Potential theory of subordinated Brownian motions

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Beginning of potential theory

- **Newton** (1687): Law of universal gravitation, study of $F(x)$ - **force acting on a unit mass** at $x \in \mathbb{R}^d$, $d \geqslant 3$.
- **Lagrange** (1773): The above vector field (of forces) is **a gradient** of a certain function $U := U_2(x) = \mathcal{A}_{d,2}|x|^{2-d}$.
- **Green** (1828) named U **potential function**.
- **Gauss** (1840) named U **potential**.
- **Gauss**: **potential method** is suitable to resolve many complicated problems **of mathematical physics**, not only problems of gravitation or electrostatics.

More generally: for a field generated by a charge located according to a measure μ we define a **potential** of μ:

$$U_2\mu(x) = \mathcal{A}_{d,2} \int_{\mathbf{R}^d} |x-y|^{2-d}\, d\mu(y)\,, \quad \mathcal{A}_{d,2} = \frac{\Gamma(d/2-1)}{2\pi^{d/2}}\,.$$

Harmonicity of potentials

Physically, $U_2(x)$ corresponds to the potential at the point x generated by the unit charge placed at the point $0 \in \mathbb{R}^d$.
By a direct differentiation we check that the function $U_2(x-y)$ is **harmonic** for $x \in \mathbb{R}^d \setminus \{y\}$, i.e. it satisfies the Laplace equation:

$$\Delta_x U_2(x-y) = 0, \quad x \neq y,$$

where $\Delta = \sum_{i=1}^d \partial_i^2$.
More generally, $U_2\mu(x)$, potential of a measure μ, is harmonic outside the support of μ.
The same (Laplace) equation is satisfied by a stabilized temperature $T(x)$ of the body D with no inner heat sources, when heated only by the surface. To determine the temperature of the body requires to solve **the Dirichlet problem**.

Radial harmonic functions

Radial harmonic functions on $\mathbb{R}^d \setminus \{0\}$, $d \geqslant 1$, (depending only on $|x|$) are of the form

$$\begin{aligned} C_1|x| + C_2 &\qquad \text{in } \mathbb{R}^1, \\ C_1 \ln|x| + C_2 &\qquad \text{in } \mathbb{R}^2, \\ C_1|x|^{2-d} + C_2 &\qquad \text{in } \mathbb{R}^d, d \geqslant 3. \end{aligned}$$

To justify this statement, we write the Laplace equation for the function $h(r)$, where $r = |x|$. By a direct differentiation, we obtain

$$\frac{d^2h}{dr^2} + \frac{d-1}{r}\frac{dh}{dr} = 0$$

Solving this differential equation, we obtain the conclusion.

Equivalent definitions of harmonicity

Let D be a domain (i.e. connected open subset) in $\mathbb{R}^d$, $d \geqslant 1$. A Borel function f, defined on $\mathbb{R}^d$ is called **harmonic** on D if $f \in C^2(D)$ and $\Delta f \equiv 0$ on D.

Equivalent definition:

A Borel function f on $\mathbb{R}^d$, $|f| < \infty$, is harmonic on a domain D iff it satisfies **mean value property** on D; that is for every ball $B(x,r) \subset\subset D$ we have

$$f(x) = \int_{S(x,r)} f(y)\, \sigma_r(dy) = \int_{S(0,1)} f(x+ry)\, \sigma_1(dy).$$

Here σ_r is the (normalized) uniform surface measure on the sphere $S(x,r) = \partial B(x,r)$; analogously σ_1 - on the unit sphere $S(0,1)$.

Remark: spherical integration over $S(x,r)$ can be replaced by integration over $B(x,r)$ with respect to the Lebesgue measure.

Basic properties of harmonic functions

- **Maximum Principle.** Let f be harmonic in a domain $D \subset \mathbb{R}^d$ and continuous in $\overline{D}$. Then either $f(x) < sup_{u \in D} f(u)$, for $x \in D$, or $f(x) \equiv const.$ over the whole set D;
- **Harnack Inequality.** Let f be a positive harmonic in a domain $D \subset \mathbb{R}^d$. Then for every compact subset $K \subset D$ there is a constant $C > 0$ such that for every $x_1, x_2 \in K$ we have

$$C^{-1} f(x_1) \leqslant f(x_2) \leqslant C f(x_1) \, .$$

- **Harnack Theorem.** Let f_n be an increasing sequence of harmonic functions in a domain $D \subset \mathbb{R}^d$. Then either f_n is convergent to a harmonic function on D, uniformly on compact subsets of D, or f_n is everywhere divergent to $+\infty$ on D.

Dirichlet Problem (1850)

D – domain in $\mathbb{R}^d$,

φ - continuous function on ∂D (boundary of D).

Problem: Find a function $f : D \to \mathbb{R}^d$ which

- is harmonic in the domain D, that is, for $x \in D$ it satisfies

$$\Delta f(x) = \sum_{i=1}^{d} \frac{\partial^2 f(x)}{\partial x_i^2} = 0,$$

- f is continuous on $\overline{D}$ and such that $f|_{\partial D} = \varphi$.
- Solution (if it exists) is unique - Maximum principle!
- Remark: not for all domains such a function exists.

Solution of the Dirichlet Problem

If the boundary of the set D is „smooth", then there exists a function of two variables $G_D(x, y)$ such that **the solution** $f(x)$ can be expressed in the form

- $$f(x) = \int_{\partial D} \varphi(y) \frac{\partial G_D(x, y)}{\partial \vec{n}_y} \, d\sigma(y),$$
- $G_D(x, y)$ is **the Green function** of the set D,
- $\vec{n}_y$ is **normal vector** at the point y of the boundary,
- σ is **the normalized surface measure** on ∂D.
- function
  $$P_D(x, y) = \frac{\partial G_D(x, y)}{\partial \vec{n}_y}$$
  is **the Poisson kernel** of the set D.

Properties of Green function

A function $G_D(x, y)$ defined on $\overline{D} \times D$, for a domain $D \subset \mathbb{R}^d$ is called **the Green function of** D if it satisfies

- $G_D(\cdot, y)$ is harmonic on $D \setminus \{y\}$,
- $G_D(\cdot, y)$ is continuous on $\overline{D} \setminus \{y\}$ and vanishes on ∂D,
- $G_D(\cdot, y) - U_2(\cdot, y)$ remains harmonic at the point $\{y\}$.

Remark. If the Green function for a domain D exists, it is unique. Indeed, for a fixed $y \in D$ the function

$$w_y(x) = G_D(x, y) - U_2(x, y)$$

is a solution of the Dirichlet problem

$$\Delta w_y = 0 \quad \text{in} \quad D, \quad w_y(x) = U_2(x, y) \quad \text{in} \quad \partial D.$$

Physically, $G_D(x, y)$ is the potential at the point x generated by the unit charge placed at $y \in D$ and the charge on the grounded (potential 0) conducting surface ∂D.

Potentials, case $D = \mathbb{R}^d$

When $d \geqslant 3$, then **the Green function** of the whole space (traditionally called **potential** and denoted by U) is given by the formula

$$U(x,y) = \frac{\Gamma(\frac{d-2}{2})/2\pi^{d/2}}{|x-y|^{d-2}}.$$

When $d = 2$ **potential** $U(x,y) = -\frac{1}{\pi} \log |x-y|$.

When $d = 1$ **potential** $U(x,y) = -|x-y|$.

Green function - halfspace

Green function and **Poisson kernel** are expressed by explicit formulas also for D being a **halfspace** or a **ball** in $\mathbb{R}^d$. Let $D = H$, $H = \{x \in R^d \ : \ x_d > 0\}$. For $y = (y_1, .., y_{d-1}, y_d) \in H$ put

$$y^* = (y_1, .., y_{d-1}, -y_d) \qquad \text{(symmetry with respect } \{y_d = 0\}).$$

Green function of halfspace: for $x, y \in H$ and $d \geqslant 3$

- $$G_H(x,y) = \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}} \left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|x-y^*|^{d-2}} \right).$$

We subtract another unit charge placed at such a point that the resulting potential at ∂H is 0. The same apply to the case of a ball. We check that $G_H(x,y)$ is harmonic for $x \in H \setminus \{y\}$, vanishes at ∂H and $G_H(x,y) - \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}} \frac{1}{|x-y|^{d-2}}$ is harmonic for all $x \in H$.

Green function for $D = B(0,1)$

If $y \in B(0,1)$, $y \neq 0$, put $y^* = y/|y|^2$ - inversion with respect to sphere $\{|x| = 1\}$. We have $|y|/|y^*| = 1$ and $y/|y| = y^*/|y^*|$.

Green function of $B(0,1)$: for $x, \; y \in B(0,1)$, $y \neq 0$, and $d \geqslant 3$

- $$G_{B(0,1)}(x,y) = \frac{\Gamma(\frac{d-2}{2})}{2\pi^{d/2}} \left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|y|^{d-2}|x-y^*|^{d-2}} \right).$$

We put $G_{B(0,1)}(x,0) = \Gamma(\frac{d-2}{2})/2\pi^{d/2}(|x|^{2-d} - 1)$. We have $|y|^2|x-y^*|^2 = |y|^2|x|^2 - 2(x, y^*/|y^*|)|y^*||y|^2 + |y|^2|y^*|^2 = |y|^2|x|^2 - 2(x, y/|y|)|y|^2|y^*| + |y|^2|y^*|^2 = |y|^2|x|^2 - 2(x,y) + 1$. Hence, $\lim_{0 \neq y \to 0} G_{B(0,1)}(x,y) = G_{B(0,1)}(x,0)$ for $x \in B(0,1)$ at $y = 0$ and $G_{B(0,1)}(x,y) = 0$, for $|x| = 1$. It also satisfies all the remaining conditions.

Properties of Poisson kernel

A positive and continuous function function $K(x, y)$ defined on $D \times \partial D$, for a domain $D \subset \mathbb{R}^d$, is called **the Poisson kernel for** D if it satisfies

- $K(\cdot, z)$ is harmonic in D, for every $z \in \partial D$,
- $\int_{\partial D} K(x, z)\, \sigma(dz) = 1$, for every $x \in D$,
- $\lim_{D \ni x \to w} \int_{\partial D \cap B(w,\delta)^c} K(x, z)\, \sigma(dz) = 0$, for every $w \in \partial D$ and $\delta > 0$.

Here σ denotes the normalized surface measure on ∂D.
Remark. If the Poisson kernel for a domain D exists, it is unique (again, it is the unique solution of of the Dirichlet problem with the given boundary condition). If the Poisson kernel for a bounded domain D exists, then the solution of the Dirichlet problem with a boundary value $f \in C(\partial D)$ can be expressed by

$$f(x) = \int_{\partial D} K(x, z)\, f(z)\, \sigma(dz) .$$

Poisson kernel

For $x = (x_1, ..., x_d)$ put $\tilde{x} = (x_1, ..., x_{d-1})$.

For $x \in H$, $y \in \partial H$ we have the formula for Poisson kernel for H:

- $$P_H(x, y) = \frac{\Gamma(d/2)}{\pi^{d/2}} \frac{x_d}{(x_d^2 + |\tilde{y} - \tilde{x}|^2)^{\frac{d}{2}}}.$$
- When $d = 2$, then $P_H(x, y)$ is the density of the Cauchy distribution on the line $\{(x, y) : y = 0\}$.

For **a ball** $B := B(0, r)$ in $\mathbb{R}^d$, $d \geqslant 1$, **Poisson kernel** is determined by the formula:

For $x \in B(0, r)$ and $z \in \partial B(0, r)$, i.e. $|z| = r$ we obtain

$$P_B(x, z) = \frac{\Gamma(d/2)}{\pi^{d/2} r} \frac{r^2 - |x|^2}{|x - z|^d}.$$

Solution of the Dirichlet problem in a ball

The explicit formula for the Poisson kernel in a ball gives us the possibility of write down the form of the solution for the Dirichlet problem.

Solution of the Dirichlet problem in $B(x_0, r)$ with the boundary value f is given by the formula:

$$u(y) = \int_{\partial B(x_0,r)} f(x) \frac{r^2 - |y - x_0|^2}{r|y - x|^d} \, d\sigma(x) \, ,$$

where σ is the normed uniform surface measure on $\partial B(x_0, r)$, and f is defined and continuous on $\partial B(x_0, r)$.

A direct consequence of the above formula is the Harnack Inequality and Harnack Theorem for a ball and, consequently, for compact subsets.

Brownian motion and the Dirichlet problem

In 40-ties of XX century **S.Kakutani**, and in 50-ties **J.L.Doob** explained how to solve **the Dirichlet problem** in terms of **Brownian motion** . Foundations of the contemporary potential theory of Markov processes are due to **G.Hunt** (1957, 1958).

let $W(t)$ be the Brownian motion starting from $\mathbb{R}^d$ and let D be a (regular) domain in $\mathbb{R}^d$. Assume that Brownian motion starts from the point $x \in D$ and put

- $\tau_D = \inf\{t > 0 : \ W_t \notin D\}$ — the first exit time from the set D.
- Function

$$f(x) = \mathbb{E}^x \left(\varphi(W_{\tau_D})\right)$$

- is the solution of the Dirichlet problem for D and φ.

Stopping time

Let (Ω, Σ, P) be a probability space, $\Omega_\tau \subseteq \Omega$, T - an interval $\overline{\mathbb{Z}}$ or $\overline{\mathbb{R}}$; $\{\mathcal{F}_t; t \in T\}$ - increasing family of sub σ-algebras Σ.

Definition. A positive random variable $\tau : \Omega_\tau \longrightarrow T$ is called a stopping time, (Markov time) if $\{\tau \leqslant t\} \in \mathcal{F}_t$, $t \in T$

We also define
$\mathcal{F}_\tau := \{A \subseteq \Omega_\tau; A \cap \{\tau \leqslant t\} \in \mathcal{F}_t, \text{ for every } t \in T\}$.

Remark. 1. When τ is countably valued then the above definition is equivalent to the following: τ is stopping time with respect to $\{\mathcal{F}_n\}$ if for every n the following holds: $\{\tau = n\} \in \mathcal{F}_n$.
2. $\mathcal{F}_\tau$ is a σ-algebra $\subseteq \Omega_\tau \cap \Sigma$.
3. $\tau : (\Omega_\tau, \mathcal{F}_\tau) \longrightarrow (T, \mathcal{B}_T)$ is measurable.

Markov property of the process $X = \{X_t; t \in T\}$

Let $\theta_s : (\Omega, \Sigma) \longrightarrow (\Omega, \Sigma)$ acts as a "shift" on the basic probability space according to the rule: $X_t \circ \theta_s = X_{t+s}$. The easiest way to perceive these operators is to work on the standard probability space $(\mathbb{R}^{[0,\infty)}, \otimes_{t\geqslant 0}\mathcal{B}_{\mathbb{R}}, \mu)$, where μ is the distribution of the process X. Then $X_t(\omega) = \omega(t)$ and $X_t(\omega) \circ \theta_s = \omega(t+s)$. Further, we consider the process with the initial distribution $X(0) = Y$ - an arbitrary random variable. Conditional expectation (probability) with respect to a process with the initial distribution Y we denote by $\mathbb{E}^Y[\cdot]$, $(P^Y(\cdot))$. When $Y = x \in \mathbb{R}^d$ we write $\mathbb{E}^x[\cdot]$, $(P^x(\cdot))$.

Markov property of $\{X_t; t \geqslant 0\}$: for $Z \geqslant 0$, $\mathcal{F}_\infty$-measurable

$$\mathbb{E}^x[Z \circ \theta_t | \mathcal{F}_t] = \mathbb{E}^{X_t}[Z],$$

where $\mathcal{F}_t = \sigma\{X_s; s \leqslant t\}$, $\mathcal{F}_\infty = \sigma\{X_s; s \geqslant 0\}$.

Strong Markov property of the process X

For τ - $\mathcal{F}_t$-stopping time and $Z \geqslant 0$, $\mathcal{F}_\infty$-measurable random variables, we have

$$\mathbb{E}^x[Z \circ \theta_\tau | \mathcal{F}_\tau] = \mathbb{E}^{X_\tau}[Z],$$

where $\mathcal{F}_t = \sigma\{X_s; s \leqslant t\}$, $\mathcal{F}_\infty = \sigma\{X_s; s \geqslant 0\}$.

Remark. When $\{X_t; \mathcal{F}_t; t \geqslant 0\}$ has a Markov property and $\mathcal{F}_t$, $\mathcal{F}_t$ is right-continuous, i.e. $\mathcal{F}_t = \mathcal{F}_{t+} = \bigcap_{s>t} \mathcal{F}_t$ and complete, and X_t is a normal Markov process, then $\{X_t; \mathcal{F}_t; t \geqslant 0\}$ **has the strong Markov property**.

Normal Markov process - phase space S is compact, metric and separable, process has a Feller property and P_t defined by $P_t f(x) = \int f(y) P_t(x, dy) = \mathbb{E}^x f(X_t)$ acts on $C(S)$ as a strongly continuous contraction semigroup (process is stochastically continuous).

Regular points of the process

Let $\{X_t\}_{t\geqslant 0}$ be a stochastic process with values in $\mathbb{R}^d$ and $D\subseteq\mathbb{R}^d$ - a Borel subset. Define the first exit time from D:

$$\tau_D=\inf\{t>0; X_t\notin D\}.$$

Definition. The point $x\in\mathbb{R}^d$ is called regular for D when $P^x(\tau_D=0)=1$.

We further assume that $X=W$ is a Wiener process in $\mathbb{R}^d$ and $\mathcal{F}_t=\sigma\{W_s; s\leqslant t\}$. We note that $\{\tau_D=0\}\in\mathcal{F}_{0+}$ so by the $0-1$ Blumenthal law we have either $P^x(\tau_D=0)=0$ or 1. The set of all regular points of the set D is denoted by D^r. When $x\in Int(D^c)$ then $x\in D^r$. When $x\in Int(D)$ then the Wiener process remains certain time in $Int(D)$ so $Int(D)\subseteq(D^r)^c$. The only problem is to determine what is the behaviour of the process at $x\in\partial D$. Typically, the process oscilates wildly in the vicinity of the point $x\in\partial D$, hence it leaves immediately from D.

Exterior cone property

Definition. Let $V_a = \{(x_1, \ldots, x_d); x_1 > 0; |(x_2, \ldots, x_d)| < ax_1\}$.
A cone V in $\mathbb{R}^d$ is a translation and a rotation of V_a.

Let $z \in \partial D$. If there exists a cone V with the vertex z such that $V \cap B(z, r) \subseteq D^c$ for a $r > 0$, then z is regular.

Proof. Put $C = \frac{\sigma_r(V \cap \mathbb{S}_r(z))}{\sigma_r(\mathbb{S}_r(z))}$ and $B_n = B(z, r/n)$,
$V_n = V \cap \mathbb{S}_{r/n}(z)$. By the rotational invariance (with respect to the starting point) of the distribution of the Wiener process, $W_{\tau_{B(x,r)}}$ is also rotationally invariant hence it is the normed spherical measure on $\mathbb{S}_r(x)$. Hence $P^x(W_{\tau_{B(x,r)}} \in V) = C$. At the same time,

$$P^z(\tau_D = 0) \geqslant P^z(\limsup\{W_{\tau_{B_n}} \in V_n\}) \geqslant \limsup P^z(\{W_{\tau_{B_n}} \in V_n\})$$

so $P^z(\tau_D = 0) \geqslant C > 0$ thus $0 - 1$ Blumenthal law implies $P^z(\tau_D = 0) = 1$.

Probabilistic solution of the Dirichlet problem

Let D be a bounded in $\mathbb{R}^d$ and $f \in L^\infty(\partial D)$. The function $H_D f(x)$ defined by

$$H_D f(x) = \mathbb{E}^x[\tau_D < \infty; f(W_{\tau_D})]$$

is harmonic in D. If z is regular and f - continuous at $z \in \partial D$ then $\lim_{D \ni x \to z} H_D f(x) = f(x)$

Proof of harmonicity. Let $x \in D$, $B \subset\subset B(x,r)$. Since $\tau_B < \tau_D$ - P^x a.s. so $\tau_D = \tau_B + \tau_D \circ \theta_{\tau_B}$. Moreover, $W_{\tau_D} \circ \theta_{\tau_B} = W_{\tau_B + \tau_D \circ \theta_{\tau_B}}$ $= W_{\tau_D}$. Let $\Psi = \mathbf{1}_{\{\tau_D < \infty\}} f(W_{\tau_D})$. It holds $\Psi \circ \theta_{\tau_B} = \Psi$ hence $\mathbb{E}^x[\Psi] = \mathbb{E}^x[\mathbb{E}^x[\Psi \circ \theta_{\tau_B} | \mathcal{F}_{\tau_B}]] = \mathbb{E}^x[\mathbb{E}^{W_{\tau_B}}[\Psi]] = \mathbb{E}^x[H_D f(W_{\tau_B})]$. Since the distribution of W_{τ_B} is the uniform normalized spherical measure, therefore $H_D f(x) = \frac{1}{r^{d-1}\omega_d} \int_{\mathbb{S}_r(x)} H_D f(y) \sigma_r(dy)$, and $H_D f$ has the mean value property in D, so it is harmonic in D.

Convergence at regular points - auxiliary lemmas

Lemma 1. If $f \in L^\infty(\mathbb{R}^d)$ or $f \in L^1(\mathbb{R}^d)$ then $P_t f(\cdot) \in C(\mathbb{R}^d)$
for every $t > 0$.

Proof. For $f \in L^\infty(\mathbb{R}^d)$ and $x_n \longrightarrow x$ we obtain $|P_t f(x_n) - P_t f(x)| \leqslant ||f||_\infty \int_{\mathbb{R}^d} |p_t(x_n, y) - p_t(x, y)| dy$ and the integral on the left-hand side converges since $\int_{B(0,r)^c} |\ldots| < \varepsilon$, for large r. We also have $\int_{B(0,r)} |\ldots| \longrightarrow 0$, by bounded convergence theorem. For $f \in L^1(\mathbb{R}^d)$ we again apply bounded convergence theorem. Here $p_t(x, y) = \frac{1}{(2\pi t)^{d/2}} e^{-||x-y||^2/2t}$ - the transition density of the Wiener process in $\mathbb{R}^d$.

Corollary. Proces X_t has both Feller and strong Feller property, i.e. $P_t : C_0(\mathbb{R}^d) \longrightarrow C_0(\mathbb{R}^d)$ and $P_t : L^\infty(\mathbb{R}^d) \longrightarrow C(\mathbb{R}^d)$

Convergence at regular points - auxiliary lemmas

Feller property of the process.
We show that $P_t : C_0(\mathbb{R}^d) \longrightarrow C_0(\mathbb{R}^d)$. To do this, observe that if $f \in C_0(\mathbb{R}^d)$ then for every $\varepsilon > 0$ there exists $r > 0$ such that $|f(y)| < \varepsilon$ if only $|y| > r$. Then we have

$$|P_t f(x)| \leqslant \varepsilon + ||f||_\infty \int_{B(0,r)} p(t; x, y)\, dy\, .$$

The integral on the right-hand side tends to 0 when $x \longrightarrow \infty$.
The property that $\lim_{t\to 0} ||P_t f - f||_\infty = 0$, for $f \in C_0(\mathbb{R}^d)$ follows from the uniform continuity of functions in $C_0(\mathbb{R}^d)$. This finishes the proof of the Feller property of the process.

Lemma 2. The function $x \longrightarrow \mathbb{E}^x[\kappa \circ \theta_t]$ is continuous on $\mathbb{R}^d$ for $t > 0$ and κ bounded and $\mathcal{F}_\infty$-measurable.

Convergence at regular points - auxiliary lemmas

Proof. Let $f(x) = \mathbb{E}^x[\kappa]$. It holds $f \in L^\infty(\mathbb{R}^d)$. Applying the Markov property: $\mathbb{E}^x[\kappa \circ \theta_t] = \mathbb{E}^x[\mathbb{E}^x[\kappa \circ \theta_t | \mathcal{F}_t]] = \mathbb{E}^x[\mathbb{E}^{W_t}[\kappa]] = \mathbb{E}^x[f(W_t)] = P_t f(x) \in C(\mathbb{R}^d)$.

Remark. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is called upper semicontinuous if it is a decreasing limit of continuous functions. We have $\limsup_{x \to x_0} \phi(x) \leqslant \phi(x_0)$.

Lemma 3. The function $\phi : x \longrightarrow P^x(\tau_D > t)$ is upper semicontinuous on $\mathbb{R}^d$ for $t > 0$ and any D open in $\mathbb{R}^d$.

Proof. We show that $P^x(\tau_D > t) = \lim_{s \downarrow 0} \downarrow P^x(\tau_D \circ \theta_s > t - s)$ $= \lim_{s \downarrow 0} \mathbb{E}^x[\mathbf{1}_{(t-s,\infty)}(\tau_D) \circ \theta_s]$. We note that $\inf\{t > s; W_t \notin D\}$ $= s + \tau_D \circ \theta_s$. Let $x \in D^r$, i.e. $P^x(\tau_D = 0) = 1$. There exists a sequence $s_n \downarrow 0$ such that $W_{s_n} \in D^c$ thus for $s < s_n$ it holds $s + \tau_D \circ \theta_s < s_n$. Now, $\{s + \tau_D \circ \theta_s > t\}_{0<s<t}$ increases in s, so

Convergence at regular points

$\lim_{s\downarrow 0} \downarrow P^x(\tau_D \circ \theta_s > t - s) = P^x(0 > t) = 0 = P^x(\tau_D > t)$.
Let now $x \notin D^r$, i.e. $\tau_D > 0$ P^x a.e. For $s < t$ it holds $\{\tau_D > s\} \supset \{\tau_D > t\}$. If $\tau_D > s$ then $\tau_D = s + \tau_D \circ \theta_s$ hence $\tau_D \circ \theta_s > t - s$. Moreover, $\tau_D \circ \theta_s > t - s$, for $s < t$, which with $\tau_D > s_0$, for some $s_0 < t$ $(\tau_D > 0)$ yields $\tau_D > t$. This justifies our formula, hence ϕ is a decreasing limit of continuous functions (Lemma 2) - consequently, it is upper semicontinuous.
Proof of convergence at regular points. Let z be a regular point from ∂D and let f be continuous at z. For $\varepsilon > 0$ there exists $\delta > 0$ such that for $w \in \partial D \cap B(z, \delta)$ we have $|f(w) - f(z)| < \varepsilon/2$. Put $M = ||f||_\infty$. Since $P^x(\tau_{B(x,\delta/2)} > 0) = P^0(\tau_{B(0,\delta/2)} > 0) = 1$ we see that there exists $s > 0$ such that for every x it holds $P^x(\tau_{B(x,\delta/2)} \leqslant s) < \varepsilon/8M$. By Lemma 3 $\limsup_{x\to z} P^x(\tau_D > s) \leqslant P^z(\tau_D > s) = 0$. Thus, there exists $\delta' > 0$ such that if $|x - z| < \delta'$ then $P^x(\tau_D > s) < \varepsilon/8M$.

Convergence at regular points

Moreover, $P^x(\tau_{B(x,\delta/2)} \leqslant \tau_D) \leqslant P^x(\tau_{B(x,\delta/2)} \leqslant s) + P^x(\tau_D > s)$.
Therefore, when $|x - z| < \delta/2$ to $\tau_{B(x,\delta/2)} \leqslant \tau_{B(z,\delta)}$ then
$P^x(\tau_{B(z,\delta)} \leqslant \tau_D) \leqslant P^x(\tau_{B(x,\delta/2)} \leqslant \tau_D) \leqslant \varepsilon/8M + \varepsilon/8M = \varepsilon/4M$.
If $x \in \overline{D}$ and $|x - z| < \delta' \wedge (\delta/2)$ we obtain
$\mathbb{E}^x[\tau_D < \infty; |f(X_{\tau_D}) - f(z)|] \leqslant$
$P^x(\tau_D < \tau_{B(z,\delta)})\varepsilon/2 + P^x(\tau_{B(z,\delta)} \leqslant \tau_D)2M \leqslant \varepsilon/2 + (\varepsilon/4M)2M$
and this concludes the proof.
As an application, for the solution $u_{1,0}$ of the Dirichlet problem with boundary values $f \equiv 1$ on $\mathbb{S}_\delta(0)$ and $f \equiv 0$ on $\mathbb{S}_R(0)$ we obtain

- **(i)** $u_{1,0}(x) = \frac{\ln R - \ln ||x||}{\ln R - \ln \delta}$, $x \in \overline{D}$, **for** $d = 2$,
- **(ii)** $u_{1,0}(x) = \frac{||x||^{2-d} - R^{2-d}}{\delta^{2-d} - R^{2-d}}$, $x \in \overline{D}$, **for** $d \geqslant 3$.

Recurrence and transitivity of the Wiener process

From the uniqueness of the solution of Dirichlet problem, we obtain $u_{1,0}(x) = P^x(||W_{\tau_D}|| = \delta) = P^x(W_t \text{ hits } \mathbb{S}_\delta \text{ before hitting } \mathbb{S}_R)$. Fix $\delta > 0$ and let $R \to \infty$. For $\mathbb{R}^2$ (case (i)): $\lim_{R\to\infty} u_{1,0}(x) = 1$. When $d \geqslant 3$ (case (ii)): $\lim_{R\to\infty} u_{1,0}(x) = (\delta/||x||)^{d-2}$. Hence

- $d = 2$: $P^x(W_t \text{ hits } \mathbb{S}_\delta \text{ , for some } t > 0) = 1$.
- $d \geqslant 3$: $P^x(W_t \text{ hits } \mathbb{S}_\delta \text{ , for some } t > 0) = (\delta/||x||)^{d-2}$.

Let now $d = 2$, $R > 0$ and $\delta \to 0$. We obtain $\lim_{\delta\to 0} u_{1,0}(x) = 0$. so $P^x(W_t \text{ hits } 0) = 0$. We repeat the same arguments for every translation of D so we obtain

Two-dimensional Wiener process, starting from x, 1 does not hit any fixed point $y \neq x$, almost surely.

Killed process

Let X_t be a Markov process with the transition density function $p(t; x, y)$. Define the first exit time from the set D:

$$\tau_D = \inf\{t > 0 : X(t) \notin D\}$$

and the process killed at the time of first exit from D:

$$X_D(t) = \begin{cases} X(t), & \text{gdy } 0 \leqslant t < \tau_D, \\ \partial, & \text{gdy } t \geqslant \tau_D. \end{cases}$$

where ∂ is a "cemetery" – a certain, isolated state of the space of the values of the process X. Its transition function is of the form

$$P_t^D(x, A) = P^D(t; x, A) = P^x(t < \tau_D; X_t \in A),\ t > 0,\ x \in D,\ ;$$

and its transition density (if X has one) is given by

Hunt's Formula

$$p^D(t; x, y) = p(t; x, y) - \mathbb{E}^x[\tau_D < t; p(t - \tau_D; X_{\tau_D}, y)].$$

Basic properties of killed process

Justification of the Hunt's Formula

For a bounded Borel function f we obtain

$$\begin{aligned}
&\int_D \mathbb{E}^x[\tau_D < t; p(t-\tau_D; X_{\tau_D}, y)]\, f(y)\, dy \\
= \quad & \mathbb{E}^x[\tau_D < t; \int_D p(t-\tau_D; X_{\tau_D}, y)\, f(y)\, dy] \\
= \quad & \mathbb{E}^x[\tau_D < t; \mathbb{E}^{X_{\tau_D}}[f(X_s)]|_{s=t-\tau_D}] \\
= \quad & \mathbb{E}^x[\mathbb{E}^x[\tau_D < t; f(X_{s+\tau_D})|\mathcal{F}_{\tau_D}]|_{s=t-\tau_D}] \\
= \quad & \mathbb{E}^x[\mathbb{E}^x[\tau_D < t; f(X_t)|\mathcal{F}_{\tau_D}]] = \mathbb{E}^x[\tau_D < t; f(X_t)]\,.
\end{aligned}$$

Subtracting from the first part of the formula, with $f(y)$ integrated over D, we obtain

$$\mathbb{E}^x[f(X_t)] - \mathbb{E}^x[\tau_D < t; f(X_t)] = \mathbb{E}^x[t < \tau_D; f(X_t)]\,.$$

Feller properties of killed process

Theorem. For regular D the killed process has Feller and strong Feller property

Semigroup, Feller and strong Feller properties. For $f \in L^\infty(D)$ and $0 < s < t$

$$\begin{aligned} P_t^D f(x) &= \mathbb{E}^x[s < \tau_D; \mathbb{E}^{X_s}[t-s < \tau_D; f(X_{t-s})]] \\ &= P_s^D P_{t-s}^D f(x) = \mathbb{E}^x[s < \tau_D; \phi_{t-s}(X_s)] = P_s^D \phi_{t-s}(x), \end{aligned}$$

where $\phi_s(x) = \mathbb{E}^x[s < \tau_D; f(X_s)]]$. This proves the semigroup property of P_t^D. Furthermore, $P_s \phi_{t-s} \in C_b(\mathbb{R}^d)$ and

$$|P_s \phi_{t-s}(x) - P_t^D f(x)| = |P_s \phi_{t-s}(x) - P_s^D \phi_{t-s}(x)| \leqslant P^x(\tau_D \leqslant s)\, ||f||_\infty .$$

We show that $P^x(\tau_D \leqslant s)$ converges uniformly to zero, as $s \to 0$, on any compact subset of D. This will show that $P_t^D f$ is continuous in D, so $P_t^D f \in C_b(D)$.

Feller property of killed process

$P^x(\tau_D \leqslant s)$ **converges uniformly to zero, as** $s \to 0$**, on any compact subset of** D.

Indeed, for $x \in D$ and small $r > 0$ we have $\tau_{B(x,r)} \leqslant \tau_D$ hence $\{\tau_D \leqslant s\} \subseteq \{\tau_{B(x,r)} \leqslant s\}$ and we obtain, as $s \to 0$,

$$\begin{aligned} & P^x(\tau_D \leqslant s) \leqslant P^x(\tau_{B(x,r)} \leqslant s) \\ = \quad & P^0(\tau_{B(0,r)} \leqslant s) \to 0\,. \end{aligned}$$

By compactness arguments, we obtain the conclusion.

Now, by lower semicontinuity of $x \to P^x(\tau_D > t)$ we obtain for any $z \in \partial D$

$$\begin{aligned} \limsup_{x \to z} P_t^D f(x)| & \leqslant ||f||_\infty \limsup_{x \to z} P_t^x(\tau_D > t) \\ & \leqslant ||f||_\infty P_t^z(\tau_D > t) \end{aligned}$$

and the last expression is 0 if z is regular. This, along with the strong continuity of the semigroup, proves the Feller property.

Killed process

Stopping or killing the process

- $\tau_D = \inf\{t > 0 : X(t) \notin D\}$ - first exit time (from D)
- $X_{\tau_D \wedge t}$ - stopped process (when exiting from D)
- $X_t, t < \tau_D$ - killed process (when exiting from D)

The simplest (conceptually) object - first exit time τ_D. The most widely used object - X_{τ_D} - the stopped process (at the first hitting time). The density of distribution of X_{τ_D} - called Poisson kernel of the set D - provides the solution of the Dirichlet problem. Killed process - very difficult to investigate.
The Hunt's formula indicates that if we know the distribution of (τ_D, X_{τ_D}) then we are able to determine the transition density of the killed process. The basic example - Brownian motion and D - a halfspace . The starting point – reflection principle for Brownian motion.

Reflection Principle for Brownian motion

Let $W = (W_t)_{t \geqslant 0}$ be a Brownian motion in $\mathbb{R}^1$ (starting from 0) and τ - a stopping time with respect to W. Put

$$\rho_\tau W_t = \begin{cases} W_t, & t \leqslant \tau, \\ 2W\tau - W_t, & t > \tau. \end{cases}$$

Reflection Principle: $\rho_\tau W_t$ is a Brownian motion

Corollary. $P(\max_{s \leqslant t} W_s > a) = 2P(W_t > a) = P(|W_t| > a)$

Remark. We apply Reflection Principle to compute the distribution of the first exit time from the halfspace (a, ∞), where $a > 0$ and the process starts from 0. We denote $\tau_a := \tau_{(a,\infty)}$.

Density function of distribution of τ_a

We have

$$\{\max_{s\leqslant t} W_t > a\} = \{\tau_a \leqslant t\}$$

We compute the density function of the random variable τ_a:

$$\begin{aligned}
\frac{d}{dt}P\{\tau_a \leqslant t\} &= \frac{d}{dt}P\{\max_{s\leqslant t} W_t > a\} = \\
\frac{d}{dt}\Big[\sqrt{\frac{2}{\pi t}}\int_a^\infty e^{-x^2/2t}dx\Big] &= -\frac{1}{2}\sqrt{\frac{2}{\pi t^3}}\int_a^\infty e^{-x^2/2t}dx + \\
\sqrt{\frac{2}{\pi t}}\int_a^\infty \frac{x^2}{2t^2}e^{-x^2/2t}dx &\overset{int.by\,parts}{=} \frac{a}{\sqrt{2\pi t^3}}e^{-a^2/2t}
\end{aligned}$$

Laplace transform of τ_a

$$\mathbb{E}[e^{-\lambda^2\tau_a}] = \frac{a}{\sqrt{2\pi}}\int_0^\infty e^{-\lambda^2 u}e^{-a^2/2u}u^{-3/2}du$$

$$= \frac{a}{\sqrt{2\pi}}2\left(\frac{a^2}{2\lambda^2}\right)^{-1/4}\mathbf{K}_{-1/2}(\sqrt{2}\lambda a)$$

$$= \frac{a}{\sqrt{2\pi}}2\sqrt{\frac{\sqrt{2}\lambda}{a}}\sqrt{\frac{\pi}{2\sqrt{2}a\lambda}}\,e^{-\sqrt{2}\lambda a} = e^{-\sqrt{2}\lambda a}.$$

where $\mathbf{K}_\vartheta$ - the modified Bessel function of second kind:

$$\int_0^\infty e^{-au}e^{-b/u}u^{\nu-1}du = 2(b/a)^{\nu/2}\mathbf{K}_\nu(2\sqrt{ab})$$

Moreover, $\mathbf{K}_{-1/2}(x) = \mathbf{K}_{1/2}(x) = \sqrt{\frac{\pi}{2x}}e^{-x}$.

Transition density of the killed process

The most fundamental object in potential theory - transition probability P^D of the process $X_D(t)$ killed at the first exit time from the set D: for $x, y \in D$ we put

$$P^D(x; A) = P^x[t < \tau_D; X_t \in A].$$

When X has the transition density $p(t; x, y)$ then the transition density of the killed process can be expressed by the formula:

$$p^D(t; x, y) = p(t; x, y) - \mathbb{E}^x[\tau_D < t; p(t - \tau_D; X_{\tau_D}, y)].$$

Kelvin's symmetry principle gives us $p^D(t; x, y)$ for halfspace $D = H = \{x \in \mathbb{R}^d \,:\, x_d > 0\}$. For $y = (y_1, .., y_{d-1}, y_d) \in H$ put

$$y^* = (y_1, .., y_{d-1}, -y_d).$$

We then obtain

$$p^D(t; x, y) = p(t; x, y) - p(t; x, y^*).$$

Transition density of the killed process (optional)

As an exercise we compute $p^D(t;x,y)$ directly from Hunt's formula. The time τ_D is determined by the last coordinate of the process; consequently, it does not depend on the first $(d-1)$ coordinates of the process. Now, $y \to \tilde{y}$ denotes the projection onto first $(d-1)$ coordinates. We thus obtain

$$\begin{aligned}
&\mathbb{E}^x[\tau_D < t; p(t-\tau_D; X_{\tau_D}, y)] \\
&= \mathbb{E}^x\Big[\int_0^t \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}}\, p(t-s,(\tilde{X}_s,0),y)\, ds\Big] \\
&= \int_{\mathbb{R}^{d-1}} \int_0^t \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}} \frac{e^{-|z-\tilde{x}|^2/2s}}{(2\pi s)^{(d-1)/2}} \frac{e^{-(|z-\tilde{y}|^2+y_d^2)/2(t-s)}}{(2\pi(t-s))^{d/2}}\, ds\, dz \\
&= \int_0^t \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}} \frac{e^{-y_d^2/2(t-s)}}{\sqrt{2\pi(t-s)}} \\
&\quad \left\{ \int_{\mathbb{R}^{d-1}} \frac{e^{-|z-\tilde{x}|^2/2s}}{(2\pi s)^{(d-1)/2}} \frac{e^{-|z-\tilde{y}|^2/2(t-s)}}{(2\pi(t-s))^{(d-1)/2}}\, dz \right\} ds\,.
\end{aligned}$$

Transition density of the killed process (optional)

Now, the expression in parentheses is the convolution of two $(d-1)$-dimensional Gaussian densities hence is equal to

$$\frac{e^{-|\tilde{x}-\tilde{y}|^2/2t}}{(2\pi t)^{(d-1)/2}} .$$

To compute the remaining expression

$$\int_0^t \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}} \frac{e^{-y_d^2/2(t-s)}}{\sqrt{2\pi(t-s)}}\, ds$$

we take the Laplace transform and, after changing order of integration and variables we obtain

$$\int_0^\infty \frac{x\, e^{-x^2/2s}}{\sqrt{2\pi s^3}}\, e^{-\lambda\, s}\, ds \int_0^\infty e^{-\lambda\, u}\, \frac{e^{-y_d^2/2u}}{\sqrt{2\pi u}}\, du$$

Transition density of the killed process (optional)

The last expression can be expressed in terms of modified Bessel function $\mathbf{K}_{1/2}$ of the second order as follows

$$\frac{2}{\sqrt{2\pi}}\left(\frac{y_d^2}{2\lambda}\right)^{1/4}\mathbf{K}_{1/2}(\sqrt{2\lambda}\,y_d)=\frac{e^{-\sqrt{2\lambda}\,y_d}}{\sqrt{2\lambda}}$$

while the first one is of the form

$$\frac{2\,x_d}{\sqrt{2\pi}}\left(\frac{x_d^2}{2\lambda}\right)^{-1/4}\mathbf{K}_{-1/2}(\sqrt{2\lambda}\,x_d)=e^{-\sqrt{2\lambda}\,y_d}\,.$$

Hence, after multiplication we obtain

$$\frac{e^{-\sqrt{2\lambda}\,(x_d+y_d)}}{\sqrt{2\lambda}}=\mathcal{L}\left(\frac{e^{-(x_d+y_d)^2/2t}}{\sqrt{2\pi t}}\right)\,.$$

Thus, the whole expression is of the form $(2\pi t)^{-d/2}\,e^{-|x-y^*|^2/2t}$ where $y^*=(y_1,y_2,\ldots,y_{d-1},-y_d)$.

Green function and Poisson kernel

Poisson kernel and **Green function** of the set D have simple explanations in terms of the process killed or stopped when exiting D:

- $$P_D(x,y) = P^x\left(X_{\tau_D} \in dy\right)$$
  the density of the distribution of hitting the boundary of the set D;
- $$G_D(x,y) = \int_0^\infty p^D(t;x,y)\,dt$$
  „density" of occupying time of the process at y.

Green operator

For bounded Borel functions $f : \mathbb{R}^d \to \mathbb{R}$ and domain D put **Green operator**:

- $$G_D f(x) = \mathbb{E}^x \left(\int_0^{\tau_D} f(X_t)\, dt \right).$$
- Green function is the kernel of this operator:
$$G_D f(x) = \int_D G_D(x, y)\, f(y)\, dy.$$
- In particular, when $f = 1_A$, we obtain
$$G_D 1_A(x) = \mathbb{E}^x \left[\int_0^{\tau_D} 1_A(X_t)\, dt \right] = \int_D G_D(x, y)\, 1_A(y)\, dy$$
is the mean occupying time of the process, starting from x, within the set A .

Properties of Green function

For $d \geqslant 3$ we have

$$G_D(x,y) = U_2(x,y) - \mathbb{E}^x[\tau_D < \infty; U_2(X_{\tau_D}, y)] \,.$$

Indeed, denote

$$r^D(t;x,y) = \mathbb{E}^x[\tau_D < t; p(t-\tau_D; X_{\tau_D}, y);] \,.$$

We obtain, for $x \neq y$,

$$\int_0^\infty r^D(t;x,y)\,dt = \mathbb{E}^x[\tau_D < \infty; \int_0^\infty p(u; X_{\tau_D}, y)\,du] \,.$$

which justifies the formula, if we show that this expression is finite. Putting $\delta = \rho(y, \partial D)$ we obtain

$$\mathbb{E}^x[\tau_D < \infty; U(X_{\tau_D}, y)] \leqslant U(\delta) < \infty \,.$$

For $d = 1,2$ the corresponding formulas are also valid but in terms of compensated potentials instead.

Potential operator

When $D = \mathbb{R}^d$, $d \geqslant 3$, computing as before, we obtain **the potential operator** and **the potential function**:

- $$U_2 f(x) = \int_0^\infty \mathbb{E}^x[f(X_t)]\,dt.$$
- Potential function is the kernel of this operator:
$$U_2 f(x) = \int_{\mathbb{R}^d} U_2(x, y)\, f(y)\, dy.$$
- We obtain
$$U_2(x, y) = \int_0^\infty \frac{1}{(2\,\pi\, t)^{d/2}}\, e^{-|x-y|^2/2t}\, dt = \frac{1}{2\pi^{d/2}} \frac{\Gamma(d/2-1)}{|x-y|d-2}.$$

In this way, we obtained the same object as at the beginning, thus exemplifying the connection between analytical and probabilistic theories. For $d = 1, 2$ analogous formulas are valid, but with different proofs.

Brownian motion: $\mathfrak{U}f = \frac{1}{2}f''$ on $\mathfrak{D}_{\mathfrak{U}} = C^{(2)}$

Generator of Brownian motion.

$S = \mathbf{R}^*$, $P_t(x,E) = \frac{1}{\sqrt{2\pi t}} \int_{E-x} e^{-y^2/2t} dy$, $P_t(\infty,\infty) = 1$. We have $P_t f(x) = \frac{1}{\sqrt{2\pi t}} \int f(y) e^{-(y-x)^2/2t} dy$, f -bounded Borel. If $f \in C^{(2)}$ then $\frac{1}{t}[T_t f(x) - f(x)] = \frac{1}{\sqrt{2\pi t}} \int e^{-y^2/2t} \frac{f(x+y)-f(x)-f'(x)\,y}{t} dy = \frac{1}{\sqrt{2\pi t}} \int e^{-y^2/2t} \frac{f''(x+\theta y)}{2t} y^2 dy = \frac{1}{2\sqrt{2\pi t}} \int e^{-u^2/2} f''(x+\theta u\sqrt{t}) u^2 du \to \frac{1}{2} f''(x)$, for every x, when $t \to 0$. We thus have $f \in C^{(2)} \subseteq \mathfrak{D}_{\mathfrak{U}}$ and $\mathfrak{U}f = \frac{1}{2}f''$ on $C^{(2)}$. Furthermore, $\mathfrak{D}_{\mathfrak{U}} = C^{(2)}$.

For $g \in C$, we solve in f: $\lambda f - \frac{1}{2}f'' = g$. We obtain $f(x) = \mathfrak{R}_\lambda g(x) = \frac{1}{\sqrt{2\pi}} \int [\int_0^\infty e^{-\lambda t} e^{-(y-x)^2/2t} t^{-1/2} dt] g(y) dy = \frac{1}{\sqrt{2\lambda}} \int g(y) e^{-\sqrt{2\lambda}|x-y|} dy$, since $\int_0^\infty e^{-au} e^{-b/u} u^{\nu-1} du = 2(b/a)^{\nu/2} \mathbf{K}_\nu(2\sqrt{ab})$ (modified Bessel function of 2-nd order), and we have $\mathbf{K}_{\frac{1}{2}}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$. $f \in C^{(2)}$, it solves $\lambda f - f''/2 = g$.

Fundamental solution of $\frac{1}{2}\Delta = \delta_0$

For $f \in C_c(\mathbb{R}^d)$ the following holds

$$\frac{1}{2}\Delta\, U_2 f(x) = -f(x)\,.$$

Proof. We obtain

$$P_t\, U_2\, f(x) = \mathbb{E}^x[\int_0^\infty \mathbb{E}^{X_t}[f(X_s)]\, ds]$$

$$= \int_0^\infty \mathbb{E}^x[\mathbb{E}^{X_t}[f(X_s)]]\, ds = \int_0^\infty \mathbb{E}^x[f(X_{t+s})]\, ds$$

$$= \int_t^\infty \mathbb{E}^x[f(X_u)]\, du\,.$$

We thus have obtained

$$P_t\, U_2\, f(x) - U_2\, f(x) = -\int_0^t \mathbb{E}^x[f(X_u)]\, du\,.$$

After dividing by t, we obtain the conclusion when $t \to 0$.

$\Delta(G_D\phi) = -2\phi$ for $\phi \in C_c(D)$

We use the following representation of the Green function

$$G_D(x,y) = U_2(x,y) - \mathbb{E}^x[\tau_D < \infty; U_2(X_{\tau_D}, y)]\,.$$

From the previous result we obtain

$$\Delta\, U_2\phi(x) = -2\phi(x)\,.$$

However, the second term in the representation of G_D, acting on ϕ, gives the harmonic function so the result follows.
If $\phi \in C_c^\infty(D)$ then $G_D\phi \in C_c^\infty(D)$ and we also have

$$G_D(\Delta\phi) = -2\phi\,.$$

For $d = 1$ and $d = 2$ additional assumptions are required, e.g. boundedness of the domain D.

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