A Logic with Kolmogorov Style Conditional Probabilities*

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Abstract. In this paper we investigate a probability logic with conditional probability operators. The logic (denoted LPP) allows making statements such as $CP_{\geq s}(\alpha \mid \beta)$, with the intended meaning "the conditional probability of α given β is at least s". Conditional probabilities are defined in the usual Kolmogorv style: $P(\alpha \mid \beta) = \frac{P(\alpha\wedge\beta)}{P(\beta)}$, $P(\beta) > 0$. A possible-world approach is used to give semantics to probability formulas. An infinitary axiomatic system for our logic is given and the corresponding strong completeness theorem is proved. It is proved that the logic is decidable.

1 Syntax

The language $\mathcal{L}$ of LPP consists of a countable set $I = \{p_1, p_2, \dots\}$ of propositional letters, classical connectives $\wedge$ and $\neg$, a list of unary probabilistic operators $P_{\geq s}$ for every rational number $s \in [0, 1]$, a list of binary probability operators $CP_{\geq s}$ for every rational number $s \in [0, 1]$, and a binary probability operator $CP_{\leq 0}$.

The set For^C_{LPP} of all classical propositional formulas is defined inductively as the smallest set X containing propositional letters and closed under the usual formation rules: if α and β belong to X, then $\neg\alpha$, $\alpha \wedge \beta$ are in X. Elements of For^C_{LPP} will be denoted by $\alpha, \beta, \dots$ The set For^P_{LPP} of all probability formulas is the smallest set Y containing all formulas of the forms: $P_{\geq s}\alpha$, $CP_{\geq s}(\alpha \mid \beta)$, $CP_{\leq 0}(\alpha \mid \beta)$, for all $\alpha, \beta \in For^C_{LPP}$ and each rational number s from $[0, 1]$, and closed under the formation rules: if A and B belong to Y, then $\neg A$ and $A \wedge B$ are in Y. The formulas from For^P_{LPP} will be denoted by $A, B, \dots$ Let $For_{LPP} = For^C_{LPP} \cup For^P_{LPP}$. The formulas from For_{LPP} will be denoted by Φ, Ψ, ...

As it can be seen, neither mixing of pure propositional formulas and probability formulas, nor nested probability operators are allowed. For example,

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$\neg P_{\geq 0.5}p_1 \wedge CP_{\geq 1}(p_1 \rightarrow p_2 \mid p_2)$ is a syntactically correct formula of the LPP, while $CP_{\geq 0.5}(p_1 \mid CP_{\geq 0.5}(p_1 \mid P_{\geq 1}p_1))$ and $p_1 \wedge P_{\geq 1}p_1$ are not.

We use the usual abbreviations for the other classical connectives $\vee$, $\rightarrow$, $\leftrightarrow$. For every rational number s from $[0,1]$ we denote $\neg P_{\geq s}(\alpha)$ by $P_{<s}(\alpha)$, $P_{\geq 1-s}(\neg\alpha)$ by $P_{\leq s}(\alpha)$, $\neg P_{\leq s}(\alpha)$ by $P_{>s}(\alpha)$, and $P_{\geq s}(\alpha) \wedge P_{\leq s}(\alpha)$ by $P_{=s}(\alpha)$ Also, for $s \neq 0$, we use the following abbreviations: $CP_{<s}(\alpha \mid \beta) \stackrel{\text{def}}{=} \neg CP_{\geq s}(\alpha \mid \beta)$, $CP_{\leq s}(\alpha \mid \beta) \stackrel{\text{def}}{=} CP_{\geq 1-s}(\neg\alpha \mid \beta)$, $CP_{>s}(\alpha \mid \beta) \stackrel{\text{def}}{=} \neg CP_{\leq s}(\alpha \mid \beta)$, and $CP_{=s}(\alpha \mid \beta) \stackrel{\text{def}}{=} CP_{\geq s}(\alpha \mid \beta) \wedge CP_{\leq s}(\alpha \mid \beta)$. For $\alpha \in For^C_{LPP}$, and $A \in For^P_{LPP}$, we abbreviate both $\alpha \wedge \neg\alpha$ and $A \wedge \neg A$ by $\bot$ letting the context determine the meaning.

2 Semantics

The semantics for For_{LPP} will be based on the possible-world approach.

Definition 1. *An $LPP-$model is a structure $\mathbf{M} = \langle W, H, \mu, v\rangle$ where:*

- *W is a nonempty set of objects called worlds,*
- *H is an algebra of subsets of W, and*
- *μ is a finitely additive measure, $\mu : H \rightarrow [0,1]$,*
- *$v : W \times I \rightarrow \{true, false\}$ provides for each world $w \in W$ a two-valued evaluation of the propositional letters, that is $v(w,p) \in \{true, false\}$, for each propositional letter $p \in I$ and each world $w \in W$; a truth-evaluation $v(w,\cdot)$ is extended to classical propositional formulas as usual.*

If $\mathbf{M}$ is an $LPP-$model and $\alpha \in For^C_{LPP}$, the set $\{w : v(w,\alpha) = true\}$ is denoted by $[\alpha]_{\mathbf{M}}$. We will omit the subscript $\mathbf{M}$ from $[\alpha]_{\mathbf{M}}$ and write $[\alpha]$ if $\mathbf{M}$ is clear from the context. An $LPP-$model $\mathbf{M} = \langle W, H, \mu, v\rangle$ is measurable if $[\alpha]_{\mathbf{M}} \in H$ for every formula $\alpha \in For^C_{LPP}$. In this section we focus on the class of all measurable models (denoted by LPP_{Meas}).

Definition 2. *The satisfiability relation fulfills the following conditions for every $LPP-$model $\mathbf{M} = \langle W, H, \mu, v\rangle$ and every world $w \in W$:*

- *if $\alpha \in For^C_{LPP}$, $\mathbf{M}, w \models \alpha$ iff $v(w,\alpha) = true$,*
- *if $\alpha \in For^C_{LPP}$, $\mathbf{M}, w \models P_{\geq s}\alpha$ iff $\mu([\alpha]) \geq s$,*
- *$\alpha, \beta \in For^C_{LPP}$, $\mathbf{M}, w \models CP_{\geq s}(\alpha \mid \beta)$ iff either $\frac{\mu([\alpha]\cap[\beta])}{\mu([\beta])} \geq s$ and $\mu([\beta]) > 0$, or $\mu([\beta]) = 0$,*
- *$\alpha, \beta \in For^C_{LPP}$, $\mathbf{M}, w \models CP_{\leq 0}(\alpha \mid \beta)$ iff $\mu([\alpha] \cap [\beta]) = 0$ and $\mu([\beta]) > 0$,*
- *if $A \in For^P_{LPP}$, $\mathbf{M}, w \models \neg A$ iff $\mathbf{M}, w \not\models A$,*
- *if $A, B \in For^P_{LPP}$, $\mathbf{M}, w \models A \wedge B$ iff $\mathbf{M}, w \models A$ and $\mathbf{M}, w \models B$.*

A formula $\Phi \in For_{LPP}$ is satisfiable if there is an LPP_{Meas}-model $\mathbf{M} = \langle W, H, \mu, v\rangle$, and a world $w \in W$ such that $\mathbf{M}, w \models \Phi$; Φ is valid in an LPP_{Meas}-model $\mathbf{M} = \langle W, H, \mu, v\rangle$ (denoted $\mathbf{M} \models \Phi$), if for every world $w \in W$, $\mathbf{M}, w \models \Phi$; Φ is valid if for every LPP_{Meas}-model $\mathbf{M}$, $\mathbf{M} \models \Phi$; a set of T formulas is satisfiable if there is an LPP_{Meas}-model $\mathbf{M} = \langle W, H, \mu, v\rangle$, and a world $w \in W$ such that $\mathbf{M}, w \models \Phi$ for every $\Phi \in T$.

3 Axiomatic system

The axiomatic system Ax_{LPP} for LPP contains the following axiom schemata:

1. all $For^C_{LPP}-$instances of classical propositional tautologies,
2. all $For^P_{LPP}-$instances of classical propositional tautologies,
3. $P_{\geq 0}\alpha$,
4. $P_{\leq r}\alpha \rightarrow P_{<s}\alpha$, $s > r$,
5. $P_{<s}\alpha \rightarrow P_{\leq s}\alpha$,
6. $(P_{\geq r}\alpha \wedge P_{\geq s}\beta \wedge P_{\geq 1}(\neg\alpha \vee \neg\beta)) \rightarrow P_{\geq \min(1,r+s)}(\alpha \vee \beta)$,
7. $(P_{\leq r}\alpha \wedge P_{<s}\beta) \rightarrow P_{<r+s}(\alpha \vee \beta)$, $r + s \leq 1$,
8. $CP_{\geq s}(\alpha \mid \beta) \wedge P_{\geq t}\beta \rightarrow P_{\geq s\cdot t}(\alpha \wedge \beta)$, $t > 0$,
9. $(P_{=0}(\alpha \wedge \beta) \wedge P_{>0}\beta) \leftrightarrow CP_{\leq 0}(\alpha \mid \beta)$,

and inference rules:

1. From Φ and $\Phi \rightarrow \Psi$ infer Ψ, $\Phi, \Psi \in For^C_{LPP}$ or $\Phi, \Psi \in For^P_{LPP}$,
2. From α infer $P_{\geq 1}\alpha$,
3. From $A \rightarrow P_{\geq s-\frac{1}{k}}\alpha$, for every integer $k \geq \frac{1}{s}$, infer $A \rightarrow P_{\geq s}\alpha$.
4. From $A \rightarrow (P_{\geq r}\beta \rightarrow P_{\geq r\cdot s}(\alpha \wedge \beta))$, for every rational number r from $[0, 1]$, infer $A \rightarrow CP_{\geq s}(\alpha \mid \beta)$.

Ax_{LPP} extends the axiomatic system for the (unconditional) probability logic analyzed in [7]. The new axioms 8 and 9, and Rule 4 express the standard definition of conditional probability. Rule 4 relates unary and binary probability operators. The inference rules 3 and 4 are infinitary.

Definition 3. *A formula Φ is deducible from a set T of formulas ($T \vdash \Phi$) if there is an at most countable sequence of formulas $\Phi_0, \Phi_1, \ldots, \Phi$, such that every Φ_i is an axiom or a formula from the set T, or it is derived from the preceding formulas by an inference rule.*

A formula Φ is a theorem ($\vdash \Phi$) if it is deducible from the empty set, and a proof for α is the corresponding sequence of formulas.

A set T of formulas is consistent if there is at least one formula from For^C_{LPP}, and at least one formula from For^P_{LPP} that are not deducible from T, otherwise T is inconsistent.

A consistent set T of formulas is said to be maximal consistent if the following holds:

- *for every $\alpha \in For^C_{LPP}$, if $T \vdash \alpha$, then $\alpha \in T$ and $P_{\geq 1}\alpha \in T$, and*
- *for every $A \in For^P_{LPP}$, either $A \in T$ or $\neg A \in T$.*

A set T is deductively closed if for every $\Phi \in For_{LPP}$, if $T \vdash \Phi$, then $\Phi \in T$.

4 Soundness and Completeness

Now, following the ideas from [6, 8, 10], we can prove the extended completeness theorem for the class LPP_{Meas}.

Theorem 1 (Soundness theorem). *The axiomatic system Ax_{LPP} is sound with respect to the class of LPP_{Meas}-models.*

In order to prove the completeness theorems for our logic, we follow the Henkin procedure. We begin with some auxiliary statements. Then, we describe how a consistent set T of formulas can be extended to a suitable maximal consistent set, and how a canonical model can be constructed out of such maximal consistent sets.

Theorem 2. *1.* **(Deduction theorem)** *If T is a set of formulas, Φ is a formula, and $T \cup \{\Phi\} \vdash \Psi$, then $T \vdash \Phi \to \Psi$, where Φ and Ψ are either both classical or both probability formulas.*
2. Let α, β be classical formulas. Then:
(a) $\vdash P_{\geq 1}(\alpha \to \beta) \to (P_{\geq s}\alpha \to P_{\geq s}\beta)$,
(b) $\vdash P_{\geq r}\alpha \to P_{\geq s}\alpha$, $r > s$,
(c) if $\alpha \to \beta$ is a classical tautology, then $\vdash CP_{\geq 1}(\beta \mid \alpha)$,
(d) $\vdash CP_{\geq 1}(\alpha \mid \beta) \to CP_{\geq s}(\alpha \mid \beta)$,
(e) $\vdash P_{=0}\beta \to CP_{=1}(\alpha \mid \beta)$.

Theorem 3. *Every consistent set of formulas can be extended to a maximal consistent set.*

The next theorem summarizes some obvious properties of the maximal consistent sets of formulas.

Theorem 4. *Let $\overline{T}$ be a maximal consistent set of formulas. Let Φ and Ψ be either both classical or both probability formulas, and let α be a classical formula. Then the following hold:*

1. *If $\Phi \in \overline{T}$, then $\neg\Phi \notin \overline{T}$.*
2. *$\Phi \wedge \Psi \in \overline{T}$ iff $\Phi \in \overline{T}$ and $\Psi \in \overline{T}$.*
3. *If $\overline{T} \vdash \Phi$, then $\Phi \in \overline{T}$, i.e. $\overline{T}$ is deductively closed.*
4. *If $\Phi \in \overline{T}$ and $\Phi \to \Psi \in \overline{T}$, then $\Psi \in \overline{T}$.*
5. *If $P_{\geq s}\alpha \in \overline{T}$, and $s \geq r$, then $P_{\geq r}\alpha \in \overline{T}$.*
6. *If r is a rational number and $r = \sup\{s : P_{\geq s}\alpha \in \overline{T}\}$, then $P_{\geq r}\alpha \in \overline{T}$.*

Using the maximal consistent extension $\overline{T}$ of the set T, we can define a tuple $\mathbf{M} = \langle W, H, \mu, v\rangle$, where:

- W contains all classical propositional interpretations of the set I of propositional letters,
- for every $\alpha \in For^C_{LPP}$, $[\alpha] = \{w \in W \mid w \models \alpha\}$ and $H = \{[\alpha] \mid \alpha \in For^C_{LPP}\}$,

– $\mu : H \rightarrow [0,1]$, such that $\mu([\alpha]) = \sup\{s : P_{\geq s}(\alpha) \in \overline{T}\}$,
– $v : W \times I \rightarrow \{true, false\}$ is an assignment such that for every world $w \in W$ and every propositional letter $p \in I$, $v(w,p) = true$ iff $w \models p$.

Note that, since w's are classical propositional interpretations, in the above definition of $\mathbf{M}$ we use $w \models \alpha$ to denote that the interpretation w satisfies α in the sense of classical propositional logic.

Theorem 5. *The above defined structure* $\mathbf{M}$ *is an* LPP_{Meas}*-model.*

Theorem 6 (Completeness theorem). *Every consistent set* T *of formulas has a model from* LPP_{Meas}*.*

5 Decidability

Since, it is well known that there is a procedure to decide whether a classical propositional formula is satisfiable, to prove decidability of LPP, it is enough to show that satisfiability problem for probability formulas is decidable. We will use the linear programming theory to show that.

Let $A \in For^{P}_{LPP}$ be a probability formula and $p_1, \ldots, p_n$ be a list of all propositional letters from A. An atom a of A is a formula $\pm p_1 \wedge \ldots \pm p_n$, where $\pm p_i$ is either p_i, or $\neg p_i$. For different atoms a_i and a_j we have $\vdash a_i \rightarrow \neg a_j$. Thus, in every LPP_{Meas}−model $\mu(a_i \vee a_j) = \mu(a_i) + \mu(a_j)$. Using propositional reasoning and the fact that if $\vdash \alpha \leftrightarrow \beta$, then $\vdash P_{\geq s}\alpha \leftrightarrow P_{\geq s}\beta$, it is easy to show that every probability formula A is equivalent to a formula:

$$DNF(A) = \bigvee_{i=1}^{m} \bigwedge_{j=1}^{k_i} X_{i,j}(p_1, \ldots, p_n)$$

called a disjunctive normal form of A, where: $X_{i,j}$ is a probability operator from the set: $\{P_{\geq s_{i,j}}, P_{<s_{i,j}}, CP_{\geq s_{i,j}}, CP_{<s_{i,j}}, CP_{\leq 0}, \neg CP_{\leq 0}\}$ ($s_{i,j}$ is a rational number from $[0,1]$), and $X_{i,j}(p_1, \ldots, p_n)$ denotes that propositional formula which is in the scope of the probability operator $X_{i,j}$ is in the complete disjunctive normal form, i.e. the propositional formula is a disjunction of the atoms of A.

The formula A is satisfiable iff at least one disjunct from $DNF(A)$ is satisfiable. Let $\pm P_{\geq r_1}\alpha_1, \ldots, \pm P_{\geq r_a}\alpha_a$, $\pm CP_{\geq s_1}(\beta_1 \mid \gamma_1), \ldots, \pm CP_{s_b}(\beta_b \mid \gamma_b)$, $\pm CP_{\leq 0}(\delta_1 \mid \eta_1), \ldots, \pm CP_{\leq 0}(\delta_c \mid \eta_c)$, $a + b + c = k_i$, be an enumeration of all probability formulas which appear as conjuncts in some disjunct $D_i = \bigwedge_{j=1}^{k} X_{i,j}(p_1, \ldots, p_n)$ from $DNF(A)$, where $\pm P_{\geq r} \in \{P_{\geq r}, P_{<r}\}$, $\pm CP_{\geq r} \in \{CP_{\geq r}, CP_{<r}\}$, and $\pm CP_{\leq 0} \in \{CP_{\leq 0}, \neg CP_{\leq 0}\}$.

Let the measure of the atom a_i be denoted by x_i. We use $a_i \in \alpha$ to denote that the atom a_i appears in the complete disjunctive normal form of the classical propositional formula α. The disjunct D_i is satisfiable iff at least one of the following finitely many systems of linear equalities and inequalities is satisfiable:

$\sum_{i=1}^{2^n} x_i = 1$

$x_i \geq 0$, $(i = 1, \ldots, 2^n)$
$\sum_{a_i \in \alpha_k} x_i \geq r_k$, if $\pm P_{\geq r_k} = P_{\geq r_k}$, $(k = 1, \ldots, a)$
$\sum_{a_i \in \alpha_k} x_i < r_k$, if $\pm P_{\geq r_k} = P_{<r_k}$, $(k = 1, \ldots, a)$
$\sum_{a_i \in \gamma_k} x_i > 0$ and $\sum_{a_i \in \beta_k \wedge \gamma_k} x_i \geq s_k \sum_{a_i \in \gamma_k} x_i$, or $\sum_{a_i \in \gamma_k} x_i = 0$, if $\pm CP_{\geq s_k} = CP_{\geq s_k}$, $(k = 1, \ldots, b)$
$\sum_{a_i \in \gamma_k} x_i > 0$ and $\sum_{a_i \in \beta_k \wedge \gamma_k} x_i < s_k \sum_{a_i \in \gamma_k} x_i$, if $\pm CP_{\geq s_k} = CP_{<s_k}$, $(k = 1, \ldots, b)$
$\sum_{a_i \in \eta_k} x_i > 0$ and $\sum_{a_i \in \delta_k \wedge \eta_k} x_i = 0$, if $\pm CP_{\leq 0} = CP_{\leq 0}$, $(k = 1, \ldots, c)$
$\sum_{a_i \in \eta_k} x_i > 0$ and $\sum_{a_i \in \delta_k \wedge \eta_k} x_i > 0$, or $\sum_{a_i \in \eta_k} x_i = 0$, if $\pm CP_{\leq 0} = \neg CP_{\leq 0}$, $(k = 1, \ldots, c)$

Since the problem of satisfiability of A is reduced to the linear systems solving problem, we have:

Theorem 7. *The logic LPP is decidable.*

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