]$ with even cardinality to those with odd cardinality.

If $G(I,\Delta;-1) \neq 0$, then a similar task arises. If, for example, $G(I,\Delta;-1) = c > 0$, this equation can be rendered more salient by identifying a subset Δ^* of Δ_0 having cardinality c, along with a bijection from $\Delta_0 - \Delta^*$ to Δ_1. The results in section 4 above present many interesting problems of this type, perhaps the most intriguing of which is posed by the equation $\tilde{B}_{-1}(n) = F_n$.

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