

ROSE-  
HULMAN  
UNDERGRADUATE  
MATHEMATICS  
JOURNAL

DOMINANCE OVER  $\mathbb{N}$

Tyler Ball<sup>a</sup>      Daniel Juda<sup>b</sup>

VOLUME 14, NO. 2, FALL 2013

Sponsored by

Rose-Hulman Institute of Technology

Department of Mathematics

Terre Haute, IN 47803

Email: [mathjournal@rose-hulman.edu](mailto:mathjournal@rose-hulman.edu)

<http://www.rose-hulman.edu/mathjournal>

---

<sup>a</sup>Pacific Lutheran University

<sup>b</sup>University of Arkansas

## DOMINANCE OVER $\mathbb{N}$

Tyler Ball

Daniel Juda

**Abstract.** This paper provides an overview of the  $b$ -dominance order over the natural numbers,  $\mathbb{N}$ , using the base  $b$  expansion of natural numbers. The  $b$ -dominance order is an accessible partially-ordered set that is less complex than the divisor relation but more complex than  $\leq$ ; thus, it supplies a good medium through which an undergraduate can be exposed to the subject of order theory. Here we discuss many ideas in order theory, including the Poincaré polynomial and the Möbius function.

---

**Acknowledgements:** The authors thank the M.J. Murdock Charitable Trust and the Pacific Lutheran University Division of Natural Sciences for their generous support. They would also like to extend their thanks to Dr. Tom Edgar for the project idea and all his help throughout their summer program.

# 1 Introduction

Partial orders and partially ordered sets have proven to be very useful combinatorial tools in many areas of mathematics. Methods from order theory have been employed heavily in set theory, number theory, the study of Lie groups and Lie algebras, and the list continues. We take this opportunity to introduce the reader to the world of partially ordered sets (posets for short) by studying a family of overlooked partial orders on the set of natural numbers,  $\mathbb{N}$ . The reader interested in partial orders in general should consult [4].

The relation on  $\mathbb{N}$  known as  $b$ -dominance provides an interesting case study for the subject of partially-ordered sets. We begin by formally defining the idea of the base  $b$  expansion of a natural number and some properties of this expansion. We then proceed to describe elementary characteristics of  $b$ -dominance as a partially-ordered set, such as its structure as a lattice and formulae for the greatest lower bound and least upper bound. Later we will further illustrate why this poset provides a good case study for order theory by providing a formula for both the Poincaré polynomial and the Möbius function associated with  $b$ -dominance.

Although it is possible to think of  $b$ -dominance simply as a subposet of an infinite product of chains, many of the proofs we present here do not use this characterization. The proofs we have included were chosen because of the interesting connections they provide to various topics in combinatorics and number theory. Thus although the proofs are less traditional, they provide additional insight into the structure of the  $b$ -dominance order and better highlight some of the properties of posets that are interesting to study.

In Section 2, we familiarize the reader with the key background information about the base  $b$  expansion of a number which will be used throughout the paper. In Section 3 we introduce the  $b$ -dominance relation and prove both that it is a lattice. Further, we discuss the rank function for our poset and provide some examples which aid in the understanding of the proven results. In Sections 4 and 5, we give formulas for the Poincaré polynomial and the Möbius function. In Section 6 we introduce a formal definition of what a carry is when adding two numbers and describe a connection between this idea and  $b$ -dominance. Finally, Section 7 is a list of possible future investigations regarding the  $b$ -dominance relation using some of the results proven throughout the paper. As a final note, this paper provides the background necessary for the reader to understand the results presented in [2].

## 2 Base $b$ Expansion of $\mathbb{N}$

Let  $b \in \mathbb{N}$  with  $b \geq 2$ . It is known that every number has a unique base  $b$  expansion; for the purposes of this paper we will formalize this here. We will also present a simple method of determining the  $i$ th digit of the base  $b$  expansion for any  $n \in \mathbb{N}$  along with some other useful characteristics of base  $b$  expansions. Let  $A_b = \{0, \dots, b-1\}$  and

$$S_b = \{(a_0, a_1, a_2, \dots) \mid a_i \in A_b \text{ and } a_i = 0 \text{ for all but finitely many } i\}.$$

It is clear then that there is a natural bijection between  $S_b$  and  $\mathbb{N}$  given by

$$(a_0, a_1, a_2, \dots) \longleftrightarrow \sum_{i=0}^{\infty} a_i b^i. \quad (1)$$

For  $n \in \mathbb{N}$ , let  $n_{(b)} \in S_b$  be the base  $b$  expansion given by the bijection in equation (1). Furthermore, we define the  $i$ th digit of the base  $b$  expansion of  $n$  where  $n_{(b)} = (n_0, n_1, \dots, n_k)$  by  $n_b(i) := n_i$ . We also find it useful to define a length function of  $n$  in base  $b$  by  $\text{len}_b(n) := k$  where  $k$  is index of the last non-zero entry of  $n$ , i.e.  $k = \max\{i \mid n_b(i) \neq 0\}$ .

Before discussing the method for determining  $n_b(i)$  as promised above, we first introduce some simple results useful in the proof of the known formula.

**Lemma 2.1.** *Let  $b \in \mathbb{N}$  with  $b \geq 2$ . Then  $\sum_{i=1}^{\infty} \frac{b-1}{b^i} = 1$ .*

*Proof.* This is a geometric series. □

**Corollary 2.2.** *Let  $n_0, \dots, n_{l-1} \in A_b$ . Then  $\frac{n_0}{b^l} + \frac{n_1}{b^{l-1}} + \dots + \frac{n_{l-1}}{b} < 1$ .*

*Proof.* Notice that each  $n_i \leq (b-1)$ . Thus,  $\frac{n_0}{b^l} + \frac{n_1}{b^{l-1}} + \dots + \frac{n_{l-1}}{b} < \sum_{i=1}^{\infty} \frac{b-1}{b^i} = 1$ . □

Recall that if  $a \in \mathbb{Z}$  and  $r \in \mathbb{R}$  with  $0 \leq r < 1$ , then  $[r+a] = a$ . Now we introduce the formula for determining the  $i$ th coefficient of the base  $b$  expansion of  $n$ .

**Proposition 2.3.** *Let  $b, n \in \mathbb{N}$  with  $b \geq 2$ . Then  $n_b(i) \equiv \left\lfloor \frac{n}{b^i} \right\rfloor \pmod{b}$  for all  $i \in \mathbb{N}$ .*

*Proof.* Let  $b, n \in \mathbb{N}$  with  $b \geq 2$ . Let  $i \in \mathbb{N}$  such that  $0 \leq i \leq \text{len}_b(n) =: k$ . Define  $q = n_{i+1} + n_{i+2}b + n_{i+3}b^2 + \dots + n_k b^{k-i-1} \in \mathbb{Z}$ . Then,  $bq = n_{i+1}b + n_{i+2}b^2 + \dots + n_k b^{k-i} + n_i - n_i$ . By Corollary 2.2, we know  $\frac{n_0}{b^i} + \frac{n_1}{b^{i-1}} + \frac{n_2}{b^{i-2}} + \dots + \frac{n_{i-1}}{b} < 1$ , and by direct calculation  $\left\lfloor \frac{n_0}{b^i} + \frac{n_1}{b^{i-1}} + \dots + \frac{n_{i-1}}{b} + n_i + n_{i+1}b + \dots + n_k b^{k-i} \right\rfloor - n_i = bq$ . Then,

$$\left\lfloor \frac{1}{b^i} (n_0 + n_1 b + n_2 b^2 + \dots + n_k b^k) \right\rfloor - n_b(i) = bq$$

and so  $\left\lfloor \frac{n}{b^i} \right\rfloor - n_b(i) = bq$ . Therefore,  $n_b(i) \equiv \left\lfloor \frac{n}{b^i} \right\rfloor \pmod{b}$  for all  $i$ . □

Let  $n = 33$ . Using the formula described in Proposition 2.3 we have,

$$33_3(3) \equiv \left\lfloor \frac{33}{3^3} \right\rfloor \equiv [1.222] \equiv 1 \equiv 1 \pmod{3}$$

Thus,  $33_3(3) = 1$ .

To determine the full sequence  $n_{(b)}$  for any number  $n$ , we can continuously divide by  $b$  and take the remainder at each step as the coefficient of the base  $b$  expansion of  $n$ . So for 33, we have

$$\begin{aligned} 33 &= 3(11) + 0, \\ 11 &= 3(3) + 2, \\ 3 &= 3(1) + 0, \\ 1 &= 3(0) + 1. \end{aligned}$$

Thus  $33_{(3)} = (0, 2, 0, 1)$ .

It is also possible, given the base  $b$  expansion of  $n$ , to determine the base  $b^\alpha$  expansion for  $\alpha \in \mathbb{N}$  using the following result.

**Theorem 2.4.** *Let  $b, n \in \mathbb{N}$  with  $b \geq 2$ . Then*

$$n_{b^\alpha}(j) = \sum_{i=0}^{\alpha-1} n_b(j\alpha + i)b^i,$$

for all  $\alpha, j \in \mathbb{N}$ .

*Proof.* From Proposition 2.3, we know that  $n_{b^\alpha}(j) \equiv \left\lfloor \frac{n}{b^{j\alpha}} \right\rfloor \pmod{b^\alpha}$  for all  $j$  and  $\alpha$ . For the purposes of this proof we will use  $n_i$  in place of  $n_{b^\alpha}(i)$ . Now

$$\begin{aligned} \left\lfloor \frac{n}{b^{j\alpha}} \right\rfloor &= \left\lfloor \frac{n_0 + n_1b + \dots + n_k b^k}{b^{j\alpha}} \right\rfloor \\ &= (n_{j\alpha} + n_{j\alpha+1}b + n_{j\alpha+2}b^2 + \dots + n_k b^{k-j\alpha}) \\ &\equiv n_{j\alpha} + n_{j\alpha+1}b + n_{j\alpha+2}b^2 + \dots + n_{j\alpha+\alpha-1}b^{\alpha-1} \pmod{b^\alpha}. \end{aligned}$$

Therefore since  $n_{j\alpha} + n_{j\alpha+1}b + n_{j\alpha+2}b^2 + \dots + n_{j\alpha+\alpha-1}b^{\alpha-1} = \sum_{i=0}^{\alpha-1} n_b(j\alpha + i)b^i$ , we have  $n_{b^\alpha}(j) \equiv$

$\sum_{i=0}^{\alpha-1} n_b(j\alpha + i)b^i \pmod{b^\alpha}$ . However, since both of these terms are less than  $b^\alpha$ , it follows that

$$n_{b^\alpha}(j) = \sum_{i=0}^{\alpha-1} n_b(j\alpha + i)b^i. \quad \square$$

Let  $b = 3$ ,  $n = 161$ , and  $\alpha = 2$ . Then  $161_{(3)} = (2, 2, 2, 2, 1)$ . Suppose we want to find  $161_{3^2}(2)$ . Applying the formula from Theorem 2.4, we have

$$161_{3^2}(2) = \sum_{i=0}^1 161_3(2(2) + i)3^i = 161_3(4)3^0 + 161_3(5)3^1 = 1(3^0) + 0(3^1) = 1.$$

Thus  $161_9(2) = 1$ . The full sequence for 161 in base 9 is given by  $161_{(9)} = (8, 8, 1)$ .

Another property of the base  $b$  expansion that we find useful is the well known sum-of-digits function defined by  $\text{sum}_b(n) := \sum_{i=0}^{\text{len}_b(n)} n_b(i)$ . This function plays an important role in the combinatorics of the partial order we intend to discuss in the next section.

### 3 $b$ -dominance

When attempting to order the natural numbers, it is most natural to consider the relation  $\leq$ . Here we present a different method of ordering  $\mathbb{N}$  called  $b$ -dominance, denoted  $\ll_b$ , which depends on the base  $b$  expansion.

**Definition 3.1.** Let  $b, n, m \in \mathbb{N}$  with  $b \geq 2$ . We say  $n \ll_b m$  if and only if  $n_b(i) \leq m_b(i)$  for all  $i$ . In this case, we will say either  $m$   $b$ -dominates  $n$  or  $n$  is  $b$ -dominated by  $m$ .

**Example 3.2.** Let  $m = 104922$  and  $n = 103873$ . Then,  $103873 \ll_8 104922$  since  $103873_{(8)} = (1, 0, 7, 2, 1, 3)$  and  $104922_{(8)} = (2, 3, 7, 4, 1, 3)$ . On the other hand,  $103873_{(5)} = (3, 4, 4, 0, 1, 3, 1, 1)$  and  $104922_{(5)} = (2, 4, 1, 4, 2, 3, 1, 1)$ . Since  $0 < 4$  ( $n_5(3) < m_5(3)$ ) and  $3 > 2$  ( $n_5(0) > m_5(0)$ ),  $103873 \not\ll_5 104922$ . See Figure 1 for an example of the Hasse diagram for 5-dominance up to  $n = 24$ .

Recall that when discussing a relation on  $\mathbb{N}$ , we call the relation a partial-order when it satisfies three properties; the relation must be reflexive, anti-symmetric, and transitive. It is clear the  $b$ -dominance relation is a poset. This follows from the fact that  $(A_b, \leq)$  is a total order. For further details, see Chapter 10 in [4].

Each pair of elements in a poset can have a unique least upper bound, a unique greatest lower bound, both, or neither. We call these a supremum, or *join*, and infimum, or *meet* respectively. We say a poset is a *lattice* if and only if any two elements have a join and meet. A poset is called a *complete lattice* if and only if any subset of the poset has a join and a meet [4]. We will prove that  $b$ -dominance does indeed form a lattice. This provides a particularly nice characterization, as not all partially-ordered sets are lattices.

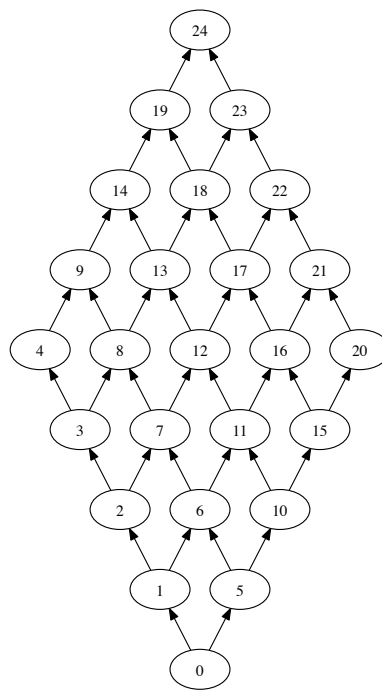


Figure 1: The Hasse diagram of the 5-dominance order up to 24.

**Theorem 3.3.** *For all  $b \in \mathbb{N}$  with  $b \geq 2$ , the partially-ordered set  $(\mathbb{N}, \ll_b)$  is a lattice with meet,  $M$ , and join,  $J$  given by  $M_b(i) = \min\{m_b(i), n_b(i)\}$  and  $J_b(i) = \max\{m_b(i), n_b(i)\}$  for all  $i$ .*

*Proof.* Let  $b \in \mathbb{N}$  with  $b \geq 2$ . Let  $m, n \in \mathbb{N}$ . Define  $M_b$  by  $M_b(i) = \min\{n_b(i), m_b(i)\}$  for all  $i$ , and note  $\text{len}_b(M) = \min\{\text{len}_b(n), \text{len}_b(m)\}$ , so  $M \in \mathbb{N}$ . Then  $M_b(i) \leq n_b(i)$  and  $M_b(i) \leq m_b(i)$  for all  $i$ . So  $M \ll_b n$  and  $M \ll_b m$ . Let  $a \in \mathbb{N}$  such that  $a \ll_b n$  and  $a \ll_b m$ . Then  $a_b(i) \leq \min\{n_b(i), m_b(i)\} = M_b(i)$  for all  $i$ . So  $a \ll_b M$ , and thus  $M$  is the meet for  $n$  and  $m$ . Now define  $J$  by  $J_b(i) = \max\{n_b(i), m_b(i)\}$  for all  $i$ , and note  $\text{len}_b(J) = \max\{\text{len}_b(n), \text{len}_b(m)\}$ , so  $J \in \mathbb{N}$ . Then  $n_b(i) \leq J_b(i)$  and  $m_b(i) \leq J_b(i)$  for all  $i$ . So  $n \ll_b J$  and  $m \ll_b J$ . Let  $c \in \mathbb{N}$  such that  $n \ll_b c$  and  $m \ll_b c$ . Then  $c_b(i) \geq \max\{n_b(i), m_b(i)\} = J_b(i)$  for all  $i$ . Thus  $J \ll_b c$ , and  $J$  is the join for  $n$  and  $m$ . Since we have shown that there is a meet and join for all  $n$  and  $m$ , the partially-ordered set  $(\mathbb{N}, \ll_b)$  forms a lattice.  $\square$

Let  $n = 21987$  and  $m = 52196$ . Then,  $21987_{(7)} = (0, 5, 0, 1, 2, 1)$  and  $52196_{(7)} = (4, 1, 1, 5, 0, 3)$ . Then, applying the formula described in Theorem 3.3,

$$G = (\min\{0, 4\}, \min\{5, 1\}, \min\{0, 1\}, \min\{1, 5\}, \min\{2, 0\}, \min\{1, 3\}) = (0, 1, 0, 1, 0, 1)$$

and

$$L = (\max\{0, 4\}, \max\{5, 1\}, \max\{0, 1\}, \max\{1, 5\}, \max\{2, 0\}, \max\{1, 3\}) = (4, 5, 1, 5, 2, 3).$$

See Figure 2 for an example of a poset with an illustration of a meet and a join. In the figure, 17, colored orange, is the least upper bound of 2 and 16. Similarly, 1, colored red, is the greatest lower bound of 2 and 16.

Although we have shown that  $b$ -dominance forms a lattice, it does not form a complete lattice. For a simple counterexample, consider the subset of all the powers of  $b$ . A join of this subset would be the sequence  $(1, 1, 1, \dots)$ , but the infinite sequence of 1's does not correspond to a natural number.

We also will find it useful, given  $n \in \mathbb{N}$ , to be able to describe its *upper covers* and *lower covers*. Given a poset  $P$ , and  $m, n \in P$ , we say that  $m$  is an upper cover of  $n$ , or  $m$  covers  $n$ , if and only if  $n < m$  and for all  $z \in P$ ,  $n \leq z \leq m$  implies  $z \in \{n, m\}$  ([4]). On the other hand, given  $m, n \in P$  we say that  $n$  is a lower cover or  $m$  if and only if  $n < m$  and for all  $z \in P$ ,  $n \leq z \leq m$  implies  $z \in \{n, m\}$ . Notice these are the elements of the poset which can be thought of as either directly below or directly above  $n$ , that is,  $n_b(i) = m_b(i)$  for all but one coefficient  $i$ ; moreover, in the differing coefficient,  $|m_b(i) - n_b(i)| = 1$ . See Figure 3 for an illustration of the idea of upper and lower covers. In the figure, the lower covers are



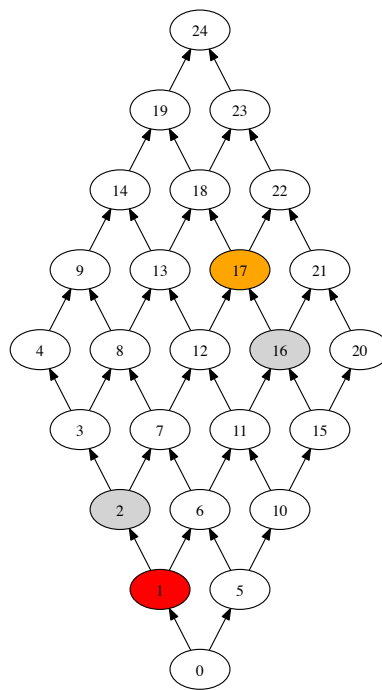


Figure 2: The 5-dominance poset with an example of a meet and join.

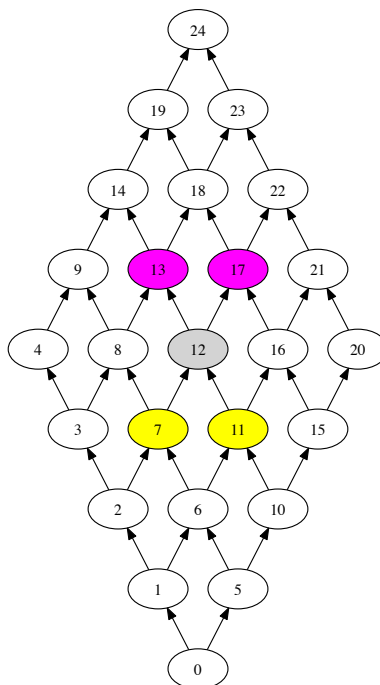


Figure 3: Examples of upper and lower covers in the 5-dominance ordering to 24.

shaded yellow and the upper covers are shaded magenta. In the case of the examples shown,  $13 = 12 + 5^0$ ,  $17 = 12 + 5^1$ ,  $11 = 12 - 5^0$ , and  $7 = 12 - 5^1$ . Thus, as described above, each upper and lower covers' base  $b$  expansion differs in only one  $n_b(i)$  by a magnitude of 1. Formally we define the notion of upper and lower covers as follows.

**Theorem 3.4.** *Let  $n \in \mathbb{N}$ . Then the upper covers of  $n$ , under  $\ll_b$ , are given by the set  $UC_b(n) = \{n + b^c \mid c \in \mathbb{N} \wedge n_b(c) \neq b - 1\}$  and the lower covers of  $n$  are given by the set  $LC_b(n) = \{n - b^c \mid c \in \mathbb{N} \wedge n_b(c) \neq 0\}$ .*

*Proof.* Let  $m, n \in \mathbb{N}$ . Suppose  $m$  covers  $n$ . So  $n \ll_b m$  and  $n \neq m$ . Therefore  $n_b(k) < m_b(k)$  for some  $k$ . Let  $p = m_b(k) - n_b(k) > 0$ . Suppose  $p > 1$ . Let  $z$  be given by  $z_b(i) = n_b(i)$  for all  $i \neq k$ , and  $z_b(k) = n_b(k) + 1$ . Then clearly  $n \ll_b z$ ,  $z \ll_b m$ , and  $z \notin \{n, m\}$ . But this contradicts that  $m$  is an upper cover, so  $p = 1$ . Thus  $m_b(k) = n_b(k) + 1$ . Note, since  $m_b(k) \leq b - 1$ ,  $n_b(k) \leq b - 2$ .

Suppose there exists  $c \in \mathbb{N}$  with  $c \neq k$  where  $n_b(c) \neq m_b(c)$ . Since  $n \ll_b m$ ,  $m_b(c) > n_b(c)$ . Let  $q = m_b(c) - n_b(c) > 0$ . Let  $w$  be given by  $w_b(i) = m_b(i)$  for all  $i \neq c$  and  $w_b(c) = n_b(c)$ . So  $n \ll_b w$ , and  $w \ll_b m$ , and  $w \notin \{m, n\}$ . But this also contradicts that  $m$  is an upper cover,

so  $m_b(c) = n_b(c)$  for all  $c \neq k$ . Therefore, we have shown that  $m = n + b^k$  for some  $k$ , and  $n_b(k) \neq b - 1$ .

Now let  $m = n + b^c$  for some  $c \in \mathbb{N}$  where  $n_b(c) \neq b - 1$ . Clearly  $n \ll_b m$ . Let  $z \in \mathbb{N}$  be arbitrary with  $n \ll_b z \ll_b m$ . Then  $n_b(i) \leq z_b(i) \leq m_b(i)$  for all  $i$ . Since  $n_b(i) = m_b(i)$  for all  $i \neq c$ ,  $n_b(i) = z_b(i) = m_b(i)$ . Furthermore,  $n_b(c) \leq z_b(c) \leq m_b(c) = n_b(c) + 1$ . Then  $z_b(c) = n_b(c)$  making  $z = n$ , or  $z_b(c) = n_b(c) + 1$  making  $z = m$ . So  $z \in \{m, n\}$ , meaning  $m$  is an upper cover of  $n$ .

A dual argument of the upper covers proof suffices for the proof of lower covers.  $\square$

There are always infinitely many upper covers, whereas there are finitely many lower covers. Therefore  $(\mathbb{N}, \ll_b)$  is called lower finite.

Another important feature of partially-ordered sets is what is called a rank function. The rank function of a poset is defined recursively as follows.

**Definition 3.5.** Let  $(P, \leq)$  be a poset. If  $p \in P$  is minimal, let  $\text{rank}(p) := 0$ . If the elements of rank  $< n$  have been determined and  $p$  is minimal in the ordered set  $P \setminus \{q \in P : \text{rank}(q) < n\}$  we set  $\text{rank}(p) := n$ . ([4])

In particular, for the  $b$ -dominance relation we show here that the rank function is actually the sum of digits function mentioned above. To do this we use the following lemma.

**Lemma 3.6.** Let  $(P, \leq)$  be a poset. Let  $f : (P, \leq) \rightarrow (\mathbb{N}, \leq)$  satisfying:

1.  $f(x) = 0$  for all  $x \in P$  with  $x$  minimal,
2.  $f(x) = f(y) + 1$  for all lower covers  $y$  of  $x$ .

Then  $f$  is the rank function for  $(P, \leq)$ .

*Proof.* This is a straightforward induction which follows from the definition of rank above.  $\square$

Recall  $\text{sum}_b(n) = \sum_{i=0}^{\text{len}_b(n)} n_b(i)$  for all  $n \in \mathbb{N}$ . With this and the basic understanding of the rank function of a poset presented above, we can define the rank function for  $b$ -dominance as follows.

**Proposition 3.7.** Let  $b \in \mathbb{N}$  with  $b \geq 2$ . Then  $\text{sum}_b$  is the rank function, denoted  $\text{rank}_b$ , for the partially-ordered set  $(\mathbb{N}, \ll_b)$ .

*Proof.* Note that 0 is the only minimal element in  $(\mathbb{N}, \ll_b)$  and  $\text{sum}_b(0) = 0$ . Next Theorem 3.4 implies that for  $m \in \mathbb{N}$ ,  $\text{sum}_b(m) = \text{sum}_b(n) + 1$  for all  $n \in LC_b(m)$ . Thus the result follows from Lemma 3.6.  $\square$

## 4 The Poincaré Polynomial of $b$ -Dominance

Now that we have discussed the basic characteristics of  $b$ -dominance, we move to a more advanced topic. For any ranked poset  $(Q, \leq)$  and  $a, c \in Q$ , we let  $[a, c] := \{x \in Q \mid a \leq x \leq c\}$  be the *interval* from  $a$  to  $c$ . Then for any interval  $[a, c]$ , we define the Poincaré polynomial of the interval by,

$$P([a, c], q) := \sum_{x \in [a, c]} q^{\text{rank}(x) - \text{rank}(a)},$$

if this polynomial exists. Note, this polynomial may not exist because there may be a rank with infinitely many elements. The coefficient of  $q^i$  counts the number of elements in  $[a, c]$  whose rank is  $i + \text{rank}(a)$ . Essentially, in well-behaved posets, this counts the number of elements at a fixed rank (above  $a$ ). For the purposes of this paper we let  $P(k, q) := P([0, k], q)$ . We define  $[0, k] =: (\ll_b k)$  to be the *down set* of  $k$ , or the set of all elements  $b$ -dominated by  $k$ .

Poincaré polynomials are well-studied in the field of topology. Here, however, we study them in a combinatorial sense because they lead to discovery of some interesting connections between the  $b$ -dominance order and the partitions of integers. A partition of  $n \in \mathbb{N}$  can be described as finding a string of integers whose sum is  $n$ . We formalize this ideas as follows.

**Definition 4.1.** For any  $m \in \mathbb{N} \setminus \{0\}$ , a *partition* of  $m$  is a sequence  $(\lambda_0^{i_0}, \lambda_1^{i_1}, \dots, \lambda_k^{i_k})$ , where  $\lambda_s \in \mathbb{N} \setminus \{0\}$ ,  $i_s \in \mathbb{N} \setminus \{0\}$  for all  $0 \leq s \leq k$ ,  $\lambda_0 > \lambda_1 > \dots > \lambda_k > 0$ , and  $m = i_0 \lambda_0 + i_1 \lambda_1 + \dots + i_k \lambda_k$ . Let  $P_m$  represent the set of all partitions of  $m$ .

This is not the standard notation but is equivalent to the standard definition.

As an example, suppose  $m = 7$ . Notice we can rewrite  $m$  as  $m = 3(2) + 1(1)$ . Then we have  $\lambda_0 = 2$ ,  $i_0 = 3$ ,  $\lambda_1 = 1$ , and  $i_1 = 1$ . Therefore  $(2^3, 1^1)$  is a partition of 7. This is only one possible partition of 7; there are 14 others.

Next, we introduce a family of sets associated with the base  $b$  expansion of  $n$  that will allow us to describe the coefficients in the Poincaré polynomial.

**Definition 4.2.** For any  $b, j, n \in \mathbb{N}$  with  $b \geq 2$ , we define  $I_{b,n,j} := \{i \in \mathbb{N} \mid n_b(i) \geq j\}$ , the set of all indices where  $n_b(i)$  is greater than or equal to a fixed integer  $j$ . When it is clear from the context we will drop the  $b$  and the  $n$  subscripts.

Let  $b \in \mathbb{N}$  with  $b \geq 2$  and let  $a \in \mathbb{N}$ . For  $l \in \mathbb{N}$  and  $x = (\lambda_0^{i_0}, \dots, \lambda_k^{i_k}) \in P_l$ , we define  $A_{\lambda_p} := I_{b,a,\lambda_p}$  for all  $0 \leq p \leq k$ . With this notation we can provide the following formula for the Poincaré polynomial of  $b$ -dominance.

**Theorem 4.3.** *Let  $a, b, l \in \mathbb{N}$  with  $b \geq 2$ , and  $l \leq \text{rank}_b(a)$ . Let  $P(a, q) = \sum_{m \in [0, a]} q^{\text{rank}_b(m)} =$*

*$\sum_{i=0}^{\text{rank}_b(a)} c_i q^i$  be the Poincaré polynomial for  $a$ . Then*

$$c_l = \sum_{(\lambda_0^{i_0}, \lambda_1^{i_1}, \dots, \lambda_k^{i_k}) \in P_l} \prod_{j=0}^k \binom{|A_{\lambda_j}| - \left(\sum_{s=0}^{j-1} i_s\right)}{i_j}.$$

Before we can prove this theorem, we need to introduce some necessary combinatorial objects. Let  $a \in \mathbb{N}$ , and let  $l$  be arbitrary with  $0 \leq l \leq \text{rank}_b(a)$ . Suppose  $x = (\lambda_0^{i_0}, \dots, \lambda_k^{i_k}) \in P_l$ . Then define  $B_x$  by

$$B_x := \left\{ (B_0, \dots, B_k) \mid (B_j \subseteq A_{\lambda_j} \setminus \bigcup_{r=0}^{j-1} B_r) \wedge (|B_j| = i_j) \right\}. \tag{2}$$

Next, if  $B_x \neq \emptyset$ , let  $f_x : B_x \mapsto (\ll_a)$  be given by  $f_x((B_0, B_1, \dots, B_k)) = (y_0, y_1, \dots, y_{\text{len}_b(a)})$

where  $y_i = \begin{cases} \lambda_j & \text{if } i \in B_j \\ 0 & \text{otherwise} \end{cases}$ .

Since these definitions are relatively complex, we provide an example of their application here. Let  $a = 3695$ ,  $b = 3$ , and  $l = 7$ . Note  $3695_{(3)} = (2, 1, 2, 1, 0, 0, 2, 1)$ . Thus  $\text{rank}_3(3695) = 9$ , and so  $0 \leq l \leq \text{rank}_3(3695)$ . One possible partition of  $l$  is  $l = 2(2) + 3(1)$ . Rewriting this as a sequence as defined above we have  $(2^2, 1^3)$ . Now define  $x := (2^2, 1^3)$ . Thus  $x \in P_7$  with  $\lambda_0 = 2$  and  $\lambda_1 = 1$ . So we have  $A_{\lambda_0} = A_2 = \{i \in \mathbb{N} \mid a_b(i) \geq 2\} = \{0, 2, 6\}$  and  $A_{\lambda_1} = A_1 = \{i \in \mathbb{N} \mid a_b(i) \geq 1\} = \{0, 1, 2, 3, 6, 7\}$ . Now  $B_x$  is the set containing all the different ways we can remove the partition  $x$  from  $a$ . As one example,  $(\{0, 2\}, \{1, 3, 7\}) \in B_{(2^2, 1^3)}$ . When considering which components of  $a$  to remove 1 from, we were considering the set  $\{0, 1, 2, 3, 6, 7\} \setminus \{0, 2\}$ . We have chosen two places ( $i_0$ ) to remove 2 from the components of  $a$  and three places ( $i_1$ ) to remove 1 from the components of  $a$ . Then, applying our function, we get  $f_x((\{0, 2\}, \{1, 3, 7\})) = (2, 1, 2, 1, 0, 0, 0, 1)$ .

Having defined these sets, we now show that the function  $f$  as defined above is injective and surjective.

**Lemma 4.4.** *Let  $a, b, l \in \mathbb{N}$  with  $b \geq 2$  and  $l \leq \text{rank}_b(a)$ . For all  $x \in P_l$  with  $B_x \neq \emptyset$ , the function  $f_x : B_x \rightarrow (\ll_b a)$  defined above is an injection.*

*Proof.* Let  $x = (\lambda_0^{i_0}, \dots, \lambda_k^{i_k}) \in P_l$  with  $B_x \neq \emptyset$ . Let  $(R_0, R_1, \dots, R_k), (S_0, S_1, \dots, S_k) \in B_x$ . Suppose  $(R_0, R_1, \dots, R_k) \neq (S_0, S_1, \dots, S_k)$ . Then there is an index  $i$  where  $R_i \neq S_i$ . Therefore, without loss of generality,  $R_i \setminus S_i \neq \emptyset$ . Suppose  $j \in R_i$  and  $j \notin S_i$ . Let  $y_1 = f_x((R_0, R_1, \dots, R_k))$  and  $y_2 = f_x((S_0, S_1, \dots, S_k))$ . Since  $j \in R_i$ , it follows that  $y_1(j) = \lambda_i$ . Since  $j \notin S_i$ , then either  $j \in S_w$  for some  $w \neq i$  or  $j \notin S_w$  for all  $w$ .

**Case 1.** Suppose  $j \in S_w$  for some  $w \neq i$ . Then  $y_2(j) = \lambda_w \neq \lambda_i = y_1(j)$ , and thus  $y_1 \neq y_2$ .

**Case 2.** Suppose  $j \notin S_w$  for all  $w$ . Then  $y_2(j) = 0 < \lambda_i = y_1(j)$ , and thus  $y_1 \neq y_2$ .

In either case, we have seen that  $y_1 \neq y_2$ , and so  $f_x$  is injective.  $\square$

**Lemma 4.5.** *Let  $a, b, l \in \mathbb{N}$  with  $b \geq 2$  and  $l \leq \text{rank}_b(a)$ . For all  $y \in (\ll_b a)$  such that  $\text{rank}_b(y) = l$ , there exists a partition  $x \in P_l$  and a sequence  $S \in B_x$  such that  $f_x(S) = y$ .*

*Proof.* Let  $y \in (\ll_b a)$  with  $\text{rank}_b(y) = l$ . Now define  $x := (\lambda_0^{i_0}, \dots, \lambda_k^{i_k})$  with the  $\lambda_j = \max_{I_q \setminus I_{\lambda_{j-1}} \neq \emptyset} \{q > 0\}$ , and  $i_j = |I_{\lambda_j} \setminus I_{\lambda_{j-1}}|$  where  $I_{\lambda_{-1}} := \emptyset$ . Notice  $x \in P_l$ , since  $\text{rank}_b(y) = l$ . Now for  $0 \leq j \leq k$  we let  $B_j = I_{\lambda_j} \setminus I_{\lambda_{j-1}}$ . Notice  $|B_j| = i_j$  so that  $(B_0, \dots, B_k) \in B_x$ . Let  $\xi = f_x((B_0, \dots, B_k))$ . Let  $n \in \mathbb{N}$ . Now either  $n \in B_j$  for some  $j$  or  $n \notin B_j$  for all  $j$ .

**Case 1.** Suppose  $n \in B_j$  for some  $j$ . Then  $\xi_n = \lambda_j$ . Also, since  $n \in B_j$ , we have that  $n \in I_{\lambda_j} \setminus I_{\lambda_{j-1}}$ . These imply, by construction of  $I_{\lambda_j}$ , that  $n \notin I_{(\lambda_j)+1}$ . Hence  $n \in I_{\lambda_j} \setminus I_{(\lambda_j)+1}$  which means  $y_n = \lambda_j$ .

**Case 2.** Now suppose  $n \notin B_j$  for all  $j$ . Then  $\xi_n = 0$ . Since  $n \notin B_j$  for all  $j$ ,  $n \notin I_{\lambda_j}$  for all  $j$ . Thus  $n \in I_0$  and by construction  $n \notin I_1$ . Therefore it follows that  $y_n = 0$ .

In either case,  $\xi_n = y_n$  and thus  $\xi = y$ .  $\square$

Recall that for a collection of sets  $\mathcal{F}$  the notation  $\bigcup_{X \in \mathcal{F}} X$  represents the disjoint union of these sets. With the previous lemmas in place, we can now prove Theorem 4.3.

*Proof of Theorem 4.3.* Notice that  $c_l = |\{y \in (\ll_b a) \mid \text{rank}_b(y) = l\}|$ . We define  $F : \bigcup_{x \in P_l} B_x \rightarrow \{y \in (\ll_b a) \mid \text{rank}_b(y) = l\}$ . Now  $F(x, B) = f_x(B)$ . By Lemma 4.4 and the fact that  $\text{image}(f_x) \cap \text{image}(f_y) = \emptyset$  whenever  $x \neq y$ , it follows that  $F$  is injective. By Lemma 4.5 we have that  $F$  is surjective. Thus  $F$  is a bijection and so  $c_l = \left| \bigcup_{x \in P_l} B_x \right| = \sum_{x \in P_l} |B_x|$ .

Let  $x \in P_l$ . To compute  $|B_x|$ , we count the number of sequences  $(B_0, \dots, B_k)$  satisfying the conditions given in equation (2). Therefore

$$|B_x| = \prod_{j=0}^k \binom{|A_{\lambda_j} \setminus \bigcup_{r=0}^{j-1} B_r|}{|B_j|}.$$

However, by definition  $|B_s| = i_s$  for all  $s$ . Furthermore, since  $B_r \subseteq A_{\lambda_j}$  for all  $r < j$ , and  $B_r \cap B_s = \emptyset$  for all  $r < s < j$ , it follows that

$$|A_{\lambda_j} \setminus \bigcup_{r=0}^{j-1} B_r| = |A_{\lambda_j}| - \left( \sum_{s=0}^{j-1} i_s \right).$$

The result now follows.  $\square$

This theorem provides the connection we mentioned before between the  $b$ -dominance order and the partitions of the integers. An area of interest that we have not investigated is if there is a connection between  $b$ -dominance and the dominance order on the set of partitions. Although this theorem is technical, its application is relatively straightforward if we have enumerated all of the partitions of the number desired. Thus we provide an example of an outline for its use.

Let  $a = 583$  and  $l = 4$ . Then  $583_{(7)} = (2, 6, 4, 1)$ . Notice that  $\text{rank}_7(583) = 13$ . Now  $P_4 = \{(4^1), (3^1, 1^1), (2^2), (2^1, 1^2), (1^4)\}$ . Consider  $x = (3^1, 1^1) \in P_l$ . Then  $A_{\lambda_0} = I_3 = \{1, 2\}$  and  $A_{\lambda_1} = I_1 = \{0, 1, 2, 3\}$ . So we have

$$\begin{aligned} \prod_{j=0}^1 \binom{|A_{\lambda_j}| - (\sum_{s=0}^{j-1} i_s)}{i_j} &= \binom{|A_{\lambda_0}|}{i_0} \binom{|A_{\lambda_1}| - i_0}{i_1} \\ &= \binom{2}{1} \binom{4-1}{1} \\ &= 2(3) = 6 \end{aligned}$$

Here we have considered only one partition of  $l$ , but we know that  $c_l$  is the sum over all the partitions. We leave the details of computing the values up to the interested reader. Thus we end with

$$c_l = \sum_{(\lambda_0^{i_0}, \lambda_1^{i_1}, \dots, \lambda_k^{i_k}) \in P_l} \prod_{j=0}^k \binom{|A_{\lambda_j}| - (\sum_{s=0}^{j-1} i_s)}{i_j} = 2 + 6 + 3 + 9 + 1 = 21.$$

Let  $n \in \mathbb{N}$ . Recall we refer to the set of all elements dominated by  $n$  as the down set of  $n$ . Our motivation for studying the Poincaré polynomial came from a desire to determine the number of elements of any rank less than  $\text{rank}_b(n)$  that are also in the down set of  $n$ . It is interesting to note that the coefficients  $c_l$ , referred to in the above theorem, answer this question. Also, note that  $|P(a, 1)| = |(\ll_b a)|$  when  $q = 1$ .

One useful feature of the Poincaré polynomial is that we only need to actually calculate half of the coefficients. This is because the  $b$ -dominance order exhibits what is known as *duality*. A good example of this is Theorem 3.4. The major difference in the process for constructing upper and lower covers of  $n$  is a difference in sign, that is, we are either adding one or subtracting one from a single digit of the base  $b$  expansion. The Poincaré polynomial also exhibits this idea of duality. Formally, we get the following result.

**Corollary 4.6.** *Let  $a, b \in \mathbb{N}$  with  $b \geq 2$ . If  $P(a, q) = \sum_{i=0}^{\text{rank}_b(a)} c_i q^i$ ; then for all  $l$  with  $0 \leq l \leq \text{rank}_b(a)$  we have  $c_l = c_{\text{rank}_b(a)-l}$ .*

*Proof.* For  $0 \leq i \leq \text{rank}_b(a)$ , we define  $R_i := \{x \in (\ll_b a) \mid \text{rank}_b(x) = i\}$ . By definition,  $c_i = |R_i|$ . According to [2]  $\text{rank}_b(r) + \text{rank}_b(a - r) = \text{rank}_b(a)$  if and only if  $r \ll_b a$  and  $(a - r) \ll_b a$  (this is also a consequence of Theorem 6.1). It follows that  $f : R_l \rightarrow R_{\text{rank}_b(a)-l}$  given by  $f(r) = a - r$  is a bijection. Thus  $c_l = |R_l| = |R_{\text{rank}_b(a)-l}| = c_{\text{rank}_b(a)-l}$ .  $\square$

Recall that  $b$ -dominance can be thought of as a subposet of an infinite product of chains as described in the introduction. It follows then that every interval  $[0, a]$  can be thought of as a finite product of chains. This interpretation provides us with an alternative form for the Poincaré polynomial.

Consider the example  $a = 583$  given previously. Then  $[0, 583] \cong [0, 2] \times [0, 6] \times [0, 4] \times [0, 1]$ . So  $P(583, q) = P(2, q) \cdot P(6, q) \cdot P(4, q) \cdot P(1, q)$  see [4]. Notice when  $a \leq b$ , the interval  $[0, a]$  is a chain and so  $P(a, q) = 1 + q + q^2 + \dots + q^a = \frac{q^{a+1} - 1}{q - 1}$ . Thus

$$P(583, q) = \left( \frac{q^3 - 1}{q - 1} \right) \left( \frac{q^7 - 1}{q - 1} \right) \left( \frac{q^5 - 1}{q - 1} \right) \left( \frac{q^2 - 1}{q - 1} \right).$$

This process works in general and is summarized by the following corollary.

**Corollary 4.7.** *Let  $a, b \in \mathbb{N}$  with  $b \geq 2$ . Let the Poincaré polynomial for  $a$  be defined as described in Theorem 4.3. Then,*

$$P(a, q) = \sum_{m \in [0, a]} q^{\text{rank}_b(m)} = \prod_{j=0}^{b-1} \left( \frac{q^{j+1} - 1}{q - 1} \right)^{|I_{b,a,j} \setminus I_{b,a,j+1}|}.$$

*Proof.* This follows from basic products of Poincaré polynomials and products of subsets.  $\square$

Although we introduced the above corollary as another way to generate the Poincaré polynomial for  $b$ -dominance, we can also use it to factor the polynomial if it is already known. On the other hand, the above corollary and Theorem 4.3 provide us with a closed formula that we can use to expand the polynomial found in Corollary 4.7. This will be familiar to those comfortable with Euler's formula for the number of partitions.

## 5 The Möbius Function for $b$ -Dominance

The Möbius function plays an important role in the combinatorics of posets. Given a poset  $(P, \leq)$  with  $x, y \in P$ , the Möbius function is defined by

$$\sum_{z \in [x, y]} \mu(x, z) = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$



Using this definition, we introduce the Möbius function associated with the  $b$ -dominance. One important feature of this function is to perform a process known as Möbius inversion; for specifics on the Möbius function or Möbius inversion see [6]. In [1, Section 8], the authors use the inversion process to provide a formula for the number of dismal partitions of a number. Due to the recursive nature of the Möbius function our proof will use induction. Moreover, we need the following lemma.

**Lemma 5.1.** *Let  $n \in \mathbb{N}$ . Then,  $\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} = 0$ .*

*Proof.* Let  $n \in \mathbb{N}$  be arbitrary. According to the Binomial Theorem,  $(x + y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \dots + \binom{n}{n}x^0 y^n$ , for all  $x, y \in \mathbb{R}$ . Let  $x = 1$  and  $y = -1$ . Then,  $0 = \binom{n}{0}(-1)^0 + \binom{n}{1}(-1)^1 + \dots + \binom{n}{n}(-1)^n = \sum_{i=0}^n (-1)^{n-i} \binom{n}{i}$ .  $\square$

On its own, this consequence of the Binomial Theorem is an interesting combinatorial result. We include it here because of its relationship to the Möbius function. Essentially we are alternating between adding and subtracting entries in a row of Pascal's Triangle and the resulting sum will be 0. We can now define the Möbius function for  $b$ -dominance.

**Theorem 5.2.** *Let  $b \in \mathbb{N}$  with  $b \geq 2$ . Let  $m, n \in \mathbb{N}$  with  $n \ll_b m$ . Then, the Möbius function of  $n$  and  $m$  is given by:*

$$\mu(n, m) = \begin{cases} (-1)^{\text{rank}_b(m) - \text{rank}_b(n)} & \text{where } m_b(i) - n_b(i) \leq 1 \text{ for all } i \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let  $b, n \in \mathbb{N}$  with  $b \geq 2$ . We proceed by induction.

**Base Case** By definition of the Möbius function, we know  $\mu(n, n) = 1$ .

Let  $q \in \mathbb{N}$  with  $n \ll_b q$ . Assume the above formula for the Möbius function holds for all  $x \in [n, q)$ .

**Case 1** Suppose  $q_b(i) - n_b(i) \geq 2$  for some  $i$ . Let  $d$  be the least upper bound of the upper covers of  $n$  that are dominated by  $q$ . Notice  $d \ll_b q$  and  $d \neq q$ . Then,  $0 = \sum_{x \in [n, q]} \mu(n, x) =$

$$\sum_{x \in [n, q)} \mu(n, x) + \mu(n, q). \text{ So, } -\mu(n, q) = \sum_{x \in [n, q)} \mu(n, x) = \sum_{x \in [n, d]} \mu(n, x) + \sum_{\substack{x \in [n, q) \\ x \notin [n, d]}} \mu(n, x). \text{ By}$$

definition of the Möbius function, we know  $\sum_{x \in [n, d]} \mu(n, x) = 0$ . We also know that

$$\sum_{\substack{x \in [n, q) \\ x \notin [n, d]}} \mu(n, x) = 0 \text{ by our induction step. Thus, we conclude } \mu(n, q) = 0.$$

**Case 2** Suppose  $q_b(i) - n_b(i) \leq 1$  for all  $i$ . Suppose  $\text{rank}_b(q) - \text{rank}_b(n) = k + 1$ . Then,  $0 =$

$\sum_{x \in [n, q]} \mu(n, x) = \sum_{x \in [n, q]} \mu(n, x) + \mu(n, q)$ . Therefore,  $-\mu(n, q) = \sum_{x \in [n, q]} \mu(n, x) = \sum_{i=1}^{k+1} (-1)^{k+1-i} \binom{k+1}{i}$  by the induction hypothesis. However, by Lemma 5.1, the latter sum can be replaced by  $-(-1)^{k+1} \binom{k+1}{k+1} = -(-1)^{k+1}$ . Therefore,  $\mu(n, q) = (-1)^{k+1}$  as required. Thus, we then conclude that  $\mu(n, q) = (-1)^{\text{rank}_b(q) - \text{rank}_b(n)}$ .

Hence the formula holds in either case. Since  $n$  and  $q$  were arbitrary in either case, we know that for any  $n$  and  $q$  with  $n \ll_b q$ , the Möbius function will be given by the above equation.  $\square$

Similar to the previous section, we can produce the previous result using the idea that an interval in  $b$ -dominance can be interpreted as a product of chains. Here we will present an example of the computation of the Möbius function in action.

We consider  $b = 5$  and  $n = 10$ . By our definition it is clear that  $\mu(10, 10) = 1$ . If we consider  $m = 11$ , by our definition  $\mu(10, 11) = (-1)^{\text{rank}_5(11) - \text{rank}_5(10)} = -(-1)^1 = -1$ . If we consider  $m = 12$ , since  $12_5(0) = 2$  and  $10_5(0) = 0$  we have that  $\mu(10, 12) = 0$ . Similarly, we would get  $\mu(10, 13) = 1$ ,  $\mu(10, 14) = -1$ ,  $\mu(10, 15) = -1$ ,  $\mu(10, 16) = 1$ , and for all other values of  $m$ ,  $\mu(10, m) = 0$  (see Figure 1).

## 6 Carries in Base $b$ Arithmetic

In this section we touch upon the idea of carries in base  $b$  arithmetic and their connection to  $b$ -dominance. The interested reader can find more details on this topic in [2]. There is an interesting connection between the rank function defined above and base  $b$  arithmetic. Before we introduce this connection however, we provide a rigorous definition of the base  $b$  carries when adding two natural numbers. We define the base  $b$  carries when adding  $n$  and  $m - n$ , denoted  $\epsilon_i^{m, n, b}$ , by  $\epsilon_{-1}^{m, n, b} = 0$  and for all  $i \geq 0$ ,

$$\epsilon_i^{m, n, b} = \begin{cases} 1 & \text{if } n_b(i) > m_b(i) \text{ or } n_b(i) = m_b(i) \text{ and } \epsilon_{i-1}^{m, n, b} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Note, when there is no confusion we will refer to  $\epsilon_i^{m, n, b}$  as  $\epsilon_i$  or  $\epsilon_i^b$ . Using this notation, if  $m = n + r$  then the base  $b$  expansion of  $m$  is given by  $m_b(i) = n_b(i) + r_b(i) + \epsilon_{i-1} - b\epsilon_i$ . Now, we let  $\kappa_b(m, n) = \sum_{i=0}^{\text{len}_b(m)} \epsilon_i$  be the total number of carries when adding  $n$  and  $m - n$  in base  $b$ . Bearing these definitions in mind, the following result plays an important role in the proof of a famous result known as Kummer's Theorem which is only applicable for prime bases.

**Theorem 6.1.** *Let  $b \in \mathbb{N}$  with  $b \geq 2$ . Let  $m, n \in \mathbb{N}$  with  $m = n + r$  and  $m \geq n$ . Let  $k = \text{len}_b(m)$ . Then*

$$\text{rank}_b(n) + \text{rank}_b(r) - \text{rank}_b(m) = (b - 1)\kappa_b(m, n).$$

*Proof.* Since  $m \geq n$  and  $m = n + r$ , the addition formula given above can be rewritten as  $r_b(i) = m_b(i) - n_b(i) - \epsilon_{i-1} + b\epsilon_i$ . Now,

$$\begin{aligned} \text{rank}_b(n) + \text{rank}_b(r) - \text{rank}_b(m) &= \sum_{i=0}^k (n_b(i) + r_b(i) - m_b(i)) \\ &= \sum_{i=0}^k (n_b(i) + m_b(i) - n_b(i) - \epsilon_{i-1} + b\epsilon_i - m_b(i)) \\ &= \sum_{i=0}^k (b\epsilon_i - \epsilon_{i-1}) \\ &= \sum_{i=0}^{k-1} b\epsilon_i - \sum_{i=1}^k \epsilon_{i-1} \\ &= \sum_{i=0}^{k-1} b\epsilon_i - \sum_{i=0}^{k-1} \epsilon_i \\ &= \sum_{i=0}^{k-1} \epsilon_i (b - 1) \\ &= (b - 1)\kappa_b(m, n). \end{aligned}$$

The fourth and last equalities hold since  $\epsilon_{-1} = \epsilon_k = 0$ . □

This result is well-known, but the connection to the  $b$ -dominance order appears to be new ([3]).

There are two ways to examine the carries when multiplying two numbers. The first way is to consider the multiplicative carries followed by the additive carries occurring because of ‘long’ multiplication. To do this, we have to formally define both multiplicative carries and provide an alternative definition of additive carries. Note that in the previous definition of carries for addition we were considering adding only two natural numbers and thus the maximum value carried was 1. Here we consider adding more than two numbers and thus we must define the carries more broadly.

**Definition 6.2.** Let  $b, x_0, x_1, \dots, x_k \in \mathbb{N}$  with  $b \geq 2$ . We define the value of the  $i$ th carry when adding the numbers  $x_0, x_1, \dots, x_k$  base  $b$ , by  $\epsilon_{-1}^{b, \{x_0, \dots, x_k\}} = 0$  and for all  $i \geq 0$ ,

$$\epsilon_i^{b, \{x_0, \dots, x_k\}} = \left\lfloor \frac{\epsilon_{i-1}^{b, \{x_0, \dots, x_k\}} + \sum_{j=0}^k x_j(i)}{b} \right\rfloor.$$

Note, we will drop the superscripts when there is no confusion as to  $b$  and the sequence used. Also, notice that this definition extends the previous definition of additive carries when adding two numbers.

**Definition 6.3.** Let  $b, a, d \in \mathbb{N}$  with  $b \geq 2$ . We define the value of the  $i$ th carry when multiplying  $a$  and  $d$  base  $b$ , by  $\delta_{i,-1}^{b,a,d} = 0$  for all  $i$  and for all  $i, j \geq 0$ ,

$$\delta_{i,j}^{b,a,d} = \left\lfloor \frac{a_i d_j + \delta_{i,j-1}^{b,a,d}}{b} \right\rfloor.$$

Note, we will drop the superscripts when there is no confusion as to  $a, d$ , and  $b$ .

Let  $a, d \in \mathbb{N}$  and define  $S_{a,d} = \{a_0 d, a_1 d b^1, \dots, a_k d b^k\}$ . With the previous definitions in place we define a formula for the  $i$ th component of  $(ad)$  by,

$$(ad)_i = \sum_{x+y=i} (a_x d_y + \delta_{x,y-1} - b \delta_{x,y}) + \epsilon_{i-1}^{b,S_{a,d}} - b \epsilon_i^{b,S_{a,d}}.$$

Also, since we have defined both types of carries, it makes sense to define the total number of carries. We define the total number of carries when multiplying  $d$  and  $a$  in base  $b$  as follows.

**Definition 6.4.** Let  $b, a, d \in \mathbb{N}$  with  $b \geq 2$ . Then we define the total carry value when multiplying  $a$  and  $d$  by,

$$\xi_b(ad, a) = \sum_{i \geq 0} \epsilon_i^{b,S_{a,d}} + \sum_{\substack{0 \leq i \leq \text{len}_b(a) \\ 0 \leq j \leq \text{len}_b(d)}} \delta_{i,j}^{b,a,d}.$$

Although these definitions are fairly dense, we provide a brief example to illustrate that they are simply a formal way of defining the familiar idea of multiplication.

Let  $a = 112$ ,  $d = 3096$ , and  $b = 5$ . Note that  $112_{(5)} = (2, 2, 4)$  and  $3096_{(5)} = (1, 4, 3, 4, 4)$ . Suppose we want to calculate  $\delta_{2,2}$ . Then we have

$$\delta_{2,2} = \left\lfloor \frac{a_2 d_2 + \delta_{2,1}}{5} \right\rfloor = \left\lfloor \frac{4(3) + 3}{5} \right\rfloor = 3.$$

The following table gives all the  $\delta$  carries as defined above.

$i/j$	-1	0	1	2	3	4
0	0	0	1	1	1	1
1	0	0	1	1	1	1
2	0	0	3	3	3	3

Now suppose we want to calculate  $\epsilon_1$ . Then we have

$$\epsilon_1^{5, \{a_0d, a_1db\}} = \left\lfloor \frac{\epsilon_{-1}^{5, \{a_0d\}} + \sum_{j=0}^2 a_j db^j(1)}{5} \right\rfloor = \left\lfloor \frac{0 + (3 + 2 + 0)}{5} \right\rfloor = 1.$$

The following table gives all the  $\epsilon$  carries as defined above.

$i$	-1	0	1	2	3	4	5	6	7
$\epsilon_i$	0	0	1	2	1	1	2	1	0

Now we can apply the formula for  $(ad)_5(i)$  as defined above for any  $i$ . For  $i = 3$  we have

$$\begin{aligned} (ad)_5(3) &= \sum_{x+y=3} (a_x d_y + \delta_{x,y-1} - 5\delta_{x,y}) + \epsilon_2 - 5\epsilon_3 \\ &= [a_0 d_3 + \delta_{0,2} - 5\delta_{0,3} + a_1 d_2 + \delta_{1,1} - 5\delta_{1,2} + a_2 d_1 + \delta_{2,0} - 5\delta_{2,1}] + \epsilon_2 - 5\epsilon_3 \\ &= [8 + 1 - 5(1) + 6 + 1 - 5(1) + 16 + 0 - 5(3)] + 2 - 5(1) \\ &= 4. \end{aligned}$$

Notice we have considered only one particular  $i$ , we leave the rest to the reader. We end by noting that  $ad_{(5)} = (2, 0, 0, 4, 4, 0, 2, 4)$ .

Now we can present a theorem relating the number of carries when multiplying  $d$  and  $a$  to the rank of  $a$ ,  $d$ , and  $ad$ .

**Theorem 6.5.** *Let  $b, a, d \in \mathbb{N}$  with  $b \geq 2$ . Then*

$$\text{rank}_b(a) \text{rank}_b(d) - \text{rank}_b(ad) = (b-1)\xi_b(ad, a).$$

*Proof.* This proof follows similar to Theorem 6.1. □

We find it interesting that this theorem is so similar to Theorem 6.1 for adding  $n$  and  $m-n$  in base  $b$ . One avenue we feel this idea could be used to explore, which we leave open, is the idea of ‘base-free’ multiplication, that is multiplication which yields the same answer in any base. As an example  $10 \times 10 = 100$  regardless of which base you are operating in (see Section 7).

While not formally correct, for our purposes we will alternatively consider multiplication over  $\mathbb{N}$  as repeated addition. For example,  $5 \times 5$  can be thought of as  $5 + 5 + 5 + 5 + 5$ . This idea changes the number of carries that occur when multiplying two natural numbers which we also find interesting. If we consider this method of multiplication, then we have the  $i$ th component when multiplying  $a$  and  $d$  by

Formally, we define the carries when using this different method for performing multiplication as follows.

**Definition 6.6.** Let  $b, a, d \in \mathbb{N}$  with  $b \geq 2$ . Then we define the total carry value when adding  $d$  copies of  $a$  by,

$$\mathcal{U}_b(ad, a) = \sum_{i \geq 0} \epsilon_i^{b, \overbrace{\{a, a, \dots, a\}}^{d\text{-copies}}} = \mathcal{U}_b(a(d-1), a) + \kappa_b(ad, a).$$

Note, the second equality can be checked using the definition of the  $\epsilon_i$ .

Using this definition we can now present the following theorem regarding the number of carries when adding  $d$  copies of  $a$  related to our rank function.

**Theorem 6.7.** Let  $b, a, d \in \mathbb{N}$  with  $b \geq 2$ . Then

$$d \operatorname{rank}_b(a) - \operatorname{rank}_b(ad) = (b-1)\mathcal{U}_b(ad, a).$$

*Proof.* We proceed using mathematical induction.

**Base Case** Let  $d = 2$ . Then

$$\mathcal{U}_b(ad, a) = \mathcal{U}(a, a) + \kappa_b(2a, a) = 0 + \frac{\operatorname{rank}_b(a) + \operatorname{rank}_b(a) - \operatorname{rank}_b(ad)}{b-1} = \frac{d \operatorname{rank}_b(a) - \operatorname{rank}_b(ad)}{b-1}.$$

**Induction Step** Let  $d \geq 2$ . Suppose  $\mathcal{U}_b(ad, a) = \frac{d \operatorname{rank}_b(a) - \operatorname{rank}_b(ad)}{b-1}$ . Then

$$\begin{aligned} \mathcal{U}_b(a(d+1), a) &= \mathcal{U}_b(ad, a) + \kappa_b(a(d+1), a) \\ &= \frac{d \operatorname{rank}_b(a) - \operatorname{rank}_b(ad)}{b-1} + \frac{\operatorname{rank}_b(a) + \operatorname{rank}_b(ad) - \operatorname{rank}_b(a(d+1))}{b-1} \\ &= \frac{(d+1) \operatorname{rank}_b(a) - \operatorname{rank}_b(a(d+1))}{b-1}. \end{aligned}$$

Thus the result follows by induction. □

We believe that this could be used as a stepping stone to investigate when multiplication can be done ‘base-free’.

## 7 Future Directions

In the previous section, we mentioned the idea of ‘base-free’ multiplication. Here we define tools we believe are necessary for investigating this idea, followed by a conjecture.

**Definition 7.1.** Let  $b, c \in \mathbb{N}$  with  $b, c \geq 2$ . Let  $n \in \mathbb{N}$ . Let  $g_{b,c} : \mathbb{N} \rightarrow \mathbb{N}$  is defined by,

$$g_{b,c}(n) = \sum_{i=0}^{\text{len}_b(n)} \min\{n_b(i), c-1\}c^i.$$

Note, when  $b$  and  $c$  are fixed, we will drop the subscripts. Furthermore, when  $b \leq c$ , the formula is simpler:

$$g_{b,c}(n) = \sum_{i=0}^{\text{len}_b(n)} n_b(i)c^i.$$

**Proposition 7.2.** Let  $b, c, n, m \in \mathbb{N}$  with  $b, c \geq 2$ . If  $n \ll_b m$  then  $g(n) \ll_c g(m)$ .

*Proof.* Suppose  $n \ll_b m$ . Then  $n_b(i) \leq m_b(i)$  for all  $i$ . Now let  $i \in \mathbb{N}$ . Then either  $m_b(i) \leq c-1$  or  $m_b(i) > c-1$ .

**Case 1** Suppose  $m_b(i) \leq c-1$ . Notice, then  $n_b(i) \leq c-1$ . So  $g(m)_c(i) = \min\{m_b(i), c-1\} = m_b(i)$  and  $g(n)_c(i) = \min\{n_b(i), c-1\} = n_b(i)$ . Thus  $g(n)_c(i) \leq g(m)_c(i)$ .

**Case 2** Suppose  $m_b(i) > c-1$ . Then  $g(m)_c(i) = \min\{m_b(i), c-1\} = c-1$ . Now either  $n_b(i) \leq c-1$  or  $n_b(i) > c-1$ . When  $n_b(i) \leq c-1$ ,  $g(n)_c(i) = \min\{n_b(i), c-1\} = n_b(i)$ . So  $c-1 = g(n)_c(i) \leq g(m)_c(i) = c-1$ . When  $n_b(i) > c-1$ ,  $g(n)_c(i) = \min\{n_b(i), c-1\} = c-1$ . Thus  $g(n)_c(i) \leq g(m)_c(i)$ .

Therefore since  $g(n)_c(i) \leq g(m)_c(i)$  for all  $i$ ,  $g(n) \ll_c g(m)$  and therefore  $g$  is an order-preserving map from  $(\mathbb{N}, \ll_b) \rightarrow (\mathbb{N}, \ll_c)$ .

□

With the previous definition and proposition in mind, we present the following conjecture as an interesting area of study open for exploration.

**Conjecture 7.3.** Let  $a, b, c, d \in \mathbb{N}$  with  $2 \leq b < c$ . Then  $g_{b,c}(ad) = g_{b,c}(a)g_{b,c}(d)$  if and only if  $\xi_b(ad, a) = 0$ .

One direction of this conjecture is clear, but the other direction seems to be difficult.

There are possibly other interesting connections between base  $b$  arithmetic and the  $b$ -dominance order. Also, as mentioned in Section 5, there seems to be an interesting connection between  $b$ -dominance and dismal arithmetic [1, Section 9]. It is fascinating that such a seemingly unremarkable order can have so many remarkable connections to a plethora of areas of mathematics. As a final note, all images were created using Sage [5] and GraphViz.

## References

- [1] David Applegate, Marc Lebrun, and N.J.A. Sloane, Dismal arithmetic, 2011, arXiv:1107.1130v2.
- [2] Tyler Ball, Tom Edgar, and Daniel Juda, Generalized binomial coefficients via dominance orders on the natural numbers., submitted.
- [3] Andrew Granville, Arithmetic properties of binomial coefficients. I. Binomial coefficients modulo prime powers, Organic mathematics (Burnaby, BC, 1995), CMS Conf. Proc., vol. 20, Amer. Math. Soc., Providence, RI, 1997, pp. 253–276. MR 1483922 (99h:11016)
- [4] Bernd S. W. Schröder, Ordered sets, Birkhäuser Boston Inc., Boston, MA, 2003, An introduction. MR 1944415 (2003k:06001)
- [5] W. A. Stein et al., Sage Mathematics Software (Version 4.8), The Sage Development Team, 2012, <http://www.sagemath.org>.
- [6] J. H. van Lint and R. M. Wilson, A course in combinatorics, second ed., Cambridge University Press, Cambridge, 2001. MR 1871828 (2002i:05001)